Stochastic local operations with classical communication of absolutely maximally entangled states

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Absolutely maximally entangled (AME) states are maximally entangled for every bipartition of the system. They are crucial resources for various quantum information protocols. We present techniques for verifying that two AME states are equivalent concerning stochastic local operations and classical communication (SLOCC). The conjecture that for a given multipartite quantum system all AME states are SLOCC equivalent is proven false. We also show that the existence of AME states with minimal support of six or more particles results in the existence of infinitely many such non-SLOCC-equivalent states. Moreover, we present AME states which are not SLOCC equivalent to the existing AME states with minimal support.

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I. INTRODUCTION

Entanglement of bipartite states is a widely discussed problem and, in fact, already very well understood [1]. However, quantification of entanglement for multipartite states remains a challenge [2]. In particular, according to different entanglement measures for multipartite states (like the tangle, the Schmidt measure, the localizable entanglement, or geometric measure of entanglement), the states with the largest entanglement do not overlap in general.

We discuss AME states which are maximally entangled for every bipartite of the system. AME states are being applied in several branches of quantum information theory: in quantum secret sharing protocols [3], in parallel open-destination teleportation [4], in holographic quantum error correcting codes [5], among many others. Different families of AME states have been introduced [6,7] and the problem of their existence is being investigated [8–10]. It has been demonstrated that the simplest class of AME states, namely, AME states with the *minimal support*, is in one-to-one correspondence with the classical error correction codes [11] and combinatorial designs known as *orthogonal arrays* (OAs) [12]. Henceforward, two-way interaction with combinatorial designs and quantum error correction codes is observed [8,13]. AME states are special cases of *k-uniform* states characterized by the property that all of their reductions to k parties are maximally mixed [14].

Since the entanglement quantification becomes an ambitious project while the number of parties in a system increases, the problem of satisfactory classification of states turns out to be essential [2]. The state space might be partitioned into equivalence classes with respect to a selected class of local operations [15]. Any two states from one class are interconvertible by an adequate local operator, while such a transformation cannot be provided for states from different classes. Nevertheless, it is not obvious which class of local operators provides the ultimate division of state space. One of the reasonable choices was the division according to local unitary (LU) operations [2]. Two states $|\psi\rangle$ and $|\phi\rangle$ belong to the same LU class if and only if there exists a local unitary operator transforming one into the other: $|\phi\rangle = U_1 \otimes \cdots \otimes U_N|\psi\rangle$.

The fact that entanglement is used for the transmission of information between parties far apart restricts us to the LU operations. Nevertheless, we may also allow classical information to be transmitted between the distant parties. This leads us to supersede the class of LU operations with the local operators and classical communications (LOCC) [16,17]. It is known that if the state $|\psi\rangle$ can be transformed into $|\phi\rangle$ by using LOCC operations only, $|\psi\rangle$ possesses at least as much entanglement as $|\phi\rangle$. In general, this transformation cannot be inverted, and hence LOCC imposes the partial order on the state space. Nevertheless, one may study whether two states are LOCC equivalent with a nonvanishing probability of success. Such operations are known as stochastic LOCC (SLOCC). Mathematically, two states $|\psi\rangle$ and $|\phi\rangle$ belong to the same SLOCC class if and only if there exists a local invertible operator transforming one into the other: $|\phi\rangle = O_1 \otimes \cdots \otimes O_N|\psi\rangle$ [18]. Thus, the state space might be partitioned into SLOCC classes.

The number of SLOCC classes rapidly increases with the number of parties N and the local dimension d of a given system. For instance, all bipartite pure states are equivalent by SLOCC [15]. In the simplest multipartite system, namely, the three qubit system, there are precisely six distinct SLOCC classes (only two of them among fully entangled states, represented by Greenberger-Horne-Zeilinger (GHZ) and W states, respectively) [18]. Those systems are the last ones with a finite number of SLOCC classes; for $N > 3$ or $d > 2$ there are infinitely many SLOCC classes [18]. Despite this fact, all four-partite qubit states were classified into nine families, some of them with infinitely many SLOCC classes but of a similar structure [19]. This result was later corrected to eight such classes, while one of the proposed families turned out to be not fully entangled [20].

For larger quantum systems a comprehensive and satisfactory description of SLOCC-class structure has not been established yet. Several studies in this area lead us in constructing invariants under LU and SLOCC transformations for three parties in [21], multipartite pure quantum systems in [22–24], and mixed multipartite states in [25]. It is known that that polynomial invariants are completely characterizing LU equivalence classes [26] and that the number of nonzero polynomials is a function of a local dimension d. SLOCC invariant polynomials were found for three qubit systems [27,28], for four qubit systems [29,30], and generally for multiqubit systems [31,32]. In the qudit case, an interesting attempt to provide polynomial invariants has been also made [33]. Despite many attempts of classification of these polynomials [34], and enhancing them with physical meaning, their structure remains inscrutable. Many efforts have also focused on LU and SLOCC equivalence of stabilizer states [35], matrix-product states and projected entangled pair states [36], Gaussian states [37,38], locally maximally entangleable states [39], or generalized Bell states [40].

The initial motivation for our paper was the question of whether different constructions of AME states are equivalent by any local transformation. It was already shown that some k-uniform states are not LU (and in fact not SLOCC) equivalent [41]. This result was based on a comparison of the ranks of reduced density matrices [42]. Nevertheless, in some specific cases, the aforementioned rank argument is never conclusive. This is the case when two k-uniform states are of minimal support, or when the bound on k is saturated, i.e., in the case of AME states. In this paper, we develop techniques of SLOCC verification between such states. We provide methods of SLOCC equivalence verification for all k-uniform states with minimal support. We show that the conjecture of all AME states being LU and SLOCC equivalent does not hold. In particular, we show that some AME states cannot be transformed into existing minimal support form by any local invertible operation. Moreover, the class of LU and SLOCC transformation is widely investigated and a systematic method for verification of LU and SLOCC equivalence of AME states and k uniformity with minimal support is provided. We expose the vital contrast between AME states of small systems (up to five parties) and larger systems. In particular, for larger systems there exist infinitely many non-LU- and non-SLOCC-equivalent AME states of minimal support differing only by phases. Additionally, we emphasize the essential difference between local transformation of k-uniform states or AME states of odd number of parties ($k < N/2$) and AME states of even number of parties ($k = N/2$). The structure of the latter is more complex and nonclassical in some sense. Despite the refined analysis of this case, obtained results are still intricate and dependent on specific cases.

The paper is organized as follows. In Sec. II, we recall construction methods of k-uniform and AME states known from the literature, and we provide several explicit examples of such states. Moreover, we discuss the relation between LU and SLOCC equivalences restricted to k-uniform and AME states. The main results of our paper are presented in the following three consecutive sections. Section III discusses local transformation of k-uniform states or AME states of odd number of parties ($k < N/2$). Similar results concerning AME states of even number of parties ($k = N/2$) are presented in Sec IV. Section V applies general results obtained in the previous sections. We present several examples of LU and SLOCC nonequivalent k-uniform states. The precise number of LU and SLOCC classes of AME states with minimal support is specified. Moreover, the nontrivial bounds on the number of such classes of general AME states are given. In Sec. VI, we discuss some classes of combinatorial designs directly related to our problem. Existence and extension of those designs turned out to be crucial to obtain the aforementioned results. This phenomenon is presented in detail. Further discussion and open problems are left for Sec. VII. Summary and conclusions are presented in Sec. VIII. Proofs of statements included in Sec. III might be found in Appendices A, B, and D, whereas claims presented in Sec. IV are justified in Appendix C.

II. NOTATION AND PRELIMINARIES

A. k-uniform states and AME states

Consider a multipartite quantum state $|\psi\rangle \in \mathcal{H}_d^{\otimes N}$ of N parties with a local dimension d each. We say that $|\psi\rangle$ is a *k-uniform* state if its reduced density matrices are maximally mixed, i.e.,

$$\rho_S(\psi) \propto \text{Id}$$

for any subsystem S of k parties ($|S| = k$). The uniformity k cannot exceed $\lfloor N/2 \rfloor$ [8]. States which saturate this bound, i.e., $\lfloor N/2 \rfloor$-uniform states, are called AME states, and are denoted by AME(N, d). Particular attention is paid to AME states of an even number of parties, which are equivalent to notions as perfect tensors [5] or multiunitary matrices [43].

The *support* of a state $|\psi\rangle$ is the number of nonzero coefficients when $|\psi\rangle$ is written in the computational basis. Note that the support of the k-uniform state is at least d^{N-k}. Indeed, the partial trace over $N - K$ particles is an identity matrix $\text{Id}_{d^{N-k}}$. k-uniform states with support equal to d^k are called *minimal support*. k-uniform states are a natural generalization of the well-established GHZ state.

Example 1. The GHZ state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

is a 1-uniform state of minimal support. Similarly, its natural generalization to N-party qudit states (each party has exactly d distinguishable energy levels)

$$\left|\text{GHZ}_d^N\right\rangle = \frac{1}{\sqrt{d}}(|0\cdots 0\rangle + \cdots + |d-1\cdots d-1\rangle)$$

is a 1-uniform state of minimal support.

It is worth mentioning that the GHZ state is maximizing entanglement properties among all three qudit states. This statement, however, is not true anymore for larger systems [44].

Example 2. The state of four qutrits

$$\begin{aligned}|\text{AME(4,3)}\rangle = \tfrac{1}{3}(&|0000\rangle + |0121\rangle + |0212\rangle \\ &+ |1110\rangle + |1201\rangle + |1022\rangle \\ &+ |2220\rangle + |2011\rangle + |2102\rangle)\end{aligned}$$

is an AME(4,3) state of minimal support [7]. It reveals larger entanglement properties than a relevant GHZ state.

AME(N, d) states are maximizing entanglement properties among all N-party states, each with d levels [3]. There is no general construction of AME(N, d) states, and, in fact, they do not exist for any numbers N and d. Indeed, it was first observed that the AME state of four qubits does not exist [9]. Nowadays, more of such negative results are known [10]. Some cases, such as AME(4,6), are believed to not exist, despite the fact that the mathematical proof is still missing [12].

We would like to finish this section with two remarkable observations. First, all known k-uniform and AME states might be written by simple closed formulas. For instance,

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i, \dots, i\rangle,$$

$$|\mathrm{AME(4,3)}\rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} |i, j, i+j, 2i+j\rangle \qquad (1)$$

are relevant to the GHZ state and AME(4,3) presented in Examples 1 and 2

Secondly, not all k-uniform states are of minimal support. It is rather easy to verify that AME states with minimal support of five or six qubits do not exist. Nevertheless, the construction of AME(6,2) was provided [6,12]. We present one example of AME states with nonminimal support relevant to the future discussion.

Example 3. Consider the following states:

$$|\mathrm{AME}(5,d)\rangle = \frac{1}{\sqrt{d^3}} \sum_{i,j,k=0}^{d-1} \omega^{(3i+j)k} |i, j, i+j, 2i+j+k, k\rangle,$$

where ω is the dth root of unity. $|\mathrm{AME}(5,d)\rangle$ states satisfy all properties required from AME states for any integer number $d \geqslant 2$ [6,12]. They cannot be written, however, in the minimal support form.

B. Orthogonal arrays

Orthogonal arrays [45] are combinatorial arrangements, tables with entries satisfying given orthogonal properties. They were created in response to optimization problems in statistical analysis. Their most famous application can be summarized in one sentence: "Your automobile lasts longer today because of orthogonal arrays" [46].

A close connection between OAs and maximally entangled states [13], error-correcting codes [45], brought a new life for these combinatorial objects. As some OAs might be in one-to-one correspondence with k-uniform states, the concept of OA is briefly presented below.

An orthogonal array OA(r, N, d, k) is a table composed by r rows and N columns with entries taken from $0, \dots, d-1$ in such a way that each subset of k columns contains all possible combinations of symbols with the same amount of repetitions (see Fig. 1). The number of such repetitions is called *the index* of the OA and denoted by λ. One may observe that the index of the OA is related to the other parameters:

$$\lambda = \frac{r}{d^k}.$$

$$\begin{array}{cccc} 0&0&0&0\\ 0&1&1&1\\ 0&2&2&2\\ 1&0&1&2\\ 1&1&0&2\\ 1&2&0&1\\ 2&0&2&1\\ 2&1&0&2\\ 2&2&1&0 \end{array} \qquad \begin{array}{cccc} 0&0&0&0\\ 0&1&1&1\\ 0&2&2&2\\ 1&0&1&2\\ 1&1&0&2\\ 1&2&0&1\\ 2&0&2&1\\ 2&1&0&2\\ 2&2&1&0 \end{array} \qquad \begin{aligned} |\mathrm{AME(4,3)}\rangle \propto\ & \omega_{00}\ |0000\rangle\\ +\ & \omega_{01}\ |0111\rangle\\ +\ & \omega_{02}\ |0222\rangle\\ +\ & \omega_{10}\ |1012\rangle\\ +\ & \omega_{11}\ |1120\rangle\\ +\ & \omega_{12}\ |1201\rangle\\ +\ & \omega_{20}\ |2021\rangle\\ +\ & \omega_{21}\ |2102\rangle\\ +\ & \omega_{22}\ |2210\rangle \end{aligned}$$

FIG. 1. The orthogonal array of unity index OA (9, 4, 3, 2) on the left and repeated in the center. Each subset consisting of two columns contains all possible combinations of symbols. Here, two such subsets are highlighted. The relevant quantum state is obtained by forming a superposition of states corresponding to consecutive rows of the array enhanced by some phases (see the expression on the right).

The OA with $\lambda = 1$ is called the OA of *index unity*. Figure 1 presents an example of an index unity OA.

A pure quantum state consisting of r terms might be associated with OA(r, N, d, k), simply by reading all rows of the OA [13,45]. With a little more effort, one may adjust phases $\omega_1, \dots, \omega_r$ in front of any term (see Example 4). Intriguingly, this relevance provides a one-to-one correspondence between k-uniform states of minimal support and OAs of index unity.

Proposition 1. There is one-to-one correspondence between k-uniform states with the minimal support of N qudits and OA OA(d^k, N, d, k) enhanced with the phase vectors

$$(\omega_1, \dots, \omega_{d^k}),$$

where $|\omega_i| = 1$.

From OA(d^k, N, d, k) the k-uniform state is created by reading all terms and adjusting them with the relevant phases ω_i. Conversely, from the AME state the OA of index unity might be built, simply by erasing phases and adjusting all terms one above the other.

Example 4. For any phases $|\omega_{i,j}| = 1$ the following state is 2-uniform:

$$|\mathrm{AME}(4,3)_\omega\rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} \omega_{i,j} |i, j, i+j, 2i+j\rangle.$$

It is defined uniquely up to global phase.

It is known that for any number N and k there exists OA(d^k, N, d, k) for a local dimension d being sufficiently large (in fact, such construction is given for d being a prime power satisfying $d > k$ and $N - 1$) [47]. Hence, for any number of parties N the k-uniform state with minimal support might be created where the component systems have a sufficiently large number of levels. The problem of existence and classification of OAs (in particular OAs of index unity) has been extensively studied [48,49]. We refer to the web page of Sloane for tables of OAs [50].

Example 5. The following states

$$|\mathrm{AME}(5,d)'\rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} |i, j, i+j, 2i+j, 3i+j\rangle$$

are AME states with minimal support for all $d \geqslant 5$ being prime numbers [11].

In fact, with a little more effort such states might be constructed for all prime powers $d \geqslant 4$ [11]. For instance, the state

$$|\mathrm{AME}(4,4)\rangle = \frac{1}{4}\sum_{i,j=0}^{d-1} \left|i, j, M^1_{i,j}, M^2_{i,j}\right\rangle$$

where

$$M^1 := \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}, \quad M^2 := \begin{pmatrix} 0 & 2 & 3 & 1 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 3 & 1 & 0 & 2 \end{pmatrix}$$

is an AME(4,4) state of minimal support (rows and columns of M^1 and M^2 are indexed by $i, j = 0, \ldots, 3$) [11]. Construction of matrices M^1 and M^2 comes from the multiplication structure in the Galois field GF(4), which might be seen as a multiplication of irreducible polynomials of degree 2 [11]. In fact, matrices M^1 and M^2 form a mutually orthogonal Latin square MOLS(4) (see Definition 3 for details).

As we already mentioned in Example 3, not all k-uniform states are of minimal support, which simply means that not all AME states are obtained from OAs of index unity.

C. Composed systems

For any two k-uniform states $|\psi_1\rangle$ and $|\psi_2\rangle$, one may consider the composed system $|\psi_1\rangle \otimes |\psi_2\rangle$, which inherits the property of being k uniform. For instance, the state

$$|\mathrm{AME}(4,9)_{3\times 3}\rangle = \frac{1}{9}\sum_{\substack{i,j=0\\ k,\ell=0}}^{2} |(i,k),(j,\ell),(i+j,k+\ell),(i+2j,k+2\ell)\rangle \tag{2}$$

is a composition of two $|\mathrm{AME}(4,3)\rangle$ states from Eq. (1). Each pair (i,k) is identified with a number $0,\ldots,8$ written in the ternary numeral system, i.e., $(i,k) \cong 3i + j$.

D. Local transformations

Two N qudit states $|\psi\rangle$ and $|\phi\rangle$ are LU equivalent if one can be transformed into another by local unitary operators, i.e.,

$$|\phi\rangle = U_1 \otimes \cdots \otimes U_N |\psi\rangle.$$

The LU equivalence of $|\psi\rangle$ is referred to in the text as an *automorphism*.

Mathematically, two states $|\psi\rangle$ and $|\phi\rangle$ are SLOCC equivalent if and only if there exists a *local invertible* operator connecting those states [18]:

$$|\phi\rangle = O_1 \otimes \cdots \otimes O_N |\psi\rangle.$$

Since LU and SLOCC equivalences are equivalence relations, the state space might be naturally partitioned into *LU classes* and *SLOCC classes*, respectively.

E. The structure of SLOCC classes

We present a brief outline of some algebraic invariant methods interconnected to the SLOCC partition problem of multipartite entangled states. We introduce the notion of critical states and we discuss consequences of the Kempf-Ness theorem [51] for multipartite systems. We refer to [52] for more details.

The state ρ is called a *critical state* if all its reduced density matrices ρ_i are proportional to the identity. In particular, the class of critical states contains stabilizer states, cluster states, and all k-uniform states, among many others [52].

Notice that the critical states were initially defined differently, via an action of the Lie group associated with the state space. By applying the Kempf-Ness theorem [51] it was later observed that, indeed, states are critical if and only if they are maximally entangled [53].

The Kempf-Ness theorem has one more significant consequence for multipartite quantum states. It follows that within one SLOCC class the critical states are unique up to LU equivalences. Therefore, such classes possess the canonical representative.

Notice that not all SLOCC classes contain a critical state, and hence there is no one-to-one correspondence between SLOCC classes and maximally mixed states. More precisely, each SLOCC class is topologically closed (equivalently closed with respect to Zariski topology) if and only if it contains a critical state [54]. In fact, closed SLOCC classes are dense in a state space [55].

We conclude this discussion with the following corollary.

Corollary 1. Two critical states are in the same SLOCC class if and only if they are LU equivalent. Notice that all k-uniform states are critical states.

Therefore, verification of LU equivalence between two k-uniform states is equivalent to verification of SLOCC equivalence between them.

III. LOCAL EQUIVALENCES, CASE $2k < N$

We introduce another class of local unitary operations, essential for the classification problem of k-uniform states.

Definition 1. A unitary matrix M is called a *unitary monomial matrix* if one of the following holds.

(1) M has exactly one nonzero entry in each row and each column.

(2) M is a product of a permutation and diagonal matrix.

(3) M does not change the support of any quantum state.

For multiparty systems, the local monomial operation will be denoted as *LM* equivalency.

One can see that all conditions 1–3 are, indeed, equivalent. Obviously, each local monomial operation provides the LU equivalence between two states of minimal support. Indeed, since it is a local unitary operation, it does not change the entanglement properties of a state; moreover, it preserves the number of elements in the support of a state. As we shall see, the reverse statement is also true for k-uniform states where $2k < N$. In other words, searching for LU equivalence between two such k-uniform states of minimal support might be restricted to the search within the LM class.

Proposition 2. For $2k < N$, each LU or SLOCC equivalency between two k-uniform states of minimal support is in fact LM equivalency.

Corollary 2. For $2k < N$, two k-uniform states of minimal support are LU or SLOCC equivalent if and only if they are LM equivalent.

We shall prove the above statement in a slightly enhanced version in Appendix A.

With the strengthened version of Proposition 2 at hand (see Appendix A), we have shown that not all AME states are equivalent, which was an initial motivation for our research.

Proposition 3. Two families of AME(5,d) states, $|\text{AME}(5,d)\rangle$ and $|\text{AME}(5,d)\text{'}\rangle$ presented in Examples 3 and 5, respectively, are not LU equivalent for any prime local dimension d.

Therefore, as an immediate conclusion from Corollary 1, states $|\text{AME}(5,d)\rangle$ and $|\text{AME}(5,d)\text{'}\rangle$ belong to different SLOCC classes. We refer to Appendix D for the proof of the above statement.

Both states $|\text{AME}(5,d)\rangle$ and $|\text{AME}(5,d)\text{'}\rangle$ belong to special classes of quantum states: *stabilizer* states and *graph* states. A comprehensive introduction to this topic might be found in [56]. The class of natural local operation among stabilizer and graph states is called *local Clifford* (LC) operations. The verification of LC equivalence between two stabilizer states is rather a simple problem and might be the polynomial-time algorithm [57]. Nevertheless, LU-or SLOCC-equivalence verification of such states is generally a challenging problem. Surprisingly, it was shown that there are LU equivalences of graph and stabilizer states which are beyond LC class [58]. One can see our results as proof for lack of LU equivalence of two graph and stabilizer states.

Verification of LU equivalence

The problem of verification whether two different states are LU and SLOCC equivalent is of the most importance from the application point of view [59]. Monomial matrices are products of permutations and diagonal matrices. Although permutation matrices are easy to quantify, diagonal matrices are indexed by a real coefficient. The following statement shows that we can overcome this apparent difficulty. In fact, verification of LU equivalence between two states is restricted to testing a finite number of possible equivalences.

We introduce the following notation. Consider a k-uniform state with minimal support:

$$|\psi\rangle = \sum_{I\in\mathcal{I}} \omega_I |I\rangle,$$

where $I \in [d]^n$ are multi-indices running over the set $\mathcal{I}$ of the size $|\mathcal{I}| = d^k$. Denote by $\mathcal{I}_a^i$ all those indices with a on the ith position:

$$W_a^i := \prod_{I\in\mathcal{I}_a^i} \omega_I .$$

Similarly, $\mathcal{I}_{a_1,\dots a_\ell}^{i_1,\dots i_\ell}$ denotes the set of indices with a_i on the ith position:

$$W_{a_1,\dots a_\ell}^{i_1,\dots i_\ell} := \prod_{I\in\mathcal{I}_{a_1,\dots,a_\ell}^{i_1,\dots,i_\ell}} \omega_I .$$

Observe that any local permutation $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_n$ acts on $W_{a_1,\dots a_\ell}^{i_1,\dots i_\ell}$ by permuting relevant elements:

$$\sigma\left(W_{a_1,\dots a_\ell}^{i_1,\dots i_\ell}\right) := \prod_{I\in\mathcal{I}_{\sigma_{i_1}(a_1),\dots,\sigma_{i_\ell}(a_\ell)}^{i_1,\dots,i_\ell}} \omega_I .$$

We specify the diagonal matrices which might appear in local equivalences described in Proposition 2.

Proposition 4. Consider two k-uniform states of minimal support $|\psi\rangle$ and $|\psi'\rangle$. The eventual LU equivalence between them is of the following form:

$$\omega(\sigma_1 D_1 \otimes \cdots \otimes \sigma_n D_n)$$

where ω is a global phase, σ is a local permutation $\sigma = \sigma_1, \dots, \sigma_n$, and D_i are the following diagonal matrices:

$$D_i = \text{diag}\left(\sqrt[d]{\frac{\left(W_{0,I}^{i,S}\right)'}{\sigma\left(W_{0,I}^{i,S}\right)}}, \dots, \sqrt[d]{\frac{\left(W_{d-1,I}^{i,S}\right)'}{\sigma\left(W_{d-1,I}^{i,S}\right)}}\right),$$

with entries given by any dth root of a relevant complex number, where S is any subset of $k-2$ indices which do not contain i, and I is any multi-index $I = i_2, \dots, i_{k-1}$. In particular, for $k = 2$ it is the empty set $S \equiv \emptyset$.

Moreover, for $k > 2$ there is the following necessary condition for existence of such LU equivalence. For any $S \subset [n] \setminus \{i\}$, such that $|S| = k-2$ and any symbol ℓ

$$\frac{\left(W_{\ell,I}^{i,S}\right)'}{\sigma\left(W_{\ell,I}^{i,S}\right)} = \frac{\left(W_{\ell,I'}^{i,S}\right)'}{\sigma\left(W_{\ell,I'}^{i,S}\right)}$$

for arbitrary multi-indices $I{=}i_2, \dots, i_{k-1}$ and $I'{=}i_2', \dots, i_{k-1}'$.

The proof of Proposition 4 is given in Appendix B. We illustrate the usefulness of this criterion in two examples: 2- and 3-uniform states (see Sec. V).

IV. LOCAL EQUIVALENCES, CASE $2k = N$

For AME($2k, d$) states, the statement of Proposition 2 does not hold anymore. The following example illustrates this difference.

Example 6. The Fourier transform F_3,

$$(F_3)^{\otimes 4} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix}^{\otimes 4}, \qquad (3)$$

provides an automorphism of the AME(4,3) state from Eq. (1).

As we shall see, Fourier matrices are not the only non-monomial matrices providing the LU equivalence between AME($2k, d$) states of minimal support. On the other hand, it is not generally true that Fourier matrices preserve all AME($2k, d$) states with minimal support. Despite the exhaustive analysis performed, the general structure of LU equivalences between AME($2k, d$) states is still puzzling and remains unknown. Nevertheless, for sufficiently small values of k and d, the complexity of the problem reduces significantly, and the analog of Proposition 2 might be stated. We begin with the definition of matrices beyond monomial class, which can provide LU equivalence between AME states.

Definition 2. Let d and q be positive integers. A *Butson-type* complex Hadamard matrix of order d and complexity q is a unitary matrix in which each entry is a complex qth root of unity scaled by the factor $1/d$. The set of Butson-type matrices is denoted by $\mathrm{BH}(d, q)$.

In literature [60,61], Butson-type matrices are defined without scaling factor $1/d$, therefore they are proportional to the unitary matrices.

For our purpose, it is enough to focus on matrices of the type $\mathrm{BH}(d, d)$. As we shall see, those matrices (up to monomial matrices) define all possible LU equivalences of AME states. The problem of existence and classification of such matrices is discussed later, in Sec. IV B.

Proposition 5. Consider two AME$(2k, d)$ states of minimal support, where k and d are sufficiently small (see Remark 1). Each LU equivalency between them is of one of the following forms: (1) the tensor product of Butson-type matrices $B_i \in \mathrm{BH}(d, d)$ multiplied by LM matrices from each side or (2) the LM matrices themselves.

Similarly to the case $2k < n$, we can specify the class of diagonal matrices which might appear in LM equivalences from Proposition 5.

Proposition 6. LU equivalence between two AME(2k,d) states of minimal support, where k and d are sufficiently small (see Remark 1), is of the following form: either

$$\omega\big(\overrightarrow{D}_1^{-1} B_1 \overleftarrow{D}_1^{-1} \otimes \cdots \otimes \overrightarrow{D}_n^{-1} B_n \overleftarrow{D}_n^{-1}\big) \tag{4}$$

or

$$\omega(\sigma_1 D_1 \otimes \cdots \otimes \sigma_n D_n) \tag{5}$$

where ω is a global phase; $B_i \in \mathrm{BH}(d, d)$ are Butson-type matrices; σ_i are permutation matrices; and D, $\overrightarrow{D}_i$, and $\overleftarrow{D}_i$ are diagonal matrices.

The entries of diagonal matrices are the dth root of specified complex numbers:

$$D_i = \mathrm{diag}\left(\sqrt[d]{\frac{\big(W_{0,I}^{i,S}\big)'}{\sigma_i\big(W_{0,I}^{i,S}\big)}}, \ldots, \sqrt[d]{\frac{\big(W_{d-1,I}^{i,S}\big)'}{\sigma_i\big(W_{d-1,I}^{i,S}\big)}}\right),$$

$$\overrightarrow{D}_i = \mathrm{diag}\Big(\sqrt[d]{\big(W_{0,I}^{i,S}\big)'}, \ldots, \sqrt[d]{\big(W_{d-1,I}^{i,S}\big)'}\Big),$$

$$\overleftarrow{D}_i = \mathrm{diag}\Big(\sqrt[d]{W_{0,I}^{i,S}}, \ldots, \sqrt[d]{W_{d-1,I}^{i,S}}\Big),$$

where S is any subset of $k-2$ indices which do not contain i, I is any multi-index $I = i_2, \ldots, i_{k-1}$, and σ is a local permutation $\sigma = \sigma_1, \ldots, \sigma_n$ of levels. In particular, for $k = 2$ it is the empty set $S \equiv \emptyset$.

Moreover, for $k > 2$ there are the following necessary conditions for existence of such equivalence. Consider any $S \subset [n] \setminus \{i\}$, such that $|S| = k-2$, and any symbols ℓ, ℓ'. For arbitrary multi-indices $I = i_2, \ldots, i_{k-1}$ and $I' = i'_2, \ldots, i'_{k-1}$,

$$\frac{\big(W_{\ell,I}^{i,S}\big)'}{\sigma\big(W_{\ell,I}^{i,S}\big)} = \frac{\big(W_{\ell,I'}^{i,S}\big)'}{\sigma\big(W_{\ell,I'}^{i,S}\big)}$$

if the equivalence is of the form Eq. (4), and

$$\frac{W_{\ell,I}^{i,S}}{W_{\ell',I}^{i,S}} = \frac{W_{\ell,I'}^{i,S}}{W_{\ell',I'}^{i,S}},$$

$$\frac{\big(W_{\ell,I}^{i,S}\big)'}{\big(W_{\ell',I}^{i,S}\big)'} = \frac{\big(W_{\ell,I'}^{i,S}\big)'}{\big(W_{\ell',I'}^{i,S}\big)'}$$

if the equivalence is of the form Eq. (5).

TABLE I. Examples of $d_{\min}$ and $d_{\max}$ of AME states.

AME$(2k, d)$	$d_{\min}$	$d_{\max}$
AME(2,d)	3	3
AME(4,d)	9	9
AME(6,d)	11	16
AME(8,d)	13	25

The statement above is important from the application point of view. The enormous class of diagonal matrices is significantly restricted. In fact, the classes of matrices from Proposition 5 which may provide the LU equivalence between two states is brought to be finite. Therefore, the LU-equivalence verification problem is discretized and made finite. Moreover, the second part of Proposition 6 imposes some necessary conditions for two AME$(2k, d)$ states to be LU equivalent for $k > 2$. As we shall see, such assumptions might be easily validated, which implies the existence of non-LU- and non-SLOCC-equivalent AME$(2k, d)$ states for all $k > 2$ (see Sec. V C). For the proof of Propositions 5 and 6, we refer to Appendix C.

Remark 1. There is the following restriction on numbers d and k imposed in the statement of Propositions 5 and 6:

$$(k+1)(1 + \sqrt[k-1]{k}) \leqslant d$$

for $k > 1$ and $2 < d$ for $k = 1$. This bound is related to the necessary condition for existence and extension of combinatorial designs called *mutually orthogonal hypercubes*. We discuss them in detail in Sec. VI.

In particular, for $k = 2, 3, 4, 5, 6$ the smallest value of $d_{\min}$ which does not satisfy the bound above is presented in a Table I. The given bound is not tight. Moreover, we present $d_{\max}$ as the maximal value of a local dimension d for which Proposition 5 does not hold (we found a counterexample). In particular, for the local dimension $d_{\max}$ we found LU equivalence which is not of the form presented in Proposition 5. The origin of $d_{\max}$ is presented in Sec. IV A.

We conjecture that those values behave asymptotically as $(k-1)^2$.

In fact, assumptions on values k and d are not restrictive from the application point of view. Indeed states outside the described class are far beyond current laboratory possibilities [62,63].

A. Composed systems

Consider the AME(4,9) state being a product of two AME(4,3) states as it was described in Eq. (2). Since the Fourier transform $(F_3)^{\otimes 4}$ and the identity $\mathrm{Id}_3^{\otimes 4}$ are automorphisms of AME(4,3), obviously

$$(F_3 \otimes \mathrm{Id})^{\otimes 4} \tag{6}$$

provides an automorphism of the aforementioned AME(9,3) state. One immediately observes that Eq. (6) is not of the form postulated in Proposition 5. Indeed, according to Proposition 5 matrices providing LU equivalence between two AME states of minimal support have either 1 or d nonzero entries in each row and column, each entry of the same norm. Matrices from Eq. (6), however, have exactly three nonzero entries in each row and column. Nevertheless, k and d were assumed to be sufficiently small in Proposition 5; indeed, according to Remark 1, the statement was restricted to $d < 9$ for AME(4,d) states.

Notice that similar automorphisms might be potentially given for any product states. We conjecture that LU equivalences of such a product form are the only ones violating the statement of Proposition 5.

Conjecture 1. If the LU equivalence $U_1 \otimes \cdots \otimes U_{2k}$ of two AME states of minimal support is not of the form proposed in Proposition 5 for $k > 1$, those states are product states and U_i splits according to the composition of states into matrices postulated in Proposition 5.

The conjecture excludes AME(2,d) where the structure of LU equivalences is more abundant (see Sec. V A).

Even though Conjecture 1 seems reliable, the mathematical proof of it is out of reach at this stage of the research. In the most general case (without any assumptions on k and d), we showed that the following might be noted about LU equivalence between AME states.

Proposition 7. Consider two AME states $|\psi\rangle$ and $|\psi'\rangle$ of minimal support which are locally equivalent by $U := U_1 \otimes \cdots \otimes U_n$.

(1) Each row or column of each matrix U_i has the same number s of nonzero elements, all having the same norm $\sqrt{s}$.

(2) Consider a mutually orthogonal Latin hypercube (MOLH) and the number s that satisfies the extension property, there exists a MOLH(s) which can be further extended onto MOLH(d).

(3) Under the assumption that all phases are trivial, i.e., $\omega_I \equiv \omega'_I \equiv 1$ for all multi-indices I, all nonzero entries of matrices U_i are sth roots of unity (scaled by $\sqrt{s}$) up to global multiplication by a complex number.

For the notion of MOLH we refer to Sec. VI.

For the proof, we refer to Appendix C.

B. Classification of Butson-type matrices

Equation (4) provides a unitary equivalence of AME states with minimal support which is not a monomial matrix. We have shown that, under some restrictions on k and d (see Remark 1), such equivalences are local Butson matrices $B_i \in \mathrm{BH}(d, d)$ up to local monomial transformations. It is not difficult to show that Fourier-transform and tensor products of such are elements of the class BH(d, d). Nevertheless the class of Butson matrices is much larger and contains 1, 2, 1, 4, 1, 143, 23, 51, 1, and 449 773 3 matrices (classified up to monomial matrices) for $d = 3, \ldots, 12$ [64]. Tables of Butson matrices are available in [65,66].

Even though the class of Butson matrices BH(d, d) grows rapidly with d, it seems that the subclass of such matrices that might be involved in LU equivalences of AME states is significantly smaller. Therefore, more specific classification of matrices providing eventual LU equivalences of AME states is needed. We suspect that all such matrices are Fourier-transform and tensor products of such.

On the other hand, not all Fourier matrices might be involved in LU equivalences of AME states. For $d > 3$, we could not construct LU equivalence of AME(5,d) states from Example 5 based on Fourier matrices F_5. It is worth mentioning that the Fourier transforms $F_5, F_7,$ and F_{11} are the only matrices of type BH(p, p) for $p = 5, 7, 11$. Hence, most probably all automorphisms of AME(p, p) states from Example 5 for $p = 5, 7, 11$ are within the LM class. Nevertheless, for $d = 4$ such an equivalence might be provided by tensor product $F_2 \otimes F_2$, which is illustrated below.

Example 7. The action of

$$(F_2 \otimes F_2)^{\otimes 4}$$

on the state $|\mathrm{AME}(4{,}4)\rangle$ from Example 5 is equivalent to the following local permutation of indices:

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \otimes \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{pmatrix} \otimes \mathrm{Id} \otimes \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

and hence provides the LU equivalence between AME states of minimal support.

We shall finish this section by linking the problem of decreasing the class of Butson matrices involved in LU equivalences of AME states with two mathematical problems.

First, it has been conjectured that for prime dimensions d the Fourier matrices F_d are the only matrices in BH(d, d) [64].

Second, for some numbers n_1 and n_2 the tensor product of two Fourier matrices F_{n_1} and F_{n_2} is isomorphic to $F_{n_1 n_2}$. For instance, $F_2 \otimes F_3 \cong F_6$, while $F_2 \otimes F_2 \not\cong F_4$. The problem of determining those numbers has been solved [67].

V. EXISTENCE AND UNIQUENESS OF k-UNIFORM STATES

We apply so-far obtained results for various classes of AME states here. We present 1-, 2-, and 3-uniform state classes separately since the analysis of their LU and SLOCC equivalences differs greatly.

A. 1-uniform states

All 1-uniform states of minimal support are of the following form:

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \omega_i \big| j_i^1, \ldots, j_i^N \big\rangle,$$

where j_i^ℓ runs over all levels $0, \ldots, d-1$ for all indices ℓ. One can observe that they are equivalent to the generalized

Bell state $|\mathrm{GHZ}_d^N\rangle$ (see Example 1) and hence pairwise LU equivalent. Indeed, the local transformation

$$U_1(|j_i^1\rangle) = (\omega_i^{-1}|j_i^1\rangle), \quad U_\ell(|j_i^\ell\rangle) = (|j_i^1\rangle)$$

for systems $\ell = 2, \dots, N$ provides aforementioned LU equivalence.

Observation 1. All 1-uniform states of minimal support are LU equivalent.

This straightforward observation suggests that the structure of one-uniform states is rather simple and, in fact, not interesting. However, we point out that an intriguing property is the automorphism group of AME(2,d) states. One can see that the Fourier transform F_2 preserves the Bell state:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes 2} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Similarly, the Fourier transform $F_n \otimes \overline{F_n}$ preserves the generalized Bell state of two parties:

$$|\mathrm{AME}(2,d)\rangle := \frac{1}{\sqrt{d}}[|00\rangle + \dots + |(d-1)(d-1)\rangle]. \quad (7)$$

Interestingly, the tensor product $U \otimes \overline{U}$ of unitary matrices preserves the generalized Bell state AME(2,d). Indeed, for each i,

$$U \otimes \overline{U}|ii\rangle = \sum_{j=0}^{d-1} |u_{ij}|^2 |jj\rangle + \text{others},$$

and hence

$$\begin{aligned} U \otimes \overline{U}|\mathrm{AME}(2,d)\rangle &= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1}\sum_{j=0}^{d-1} |u_{ij}|^2 |jj\rangle + \text{others} \\ &= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \left(\sum_{i=0}^{d-1} |u_{ij}|^2\right) |jj\rangle + \text{others} \\ &= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\rangle + \text{others}. \end{aligned}$$

Since the state was normalized, all other terms on the right-hand side disappear.

This is in contrast to AME($2k, d$) states for $k > 1$, where LU equivalences were provided only by appropriate Butson type matrices $B(d, d)$ for all d sufficiently small.

B. 2-uniform states

Consider 2-uniform states with minimal support

$$|\phi_\alpha\rangle := \frac{1}{d}\left(\alpha|0, \dots, 0\rangle + \sum_{i,j \neq (0,0)} |i, j\rangle \otimes |\phi_{i,j}\rangle\right)$$

indexed by a complex numbers α, $|\alpha| = 1$. Each of such a state is LU equivalent to $|\phi_{\alpha=0}\rangle$ by the following:

$$\begin{aligned} U_1 &= \mathrm{diag}[(\overline{\omega_\alpha})^{n-1}, 1, \dots, 1], \\ U_i &= \mathrm{diag}[(\overline{\omega_\alpha})^{d-1}, \omega_\alpha, \dots, \omega_\alpha], \end{aligned}$$

for $i = 2, \dots, n$, where $\omega_\alpha = \sqrt[d(n-1)]{\alpha}$ is an arbitrary root.

Since all states from the family $|\phi_\alpha\rangle$ are LU equivalent with $|\phi_{\alpha=0}\rangle$, they are also pairwise equivalent. If the exceptional phase stands by a different term, the similar transformation of such a state onto $|\phi_{\alpha=0}\rangle$ might be given. Therefore, all 2-uniform states

$$|\phi_\omega\rangle = \frac{1}{d}\sum_{i,j} \omega_{i,j}|i, j\rangle \otimes |\phi_{i,j}\rangle$$

are LU equivalent to $|\phi_{\alpha=0}\rangle$, and hence pairwise equivalent. Indeed, the LU equivalence is a composition of the aforementioned transformations. The matrices $U_1, \dots, U_N$ are the simplest matrices satisfying restrictions given in Proposition 4, which explains how they were found. Observe that the assumption $2k < N$ was irrelevant in the presented analysis. Therefore, we conclude this discussion in the following corollary.

Corollary 3. Two 2-uniform states of minimal support which differs only with phases, i.e.,

$$|\psi\rangle = \sum_{I \in \mathcal{I}} \omega_I |I\rangle,$$

$$|\psi'\rangle = \sum_{I \in \mathcal{I}} \omega'_I |I\rangle,$$

where the sum runs over multi-index set $\mathcal{I} \subset [d]^N$ of size $|\mathcal{I}| = d^k$, are always LU equivalent (and hence belong to the same SLOCC class).

Example 8. All AME(4,3) states $|\mathrm{AME}(4,3)_\omega\rangle$ from Example 4 are LU equivalent, and hence they belong to the same SLOCC class. Similarly, the states

$$|\mathrm{AME}(5,d)_\omega\rangle = \frac{1}{d}\left(\sum_{i,j=0}^{d-1} \omega_{i,j}|i, j, i+j, 2i+j, 3i+j\rangle\right)$$

for $|\omega_{i,j}| = 1$ are LU and SLOCC equivalent.

From Corollary 3, it becomes clear that the diversity of possible phases in front of each term in the 2-uniform state with minimal support does not reflect the number of SLOCC classes. In fact, each 2-uniform state with minimal support is equivalent to the one with all phases equal to 1.

Therefore, enumeration of SLOCC classes might be restricted to phase 1 states only. In fact, it is equivalent to the classification of the relevant OA.

Corollary 4. Classification of two-uniform states with minimal support for $N > 2k$ is equivalent to the classification of the relevant OA, i.e., OA(d^k, N, d, k) up to permutation of indices on each position. Potentially, for $N = 2k$ two AME(4,d) states of minimal support might be in the same SLOCC class, even though the corresponding OA is not equivalent.

In literature, the classification of OAs is considered up to permutations of rows and columns [48,49]. Permutation of columns resembles, however, the physical operation of exchanging subsystems. Therefore, by dividing the state space into SLOCC classes one should always indicate whether such operations are considered under the division [19,20].

In particular, by the classification of OA, there exists at most one OA(d^k, N, d, k) for $d = 2, \dots, 17$ for any number N [50]. Hence, 2-uniform states with minimal support and the local dimension $d = 2, \dots, 17$ are always SLOCC equivalent or SLOCC equivalent after permutation of parties. Nevertheless, we checked that for $N = 4, 5$ and $d = 4, 5, 6, 7$ permutation of parties is not necessary for being SLOCC equivalent. We suppose this is true in general.

Conjecture 2. All 2-uniform states of minimal support are LU equivalent, and hence represent the same SLOCC class.

In all verified cases there exists only one 2-uniform state of minimal support. Nevertheless, some two-uniform states are not equivalent to the above-mentioned one. In particular, for $d = 5, 7, 11, 13, \ldots$ there are AME(5,d) states of nonminimal support belonging to different SLOCC classes (see Proposition 3).

C. 3-uniform states

We have shown that the number of LU and SLOCC classes for 2-uniform states of minimal support coincides with the number of relevant OAs which are nonisomorphic. In particular, two states which differ only with phases are always LU and SLOCC equivalent. This is in a strong contrast to the 3-uniform states.

Example 9. There exists a AME(6,4) state with minimal support [11]. In fact, this state might be obtained by reading consecutive rows of OA(64,6,4,3) from the OA table [50]. Obviously, enhancing successive terms with any phase factor $|\omega| = 1$ also yields the AME(6,4) state:

$$|\mathrm{AME}(6,d)_\omega\rangle = \frac{1}{d\sqrt{d}}\left(\sum_{i,j,k=0}^{d-1} \omega_{i,j,k}|i,j,k\rangle \otimes |\psi_{i,j,k}\rangle\right).$$

We focus our attention on states with all phases $\omega_{i,j,k} = 1$ with one exception: $\omega_{0,0,0} = \alpha$. Denote them as $|\psi_\alpha\rangle$ for simplicity. According to Proposition 6, the necessary condition for equivalence of such states $|\psi_{\alpha_1}\rangle$ and $|\psi_{\alpha_2}\rangle$ is

$$\frac{\left(W_{00}^{1,2}\right)'}{W_{\sigma_1(0),\sigma_2(0)}^{1,2}} = \frac{\left(W_{0,1}^{1,2}\right)'}{W_{\sigma_1(0),\sigma_2(1)}^{1,2}}$$

for any permutations σ_1 and σ_2. According to the form of permutations, we have the following.

(1) If $(\sigma_1(0), \sigma_2(0)) = (0, 0)$, then $\alpha_1 = \alpha_2$.

(2) If $(\sigma_1(0), \sigma_2(1)) = (0, 0)$, then $\alpha_1 = \overline{\alpha_2}$.

(3) Otherwise $\alpha_1, \alpha_2 = 1$.

Therefore, if none of those conditions is satisfied, states $|\psi_{\alpha_1}\rangle$ and $|\psi_{\alpha_2}\rangle$ cannot be LU equivalent. By simple analysis, all states $|\psi_{e^{i\phi}}\rangle$ are pairwise non-LU-equivalent for $\phi \in [0, \pi)$.

Observe that, in such a way, we obtained a continuous family of non-LU-equivalent AME(6,4) states with minimal support. We conclude this observation in the corollary below. In fact, if the necessary conditions from Proposition 6 are satisfied, the LU equivalence may be provided (similarly to the case of 2-uniform states).

Corollary 5. The AME(6,4) states

$$|\mathrm{AME}(6,d)_{e^{i\phi}}\rangle := \frac{1}{d\sqrt{d}}\left(e^{i\phi}|000000\rangle + \sum_{i,j,k\neq(0,0,0)} |i,j,k\rangle \otimes |\psi_{i,j,k}\rangle\right)$$

are pairwise in different LU and SLOCC classes for all phases $\phi \in [0, \pi)$.

Notice that for any k-uniform state with minimal support where $k > 2$ a similar construction of a continuous non-LU-equivalent family might be provided.

Corollary 6. If there exists a k-uniform state with minimal support $|\psi\rangle$ where $k > 2$,

$$|\psi\rangle = \frac{1}{\sqrt{d^k}} \sum_{i_1,\ldots,i_k=0}^{d-1} |i_1,\ldots,i_k\rangle \otimes |\psi_{i_1,\ldots,i_k}\rangle,$$

then the family of k-uniform states

$$|\psi_{e^{i\phi}}\rangle := \frac{1}{\sqrt{d^k}}\left(e^{i\phi}|0,\ldots,0\rangle \otimes |\psi_{0,\ldots,0}\rangle + \sum_{(i_1,\ldots,i_k)\neq(0,\ldots,0)} |i_1,\ldots,i_k\rangle \otimes |\psi_{i_1,\ldots,i_k}\rangle\right)$$

is pairwise non-LU- and non-SLOCC-equivalent for all phases $\phi \in [0, \pi)$.

D. Number of non-SLOCC-equivalent AME states

We summarize shortly the number of non-SLOCC-equivalent AME states and AME states with minimal support in Tables III and II, respectively.

The existence of AME states with minimal support for $N, d < 8$ was analyzed [68] based on the table of OAs and similar combinatorial designs. According to the discussion presented in the previous sections, if $N \geqslant 6$, existence of the AME(n, d) state with minimal support indicates the existence of infinitely many non-SLOCC-equivalent such states (see Corollary 6).

Verification of the existence of AME states (not necessarily with minimal support) is a far more complex problem. We refer to the tables of AME states [69], which summarize several results concerning this problem [9,10,68,70,71]. Even though the exact classification of AME states up to SLOCC equivalence is yet unobtainable, in some specific cases the nontrivial lower bound is given.

TABLE II. The exact number of not SLOCC-equivalent AME states with minimal support presented on a differently shaded blue background.

local dimension	2 qubits	3 qutrits	4	5	6	7
AME(3,d)	1	1	1	1	1	1
AME(4,d)	0	1	1	1	0	1
AME(5,d)	0	0	1	1	0	1
AME(6,d)	0	0	∞	∞	0	∞
AME(7,d)	0	0	0	0	0	∞

TABLE III. The minimal number of non-SLOCC-equivalent AME states. The question mark by zero value suggests that the existence of the relevant state is dubitative, while 0 itself emphasizes that the relevant state certainly does not exist.

local dimension	2 qubits	3 qutrits	4	5	6	7
AME(3,d)	1	1	1	1	1	1
AME(4,d)	0	1	1	1	0?	1
AME(5,d)	1	1	1	2	1	2
AME(6,d)	1	1	∞	∞	1	∞
AME(7,d)	0	1	1	1	0?	∞

VI. COMBINATORIAL DESIGNS

We discuss a class of combinatorial designs, known as mutually orthogonal Latin hypercubes. As we shall see, the existence and extension of those designs turned out to be crucial in the classification of AME states. In particular, we show the origin of restrictions on values of d and k imposed in Propositions 5 and 6. Exceeding those bounds potentially yields the existence of composed AME states. Automorphisms of such states have product form and are not described by Propositions 5 and 6. In that sense, we show that obtained bounds on d and k are tight.

In general, classical combinatorial designs (as orthogonal arrays: mutually orthogonal Latin squares, cubes, and hypercubes) are related to AME and k-uniform states of minimal support. Quantized versions of such combinatorial designs are related to arbitrary AME and k-uniform states. Our paper is restricted to minimal support states, hence the presentation of quantum combinatorial designs is not needed here. It is not our intention to provide a full picture of interactions between AME states and different combinatorial designs. For that purpose, we refer to Goyeneche and Życzkowski [13], where the comprehensive introduction to that topic is presented.

Consider a discrete hypercube $[d]^k$ of dimension k. One can relate to $[d]^k$ the lower dimensional hypercube $[d]^s$ in two natural ways. First, by choosing $k-s$ indices $S=\{s_1,\dots,s_{k-s}\}\subset[k]$ and their values $i_1,\dots,i_{k-s}\in[d]$, there is an injective map:

$$[d]^s \cong [d]^k_{s_1=i_1,\dots,s_{k-s}=i_{k-s}} \overset{i}{\hookrightarrow} [d]^d.$$

Second, for any subset $S'\subset[k]$ of indices where $|S'|=s$, one can simply forget about indices out of S'. This operation is relevant to the surjection

$$[d]^k \xrightarrow{\text{sur}} [d]^k_{|S'} \cong [d]^s.$$

Definition 3. A k-MOLH of size d and dimension k is a bijection

$$L : [d]^k \longrightarrow [d]^k$$

such that by choosing any set $S=\{s_1,\dots s_{k-s}\}\subset[k]$ of $k-s$ indices and their values $i_1,\dots,i_{k-s}\in[d]$, and any subset $S'\subset[k]$, the composition of L with above-defined injection i (on the left) and surjection sur (on the right) provides a bijection:

$$[d]^s \cong [d]^k_{s_1=i_1,\dots,s_{k-s}=i_{k-s}} \xrightarrow{i\circ L\circ \text{sur}} [d]^k_{|S'} \cong [d]^s.$$

We denote such an object as k-MOLH(d).

Example 10. Bijection

$$L : [d]^2 \longrightarrow [d]^2$$

such that in each row and on each position all elements appear exactly once constitutes a mutually orthogonal Latin square MOLS(d). Here *square* stands for MOLH dimension $k=2$.

In general, orthogonality and dimension of the MOLH might be indexed by different numbers (here both are equal and denoted by k) [12]. This distinction is, however, not needed for our purpose.

There is a one-to-one correspondence between k-MOLH(d) and orthogonal arrays OA$(d^k,k,2k,k)$ and, hence, between them and AME$(2k,d)$ states of minimal support [12].

Proposition 8. Any AME$(2k,d)$ state of minimal support is equivalent to k-mutually orthogonal Latin hypercube L:

$$L(i_1,\dots,i_k) := \left(\phi_I^1,\dots,\phi_I^k\right).$$

Proof. Consider a k-MOLH(d) L. By adjusting

$$i_1,\dots,i_k,\phi_I^1,\dots,\phi_I^k \tag{8}$$

into d^k rows (with $2k$ elements each), one obtains OA$(d^k,k,2k,k)$. Indeed, choose any set of k indices and split it into two: $S\cup S'$, where S is a $(k-s)$-elementary subset of the first half of indices, and S' is an s-elementary subset of the second half of indices. For any choice of values $i_1,\dots,i_{k-s}\in[d]$, by Definition 3, there is a bijection

$$[d]^k_{s_1=i_1,\dots,s_{k-s}=i_{k-s}} \xrightarrow{i\circ L\circ \text{sur}} [d]^k_{|S'},$$

and hence the subset $S\cup S'$ of k columns in Eq. (8) contains all possible combinations of symbols. Since the choice of k-elementary subset $S\cup S'$ was unrestricted, this is a defining property of an OA of index unity. Overturning this argument provides the reverse statement. ∎

Example 11. The AME(4,3) state from Eq. (1) is equivalent to the mutually orthogonal Latin square (2-MOLH):

As it is shown in Table IV, the entries of MOLS are pairs of numbers (k,ℓ) and considering this, the relevant quantum

TABLE IV. The entries of MOLS(3).

$i \backslash j$	0	1	2
0	00	11	22
1	12	20	01
2	21	02	10

state can be obtained by reading all entries as

$$|\mathrm{AME}(4,3)\rangle = \frac{1}{d}\sum_{i,j=0}^{d-1} |i, j, k, \ell\rangle.$$

The notion of MOLS was used for the construction of several AME states [70].

A. Existence of Latin designs

Proposition 9. If a mutually orthogonal Latin hypercube k-MOLH(d) for $k > 1$ exists, then indices d and k satisfy

$$k \leqslant d - 1.$$

Proof. The hyper-row $I := (i, 0, \dots, 0)$, $i \in d$ defines the following mapping:

$$L_{|I} : i \longmapsto \left(\phi_i^1, \dots, \phi_i^k\right) \in [d]^k$$

where $\phi_i^\ell := \phi_I^\ell$ for simplicity. Observe that on each position all symbols appear, i.e.,

$$\left\{\phi_i^\ell : i \in [d]\right\} = [d]. \tag{9}$$

Consider now the element

$$(j_1, \dots, j_k) := L(0, 1, 0, \dots, 0) \in [d]^k.$$

From Eq. (9), clearly

$$j_1 = \phi_{i_1}^1, \dots, \quad j_k = \phi_{i_k}^k$$

for some indices $i_1, \dots, i_k$. Observe the following.

(1) $i_1, \dots, i_k \neq 0$. Suppose the contrary, $i_s = 0$. Then $j_s = \phi_0^s$, and hence

$$L(0, 0, 0, \dots, 0) = (\dots, j_s, \dots),$$
$$L(0, 1, 0, \dots, 0) = (\dots, j_s, \dots),$$

where dotted symbols on the right are not specified. This is in contradiction to Definition 3 for S given by $i_j = 0$ for all $j \neq 1$, and $S' = \{s\}$.

(2) Indices $i_1, \dots i_k$ are pairwise different. Suppose the contrary, $i_{s_1} = i_{s_2}$. Then

$$L(i_{s_1}, 0, 0, \dots, 0) = (\dots, j_{s_1}, \dots, j_{s_2}, \dots),$$
$$L(0, 1, 0, \dots, 0) = (\dots, j_{s_1}, \dots, j_{s_2}, \dots).$$

This is in contradiction to Definition 3 for S given by $i_j = 0$ for all $j \neq 0, 1$, and $S' = \{s_1, s_2\}$.

Since all indices $i_1, \dots, i_k \in [d]$ are pairwise different and nonzero, $k \leqslant d - 1$. ■

It is worth mentioning that the condition given in Proposition 9 is only a necessary condition for the existence of the MOLH. If it is satisfied, the precise construction of the MOLH is known for all d being prime powers. This construction might be extended further by composing two MOLHs of a different size. Nevertheless, the aforementioned condition is not a sufficient one. For instance, construction of the MOLS (2-MOLH) of size $d = 6$ refers to the famous problem of *36 officers of Euler* [72], which was proven to have no solution [73].

TABLE V. Extension of MOLS(3) into MOLS(9).

$i \backslash j$	0	1	2	3	4	5	6	7	8
0	00	11	22	33	44	55	66	77	88
1	12	20	01	45	53	34	78	86	67
2	21	02	10	54	35	43	87	68	76
3	36	47	58	60	71	82	03	14	25
4	48	56	37	72	80	61	15	23	04
5	57	38	46	81	62	70	24	05	13
6	63	74	85	06	17	28	30	41	52
7	75	83	64	18	26	07	42	50	31
8	84	65	73	27	08	16	51	32	40

B. Extension of Latin designs

As we shall see, Latin designs are not only related to the construction of AME states with minimal support, but also to the local unitary relations between such. In particular, the problem of existence and extension of the k-dimensional MOLH is relevant to the description of LU equivalences between AME$(2k, d)$ states. Therefore, a short outline of the extension problem is presented below.

Definition 4. A MOLH(s) of size k and dimension s,

$$L : [s]^k \longrightarrow [s]^k,$$

might be extended to MOLH(d) if there exists MOLH(d):

$$L' : [d]^k \longrightarrow [d]^k,$$

which preserves the structure of L, i.e., $L'_{|[s]^k} \equiv L$. Moreover, we refer to L as a sub-MOLH(s) of MOLH(d).

Example 12. A MOLS(3) L might be extended into MOLS(9) presented in Table V. Indeed, one can see that entries in the square consisting of three first rows and columns are taken from 0, 1, and 2 only. In fact, this extension is relevant to a tensor product of two identical Latin squares L.

Remark 2. Consider a MOLH(d) L. If there exists a sub-hypercube $S = S_1 \times \cdots \times S_k \subseteq [k]^d$ which is mapped by L on another hypercube $S' = S'_1 \times \cdots \times S'_k \subseteq [k]^d$, then up to permutation of labels $L' := L_{|S}$ is the sub-MOLH(s) of MOLH(d). Moreover, L cannot map hyper-rectangles onto hyper-rectangles except hypercubes into hypercubes. Hence, a sub-hyper-rectangle of L does not exist.

The problem of extension of Latin designs might be traced to Ryser's theorem [74], and is a plentiful scientific problem considered in several papers [75,76].

There are the following dimension bounds for extending MOLH(s) into MOLH(d) or, equivalently, finding sub-MOLH(s) of MOLH(d).

Proposition 10. Inequality

$$s \leqslant \frac{1}{1+\sqrt[k-1]{k}}\ d$$

is a necessary condition for extending MOLH(s) into MOLH(d) for any $k > 1$.

Proof. Suppose that L' extends to L. Consider pairwise disjoint sets:

$$S_i := \underbrace{S \times S}_{i-1} \times S^c \times \underbrace{S \times S}_{k-i}$$

where $S := [s]$. Denote their sum by $S' = \cup_{i=1}^{k} S_i$. Observe that for any multi-index $I \in S'$

$$L(I) \in S^c \times \cdots \times S^c.$$

Indeed, it follows from the fact that $L(S) = S$, and hence on any position no indices from S might appear in $L(S')$. Since L is the bijection

$$|S'| \leqslant |S^c \times \cdots \times S^c|,$$

and hence

$$ks^{k-1} \leqslant (d-s)^{k-1},$$

which is equivalent to the statement of Proposition 10. ∎

As we shall see, the existence of nontrivial sub-MOLH(s) in MOLH(d) is directly related to the problem of describing the automorphisms of AME($2k, d$) states. More precisely, assumptions that either nontrivial sub-MOLH(s) does not exist or cannot be extended to MOLH(d) allows us to provide a comprehensive description of LU equivalences of AME($2k, d$) states. Therefore, the necessary conditions of existence and extension of MOLH are limiting the statements of Propositions 5 and 6. This limitation is notified in Proposition 1, which follows directly from Propositions 9 and 10.

VII. FURTHER DISCUSSION AND OPEN PROBLEMS

Ultimate LU and SLOCC classification of k-uniform states, even of minimal support, is in fact a complex project involving many open mathematical problems, such as (1) existence and extension of mutually orthogonal Latin hypercubes (see Sec. VI), (2) classification of Hadamard matrices of Butson type $B(d, d)$, and (3) classification and uniqueness of OAs of index unity (without permutation), among others. Therefore, with great conviction, we claim it to be currently out of reach. Below, we discuss three open problems regarding LU and SLOCC classification of k-uniform states with minimal support in a detailed way. We show their connections with some open mathematical problems.

First, consider two k-uniform states of minimal support $|\psi\rangle$ and $|\psi'\rangle$ with all phases equal to 1 for simplicity. With this constraint on phases, Proposition 2 shows that $|\psi\rangle$ and $|\psi'\rangle$ are LU equivalent if and only if there exist local permutation matrices relating $|\psi\rangle$ and $|\psi'\rangle$:

$$|\psi'\rangle = \sigma_1 \otimes \cdots \otimes \sigma_n |\psi\rangle.$$

States $|\psi\rangle$ and $|\psi'\rangle$ are in one-to-one correspondence with two OAs of index unity. The existence of local permutation matrices is equivalent to an isomorphism between two OAs of index unity. Hence LU classification of such states is equivalent to the classification of OAs of index unity. Such a classification is, however, an open mathematical problem. In many situations, when number of parties N, uniformity k, and local dimension d are small, it is known that all OAs of index unity are isomorphic [77–79].

Conjecture 3. All OAs of index unity are isomorphic by permutations of symbols on each level. Equivalently, all k-uniform states with minimal support and all term phases equal are LU equivalent.

Second, in Propositions 5 and 6, the form of the arbitrary LU operator between two AME($2k, d$) states with minimal support is provided for small numbers k and d. It is given by a Butson-type matrix $B(d, d)$ or an identity matrix, multiplied by local monomial matrices from both sides. We have shown that for a composed system there are local operators beyond the provided formula. Indeed, in this case, the tensor product of the Butson-type matrix and the identity matrix may provide LU equivalence. We conjecture that it is a general form for LU equivalences for all AME($2k, d$) states, and it is tightly related to the possible decomposition of a system. This supposition is stated in Conjecture 1.

Third, Proposition 2 states that any LU operator between two k-uniform states of minimal support $|\psi\rangle$ and $|\psi'\rangle$ is a local product of phase (diagonal) and permutation matrices (for $2k < N$). By considering states $|\psi\rangle$ and $|\psi'\rangle$ with terms of various phases, we showed that not all of them are LU equivalent for $k > 2$. Nevertheless, the precise description of SLOCC classes containing such states is not given. Therefore, the role of permutation matrices in LU classification is not yet absolutely clear.

Finally, in Sec. IV the basic difference between k-uniform states of minimal support where $2k < N$ and $2k = N$ is discussed. LU equivalence between two k-uniform states with $2k = N$ decomposes into multiplication of the Butson-type matrix and LM matrices from both sides. Obviously, Butson-type matrices significantly increase the class of LU equivalences between two states. Nevertheless, it is not known yet whether such LU equivalences are beyond local monomial equivalences. In fact, in all provided examples involving Butson-type matrices in LU equivalence, states were always LM equivalent. Therefore we conjecture that Corollary 2 holds true in the case $2k = N$ (even though Proposition 2 does not hold anymore).

Conjecture 4. All AME($2k, d$) states with minimal support are LU equivalent if and only if they are LM equivalent.

Notice that any attempt to prove the statement above makes sense only if Conjecture 3 is true. In such a case, the classification of Butson type $B(d, d)$, which is an open mathematical problem, and refined analysis of such are required.

VIII. CONCLUSIONS

In this paper, we develop techniques of SLOCC verification between k-uniform and AME states. In particular, we show that two k-uniform states are SLOCC equivalent if and only if they are LM equivalent. We further specify the matrices which might appear in such equivalences. These results significantly restrict the class of possible local transformations to a finite set, which makes SLOCC verification feasible.

For AME($2k, d$) states, the aforementioned statement is not true anymore. Intriguingly, SLOCC equivalences might be provided by Fourier transforms and, in general, by Butson-type matrices. This restriction is valid, however, only for small local dimensions d and number of parties N (in particular for arbitrary N and $d < 9$). The exact bound on d and N is related to the necessary condition for existence and extension of combinatorial designs called mutually orthogonal hypercubes. Despite the exhaustive analysis performed, the general structure of SLOCC equivalences between AME$(2k, d)$ states is still puzzling and remains unknown. We present evidence that exceeding this class of equivalences is possible only in composed systems. General results concerning SLOCC equivalences of AME$(2k, d)$ states are also presented.

We illustrate the usefulness of the provided criteria on various examples. First, we show that the existence of AME states with minimal support of six or more particles yields the existence of infinitely many such non-SLOCC-equivalent states. The exact number of SLOCC classes containing AME states with minimal support is given. Second, we show that some AME states cannot be locally transformed into existing AME states of minimal support. This shows that the notion of support is relevant even for AME states.

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APPENDIX A: THE PROOF OF PROPOSITION 2

We shall prove Proposition 2 in a slightly enhanced version. Notice that LU and SLOCC equivalences coincide on the class of AME states, which is an immediate conclusion from Corollary 1. Therefore, we restrict our argument to LU equivalences only. We would like to emphasize the statement below as primary and more valuable for the LU-verification procedure than the claim of Proposition 2 itself. Indeed, this extended version is used later in Appendix D for the demonstration of nonequivalence of two families of AME(5,d) states.

Proposition 11. Consider two k-uniform states $|\psi\rangle$ and $|\phi\rangle$ with minimal support. For any subsystem S consisting of $s > k$ parties, the reduced density matrices $\rho_S(\psi)$ and $\rho_S(\phi)$ are LU equivalent if and only if they are LM equivalent.

Observe that Proposition 2 is an immediate consequence of the statement above for subsystem S of all parties, i.e., $|S| = N$. Without loss of generality it is enough to prove the statement of Proposition 11 only for the smallest possible subsystems S, i.e., consisting of $k+1$ parties. Indeed, assume that the reduced states $\rho_{S'}(\psi)$ and $\rho_{S'}(\phi)$ are equivalent by a local unitary matrix U. Consider any subsystem $S \subseteq S'$ of $k+1$ parties. The local operator U splits:

$$U = U_S U_{S'\setminus S}$$

where U_S is the local operation on the S subsystem and $U_{S'\setminus S}$ is the local operation on the $S' \setminus S$ subsystem equivalently. Since the (partial) trace is invariant under cycling permutations, we have

$$\rho_S(\psi) = U_S[\rho_S(\phi)]$$

and, by Proposition 11, U_S is a local monomial operation. Since the subsystem S was chosen arbitrary, U is a local monomial operator.

Notice that the size $s > k$ of the subsystem S in Proposition 11 is the largest possible. Indeed, after taking the partial trace over the larger subsystem, both states $|\psi\rangle$ and $|\phi\rangle$ become proportional to the identity, and hence any local unitary operation provides their equivalence.

We introduce the following notation. Consider two LU-equivalent k-uniform states: $|\psi\rangle$ and $|\phi\rangle$ of minimal support form. We make use of the decomposition into support elements:

$$|\psi\rangle = \sum_{i=1}^{d^k} \alpha_i |\psi_i\rangle$$

where $|\psi_i\rangle$ are of unity support. Moreover, we denote elements of $|\psi_i\rangle$ as follows:

$$|\psi_i\rangle = \left|x_1^i \cdots x_N^i\right\rangle$$

where $x_j^i = 0, \dots, d-1$. We use similar notation for the state $|\phi\rangle$:

$$|\phi\rangle = \sum_{i=1}^{d^k} \alpha_i |\phi_i\rangle, \quad |\phi_i\rangle = \left|y_1^i \cdots y_N^i\right\rangle.$$

Lemma 1. For a partial trace over any subsystem S of $|S| \geqslant k$ parties

$$\operatorname{tr}_S |\psi\rangle\langle\psi| = \sum_{i=0}^{d^k} |\widetilde{\psi}_i\rangle\langle\widetilde{\psi}_i|$$

where $|\widetilde{\psi}_i\rangle = \operatorname{tr}_S |\psi_i\rangle$. Moreover, for any $i \neq j$, $|\psi_i\rangle$ and $|\psi_j\rangle$ coincide on at most $k-1$ positions.

Proof. For any subsystem S such that $|S| = N - k$, $\operatorname{tr}_S |\psi\rangle\langle\psi| = \mathrm{Id}_{d^k}$. Hence, for any $i \neq j$, vectors $|\psi_i\rangle$ and $|\psi_j\rangle$ coincide on at most $k-1$ positions. Indeed, suppose the contrary, i.e., they coincide on some k positions. Then, by tracing out the rest of parties, we get $\operatorname{tr}_{N-k} |\psi_i\rangle\langle\psi_i| = \operatorname{tr}_{N-k} |\psi_j\rangle\langle\psi_j|$ and by the minimality of the support $\operatorname{tr}_{N-k} |\psi\rangle\langle\psi| \neq \mathrm{Id}_{d^k}$. The statement of the lemma follows immediately from the presented observation. ■

Proof of Proposition 11. As we already discussed, it is enough to prove Proposition 11 for subsystems S of $k+1$ parties. Without loss of generality, analyze the reduction to the subsystem S of the first $k+1$ parties. Consider a local unitary operation $U := U_1 \otimes \cdots \otimes U_{k+1}$ transforming $\rho_S(\psi)$ into $\rho_S(\phi)$. Unitary operators U_i might be seen as the following change of basis:

$$|\widehat{i_1} \cdots \widehat{i_{k+1}}\rangle := |U_1(i_1) \cdots U_{k+1}(i_{k+1})\rangle. \tag{A1}$$

From Lemma 1 we have

$$\rho_S(\psi) = \sum_{i=1}^{d^k} |x_1^i \cdots x_{k+1}^i\rangle\langle x_1^i \cdots x_{k+1}^i| \tag{A2}$$

and

$$\rho_S(\phi) = \sum_{i=1}^{d^k} |y_1^i \cdots y_{k+1}^i\rangle\langle y_1^i \cdots y_{k+1}^i|. \tag{A3}$$

Observe that

$$U\rho_S(\psi)U^{-1} = \sum_{i=1}^{d^k} |\widehat{x_1^i} \cdots \widehat{x_{k+1}^i}\rangle\langle \widehat{x_1^i} \cdots \widehat{x_{k+1}^i}|,$$

and since U is the LU equivalence the two expressions above are equal. Consequently, we have equality of the following spaces:

$$\begin{aligned}&\text{span}\{|y_1^i \cdots y_{k+1}^i\rangle; i = 1, \dots, d^k\}\\&= \text{span}\{|\widehat{x_1^i} \cdots \widehat{x_{k+1}^i}\rangle; i = 1, \dots, d^k\}.\end{aligned}$$

In general, each vector from the first space is a linear combination of vectors from the second space. We will show, however, that there is a one-to-one correspondence between vectors from both spaces; namely, for any index i there exists an index j_i such that

$$|\widehat{x_1^i} \cdots \widehat{x_{k+1}^i}\rangle = |y_1^{j_i} \cdots y_{k+1}^{j_i}\rangle. \tag{A4}$$

With this observation at hand, and by Eq. (A1), the statement of Proposition 2 follows immediately.

What remains to be shown is that, indeed, Eq. (A4) holds. Consider the vector $|\widehat{x_1^i} \cdots \widehat{x_{k+1}^i}\rangle$ and present it as the following linear combination:

$$|\widehat{x_1^1} \cdots \widehat{x_{k+1}^1}\rangle = \sum_{i=1}^{d^k} \beta_i |y_1^i \cdots y_{k+1}^i\rangle. \tag{A5}$$

On the other hand,

$$|\widehat{x_1^1} \cdots \widehat{x_{k+1}^1}\rangle = |U_1(x_1^1) \cdots U_{k+1}(x_{k+1}^1)\rangle \tag{A6}$$

$$= \sum_{j_1,\dots,j_{k+1}=0}^{d-1} u^1_{x_1^1 j_1} \cdots u^{k+1}_{x_{k+1}^1 j_{k+1}} |j_1 \cdots j_{k+1}\rangle \tag{A7}$$

where $u^k_{lm} = (U_k)_{lm}$, and hence

$$\beta_i = u^1_{x_1^1 y_1^i} \cdots u^{k+1}_{x_{k+1}^1 y_{k+1}^i}.$$

Suppose now that, for $i \neq j$, $\beta_i, \beta_j \neq 0$. Consequently,

$$u^m_{x_m^1 y_m^i} \neq 0, \quad u^m_{x_m^1 y_m^j} \neq 0,$$

for $m = 1, \dots, k+1$. By Lemma 1, $|y_1^i \cdots y_{k+1}^i\rangle$ and $|y_1^j \cdots y_{k+1}^j\rangle$ differ on at least two positions; without loss of generality suppose $y_1^i \neq y_1^j$. Observe that

$$u^1_{x_1^1 y_1^i} \neq 0, \quad u^2_{x_j^1 y_2^j} \neq 0, \dots, u^{k+1}_{x_{k+1}^1 y_{k+1}^j} \neq 0$$

and hence the expression

$$|y_1^i y_2^j \cdots y_{k+1}^j\rangle\langle y_1^i y_2^j \cdots y_{k+1}^j|$$

appear on the right-hand side of Eq. (A3) with nonzero coefficient, which is in contradiction to Lemma 1.

Since there is at most one $\beta_i \neq 0$, the sum in Eq. (A5) collapses to the one term. Similar reasoning shows that $|\widehat{x_1^m} \cdots \widehat{x_{k+1}^m}\rangle$ is in general equal to $|y_1^i \cdots y_{k+1}^i\rangle$ for some $i = 1, \dots, d^k$, which should have been shown. ■

Notice that the presented argument does not hold if $2k = N$. Indeed, the smallest nontrivial reduced system of k-uniform states consists of $k+1$ parties. The proof is based on Lemma 1 to justify that vectors $|y_1^i \cdots y_{k+1}^i\rangle$ and $|y_1^j \cdots y_{k+1}^j\rangle$ differ on at least two positions. For $2k = N$, however, the argument of Lemma 1 might be used only for the trivial reduction to k parties.

APPENDIX B: THE PROOF OF PROPOSITION 4

Proof of Proposition 4. From Proposition 2 the LU equivalence between two states of minimal support is a product of permutation and diagonal matrices. Suppose that the local permutation σ was already applied to the state $|\psi\rangle$. Therefore, we may assume that $|\psi\rangle$ and $|\psi'\rangle$ are related only by diagonal operators.

Since both states $|\psi\rangle$ and $|\psi'\rangle$ are k-uniform states of minimal support related only by diagonal operators, they might be written in the following form:

$$|\psi\rangle = \sum_{I\in\mathcal{I}} \omega_I |I\rangle, \tag{B1}$$

$$|\psi'\rangle = \sum_{I\in\mathcal{I}} \omega'_I |I\rangle \tag{B2}$$

where $\mathcal{I} \subset [d]^n$ has dimension $|\mathcal{I}| = [d]^k$. Denote by

$$D^\ell = \text{diag}(u_1^\ell, \dots, u_d^\ell)$$

the diagonal operators relating $|\psi\rangle$ and $|\psi'\rangle$. Clearly,

$$\omega'_I = u^1_{i_1} \cdots u^n_{i_n} \omega_I \tag{B3}$$

for any index $I = i_1, \dots, i_n$. Denote by $U^\ell := \prod_{i\in[d]} u_i^\ell$ the product of all nonzero elements from the matrix D^ℓ.

For simplicity, let us choose $S = \{1, \dots, k-1\}$ as the set of first $k-1$ indices. We shall show the statement with respect to the first matrix D^1. Consider any multi-index $I = i_2, \dots, i_{k-1}$. By multiplying adequate expressions from Eq. (B3) by sides, one may obtain

$$(W^S_{i,I})' = (u_i^1)^d (u_{i_2}^2)^d \cdots (u_{i_k-1}^{k-1})^d (U^k \cdots U^n) W^S_{i,I}.$$

The fact that $U^k \cdots U^n$ appears on the right-hand side follows from the basic properties of OAs related to the states $|\psi\rangle$ and $|\psi'\rangle$. Hence

$$(u_0^1)^d \frac{W^S_{0,I}}{(W^S_{0,I})'} = \cdots = (u^1_{d-1})^d \frac{W^S_{d-1,I}}{(W^S_{d-1,I})'}. \tag{B4}$$

From this immediately follows that

$$D^1 = \omega_1 \text{diag}\left(\sqrt[d]{\frac{(W^S_{0,I})'}{W^S_{0,I}}}, \dots, \sqrt[d]{\frac{(W^S_{d-1,I})'}{W^S_{d-1,I}}}\right),$$

for some phase factor ω_1.

Since the multi-index I is arbitrary, from Eq. (B4) follows that

$$\frac{(W^S_{0,I})'}{W^S_{0,I}} = \frac{(W^S_{0,I'})'}{W^S_{0,I'}} \tag{B5}$$

for any other multi-index $I' = i'_2, \dots, i'_{k-1}$.

Since we assumed that the local permutation σ was already applied to the state $|\psi\rangle$, one has to consider the action of σ on the denominator in Eq. (B5) in the general case. This shows the statement of Proposition 4 for the set S of first $k-1$ indices. The same reasoning might be applied for other sets S, and further for any indices: $2, \dots, n$. The global phase is then a multiplication of obtained factors ω_i. ■

APPENDIX C: SEC. IV REVISITED

We discuss in detail LU equivalences of AME$(2k, d)$ states with minimal support. In particular, proofs of Propositions 5 to 7 are presented. We begin with the necessary notation. For simplicity, all AME$(2k, d)$ states considered in Appendix C are normalized to $\sqrt{d^k}$. In such a way, all terms in the computational basis of states with minimal support are normalized to one. For a local unitary operator $U := U_1 \otimes \dots \otimes U_\ell$ and a subset of indices $S \subseteq [\ell]$, we define its S part by

$$U_S := \bigotimes_{i\in S} U_i.$$

Observation 2. Let $U := U_1 \otimes \dots \otimes U_\ell$ be a local unitary operator transforming a state $\rho(\psi) \in \mathcal{H}^{\otimes \ell}$ onto $\rho(\psi')$, i.e.,

$$U[\rho(\psi)] = \rho(\psi').$$

Then, for any subset $S \subseteq [\ell]$ of indices,

$$U_S[\rho_S(\psi)] = \rho_S(\psi'). \tag{C1}$$

Moreover, if

$$\rho_S(\psi) = |\psi_1\rangle\langle\psi_1| + \dots + |\psi_k\rangle\langle\psi_k|,$$

$$\rho_S(\psi') = |\psi'_1\rangle\langle\psi'_1| + \dots + |\psi'_k\rangle\langle\psi'_k|,$$

where vectors $|\psi_i\rangle$ and $|\psi'_i\rangle$, respectively, are orthogonal, i.e., $\langle\psi_i|\psi_j\rangle = c_i\delta_{ij}$ and $\langle\psi'_i|\psi'_j\rangle = c'_i\delta_{ij}$, then

$$U_S|\psi_i\rangle = \sum_{j=1}^{k} v_{ij}|\psi'_j\rangle \tag{C2}$$

for some elements v_{ij}, which form a unitary matrix $V := (v_{ij})$ if and only if all the vectors $|\psi_i\rangle$ and $|\psi'_i\rangle$ have the same norm.

Proof. Equation (C1) follows immediately from basic properties of partial trace and unitary operations.

By imposing the orthogonality relations between vectors $|\psi_i\rangle$ and $|\psi'_i\rangle$, respectively, one may extend both families (up to normalization of vectors $|\psi_i\rangle$ and $|\psi'_i\rangle$) into the basis of the entire Hilbert space. Consider now the matrix U_S in this basis. Since Eq. (C1) holds and U_S is a unitary matrix, U_S has a block structure, which transfers subspace spanned by vectors $|\psi_i\rangle$ onto the subspace spanned by vectors $|\psi'_i\rangle$, which implies Eq. (C2).

Assume now that vectors $|\psi_i\rangle$ and $|\psi'_i\rangle$ are normalized. Observe that, in the aforementioned basis, V is a block matrix of U_S, and hence is a unitary matrix. ■

Each AME$(2k, d)$ state $|\psi\rangle$ might be written in the following form:

$$|\psi\rangle = \sum_{I\in[d]^k} |I\rangle \otimes |\phi_I\rangle,$$

where I is multi-index $I = i_1, \dots, i_k$ which runs over the space $[d]^k$. If $|\psi\rangle$ is of minimal support, the vector $|\phi_I\rangle \in \mathcal{H}_d^{\otimes k}$ is separable in the computational basis, i.e.,

$$|\psi\rangle = \sum_{I\in[d]^k} \omega_I |I\rangle \otimes |\phi^1_I\rangle \otimes \dots \otimes |\phi^k_I\rangle, \tag{C3}$$

where vectors $|\phi^j_I\rangle$ are from the computational basis, i.e., $|\phi^j_I\rangle = |0\rangle, \dots, |d-1\rangle$.

Lemma 2. Consider two AME states $|\psi\rangle$ and $|\psi'\rangle$ of the form

$$|\psi\rangle = \sum_{I=i_1,\dots,i_k} \omega_I |I\rangle \otimes |\phi_I\rangle,$$

$$|\psi'\rangle = \sum_{I=i_1,\dots,i_k} \omega'_I |I\rangle \otimes |\phi'_I\rangle,$$

which are local unitary equivalent by U. For any multi-index $I = i_1, \dots, i_k$

$$U(\omega_I|\phi_I\rangle) = \sum_{I'\in[d]^k} v^1_{i_1 i'_1} v^2_{i_1 i'_1}(i_1) \cdots v^k_{i_k i'_k}(i_1, \dots, i_{k-1}) \omega_{I'}|\phi_{I'}\rangle,$$

where elements $v^\ell_{ij}(i_1, \dots, i_{\ell-1})$ form a unitary matrix

$$V^\ell(i_1, \dots, i_{\ell-1}) := \left[v^\ell_{ij}(i_1, \dots, i_{\ell-1})\right]$$

for any $\ell = 1, \dots, k$ and indices $i_1, \dots, i_{\ell-1}$.

Notice that structure constants $v^\ell_{ij}(i_1, \dots, i_{\ell-1})$ depend on indices $i_1, \dots, i_{\ell-1}$, which in fact is the main obstruction for obtaining more general results as those presented in this section.

Proof. We shall use Observation 2 repetitively k times, by tracing out parties $1, \dots, k$, respectively. In fact, the order of the procedure does not matter. In each step, the orthogonality of adequate vectors is fulfilled by relations $\langle\phi_{I'}|\phi_I\rangle = \delta_{I,I'}$. We present the first two steps of the procedure in a more detailed way.

We know that $U(|\psi\rangle) = |\psi'\rangle$. Consider the partial traces over the first subsystem in both vectors $|\psi\rangle$ and $|\psi'\rangle$:

$$\rho_{1^c}(\psi) = \sum_{i_1=0}^{d-1} |\psi_{i_1}\rangle\langle\psi_{i_1}|,$$

$$\rho_{1^c}(\psi') = \sum_{i_1=0}^{d-1} |\psi'_{i_1}\rangle\langle\psi'_{i_1}|,$$

where

$$|\psi_{i_1}\rangle = \sum_{I=i_2,\dots,i_k} \omega_{i_1,I}|I\rangle \otimes |\phi_{i_1,I}\rangle, \tag{C4}$$

and similarly

$$|\psi'_{i_1}\rangle = \sum_{I=i_2,\dots,i_k} \omega'_{i_1,I}|I\rangle \otimes |\phi'_{i_1,I}\rangle.$$

From Observation 2 follows that

$$U_{1^c}(|\psi_{i_1}\rangle) = \sum_{i'_1=0}^{d-1} v^1_{i_1 i'_1}|\psi'_{i'_1}\rangle, \tag{C5}$$

and the elements $v^1_{i_1 i'_1}$ form the unitary matrix $V^1 := (v^1_{i_1 i'_1})$.

We shall consider vectors $|\psi_{i_1}\rangle$ separately. For an arbitrary index i_1, consider a partial trace over the second party of $|\psi_{i_1}\rangle$. From Eq. (C4)

$$\rho_{2^c}(\psi_{i_1}) = \sum_{i_2=0}^{d-1} |\psi_{i_1,i_2}\rangle\langle\psi_{i_1,i_2}|,$$

where

$$|\psi_{i_1,i_2}\rangle = \sum_{I=i_3,\dots,i_k} \omega_{i_1,i_2,I}|I\rangle \otimes |\phi_{i_1,i_2,I}\rangle.$$

The partial trace over the second party of the right-hand side of Eq. (C5) is equal to

$$\sum_{i_2=0}^{d-1} |\psi'_{i_1,i_2}\rangle\langle\psi'_{i_1,i_2}|,$$

where

$$|\psi'_{i_1,i_2}\rangle = \sum_{i'_1} v^1_{i_1 i'_1} \sum_{I=i_3,\dots,i_k} \omega'_{i_1,i_2,I}|I\rangle \otimes |\phi'_{i_1,i_2,I}\rangle,$$

and hence by Observation 2 applied to both sides of Eq. (C5) we have

$$U_{\{1,2\}^c}(|\psi_{i_1,i_2}\rangle) = \sum_{i'_2=0}^{d-1} v^2_{i_2 i'_2}(i_1)|\psi'_{i_1,i_2}\rangle, \tag{C6}$$

for elements $v^2_{i_2 i'_2}(i_1)$, which form a unitary matrix. Notice that the matrix $V^2(i_1) := [v^2_{i_2 i'_2}(i_1)]$ is in particular dependent on the chosen index i_1.

We repeat the presented procedure for an arbitrary pair of indices i_1 and i_2; then for i_1, i_2, and i_3; and in general k times up to $i_1,\dots,i_k$. Finally, we obtain

$$U_{\{1,\dots,k\}^c}(|\psi_{i_1,\dots,i_k}\rangle) = \sum_{i'_k=0}^{d-1} v^k_{i_k i'_k}(i_1,\dots,i_{k-1})|\psi'_{i_1,\dots,i_k}\rangle \tag{C7}$$

where elements $v^k_{i_k i'_k}(i_1,\dots,i_{k-1})$ form a unitary matrix. Notice that $|\psi_{i_1,\dots,i_k}\rangle = \omega_I|\phi_{i_1,\dots,i_k}\rangle$. On the other hand

$$|\psi'_{i_1,\dots,i_k}\rangle = \sum_{i'_1,\dots,i'_{k-1}} v^1_{i_1 i'_1}\cdots v^{k-1}_{i_{k-1}i'_{k-1}}(i_1,\dots,i_{k-2}) \omega'_{i_1,\dots,i_k}|\phi'_{i_1,\dots,i_k}\rangle.$$

By the analysis of the recursion, substitution of this formula to Eq. (C7) proves the proposition. ∎

Corollary 7. For two AME states of minimal support

$$|\psi\rangle = \sum_{I=i_1,\dots,i_k} \omega_I|I\rangle \otimes |\phi^1_I\rangle \otimes\cdots\otimes |\phi^k_I\rangle,$$

$$|\psi'\rangle = \sum_{I=i_1,\dots,i_k} \omega'_I|I\rangle \otimes |\phi^{1\prime}_I\rangle \otimes\cdots\otimes |\phi^{k\prime}_I\rangle,$$

which are local unitary equivalent by U, the equivalence is of the following form:

$$U_{[k]}(\omega_I|I\rangle) = \sum_{I'=i'_1,\dots,i'_k} v^1_{\phi^1_I\phi^{1\prime}_{I'}} v^2_{\phi^2_I\phi^{2\prime}_{I'}}(\phi^1_I)\cdots \times v^k_{\phi^k_I\phi^{k\prime}_{I'}}(\phi^1_I,\dots,\phi^{k-1}_I)\omega_{I'}|I'\rangle. \tag{C8}$$

Proof. The proof follows immediately from Lemma 2 applied to the second half of indices. ∎

The above obtained statements are rather technical. We shall demonstrate their effectiveness.

Proof of Proposition 7. Fix a multi-index $I = i_1,\dots,i_k$. By the definition,

$$U_{[k]}(|I\rangle) = \sum_{I'=i'_1,\dots,i'_k} u^1_{i_1 i'_1}\cdots u^k_{i_k i'_k}|I'\rangle. \tag{C9}$$

On the other hand, $U_{[k]}$ might be expressed in the form Eq. (C8). For simplicity of the proof, we use the following notation:

$$u^\ell_{i'_\ell} := u^\ell_{i_\ell i'_\ell}, \quad v^\ell_{\phi^{\ell\prime}_{I'}} := v^\ell_{\phi^\ell_I\phi^{\ell\prime}_{I'}}(\phi^1_I,\dots,\phi^{\ell-1}_I),$$

which is correctly defined once the index $I = i_1,\dots,i_k$ is fixed. Furthermore, we define

$$\ell_{I'} := u^1_{i'_1}\cdots u^k_{i'_k}, \quad p_{I'} := v^1_{i'_1}\cdots v^k_{i'_k},$$

for any multi-index $I' = i'_1,\dots,i'_k$.

By comparison of Eqs. (C8) and (C9), we have

$$\ell_{I'} = p_{\phi'_{I'}}\frac{\omega'_{I'}}{\omega_I}, \tag{C10}$$

where $\phi'_{I'} := (\phi^{1\prime}_{I'},\dots,\phi^{k\prime}_{I'})$ for any $I' = i'_1,\dots,i'_k$.

Consider now the mutually orthogonal Latin hypercube MOLH(d) (see Proposition 8 for details):

$$L(i'_1\dots i'_k) := (\phi^{1\prime}_{I'},\dots,\phi^{k\prime}_{I'}).$$

On the one hand, the set of indices $I' = i'_1,\dots,i'_k$ for which $\ell_{I'} \neq 0$ forms a rectangle $S := S_1\times\cdots\times S_k \subseteq [d]^k$. Indeed, it follows from the product form of $\ell_{I'}$. On the other hand, the set of indices $I'' = i''_1,\dots,i''_k$ for which $p_{I''}\neq 0$ also forms a rectangle $S' := S'_1\times\cdots\times S'_k \subseteq [d]^k$, which follows from the product form of $p_{I''}$. From the equality in Eq. (C10) follows that L maps S onto S'. From Remark 2, S and S' are hypercubes and $L_{|S}$ is a MOLH(s), where $s := |S_1| = \cdots = |S_k| = |S'_1| = \cdots = |S'_k|$. This proves the second statement of Proposition 7.

So far, we showed that for any fixed multi-index $I = i_1,\dots,i_k$ the relevant rows of matrices $U_1,\dots,U_k$, i.e., vectors

$$(u^1_{i_1 i})^{d-1}_{i=0},\dots,(u^k_{i_k i})^{d-1}_{i=0},$$

have the same number s of nonzero elements. Observe that this is also true for any other multi-index $I' = i'_1, \dots, i'_k$. Indeed, the analogous reasoning, for other multi-indices $I'_0 = i_1, i'_2, \dots, i'_k$ (I' and I'_0 are equal on the first position), ensures us that the vectors

$$\left(u^1_{i_1 i}\right)_{i=0}^{d-1}, \quad \left(u^2_{i'_2 i}\right)_{i=0}^{d-1} \dots, \left(u^k_{i'_k i}\right)_{i=0}^{d-1}$$

have the same number of nonzero elements, equal to s. From here, one can deduce it for arbitrary $I' = i'_1, \dots, i'_k$. Since the inverse of the unitary matrix is its conjugate transpose, the matrices $U_1^\dagger, \dots, U_k^\dagger, \dots$ provide the local unitary equivalence between $|\psi'\rangle$ and $|\psi\rangle$. Reasoning similar to the above proves that those matrices have the same number of nonzero elements in each row. This is equivalent to the fact that $U_1, \dots, U_k$ have the same number of nonzero elements in each column. This proves the first statement of Proposition 7 for matrices $U_1, \dots, U_k$ (without stating the equality of the element's norms). Observe that by taking another set of indices $S \subset [n]$ one may extend this reasoning to all matrices $U_1, \dots, U_k, \dots, U_{2k}$.

In fact, more detailed analysis of matrices $U_1, \dots, U_k$ might be performed. Once more, fix $I = i_1, \dots, i_k$, and keep the notation introduced before in the proof. For each $\bar{I}' = i'_1, \dots, i'_{k-1} \in S_1 \times \dots \times S_{k-1}$, we have

$$\prod_{i'_k \in S_k} \ell_{\bar{I}' i'_k} = \left(u^1_{i'_1}\right)^s \cdots \left(u^{k-1}_{i'_{k-1}}\right)^s C_1 \tag{C11}$$

where $C_1 = \prod_{i'_k \in S_k} u^k_{i'_k}$. By Eq. (C10), the left-hand side of Eq. (C11) is equal to

$$\prod_{i'_k \in S_k} p_{\phi'_{\bar{I}' i'_k}} \frac{\omega'_{\bar{I}' i'_k}}{\omega_I} = \frac{1}{\omega_I} W^{[k-1]}_{\bar{I}'} \prod_{i'_k \in S_k} v^1_{L^1_{\bar{I}' i'_k}} \cdots v^k_{L^k_{\bar{I}' i'_k}}. \tag{C12}$$

Here, we use the notation of $W^{[k-1]}_{\bar{I}'}$ introduced in Sec. III A with a slight modification. Namely, the product

$$W^{[k-1]}_{\bar{I}'} := \prod_{i'_k \in S_k} \omega_{\bar{I}' i'_k}$$

runs only over all nonzero elements $\omega_{\bar{I}' i'_k}$, which is exactly s. One of the basic properties of a MOLH is that in each row and on each position all elements appear exactly once (see Definition 3 for details). Hence, Eq. (C12) is equal to

$$\frac{1}{\omega_I} W^{[k-1]}_{\bar{I}'} C_2, \quad \text{where} \quad C_2 = \prod_{I'' \in S'} v^1_{i''_1} \cdots v^k_{i''_k}. \tag{C13}$$

Equations (C11)–(C13) combine to the following:

$$\left(u^1_{i'_1}\right)^s \cdots \left(u^{k-1}_{i'_{k-1}}\right)^s \frac{1}{W^{[k-1]}_{\bar{I}'}} = \frac{1}{\omega_I} \frac{C_2}{C_1} \tag{C14}$$

for each $\bar{I}' = i'_1, \dots, i'_{k-1} \in S_1 \times \dots \times S_{k-1}$, where constants C_1 and C_2 are independent of $\bar{I}'$. From Eq. (C14), one can deduce the proportions of nonzero elements in rows of U_1. Indeed, choose a multi-index $\widetilde{I}' \in S_2 \times \dots \times S_{k-1}$ and two indices $i'_1, i''_1 \in S_1$. From Eq. (C14) applied to $\bar{I}' = i'_1 \widetilde{I}'$ and $i''_1 \widetilde{I}'$ follows that

$$\left(u^1_{i'_1}\right)^s \frac{1}{W^{[k-1]}_{i'_1 \widetilde{I}'}} = \left(u^1_{i''_1}\right)^s \frac{1}{W^{[k-1]}_{i''_1 \widetilde{I}'}}. \tag{C15}$$

Since $W^{[k-1]}_{i'_1 \widetilde{I}'}$ and $W^{[k-1]}_{i''_1 \widetilde{I}'}$ have the same norms, there is the following equality: $|u^1_{i'_1}| = |u^1_{i''_1}|$. Those, however, under the introduced notation denote elements of the local unitary matrix: u_{i_1,i'_1} and u_{i_1,i''_1}. We conclude that in the i_1th row of the matrix U_1 there are exactly s nonzero elements, all having the same norm. Since U_1 is a unitary matrix (in particular, it preserves the norms), all nonzero elements from the i_1th row have norm equal to $1/\sqrt{s}$. Analogous reasoning might be performed with respect to each row of matrix U_1 and further to each matrix U_i, $i = 1, \dots, 2k$. Therefore, we affirm the equality of the element's norms in the first statement of Proposition 7.

In fact, Eq. (C15) is the last general result concerning the characteristic of local unitary matrices U_i. We restrict now to the case when $\omega_I \equiv \omega'_I \equiv 1$ for all multi-indices $I = i_1, \dots i_k$. Obviously, $W^{[k-1]}_{i'_1 \widetilde{I}'} = 1$ for all multi-indices $i'_1 \widetilde{I}' \in S_1 \times \dots \times S_{k-1}$. Thus, from Eq. (C15) follows that all nonzero entries of the i_1th row have not only equal norms but also equal sth powers. In other words, all nonzero entries of the i_1th row are sth roots of unity up to some scaling complex number w_{i_1}. Analogous reasoning might be performed with respect to each row of matrix U_1. In such a way we obtain scaling factors w_i for $i = 1, \dots, d$. Observe that matrix U_1 multiplied by the diagonal matrix

$$\sqrt{s}\, \mathrm{diag}(w_1, \dots, w_d)$$

consists only of zeros and sth roots of unity. Similarly, one can show the same property for local matrices $U_2, \dots, U_{2k}$. This proves the third statement of Proposition 7. ∎

We conjecture that the matrices $U_1, \dots, U_k, \dots, U_{2k}$ from Proposition 7 have the block structure

$$U_\ell = \begin{bmatrix} S_{\ell,1} & 0 & \cdots & 0 \\ 0 & S_{\ell,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{\ell,d/s} \end{bmatrix},$$

where matrices $S_{i,j}$ are $s \times s$ unitary matrices with all entries of the same norm. Unfortunately, coefficients $v^\ell_{ij}(i_1, \dots, i_{\ell-1})$ depend on indices $i_1, \dots, i_{\ell-1}$, which impose selection of the index I at the beginning of the proof above. This enables us to deduce the general block structure of matrices U_i. Notice that not all unitary matrices having the same number of nonzero elements in each row and column of the same norm each are necessarily of the block structure.

Concluding the block structure of matrices U_ℓ is not within our reach yet. Nevertheless, in two specific cases when $s = 1$ or d the block structure is obvious. Interestingly, those two values of s are the only possibilities for most small dimensional AME$(2k, d)$ states (see Remark 1). This follows from the requirement of appropriate dimensional MOLH extension.

Proof of Propositions 5 and 6. Suppose that the matrix $U := U_1 \otimes \ldots \otimes U_{2k}$ provides an LU equivalence between two AME$(2k, d)$ states with minimal support. Assumption of k and d being small enough is equivalent to the fact that the only possible values of s are simply 1 and d.

In the first case when $s = 1$, matrices U_i are, by the definition, monomial matrices. This gives the second possibility in Proposition 5. Each monomial matrix is a product of permutation and the diagonal matrix, hence the form of Eq. (5) in Proposition 6 follows. Similarly to the proof of Proposition 4 presented in Appendix B, it may be shown that the form of diagonal matrices D_i from Eq. (5) are exactly as it is indicated in Proposition 6.

The second case, when $s = d$, is new and goes beyond the analysis performed so far. We investigate matrix U_1. Let us recall the last general formula in our analysis, namely, Eq. (C15). By descrambling the notation introduced in the proof of Proposition 7, Eq. (C15) takes the following form:

$$\left(u^1_{ij}\right)^d \frac{1}{W^{[k-1]}_{jI}} = \left(u^1_{ij'}\right)^d \frac{1}{W^{[k-1]}_{j'I}}, \tag{C16}$$

where $i \in [d]$ and $j, j' \in [d]$ are arbitrary indices, and $I \in [d]^{k-2}$ is an arbitrary multi-index. This equation describes the proportion of dth powers of elements in the ith row of the matrix U_1. Observe that they are independent of the index i. Indeed, multiply the matrix U_1 on the right-hand side by the following diagonal matrix:

$$\overleftarrow{D}_1 = \mathrm{diag}\left(\sqrt[d]{\left(W^{[k-1]}_{0,I}\right)}, \ldots, \sqrt[d]{\left(W^{[k-1]}_{d-1,I}\right)}\right).$$

Observe that the entries of $\bar{U}_1 := U_1 \overleftarrow{D}_1$ satisfy

$$\left(\bar{u}^1_{ij}\right)^d = \left(\bar{u}^1_{ij'}\right)^d,$$

for any number i indexing the rows and any pair of indices $j, j' \in [d]$. As we already discussed while proving Proposition 7, the Hermitian-conjugate matrix $U^\dagger$ provides the reverse LU equivalence. One can analyze the proportions of elements in rows of $U_1^\dagger$ in the same way as we did for U_1. The analog of Eq. (C16) yields the following conclusions on the columns of matrix U_1:

$$\left(u^1_{ij}\right)^d \frac{1}{\left(W^{[k-1]}_{iI}\right)'} = \left(u^1_{i'j}\right)^d \frac{1}{\left(W^{[k-1]}_{i'I}\right)'}, \tag{C17}$$

where $i, i', j \in [d]$ are arbitrary indices and $I \in [d]^{k-2}$ is an arbitrary multi-index. Therefore, by multiplying the matrix $\bar{U}_1$ from the left-hand side by the diagonal matrix

$$\overrightarrow{D}_i = \mathrm{diag}\left(\sqrt[d]{\left(W^{[k-1]}_{0,I}\right)'}, \ldots, \sqrt[d]{\left(W^{[k-1]}_{d-1,I}\right)'}\right),$$

we obtain the matrix

$$\widetilde{U}_1 := \overrightarrow{D}_i U_1 \overleftarrow{D}_1$$

with the following property:

$$\left(\widetilde{u}^1_{ij}\right)^d = \left(\widetilde{u}^1_{i'j'}\right)^d,$$

for any indices $i, i' j, j' \in [d]$. Up to some global factor ω_1, all entries of the matrix $\widetilde{U}_1$ are dth roots of unity. By the definition $\omega_1 \widetilde{U}_1$ is a Butson-type matrix.

We have shown that under the assumption $s = d$ the statement in Eq. (4) of Proposition 6 holds for the matrix U_1 and the set S of the consecutive next $k-2$ indices. The same reasoning might be applied for another matrix U_i and set S. The global phase in Eq. (4) is then a multiplication of obtained factors ω_i.

What is left for the analysis is the necessary condition for existence of LU equivalence. Consider Eq. (C16). Since $I \in [d]^{k-2}$ is an arbitrary multi-index and the ratio of two matrix elements $u^1_{ij}/u^1_{ij'}$ is constant, we immediately obtain

$$\frac{W^{[k-1]}_{jI}}{W^{[k-1]}_{j'I}} = \frac{W^{[k-1]}_{jI'}}{W^{[k-1]}_{j'I'}}$$

for any multi-indices $I, I' \in [d]^{k-2}$. Similarly, from Eq. (C17), we have

$$\frac{\left(W^{[k-1]}_{jI}\right)'}{\left(W^{[k-1]}_{j'I}\right)'} = \frac{\left(W^{[k-1]}_{jI'}\right)'}{\left(W^{[k-1]}_{j'I'}\right)'}.$$

This ends the proof of Proposition 6. Obviously, Proposition 5 is an immediate consequence of Proposition 6. ■

APPENDIX D: THE PROOF OF PROPOSITION 3

We begin this section manifesting the problem of LU verification of two given AME states. In general, for two given k-uniform states, one can compare ranks of reduced density matrices in order to exclude a local equivalence between them [42]. We illustrate this phenomenon in the following example of two 1-uniform states of four qubits:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \sum_{i=0}^{1} |i, i, i, i\rangle, \quad |\psi_2\rangle = \frac{1}{2} \sum_{i,k=0}^{1} |i, k, k, i+k\rangle.$$

Observe that $\mathrm{rank}\rho_{12}(\psi_1) \neq \mathrm{rank}\rho_{12}(\psi_2)$, hence the states $|\psi_1\rangle$ and $|\psi_1\rangle$ are not LU equivalent. Nevertheless, this simple argument is never conclusive for both states being AME. Indeed, all reduced density matrices of AME state $|\psi\rangle$ have precisely determined ranks: $\mathrm{rank}\rho_S(\psi) = \min\{|S|, |S^c|\}$.

Our initial attempt for showing that states $|\mathrm{AME}(5,d)\rangle$ and $|\mathrm{AME}(5,d)\text{'}\rangle$ presented in Examples 3 and 5 are not locally equivalent was to reduce this problem to smaller subsystems. We investigated the reduced density matrices:

$$\rho_{345}[\mathrm{AME}(5,d)\text{'}] = \sum_{i,j=0}^{d-1} |i+j, i+2j, i+3j\rangle\langle i+j, i+2j, i+3j|$$

and

$$\rho_{345}[\mathrm{AME}(5,d)] = \sum_{i,j=0}^{d-1} \left(\sum_{k,k'=0}^{d-1} \omega^{(i+3j)(k-k')} |i+j, k+i+2j, k\rangle \langle i+j, k'+i+2j, k'| \right),$$

where ω is the dth root of unity. Even though $\rho_{345}[\mathrm{AME}(5,d)\text{'}]$ and $\rho_{345}[\mathrm{AME}(5,d)]$ have the same rank, we attempt to show that they are not LU equivalent. Surprisingly, it turned out that they are. We present both reduced density matrices for a local dimension $d = 3$ in Figs. 2 and 3, respectively.

It is not a straightforward observation that both states are actually LU equivalent. Standard procedures of diagonalizing

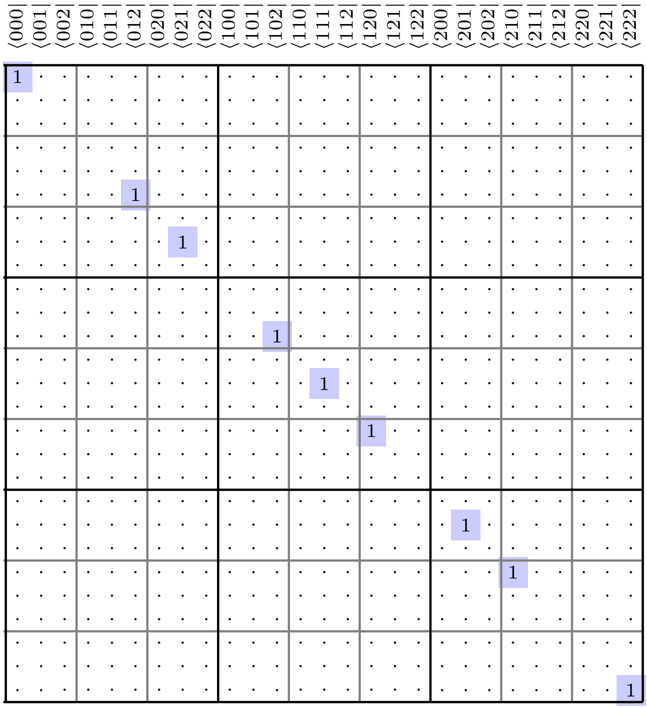

FIG. 2. The density matrix ρ_{345}[AME(5,d)].

the density matrix ρ_{345}[AME(5,d)] lead to nonlocal operations. Attentive analysis of low local dimensions ($d = 3, 5$) showed that diagonalization might be performed in a local way. Indeed, for a local dimension $d = 3$, the unitary matrices

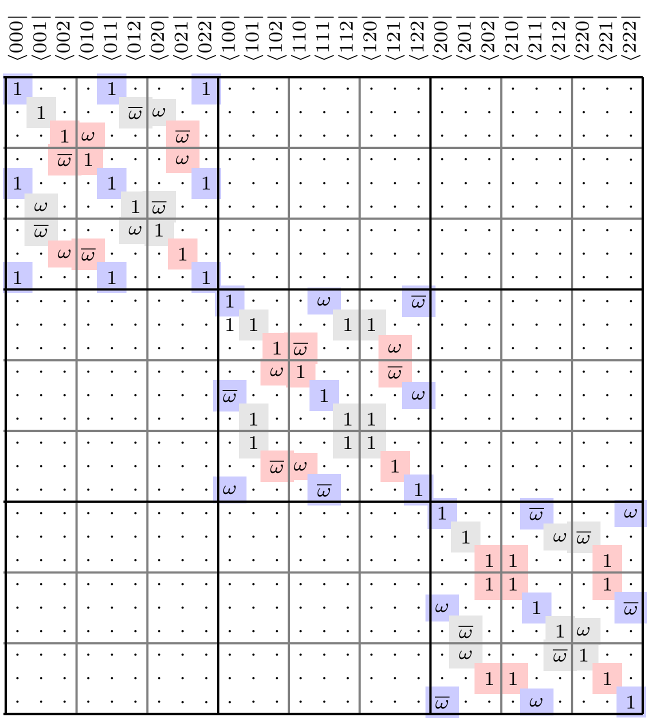

FIG. 3. The density matrix ρ_{345}[AME(5,d)'].

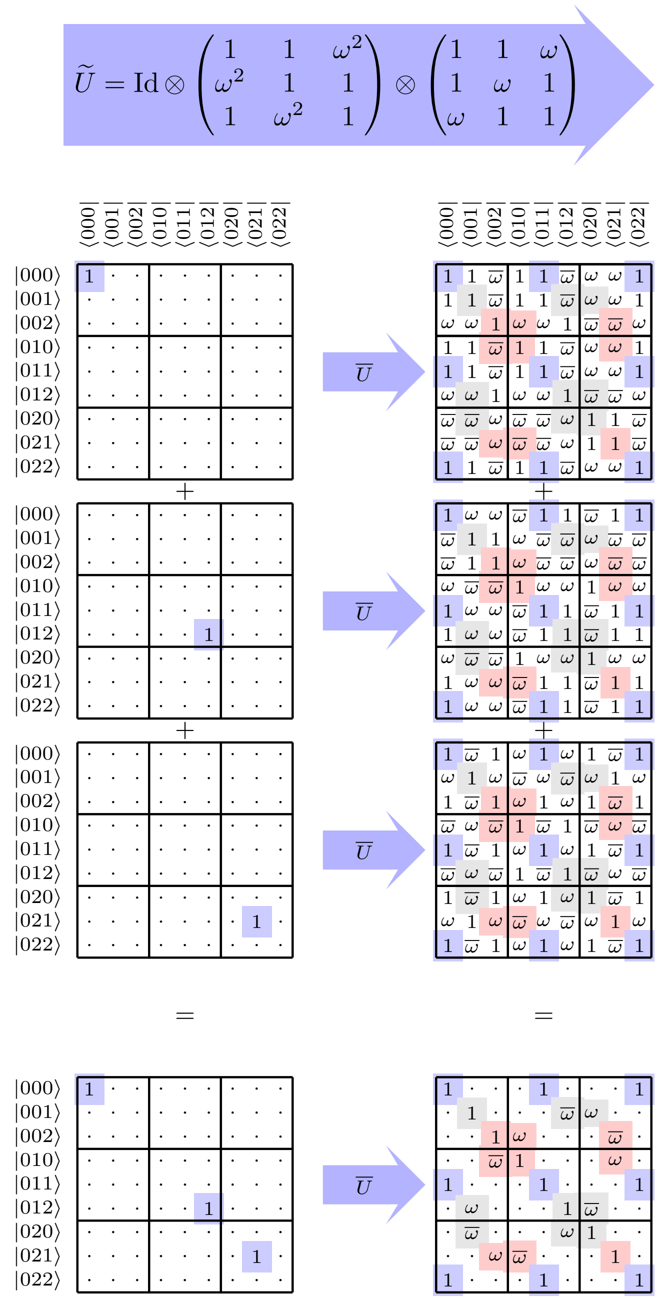

FIG. 4. The matrix Id $\otimes \widetilde{U}_4 \otimes \widetilde{U}_4$ transforms $\rho_{cde}(\psi)$ into $\rho_{cde}(\phi)$. The figure depicts how it acts on the first block of Fig. 2.

$$\widetilde{U}_3 = \mathrm{Id}_3,$$

$$\widetilde{U}_4 = \frac{1}{3}\begin{pmatrix} 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \end{pmatrix}, \quad \widetilde{U}_5 = \frac{1}{3}\begin{pmatrix} 1 & 1 & \omega \\ 1 & \omega & 1 \\ \omega & 1 & 1 \end{pmatrix}$$

provide a local equivalence between ρ_{345}[AME(5,3)'] and ρ_{345}[AME(5,3)]. This equivalence is illustrated in Fig. 4.

Similarly, for a local dimension $d = 5$, matrices $\widetilde{U}_3 = \mathrm{Id}_5$,

$$\widetilde{U}_4 = \frac{1}{5}\begin{pmatrix} 1 & 1 & \omega^4 & \omega^2 & \omega^4 \\ \omega^4 & 1 & 1 & \omega^4 & \omega^2 \\ \omega^2 & \omega^4 & 1 & 1 & \omega^4 \\ \omega^4 & \omega^2 & \omega^4 & 1 & 1 \\ 1 & \omega^4 & \omega^2 & \omega^4 & 1 \end{pmatrix},$$

and

$$\widetilde{U}_5 = \frac{1}{5}\begin{pmatrix} 1 & 1 & \omega & \omega^3 & \omega \\ \omega & \omega^3 & \omega & 1 & 1 \\ \omega & 1 & 1 & \omega & \omega^3 \\ 1 & \omega & \omega^3 & \omega & 1 \\ \omega^3 & \omega & 1 & 1 & \omega \end{pmatrix}$$

provide a local equivalence between ρ_{345}[AME(5,5)'] and ρ_{345}[AME(5,5)]. In order to provide the general formula, for any odd local dimension d, we introduce the following recursive construction of two $(d \times d)$-dimensional matrices:

$$W = (w_{ij}) \text{ and } V = (v_{ij})$$

with coefficients in GF(d).

(a) $w_{00} = 0$, $v_{00} = 0$.

(b) $w_{0(j+1)}$ is defined by the formula $w_{0(j+1)} = w_{0j} + 2j$; similarly, $v_{0(j+1)} = v_{0j} - 2j$ (definition of first rows).

(c) $w_{ij} := w_{(i-1)(j-1)}$; similarly, $v_{ij} := v_{(i-2)(j-1)}$ (definition of succeeding rows).

Notice that in order to define rows of the matrix V correctly we impose $2 \nmid d$. The matrices $\widetilde{U}_4$ and $\widetilde{U}_5$ are of the following form:

$$\widetilde{U}_4 = (\omega^{w_{ij}}) \text{ and } \widetilde{U}_5 = (\omega^{v_{ij}}).$$

This construction overlaps with the aforementioned constructions in dimensions $d = 3, 5$. In fact, there is a close formula for entries of matrices W and V (and hence for matrices $\widetilde{U}_4$ and $\widetilde{U}_5$) given in terms of triangular numbers:

$$w_{ij} = 2t_{j-i-1}, \quad v_{ij} = \begin{cases} -2t_{j-i/2-1} & \text{for } 2 \mid i \\ -2t_{j-(i+d)/2-1} & \text{for } 2 \nmid i \end{cases}$$

where $t_i = 0, 1, 3, 6, 10, 15, \ldots$ are consecutive triangular numbers defined as

$$t_k = \sum_{i=0}^{k} i = \frac{(k+1)k}{2}. \tag{D1}$$

Lemma 3. For any odd local dimension d, the matrices $\widetilde{U}_4$ and $\widetilde{U}_5$ provide the LU equivalence between ρ_{345}[AME(5,d)'] and ρ_{345}[AME(5,d)], i.e.,

$$\rho_{345}[\mathrm{AME}(5,d)'] = \mathrm{Id} \otimes \widetilde{U}_4 \otimes \widetilde{U}_5\{\rho_{345}[\mathrm{AME}(5,d)]\}.$$

Proof. Observe that

$$\sum_{s,j=0}^{d-1} |s, s+j, s+2j\rangle\langle s, s+j, s+2j| \xrightarrow{\mathrm{Id}_3 \otimes \widetilde{U}_4 \otimes \widetilde{U}_5}$$

$$\sum_{s=0}^{d-1} \sum_{\substack{m,m',k,\\ k'=0}}^{d-1} \left[\sum_{j=0}^{d-1} \omega^{(w_{(s+j)m}\overline{w_{(s+j)m'}}v_{(s+2j)k}\overline{v_{(s+2j)k'}})} |s, m, k\rangle\langle s, m', k'| \right].$$

We examine the coefficient by

$$|s, k+s+j, k\rangle\langle s, k'+s+j', k'|$$

in the expression above:

$$\frac{1}{d}\sum_{i=0}^{d-1} \omega^{(w_{(s+i)(k+j+s)}\overline{w}_{(s+i)(k'+j'+s)}v_{(s+2i)k}\overline{v}_{(s+2i)k'})}$$

$$= \frac{1}{d}\sum_{i=0}^{d-1} \omega^{2(t_{k+i-j-1} - t_{k'+j'-i-1} - t_{k-s/2-i-1}t_{k'-s/2-i-1})}, \tag{D2}$$

where we assumed $2 \mid s$; the argument for the opposite case, where $2 \nmid s$, is similar to the one we present. Using Eq. (D1), after elementary transformations Eq. (D2) is equal to

$$\frac{1}{d}\omega^{[s(k-k')+2jk-2j'k']}\omega^{[j(j-1)-j'(j'-1)]}\underbrace{\sum_{i=0}^{d-1}\omega^{[2i(j-j')]}}_{d\delta_{jj'}} = \omega^{(s+2j)(k-k')},$$

which remains to be shown. ■

Proof of Proposition 3. We show the statement by contradiction. Assume that states |AME(5,d)⟩ and |AME(5,d)'⟩ are LU equivalent by some unitary matrices:

$$U_1 \otimes U_2 \otimes U_3 \otimes U_4 \otimes U_5 =: U_{12} \otimes U_{345}.$$

We keep "⊗" in our notation in order to distinguish it from the matrix multiplication. Since the (partial) trace is invariant under cycling permutations, we have

$$\begin{aligned} &\rho_{345}[\mathrm{AME}(5,d)'] \\ &= \mathrm{tr}_{12}\, |\mathrm{AME}(5,d)'\rangle\langle \mathrm{AME}(5,d)'| \\ &= \mathrm{tr}_{12}\left[U_{12} \otimes U_{345}|\mathrm{AME}(5,d)\rangle\langle \mathrm{AME}(5,d)|U_{12}^{-1} \otimes U_{345}^{-1}\right] \\ &= U_{345}\left\{\mathrm{tr}_{12}\left[U_{12}|\mathrm{AME}(5,d)\rangle\langle \mathrm{AME}(5,d)|U_{12}^{-1}\right]\right\}U_{345}^{-1} \\ &= U_{345}\{\mathrm{tr}_{12}\,[|\mathrm{AME}(5,d)\rangle\langle \mathrm{AME}(5,d)|]\}U_{345}^{-1} \\ &= U_{345}\{\rho_{345}[\mathrm{AME}(5,d)]\}U_{345}^{-1}. \end{aligned}$$

Hence, the operator $U_{345} := U_3 \otimes U_4 \otimes U_5$ provides the local equivalence between ρ_{345}[AME(5,d)] and ρ_{345}[AME(5,d)'].

Notice that in Lemma 3 we pointed out that the earlier constructed matrices $\widetilde{U}_4$ and $\widetilde{U}_5$ provide the LU equivalence between ρ_{345}[AME(5,d)'] and ρ_{345}[AME(5,d)] for any odd local dimension d. Precisely,

$$\rho_{345}[\mathrm{AME}(5,d)] = \mathrm{Id} \otimes \widetilde{U}_4 \otimes \widetilde{U}_5\{\rho_{345}[\mathrm{AME}(5,d)']\}.$$

Therefore from Proposition 2 we conclude that

$$U_3 = M_3, \quad U_4 = M_4\widetilde{U}_4, \quad U_5 = M_5\widetilde{U}_5,$$

for some monomial matrices M_3, M_4, and M_5. Indeed, $U_{345}(\mathrm{Id} \otimes \widetilde{U}_4 \otimes \widetilde{U})^{-1}$ constitutes an automorphism of ρ_{345}[AME(5,d)'] and hence, by Proposition 2, is a tensor product of monomial matrices. We shall prove that such restriction on matrices U_3, U_4, and U_5 leads to a contradiction.

To sum up the discussion so far, LU equivalence between states $|\mathrm{AME}(5,d)\rangle$ and $|\mathrm{AME}(5,d)'\rangle$ has the following form:

$$U_1 \otimes U_2 \otimes M_3 \otimes M_4\widetilde{U}_4 \otimes M_5\widetilde{U}_5$$

where U_i are arbitrary unitary matrices, while M_i is the product of diagonal and permutation matrices. Therefore

$$|\mathrm{AME}(5,d)\rangle = (U_1 \otimes U_2)|i, j\rangle \otimes M_3|i + j\rangle \otimes B_{ij} \quad \text{(D3)}$$

where

$$B_{ij} := (M_4\widetilde{U}_4 \otimes M_5\widetilde{U}_5)|i + 2j, i + 3j\rangle.$$

Observe that B_{ij} are linearly independent. Indeed, they are unitary transformed linearly independent vectors $|i + 2j, i + 3j\rangle$.

We shall show that matrices U_1 and U_2 are monomial matrices. Suppose for simplicity that $M_3 = \mathrm{Id}$. Recall that $|\mathrm{AME}(5,d)\rangle$ has the following form:

$$|\mathrm{AME}(5,d)\rangle = |i, j, i + j\rangle \otimes C_{ij}, \quad \text{(D4)}$$

where

$$C_{ij} = \omega^{(i+3j)k}|k + 2j, k\rangle.$$

We compare this expression with Eq. (D3). Suppose now that in some column of the matrix U_1 there are at least two nonzero elements: $u^1_{lk}, u^1_{l'k}$ $(l \neq l')$; consider some nonzero element u^2_{nm} of the matrix U_2. Observe that it leads to the expressions

$$|l, n\rangle \otimes |k + m\rangle \otimes B_{km} \quad \text{and} \quad |l', n\rangle \otimes |k + m\rangle \otimes B_{km}$$

in Eq. (D3). Clearly, there is an additional contribution from other nonzero elements of matrices U_1 and U_2. Since B_{ij} are linearly independent, there are the terms

$$|l, n\rangle \otimes |k + m\rangle \otimes D_{ln} \quad \text{and} \quad |l', n\rangle \otimes |k + m\rangle \otimes D_{l'n}$$

in Eq. (D3), where D_{ln} and $D_{l'n}$ are some nonzero elements. Observe that such terms might appear in Eq. (D4) only if $l + n = k + m$ and $l' + n = k + m$, which is contradictory to $l \neq l'$. Similarly, one can show that none of the matrix U_1 columns have two nonzero elements. We have shown that, indeed, matrices U_1 and U_2 are monomial under the assumption $M_3 = \mathrm{Id}$. Nevertheless, the assumption $M_3 = \mathrm{Id}$ is not essential here; a similar argument might be given for arbitrary monomial matrix M_3. Hence U_1 and U_2 are monomial matrices in general.

Observe that $\mathrm{supp}(B_{ij}) = d^2$. Indeed,

$$\mathrm{supp}(\widetilde{U}_4 \otimes \widetilde{U}_5|i + 2j, i + 3j\rangle) = d^2,$$

and the monomial operators M_4 and M_5 do not change the support. Since U_1 and U_2 are monomial matrices, the support of the right-hand side in Eq. (D3) is equal to d^4. This is contradictory to the fact that $\mathrm{supp}[|\mathrm{AME}(5,d)\rangle] = d^3$. ■

We have shown that two families of AME(5,d) states are not LU and SLOCC equivalent. Even though only a special family of states is considered here, analysis of the proof of Proposition 3 reveals the general method of LU verification of AME and k-uniform states where one of them is written with minimal support. First, the formula for LU equivalence between reduced $(k + 1)$-dimensional systems should be provided. Second, based on Proposition 2, one can classify such equivalences between those reduced states. Finally, it should be shown that none of such equivalences can be extended to the local equivalence of initial states.

[1] V. Giovannetti, S. Mancini, D. Vitali, and P. Tombesi, Phys. Rev. A **67**, 022320 (2003).
[2] B. Kraus, Phys. Rev. A **82**, 032121 (2010).
[3] W. Helwig, W. Cui, J. I. Latorre, A. Riera, and H.-K. Lo, Phys. Rev. A **86**, 052335 (2012).
[4] W. Helwig and W. Cui, arXiv:1306.2536.
[5] F. Pastawski, B. Yoshida, D. C. Harlow, and J. Preskill, J. High Energy Phys. 06 (2015) 149.
[6] E. M. Rains, IEEE Trans. Inf. Theory **45**, 1827 (1999).
[7] W. Helwig, arXiv:1306.2879.
[8] A. J. Scott, Phys. Rev. A **69**, 052330 (2004).
[9] A. Higuchi and A. Sudbery, Phys. Lett. A **273**, 213 (2000).
[10] F. Huber, O. Gühne, and J. Siewert, Phys. Rev. Lett. **118**, 200502 (2017).
[11] Z. Raissi, C. Gogolin, A. Riera, and A. Acín, J. Phys. A: Math. Theor. **51**, 075301 (2017).
[12] D. Goyeneche, Z. Raissi, S. Di Martino, and K. Życzkowski, Phys. Rev. A **97**, 062326 (2018).
[13] D. Goyeneche and K. Życzkowski, Phys. Rev. A **90**, 022316 (2014).
[14] L. Arnaud and N. J. Cerf, Phys. Rev. A **87**, 012319 (2013).
[15] M. A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).
[16] E. Chitambar, D. Leung, L. Mancinska, M. Ozols, and A. Winter, Commun. Math. Phys. **328**, 303 (2012).
[17] E. Chitambar, Phys. Rev. Lett. **107**, 190502 (2011).
[18] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[19] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A **65**, 052112 (2002).
[20] M. Gharahi Ghahi and S. Mancini, Phys. Rev. A **98**, 066301 (2018).
[21] A. Sudbery, J. Phys. A: Math. Gen. **34**, 643 (2001).
[22] J. Wang, M. Li, S.-M. Fei, and L.-J. Xian-Qing, Commun. Theor. Phys. **62**, 673 (2014).
[23] M. Grassl, M. Rötteler, and T. Beth, Phys. Rev. A **58**, 1833 (1998).
[24] P. Vrana and P. Lévay, J. Phys. A: Math. Theor. **43**, 125303 (2010).
[25] P. Vrana, J. Phys. A: Math. Theor. **44**, 225304 (2011).
[26] T.-G. Zhang, M.-J. Zhao, M. Li, S.-M. Fei, and X. Li-Jost, Phys. Rev. A **88**, 042304 (2013).
[27] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[28] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).

[29] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[30] J.-G. Luque and J.-Y. Thibon, Phys. Rev. A **67**, 042303 (2003).
[31] M. S. Leifer, N. Linden, and A. Winter, Phys. Rev. A **69**, 052304 (2004).
[32] A. Osterloh and J. Siewert, Phys. Rev. A **72**, 012337 (2005).
[33] S. Szalay, J. Phys. A: Math. Theor. **45**, 065302 (2012).
[34] G. Gour and N. R. Wallach, Phys. Rev. Lett. **111**, 060502 (2013).
[35] D. Gottesman, arXiv:quant-ph/9705052.
[36] D. Perez-Garcia, F. Verstraete, M. Wolf, and J. Cirac, Quantum Inf. Comput. **7**, 401 (2007).
[37] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).
[38] C. Spee, K. Schwaiger, G. Giedke, and B. Kraus, Phys. Rev. A **97**, 042325 (2018).
[39] C. Kruszynska and B. Kraus, Phys. Rev. A **79**, 052304 (2009).
[40] B. Wu, J. Jiang, J. Zhang, G. Tian, and X. Sun, Phys. Rev. A **98**, 022304 (2018).
[41] Z. Raissi, A. Teixido, C. Gogolin, and A. Acin, arXiv:1910.12789.
[42] W. Huang and Z. Wei, Phys. Rev. A **78**, 024304 (2008).
[43] D. Goyeneche, D. Alsina, J. I. Latorre, A. Riera, and K. Zyczkowski, Phys. Rev. A **92**, 032316 (2015).
[44] C. Spee, J. I. de Vicente, and B. Kraus, J. Math. Phys. **57**, 052201 (2016).
[45] A. S. Hedayat, N. J. A. Sloane, and J. Stufken, *Orthogonal Arrays, Theory and Applications* (Springer-Verlag, New York, 1999).
[46] R. Koselka, Forbes **118**, 114 (1996).
[47] K. A. Bush, Ann. Math. Statist. **23**, 426 (1952).
[48] S. Hedayat, J. Stufken, and G. Su, Ann. Stat. **25**, 2044 (1997).
[49] S. Yagi, K. Mimura, and M. Jimbo, J. Stat.: Planning Inference **138**, 3309 (2008).
[50] N. J. A. Sloane, Library of Orthogonal Arrays, available at http://neilsloane.com/oadir.
[51] G. Kempf and L. Ness, Lect. Notes Math. **732**, 233 (1979).
[52] G. Gour and N. Wallach, New J. Phys. **13**, 073013 (2011).
[53] G. Gour and N. Wallach, J. Math. Phys. **51**, 112201 (2010).
[54] O. Słowik, M. Hebenstreit, B. Kraus, and A. Sawicki, Quantum **4**, 300 (2020).
[55] N. R. Wallach, Lectures on quantum computing Venice CIME (2004), http://www.math.ucsd.edu/~nwallach/venice.pdf.
[56] S. Y. Looi and R. B. Griffiths, Phys. Rev. A **84**, 052306 (2011).
[57] M. Bahramgiri and S. Beigi, arXiv:quant-ph/0610267 (2016).
[58] M. Gachechiladze, N. Tsimakuridze, and O. Gühne, J. Phys. A: Math. Theor. **50**, 19LT01 (2017).
[59] E. Chitambar, R. Duan, and Y. Shi, Phys. Rev. Lett. **101**, 140502 (2008).
[60] A. T. Butson, Proc. Am. Math. Soc. **13**, 894 (1962).
[61] W. Tadej and K. Zyczkowski, Open Syst. Inf. Dyn. **13**, 133 (2006).
[62] C. Sackett, D. Kielpinski, B. King, C. Langer, V. Meyer, C. Myatt, M. Rowe, Q. Turchette, W. Itano, D. Wineland, and C. Monroe, Nature (London) **404**, 256 (2000).
[63] A. Bergschneider, V. Klinkhamer, J. Becher, R. Klemt, L. Palm, G. Zürn, S. Jochim, and P. Preiss, Nat. Phys. **15**, 640 (2019).
[64] P. H. J. Lampio, P. Östergård, and F. Szöllősi, arXiv:1707.02287.
[65] P. Lampio, Library of Butson matrices, available at https://wiki.aalto.fi/display/Butson.
[66] W. T. Wojciech Bruzda and K. Życzkowski, Library of Butson matrices, available at https://chaos.if.uj.edu.pl/~karol/hadamard/.
[67] W. Tadej, Open Syst. Inf. Dyn. **26**, 1950003 (2019).
[68] A. Bernal, Quantum Phys. Lett. **6**, 1 (2017).
[69] F. Huber and N. Wyderka, Table of AME states, available at https://www.tp.nt.uni-siegen.de/+fhuber/ame.html.
[70] T. Paterek, M. Pawlowski, M. Grassl, and C. Brukner, Phys. Scr. **T140**, 014031 (2010).
[71] F. Huber, C. Eltschka, J. Siewert, and O. Gühne, J. Phys. A: Math. Theor. **51**, 175301 (2018).
[72] L. Euler, in *Verhandelingen Uitgegeven Door Het Zeeuwsch Genootschap der Wetenschappen te Vlissingen* (Middelburg, 1782), Vol. 9, pp. 85–239; also published in *Commentationes Arithmeticae* (1849), Vol. 2, pp. 302-361, available online in The Euler Archive, http://eulerarchive.maa.org/pages/E530.html.
[73] G. Tarry, Compte Rendu de l'Association Française pour l'Avancement des Sci. **1**, 122 (1990).
[74] H. J. Ryser, Proc. Amer. Math. Soc. **2**, 550 (1951).
[75] S. Boyadzhiyska, S. Das, and T. Szabó, arXiv:1910.02753.
[76] B. McKay and I. Wanless, SIAM J. Discrete Math. **22**, 719 (2008).
[77] D. Bulutoglu and F. Margot, J. Stat.: Planning Inference **138**, 654 (2008).
[78] J. C. Wang and C. F. J. Wu, Technometrics **34**, 409 (1992).
[79] J. Stufken and B. Tang, Ann. Statist. **35**, 793 (2007).