Emergence of a geometric phase shift in planar noncommutative quantum mechanics

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Appearance of adiabatic geometric phase shift in the context of noncommutative quantum mechanics is studied using an exactly solvable model of a two-dimensional simple harmonic oscillator in the Moyal plane, where momentum noncommutativity are also considered along with spatial noncommutativity. For that we introduce a modified form of Bopp's shift, that bridges the noncommutative phase-space operators with their effective commutative counterparts, having their dependence on the noncommutative parameters, and study the adiabatic evolution in the Heisenberg picture. An explicit expression for the geometric phase shift under adiabatic approximation is then found without using any perturbative technique. Lastly, this phase is found to be related to Hannay's angle of a classically analogous system, by studying the evolution of the coherent state of this system.

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I. INTRODUCTION

Ever since Berry observed the occurrence of geometrical phase obtained in the adiabatic transport of a quantum system around a closed loop in the parameter space, the concept of Berry phase has attracted great interest both theoretically [1,2] as well as experimentally [3,4]. Particularly it has been observed in certain condensed-matter systems that this phase can give rise to effective noncommutative (NC) structure among the coordinates and thereby impact physics [5–7]. Additionally, the occurrence of noncommutativity in the lowest Landau level in the Landau problem is quite well known. Apart from these effective low-energy effects, there are very strong plausibility arguments due to Doplicher *et al.* [8,9] that this noncommutative algebra satisfied by spatiotemporal coordinates [10], when elevated to the level of operators, can naturally serve as a "deterrent" against gravitational collapse, associated with the localization of an event at the Planck scale. Here the status of the noncommutative parameters is more fundamental, as if they are new constants of nature, like $\hbar, G, c$, etc. [11], and possibly can play a vital role in the development of a future theory of quantum gravity [12]. This aspect was also corroborated in a separate study of the low-energy limit of string theory by Seiberg and Witten [13].

Besides this above-mentioned noncommutative structure among the space-time coordinates, it has also been proposed that, along with the spatial components of space-time, momentum components too can satisfy a noncommutative algebraic structure [14–16]. This was already indicated by the reciprocity theorem proposed by Born in 1938 [17]. In the same spirit it was observed in [18,19] that the noncommutative structure among the spatial and momentum components can be related to the respective curvatures in momentum and coordinate spaces, respectively. Further, it was observed in [20] that, to maintain Bose-Einstein statistics in noncommutative spaces, one needs to introduce noncommutative momenta as well. It was also indicated in the literature [21–23] that a nonrelativistic system in $(2+1)$ dimensions admitting fractional spin can exhibit Galilean symmetry through a twofold centrally extended Galilean algebra, where one involves the commutator of the boost generators K_i between themselves, which is a nonvanishing constant $[K_1, K_2] = i\hbar\kappa$, and the other involves boost and linear momentum, which gives the mass m: $[K_i, P_j] = i\hbar m\delta_{ij}$, with other commutators taking their usual forms. Here κ can be associated with the fractional spin of the anyons, as has been shown in [24,25], through certain nonrelativistic reduction of (2+1)-dimensional [(2+1)D] Poincaré algebra iso (2, 1). A minimalistic realization of this twofold centrally extended algebra in terms of the covariantly transforming particle coordinate $\hat{x}_i$ under linear or angular momenta (P_i, J) and boost K_i for zero values of the pair of Casimir operators can only be provided if $\hat{x}_i$'s satisfy a NC algebra of the form $[\hat{x}_i, \hat{x}_j] = i\theta\epsilon_{ij}$ with $\kappa = m^2\theta$ being a constant [26]. Further, if the planar system is now subjected to a static and uniform magnetic field B, then the momentum components too become noncommutative with further deformation in the $[\hat{p}_1, \hat{p}_2]$ commutator [27,28].

In fact the sheer presence of momentum noncommutativity alone can have nontrivial astrophysical consequences, like enhancing the Chandrasekhar mass limit for the white dwarf stars, as has been shown by one of the authors very recently in [29]. The phase-space noncommutative structure has also been shown to emerge naturally in certain systems in an enlarged phase-space analysis [30]. It is therefore quite natural to investigate the occurrence of Berry phase, if any, in a quantum-mechanical system where both position and momentum operators satisfy noncommutative algebra. This will then serve, in some sense, as the converse of the case where the occurrence of Berry phase can give rise to noncommutative algebra [5], as mentioned earlier. In order to undertake this study in this paper, we find it convenient to consider the

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simplest system of a harmonic oscillator lying in the Moyal plane, where the momentum components are also taken to be noncommutative along with the position coordinates. Further, to compute Berry phase, we make the mass and "spring constant" time dependent, varying adiabatically with time period T. There are precedents of such studies in the literature. For that one may cite the examples of the well-known Paul trap in [31–33] where the spring constant or frequency is time dependent and in [34,35] where the mass is also time dependent. In fact, our model (1) below was inspired by these previous works, although the Paul trap will not be directly applicable in our case, as it employs a varying magnetic field. Through a detailed analysis, we shall indeed show in this paper that occurrence of both types of noncommutativity in this quantum system plays a vital role for the existence of nonvanishing geometric phase shift for the system.

In this context, we would like to mention that some authors [36] had also considered a similar problem earlier, though in a different system involving a gravitational potential well, and found no geometric contribution in the total adiabatic phase. However, there are certain basic differences between our considerations and theirs. First of all, their original system, involving a gravitational potential well, was commutative in nature to begin with and thereby possessed time-reversal symmetry. On the other hand, it is known that the breaking of time-reversal symmetry plays a necessary (but not sufficient) role in the existence of Berry's phase from [37,38], the authors of which also introduced noncommutativity just by substituting the commutative variables by their NC counterparts, using the inverse of the Bopp shift (or the Seiberg-Witten map in the parlance of [36]). Therefore, here the occurrence of noncommutativity in the rewritten version does not have any fundamental status; rather its occurrence is a bit contrived in nature. More importantly, it has been pointed out in [39] that their particular form of noncommutative algebra and Bopp shift is equivalent to a scaled version of (A1) and its realization (A2) only for the scale factor ξ_c^{-1} as in (16). Consequently, the above-mentioned Bopp shift (A2) becomes very restrictive in nature, in the sense that it provides a realization of the one parameter (ξ) family of phase-space commutation relations (A1) in terms of commutative phase-space variables (A2), only for a specific and critical value of $\xi = \xi_c$ (A3). In contrast, our realization (A4) holds for any arbitrary value of ξ. And it is only for $\xi = \xi_c$ that these two realizations are unitarily equivalent. This has been elaborated in Appendix A. In fact, as we show in the sequel (see the fourth point in Sec. IV), the geometrical phase vanishes for that critical value of $\xi = \xi_c$, which can be shown by using any one of the realizations (A2) or (A4). And for other values of $\xi \neq \xi_c$ there is a nonvanishing geometrical phase. This can be clearly shown by making use of our realization (A4) only, which is solely responsible for the generation of a crucial dilatation term, the presence of which, as we show below, is quite indispensable for getting the desired geometrical phase shift.

Furthermore, we would also like to point out that the issue of Berry phase in noncommutative space was analyzed in [40] also, albeit in the absence of any momentum space noncommutativity; only position-position noncommutativity was considered. Additionally, there the author basically computed the first-order noncommutative correction, by expanding the Moyal star product up to $O(\theta)$, to the already existing geometrical phase shift. This is quite in contrast to our analysis, where we show how the presence of phase-space noncommutativity itself can be a source for generating nonvanishing Berry phase in an otherwise simple quantum-mechanical system, in the sense that this phase shift does not survive when the commutative limit is taken. And this could be accomplished only through the form of the Bopp shift we have introduced, as mentioned above. Additionally, we have also computed the geometric phase shift in the Heisenberg picture. This served a dual purpose as it not only enabled us to virtually read off the quantal Berry phase (we also showed how to obtain this by using the more well-known approach of obtaining the Berry phases through the one acquired by the state vectors after time evolving in the Schrödinger picture) corresponding to our system of interest, just by looking at the extra phase factor occurring over and above the dynamic phase by adiabatically transporting the ladder operators of our system Hamiltonian, but also enabled us to identify the corresponding Hannay angle in a rather straightforward manner. It therefore helps us to "kill two birds with one stone."

Moreover, an inter-relation between the extra quantal geometric phase, apart from the dynamical phase, in the wave function in the quantum description and the corresponding angle shift at classical level, was established by Berry through semiclassical torus quantization [1]. The change of the angle was found to be related to the rate of change of the extra phase with respect to the quantum number of the state which is being transported adiabatically. This can be viewed as a manifestation of Bohr's correspondence principle for phases arising through adiabatic transports in the respective quantum and classical systems. However, in the present paper, we shall establish this classical correspondence by extending Berry's analysis to nonstationary coherent states, in the spirit of [41–43], representing localized nonspreading wave packets which are being transported along classical trajectories.

The paper is organized as follows. We begin by introducing in Sec. II noncommutative phase-space operators where we consider momentum space noncommutativity, along with the spatial ones of Moyal type. Here we also compute the instantaneous energy spectrum of a two-dimensional (2D) harmonic oscillator the mass and frequency parameters of which are slowly varying with respect to time by making use of a generalized noncanonical phase-space transformation, which we refer to as generalized Bopp shift (see Appendix A), which maps the noncommutative phase-space variables to their commutative counterparts. In Sec. III we find, in the Heisenberg picture, the extra phase factor which is acquired by the creation and annihilation operators under an adiabatic excursion in parameter space of the system. We then discuss the geometric phase shift in state space of the oscillator in Sec. IV and provide a contact between quantum geometric phase shift and classical Hannay angle in Sec. V. Finally, we conclude the paper in Sec. VI. Lastly, in Appendix A we discuss different types of Bopp shifts, i.e., different realizations of noncommutative algebra and their relations with one another, and in Appendix B, apart from reviewing some of the necessary group theoretical aspects related to our model, we show that although the dilatation term in the Hamiltonian can apparently be transformed away by a time-dependent unitary transformation it nevertheless reappears in "disguise" in the dynamical term, albeit retaining its geometrical nature.

II. PLANAR NONCOMMUTATIVE SYSTEM

In this paper, we are basically considering a 2D harmonic oscillator on the Moyal plane, with time-dependent coefficients $P(t)$, $Q(t)$ varying adiabatically with period T:

$$\mathcal{H}(t) = P(t)\big(\hat{p}_1^2 + \hat{p}_2^2\big) + Q(t)\big(\hat{x}_1^2 + \hat{x}_2^2\big) \tag{1}$$

such that $P(t), Q(t) > 0$ and these time-dependent parameters are assumed to subsume all other parameters like mass and frequency as mentioned in the previous section. This model is constructed in the spirit of [31–35] where the plausibility of having time-dependent parameters $P(t)$ and $Q(t)$ in a real physical system was demonstrated. We are further assuming that the momentum components also satisfy a noncommutative algebra [44–47] in addition to the position coordinates, so that the entire noncommutative structure takes the following form:

$$[\hat{x}_i, \hat{x}_j] = i\theta\epsilon_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\eta\epsilon_{ij},$$
$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad \theta\eta < 0. \tag{2}$$

Note that we need to enforce $\theta\eta$ to be negative for consistent quantization. See, for example, [48,49] and references therein.

In order to carry out the diagonalization, we introduce below a type of Bopp shift, providing a realization of the above algebra (2) through a linear map

$[(\hat{x}_i, \hat{p}_j) \rightarrow (q_i, p_j); i, j = 1, 2]$, and we refer to this as the generalized Bopp shift (see Appendix A for other kinds of Bopp shift [36,39,50] and their relation to the one given below):

$$\hat{x}_i = q_i - \frac{\theta}{2\hbar}\epsilon_{ij}p_j + \frac{\sqrt{-\theta\eta}}{2\hbar}\epsilon_{ij}q_j,$$
$$\hat{p}_i = p_i + \frac{\eta}{2\hbar}\epsilon_{ij}q_j + \frac{\sqrt{-\theta\eta}}{2\hbar}\epsilon_{ij}p_j, \tag{3}$$

where q_i and p_i are commuting coordinates and momenta, respectively, satisfying the usual Heisenberg algebra, $[q_i, q_j] = 0 = [p_i, p_j]$, $[q_i, p_j] = i\hbar\delta_{ij}$, and are distinguished by the absence of overhead hats. Although this transformation (A7) is not a canonical one, it nevertheless helps us in diagonalizing the Hamiltonian. Substituting (3) in (1) results in the following form of the Hamiltonian:

$$\mathcal{H}(t) = \alpha(t)\big(p_1^2 + p_2^2\big) + \beta(t)\big(q_1^2 + q_2^2\big) + \delta(t)(p_iq_i + q_ip_i) - \gamma(t)(q_1p_2 - q_2p_1), \tag{4}$$

where the time-dependent coefficients $\alpha, \beta, \gamma, \delta$ are given by

$$\alpha(t) = P(t)\left\{1 - \frac{\theta\eta}{4\hbar^2}\right\} + Q(t)\left(\frac{\theta}{2\hbar}\right)^2,$$
$$\beta(t) = Q(t)\left\{1 - \frac{\theta\eta}{4\hbar^2}\right\} + P(t)\left(\frac{\eta}{2\hbar}\right)^2,$$
$$\gamma(t) = \frac{1}{\hbar}[\eta P(t) + \theta Q(t)],$$
$$\delta(t) = \left(\frac{\sqrt{-\theta\eta}}{4\hbar^2}\right)[\eta P(t) - \theta Q(t)]. \tag{5}$$

At this stage we recognize the Hamiltonian as a combination of three terms:

$$\mathcal{H}(t) = \mathcal{H}_{\mathrm{gho},1}(t) + \mathcal{H}_{\mathrm{gho},2}(t) + \mathcal{H}_{L}(t) \tag{6}$$

where $\mathcal{H}_{\mathrm{gho},i}(t)$'s ($i = 1$ or 2) are like generalized time-dependent harmonic oscillator Hamiltonians along the ith direction,

$$\mathcal{H}_{\mathrm{gho},i}(t) = \alpha(t)\big(p_i^2\big) + \beta(t)\big(q_i^2\big) + \delta(t)(p_iq_i + q_ip_i) \quad (\text{no sum on } i), \tag{7}$$

and

$$\mathcal{H}_{L}(t) = -\gamma(t)(q_1p_2 - q_2p_1), \tag{8}$$

which is like a Zeeman term. In order to diagonalize the whole Hamiltonian, we first need to diagonalize $\mathcal{H}_{\mathrm{gho},i}(t)$ of (7) for each i [51–53], so that these Hamiltonians can be brought into the form $\mathcal{H}_{\mathrm{gho},i}(t) = X(t)(\boldsymbol{a}_i^\dagger\boldsymbol{a}_i + 1)$ (no sum on i). To that end we introduce annihilation (and corresponding creation) operators $\boldsymbol{a}_1, \boldsymbol{a}_2$ with the following structure:

$$\boldsymbol{a}_j = \left(\frac{\beta}{2\hbar\sqrt{\alpha\beta - \delta^2}}\right)^{1/2}\left[q_j + \left(\frac{\delta}{\beta} + i\frac{\sqrt{\alpha\beta - \delta^2}}{\beta}\right)p_j\right],$$
$$j = 1, 2, \tag{9}$$

satisfying $[\boldsymbol{a}_i, \boldsymbol{a}_j^\dagger] = \delta_{ij}$. Note that we have $\beta > 0$ and $\alpha\beta - \delta^2 = (\frac{P\eta}{2\hbar} - \frac{Q\theta}{2\hbar})^2 + PQ > 0$, as follows from (5) and from the fact that $PQ > 0$. The entire Hamiltonian (4) then takes the following form:

$$\mathcal{H}(t) = \hbar\omega\left(\sum_{j=1,2}\boldsymbol{a}_j^\dagger\boldsymbol{a}_j + 1\right) + i\hbar\gamma(\boldsymbol{a}_1^\dagger\boldsymbol{a}_2 - \boldsymbol{a}_2^\dagger\boldsymbol{a}_1),$$
$$\omega = 2\sqrt{\alpha\beta - \delta^2}. \tag{10}$$

Noting at this stage [54] that the second nondiagonal term is like the Jordan-Schwinger representation of the J_2 angular momentum operator $[\vec{J} = \boldsymbol{a}_i{}^\dagger(\vec{\sigma})_{ij}\boldsymbol{a}_j]$, we need to carry out another additional unitary transformation of the following form, which can bring the term into the exact diagonal form of J_3, while retaining the diagonal form of the first term:

$$\begin{bmatrix}\boldsymbol{a}_1\\ \boldsymbol{a}_2\end{bmatrix} \rightarrow \begin{bmatrix}\boldsymbol{a}_+\\ \boldsymbol{a}_-\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ i & -1\end{bmatrix}\begin{bmatrix}\boldsymbol{a}_1\\ \boldsymbol{a}_2\end{bmatrix}, \tag{11}$$

$$[\boldsymbol{a}_i, \boldsymbol{a}_j^\dagger] = \delta_{ij}, \quad [\boldsymbol{a}_i, \boldsymbol{a}_j] = 0 \quad (i, j \in \{+, -\}). \tag{12}$$

Finally the diagonalized Hamiltonian in the standard quadratic form reads

$$\mathcal{H}(t) = \hbar\sum_{j=+,-}\omega_j\boldsymbol{a}_j{}^\dagger\boldsymbol{a}_j + \hbar\omega, \quad \omega_\pm = \omega \mp \gamma. \tag{13}$$

Note that here we have identified two characteristic frequencies $\omega_\pm$ of the system.

The eigenvalue equation of this Hamiltonian is

$$\mathcal{H}(t)|n_1, n_2; t\rangle = E_{n_1n_2}(t)|n_1, n_2; t\rangle, \tag{14}$$

the solution spectrum of which can virtually be read off from (13) as

$$E_{n_1n_2}(t) = \hbar\omega(n_1+n_2+1) - \hbar\gamma(n_1-n_2),$$
$$|n_1,n_2;t\rangle = \frac{(\boldsymbol{a}_+^\dagger)^{n_1}(\boldsymbol{a}_-^\dagger)^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}}|0,0;t\rangle \tag{15}$$

where n_1, n_2 are semipositive definite integers and $\boldsymbol{a}_\pm(t)|0,0;t\rangle = 0$. This reproduces the spectrum obtained in [55]. Clearly the spectrum is nondegenerate and one can safely assume that there will not be any level crossing during the adiabatic process. In this context, we would like to point out that essentially the same system was analyzed in [44] but using Bartelomi's realization. One can easily check that both the algebra and spectrum in [44] agree with (2) and (15), respectively, by making the following simple formal replacements in (A1) and (A2):

$$\theta \to \xi^{-1}\theta, \quad \eta \to \xi^{-1}\eta, \quad \hbar \to \hbar_{\text{eff}} = \hbar\xi^{-1}, \tag{16}$$

with $\xi = \xi_c$ (A3).

Finally let us write down our previous operators $\boldsymbol{a}_1, \boldsymbol{a}_2$ of (9) in short as

$$\boldsymbol{a}_i = A(t)\{q_i + [B(t) + iC(t)]p_i\} \ (i \in \{1,2\})$$
$$\text{where } A(t) = \left(\frac{\beta}{\hbar\omega}\right)^{1/2}, \quad B(t) = \frac{\delta}{\beta}, \quad C(t) = \frac{\omega}{2\beta}. \tag{17}$$

We see that $\boldsymbol{a}_\pm, \boldsymbol{a}_\pm^\dagger$ has explicit time dependence through the time dependence of A, B, C.

III. HEISENBERG EVOLUTION OF LADDER OPERATORS

In this section we will solve the adiabatic evolution in the Heisenberg picture to look for the geometric phase shift. The equation of motion for a generic operator $\hat{O}$ is given by $\frac{d\hat{O}}{dt} = \frac{1}{i\hbar}[\hat{O}, \mathcal{H}] + \frac{\partial\hat{O}}{\partial t}$. Specifically, for the ladder operators, they take the following forms:

$$\frac{d}{dt}\begin{bmatrix} a_+ \\ a_- \\ a_+^\dagger \\ a_-^\dagger \end{bmatrix} = \begin{bmatrix} X_+ & 0 & 0 & Y \\ 0 & X_- & Y & 0 \\ 0 & Y^* & X_+^* & 0 \\ Y^* & 0 & 0 & X^* \end{bmatrix}\begin{bmatrix} a_+ \\ a_- \\ a_+^\dagger \\ a_-^\dagger \end{bmatrix},$$
$$X_\pm = \frac{\dot{A}}{A} \pm i\left(\gamma \mp 2C\beta \mp \frac{(\dot{B}+i\dot{C})}{2C}\right),$$
$$Y = -\frac{(\dot{B}+i\dot{C})}{2C}. \tag{18}$$

Up to now all the expressions we have found are exact. However, here onwards we will start considering the adiabaticity of $P(t)$ and $Q(t)$. Note, from the dependence of $A, B, C, \alpha, \beta, \gamma, \delta$ on $P(t), Q(t)$ we anticipate that they also follow the same order of adiabaticity as P and Q, i.e., if $\dot{P}, \dot{Q} \approx \epsilon$, $\ddot{P}, \ddot{Q} \approx \epsilon^2 \ldots$, then $\dot{F} \approx \epsilon$, $\ddot{F} \approx \epsilon^2 \ldots$, where F collectively stands for $A, B, C, \alpha, \beta, \gamma, \delta$. We will not omit any term under adiabatic approximation right now, but will only keep track of the order of various terms. Eventually, it will be clear that it is the second- or higher-order terms which are ignorable [56,57].

We can now decouple the four coupled equations occurring in (18) by taking the derivatives of these equations and combining them suitably to get

$$\frac{d^2\boldsymbol{a}_+}{dt^2} = \frac{d\boldsymbol{a}_+}{dt}\left(X_+ + \frac{\dot{Y}}{Y} + X_-^*\right)$$
$$+ \boldsymbol{a}_+\left(\dot{X}_+ - \frac{\dot{Y}}{Y}X_+ + YY^* - X_+X_-^*\right),$$
$$\frac{d^2\boldsymbol{a}_-}{dt^2} = \frac{d\boldsymbol{a}_-}{dt}\left(X_- + \frac{\dot{Y}}{Y} + X_+^*\right)$$
$$+ \boldsymbol{a}_-\left(\dot{X}_- - \frac{\dot{Y}}{Y}X_- + YY^* - X_-X_+^*\right). \tag{19}$$

We can make some important observations here: if $Y \approx \epsilon$, then $\dot{Y} \approx \epsilon^2$, and so $\frac{\dot{Y}}{Y} \approx \epsilon$.

Now substituting X_+, X_-, Y from (18) and only retaining terms involving $\frac{\dot{Y}}{Y}$, we deduce

$$\frac{d^2\boldsymbol{a}_+}{dt^2} = \frac{d\boldsymbol{a}_+}{dt}\left(\mathfrak{P} + \frac{\dot{Y}}{Y}\right) + \boldsymbol{a}_+\left(\mathfrak{Q} - \frac{\dot{Y}}{Y}X_+\right) \tag{20}$$

where

$$\mathfrak{P} = \left(2\frac{\dot{A}}{A} + 2i\gamma + \frac{\dot{C}}{C}\right),$$
$$\mathfrak{Q} = i\left(\dot{\gamma} - 2\frac{d}{dt}(C\beta) - \gamma\frac{\dot{C}}{C} - 2\frac{\dot{A}}{A}\gamma\right)$$
$$+ \{\gamma^2 - 4C^2\beta^2 - 2\dot{B}\beta\} + \mathscr{O}(\epsilon^2). \tag{21}$$

As one can check, the differential equation satisfied by $\boldsymbol{a}_-$ also has a similar form.

To proceed further we now need to cast (20) into its so-called normal form. To that end, we define another time-dependent operator $\boldsymbol{b}(t)$ as

$$\boldsymbol{a}_+(t) = \boldsymbol{b}(t)e^{\frac{1}{2}\int_t(\mathfrak{P}+\frac{\dot{Y}}{Y})d\tau}. \tag{22}$$

In terms of $\boldsymbol{b}(t)$ Eq. (20) can now be rewritten as

$$\ddot{\boldsymbol{b}} + \boldsymbol{b}\left(\frac{\dot{\mathfrak{P}}}{2} - \frac{\mathfrak{P}^2}{4} - \mathfrak{Q}\right) + \boldsymbol{b}\left(\frac{\dot{Y}}{Y}X_+ - \frac{\mathfrak{P}}{2}\frac{\dot{Y}}{Y} + \mathscr{O}(\epsilon^2)\right) = 0. \tag{23}$$

Now, we have $\frac{\dot{Y}}{Y} = \frac{\ddot{B}+i\ddot{C}}{\dot{B}+i\dot{C}} - \frac{\dot{C}}{C}$ as follows from (18). Let us write it as

$$\frac{\dot{Y}}{Y} = Z + i\tilde{Z} - \frac{\dot{C}}{C}; \tag{24}$$

here both Z and $\tilde{Z} \approx \mathscr{O}(\epsilon)$ and correspond, respectively, to the real and imaginary part of $\frac{\ddot{B}+i\ddot{C}}{\dot{B}+i\dot{C}}$.

Then using the expressions of $\mathfrak{P}$ and $\mathfrak{Q}$ from (21) we get

$$\ddot{\boldsymbol{b}} + \boldsymbol{b}(U + iV) = 0 \tag{25}$$

where

$$U = 4C^2\beta^2 + 2\dot{B}\beta + 2\tilde{Z}C\beta + \mathscr{O}(\epsilon^2) \approx \mathscr{O}(\epsilon^0),$$
$$V = 2\frac{d}{dt}(C\beta) - 2C\beta\left(Z - \frac{\dot{C}}{C}\right) + \mathscr{O}(\epsilon^2) \approx \mathscr{O}(\epsilon). \tag{26}$$

Note that, since we are working in the adiabatic regime, the functions U and V vary very slowly with time. Hence, the formula for WKB approximation for the complex potential [58] can be applied to get the general solution of the differential equation as

$$\boldsymbol{b}(t) = \boldsymbol{b}(0)\left[\frac{C_1}{\sqrt{|\xi(t)|}}\exp\left(\int_0^t [i\xi(\tau)-\phi(\tau)]d\tau\right) + \frac{C_2}{\sqrt{|\xi(t)|}}\exp\left(\int_0^t [-i\xi(\tau)+\phi(\tau)]d\tau\right)\right] \quad (27)$$

where $\sqrt{U+iV} = \xi + i\phi$ and (C_1, C_2) are arbitrary coefficients. This result can also be derived by solving the differential equation using the method of successive approximation and considering the adiabatic variation of U and V.

In our case, this boils down to

$$\xi = \sqrt{\frac{\sqrt{U^2+V^2}+U}{2}} \approx \sqrt{U+\frac{V^2}{4U}} \approx \sqrt{U} \approx 2C\beta + \frac{\dot{B}\beta + C\beta\tilde{Z}}{2C\beta},$$
$$\phi = \sqrt{\frac{\sqrt{U^2+V^2}-U}{2}} \approx \sqrt{\frac{V^2}{4U}} \approx \frac{2\frac{d}{dt}(C\beta) - 2C\beta\left(Z-\frac{\dot{C}}{C}\right)}{4C\beta}. \quad (28)$$

Note that we have ignored second- and higher-order terms. We now observe that the solution must satisfy the boundary condition $\boldsymbol{b}(t=0) = \boldsymbol{b}(0)$. Also, the periodicity of the parameters implies $\sqrt{|\xi(0)|} = \sqrt{|\xi(T)|}$. Finally, it can be observed that only the second term with coefficient C_2 in the solution (27) yields the dynamical phase of $\boldsymbol{a}_+$ with proper sign. This will eventually be clear as we calculate $\boldsymbol{a}_+(T)$. We therefore set $C_1 = 0$ in (27). Now combining all these expressions, the particular solution of (23) is obtained as

$$\boldsymbol{b}(T) \approx \boldsymbol{b}(0)\exp\left[\int_0^T \left\{-i\left(2C\beta + \frac{\dot{B}\beta + C\beta\tilde{Z}}{2C\beta}\right) + \phi\right\}d\tau\right]. \quad (29)$$

Now, we have $\frac{\dot{Y}}{Y} = Z + i\tilde{Z} - \frac{\dot{C}}{C}$. As the latter is an exact differential, we can write

$$\int_0^T \frac{\dot{Y}}{Y}d\tau = \int_0^T \left(Z + i\tilde{Z} - \frac{\dot{C}}{C}\right)d\tau = \int_0^T Z d\tau + i\int_0^T \tilde{Z} d\tau = 0. \quad (30)$$

So, $\int_0^T Z d\tau = \int_0^T \tilde{Z} d\tau = 0$, implying that ϕ is also an exact differential.

Hence using (22), we can essentially drop the term involving just the exact derivatives and then split the respective dynamical and geometric phase shifts as

$$\boldsymbol{a}_+(T) = \boldsymbol{a}_+(0)\exp\left[-i\int_0^T \left(2C\beta + \frac{\dot{B}\beta + C\beta\tilde{Z}}{2C\beta}\right)d\tau + \frac{1}{2}\int_0^T \left(\frac{\dot{A}}{A} + 2i\gamma + \frac{\dot{C}}{C} + \frac{\dot{Y}}{Y}\right)d\tau\right]$$
$$= \boldsymbol{a}_+(0)\exp\left\{-i\int_0^T \left[(2C\beta - \gamma) + \left(\frac{\dot{B}\beta + C\beta\tilde{Z}}{2C\beta}\right)\right]d\tau\right\}. \quad (31)$$

And the solution becomes

$$\boldsymbol{a}_+(T) = \boldsymbol{a}_+(0)\exp\left\{-i\int_0^T \left[(2C\beta - \gamma) + \left(\frac{\dot{B}}{2C}\right)\right]d\tau\right\}$$
$$= \boldsymbol{a}_+(0)\exp\left[-\frac{i}{\hbar}\int_0^T (\hbar\omega - \gamma\hbar)d\tau - i\int_0^T \frac{\beta}{\omega}\frac{d}{d\tau}\left(\frac{\delta}{\beta}\right)d\tau\right], \quad (32)$$

with the two terms in the exponent representing the dynamical and the geometrical phases, respectively.

Finally, a close look into the decoupled evolution equation of $\boldsymbol{a}_-$ in (19) tell us that it is similar to the one for $\boldsymbol{a}_+$, except that $(+\gamma)$ is replaced by $(-\gamma)$. Also, γ is entering into the solution only through the substitution of (22). So, we get

$$\boldsymbol{a}_-(T) = \boldsymbol{a}_-(0)\exp\left[-\frac{i}{\hbar}\int_0^T (\hbar\omega + \gamma\hbar)d\tau - i\int_0^T \frac{\beta}{\omega}\frac{d}{d\tau}\left(\frac{\delta}{\beta}\right)d\tau\right], \quad (33)$$

which gives the correct dynamical phase for $\boldsymbol{a}_-$.

IV. GEOMETRIC PHASES

Now looking at the second phase factor in the expression of both the creation and annihilation operators $\boldsymbol{a}_\pm(T)$ in (32) and (33), the additional factor over and above the dynamical phase, obtained by leading behavior for adiabatic transport around a closed loop Γ in time T, can be identified with the Berry phase or geometric phase (more precisely geometric phase shift) in the Heisenberg picture. As pointed out earlier, the result obtained here can readily be converted to the more familiar form in terms of the phase gathered by the state vector by going over from the Heisenberg to the Schrödinger picture. The geometric phase shift Φ_G found above can be written in a more familiar form by using the transformation $\frac{d}{d\tau} = \frac{d\boldsymbol{R}}{d\tau}\cdot\nabla_{\boldsymbol{R}}$, where $\boldsymbol{R}$ represents a vector in the parameter space the components of which are time dependent. Then we can write Φ_G as a line integral over a closed loop Γ traced out in the parameter space as τ varies from zero to T, i.e., a complete period, so that it can be written as a functional of Γ

as

$$\Phi_G[\Gamma] = \oint_{\Gamma=\partial S} \frac{\beta}{\omega} \nabla_{\boldsymbol{R}}\left(\frac{\delta}{\beta}\right) \cdot d\boldsymbol{R} = \iint_S \nabla_{\boldsymbol{R}}\left(\frac{\beta}{\omega}\right) \times \nabla_{\boldsymbol{R}}\left(\frac{\delta}{\beta}\right) \cdot d\boldsymbol{S}, \quad (34)$$

where we have made use of Stoke's theorem in the second equality, to recast it as a surface integral over S. Note that S stands for any surface belonging to the equivalence class of surfaces in the parameter space having the same boundary Γ, and where any two such surfaces can be connected by smooth deformation without encountering any singularity. Now substituting $\alpha, \beta, \gamma, \delta$ from (5), the geometric phase can now be expressed in terms of our original time-dependent parameters $P(t)$, $Q(t)$ and the noncommutative parameters θ and η as

$$\Phi_G[\Gamma] = \left(\frac{\sqrt{-\theta\eta}}{4\hbar}\right) \iint_S \nabla_{\boldsymbol{R}}\left(\frac{Q\left(1-\frac{\theta\eta}{4\hbar^2}\right)+P\left(\frac{\eta}{2\hbar}\right)^2}{\sqrt{\left[P\left(\frac{\eta}{2\hbar}\right)-Q\left(\frac{\theta}{2\hbar}\right)\right]^2+PQ}}\right) \times \nabla_{\boldsymbol{R}}\left(\frac{P\left(\frac{\eta}{2\hbar}\right)-Q\left(\frac{\theta}{2\hbar}\right)}{Q\left(1-\frac{\theta\eta}{4\hbar^2}\right)+P\left(\frac{\eta}{2\hbar}\right)^2}\right) \cdot d\boldsymbol{S}. \quad (35)$$

One can rest assured at this stage that the denominator never vanishes as $PQ > 0$. Also it is worth noting that, in the absence of either of the two types of noncommutativity, i.e., if θ or $\eta = 0$, the geometric phase vanishes. So, it is the noncommutative nature of phase space, as a whole, alongside the geometry of the parameter space trajectory, which plays the crucial role in the appearance of geometric phase shift for this 2D harmonic oscillator system.

Before we proceed further, let us pause for a while and make some pertinent comments.

(1) The reason behind the appearance of this phase can be attributed, in our case, to the time-reversal symmetry breaking of the Hamiltonian (1) and (4) [37,38,59–61]. To elaborate on this matter, we need to explain in a bit more detail about what we mean by time-reversal symmetry of a generic time-dependent Hamiltonian $\mathcal{H}(t)$ having a set of time-dependent parameters. First note that this Hamiltonian $\mathcal{H}(t)$ can be regarded as a sequence of instantaneous time-independent Hamiltonians: one for each time t and each of them being a distinct Hermitian matrix (finite or infinite) with real diagonal and complex off-diagonal entries, in general. Now, time-reversal symmetry refers to the instantaneous Hamiltonians $\mathcal{H}(t_0)$, i.e., as if the parameters are frozen at their values corresponding to that instant $t = t_0$, which is not time dependent anymore. And a time-dependent Hamiltonian being time-reversal symmetric means that each such instantaneous Hamiltonian in the sequence must be real symmetric, not just complex Hermitian. In other words, if we let a system evolve by this Hamiltonian $\mathcal{H}(t_0)$ for some finite time interval after t_0 and then time reverse at any later time $t > t_0$, then the corresponding wave function is simply obtained by complex conjugation, without touching the set of parameters occurring there at all. This is because of the fact that the values of the parameters are now held fixed to their respective values at time t_0 and so are affected neither by the subsequent continuous time evolution nor under the discrete time-reversal transformation, i.e., the time arguments occurring in the parameters undergo no flipping of sign by this, which we can call more appropriately a "quasi-time-reversal" transformation. Consequently, under this quasi-time-reversal transformation, the system will retrace its own history, and if that happens regardless of which instantaneous Hamiltonian of the original time-dependent system was chosen then we say $\mathcal{H}(t)$ is time-reversal symmetric. A concrete mathematical definition of such time-reversal symmetry [37,62] would be

$$\hat{\Xi}\hat{H}(t)\hat{\Xi}^{-1} = \hat{H}(t) \quad \text{(without any change in the sign of } t\text{)}$$

where, the antilinear (quasi- or instantaneous) time-reversal operator $\hat{\Xi}$ leaves all the real time-dependent parameters intact. The latter nomenclature, i.e., instantaneous time-reversal symmetry, has been borrowed from [63], where similar circumstances were encountered in a system involving a topological insulator.

Now, in the commutative plane, $\mathcal{H}_c(t) = P(t)(\hat{p}_1^2+\hat{p}_2^2) + Q(t)(\hat{x}_1^2+\hat{x}_2^2)$, with $\hat{p}_1, \hat{p}_2, \hat{x}_1, \hat{x}_2$ satisfying ordinary Heisenberg algebra. (Instantaneous or quasi-) time-reversal transformation operates as $\hat{p}_i \to \hat{p}_i' = \hat{\Xi}\,\hat{p}_i\,\hat{\Xi}^{-1} = -\hat{p}_i$ and $\hat{x}_i \to \hat{x}_i' = \hat{\Xi}\,\hat{x}_i\,\hat{\Xi}^{-1} = \hat{x}_i$, which shows that the Hamiltonian is symmetric under time reversal, $\hat{\Xi}\,\mathcal{H}_c(t)\,\hat{\Xi}^{-1} = \mathcal{H}_c(t)$, as the parameters $P(t)$ and $Q(t)$ are not touched.

On the other hand, in the noncommutative plane, the dynamics is given by the Hamiltonian (1) with noncommutative coordinates and momenta satisfying algebra (A1) or equivalently by the Hamiltonian (4) with the mathematically commuting coordinates and momentum transforming like $p_i \to -p_i$, $q_i \to q_i$, under time reversal. Hence, the Hamiltonian $\mathcal{H}(t) = \alpha(t)(p_1^2+p_2^2) + \beta(t)(q_1^2+q_2^2) + \delta(t)(p_iq_i + q_ip_i) - \gamma(t)(q_1p_2 - q_2p_1)$ is not time-reversal symmetric: $\hat{\Xi}\,\mathcal{H}(t)\,\hat{\Xi}^{-1} \neq \mathcal{H}(t)$; the presence of the dilatation term and the Zeeman-like term breaks this symmetry [64]. Particularly, the breaking by the dilatation term is primarily responsible for getting the nonvanishing Berry phase in our case. In fact it has been shown in [37,62] that this time-reversal symmetry breaking is a necessary, but not sufficient, condition for the appearance of nonvanishing Berry phase. And it is because of this broken time-reversal symmetry, while considering noncommutative phase space [44,62,65], that there arises the possibility of obtaining a nonvanishing geometrical phase shift in our system of a 2D simple harmonic oscillator in noncommutative phase space.

(2) The Berry connection one-form $\boldsymbol{A}$ on the loop Γ that we found can be also be directly read off from (32) and (33) as

$$\boldsymbol{A} = \frac{\beta}{\omega} d\left(\frac{\delta}{\beta}\right) = -\frac{\alpha}{\omega} d\left(\frac{\delta}{\alpha}\right) - d\left[\tan^{-1}\left(\sqrt{\frac{\alpha\beta}{\delta^2}-1}\right)\right], \quad (36)$$

showing that, up to a nonsingular gauge transformation, the Berry connection can also be written as

$$\boldsymbol{A} := -\frac{\alpha}{\omega} d\left(\frac{\delta}{\alpha}\right). \quad (37)$$

This particular feature of this connection one-form is indeed quite gratifying as the symmetry between α and β, the coefficients of $\vec{p}^2$ and $\vec{q}^2$ in the Hamiltonian (4), is somehow restored with this. In fact, this form of the connection one-form

(37) occurred earlier in [1,66–68], where a Hamiltonian of the same form as (4) was used to describe a one-dimensional (1D) parametric generalized harmonic oscillator, except that there time-dependent coefficients α, β, δ were of fundamental nature by themselves, unlike in our case, where α, β, δ are not fundamental and rather given in terms of more fundamental P and Q through a set of linear relations (5). Also note that any of our time-dependent parameters $\alpha(t), \beta(t), \delta(t), \gamma(t)$ cannot vanish for all time t, otherwise we would have got $P(t) \propto Q(t)$[1] as is clear from (5). With this the closed loop Γ will collapse to a 1D line in P, Q space thereby yielding a vanishing geometric phase. This is particularly true for $\delta(t)$, which clearly plays a vital role here, and its origin can be traced back to (9) where $(\frac{\delta}{\beta})$ occurs as the real part of the coefficient p_j; in its absence we cannot get any geometrical phase, as is clear from (32) and (33).

(3) Although the $\gamma(t)$ occurring in the Zeeman term in (4) is required to be nonvanishing, in order to get a nonvanishing Berry phase, it also plays another important role by allowing us to avoid the crossings of energy levels by lifting the degeneracy, as we have mentioned already in Sec. II. Despite all this, it does not have an explicit presence in the expression of the Berry phase (34) and (37) [if we ignore the relations in (5) for the time being]; it rather appears in the dynamical phases in (32) and (33) and corroborates the general observation made by Anandan and Stodolsky [69] where the dynamical group was $U(2)$. To understand the reason behind all this, observe that, although the Hamiltonian $\mathcal{H}(t)$ (4) or (6) do not commute at different times, $[\mathcal{H}(t), \mathcal{H}(t')] \neq 0$ for $t \neq t'$, but being an element of the algebra $su(1,1) \oplus u(1)$ it splits into two commuting parts as in (80) (see Appendix B). Importantly, these two terms in (80) commute with each other at different times also. Consequently, the corresponding time evolution operator (in the Schrödinger picture) factorizes as

$$\mathcal{U} = \hat{\mathcal{T}}(e^{-\frac{i}{\hbar}\int dt[\alpha(t)(p_1^2+p_2^2)+\beta(t)(q_1^2+q_2^2)+\delta(t)(p_iq_i+q_ip_i)]}) \times \hat{\mathcal{T}}(e^{\frac{i}{\hbar}\int dt\gamma(t)(q_1p_2-q_2p_1)}) \tag{38}$$

where $\hat{\mathcal{T}}$ is the time-ordering or chronological operator. After all, it can be easily seen that $\gamma(t)$, like $\omega(t)$, occurs in the integral as $\int_0^T \gamma(t)dt$, which is not a functional of the closed loop Γ, the latter being the telltale sign of geometrical phases (34) and (36): $\Phi_G[\Gamma] = \int_\Gamma \boldsymbol{A}$.

(4) Finally, we would like to study the implication of the unitary equivalence between the two forms of Bopp shifts (A2) and (A4), as has been demonstrated in Appendix A, in our context. To begin with, note that the geometrical phase Φ_G in (35) was obtained for the scale parameter ξ, taken without loss of generality, to be $\xi = 1$. For any other value of ξ, the corresponding Φ_G can be obtained easily by replacing $\theta \to \xi\theta$ and $\eta \to \xi\eta$, i.e., making use of the Bopp shift (A4). Note that Φ_G remains invariant if and only if $\hbar$ is also scaled to $\hbar \to \xi\hbar$, as (A4) will reduce in this case to $\xi = 1$, thereby undoing the scaling operation just as in (16). At the critical point $\xi = \xi_c$ (A3), however, the counterpart of the expression (35) will be absent if one makes use of realization (A2). On the other hand, with the equivalent realization (A4) with $\xi = \xi_c$, the counterpart of (35) will definitely be present, but it can be shown that it loses its geometrical nature and in either case one finds that Φ_G vanishes: $\Phi_G = 0$. This can be shown in two different but equivalent ways by taking advantage of the above-mentioned unitary equivalence between two forms of Bopp shifts (A2) and (A4) holding for $\xi = \xi_c$ (A3) only. To this end, we first employ (A2) in (1) to obtain the Hamiltonian in the following form:

$$\mathcal{H}^1(t) = \alpha^{(1)}(t)p_i^2 + \beta^{(1)}(t)q_i^2 - \gamma^{(1)}(t)\epsilon_{ij}q_ip_j \tag{39}$$

where

$$\alpha^{(1)}(t) = \xi_c\left(P(t) + \frac{\theta^2}{4\hbar^2}Q(t)\right),$$
$$\beta^{(1)}(t) = \xi_c\left(Q(t) + \frac{\eta^2}{4\hbar^2}P(t)\right),$$
$$\gamma^{(1)}(t) = \frac{\xi_c}{\hbar}[\theta Q(t) + \eta P(t)]. \tag{40}$$

Equations (39) and (40) are the counterparts of (4) obtained by employing (3) in (1). Here, not only do we have $\delta^1(t) = 0\ \forall t$ [which is the counterpart of $\delta(t)$ in (5)], but also note the absence of any linear equation relating $\delta^{(1)}(t)$ with $P(t)$ and $Q(t)$, thereby indicating the absence of a counterpart of (35) following from (34), in this case. Consequently, one has to make use of (36) and (37), or the second terms in the exponents of (32) and (33), to find that the Berry phase vanishes: $\Phi_G = 0$, in this case.

To arrive at the same conclusion through the Bopp shift (A4), we need to write down the corresponding expressions for $\alpha, \beta, \gamma, \delta$ for $\xi = \xi_c$ which is simply obtained by replacing $\theta \to \xi_c\theta; \eta \to \xi_c\eta$ in (5), as mentioned above, to get

$$\mathcal{H}^{(2)}(t) = \alpha^{(2)}(t)p_i^2 + \beta^{(2)}(t)q_i^2 - \gamma^{(2)}(t)\epsilon_{ij}q_ip_j + \delta^{(2)}(t)(q_ip_i + p_iq_i) \tag{41}$$

where

$$\alpha^{(2)}(t) = \alpha(t;\xi_c) = P(t)\left\{1 - \frac{\xi_c^2\theta\eta}{4\hbar^2}\right\} + Q(t)\left(\frac{\xi_c\theta}{2\hbar}\right)^2,$$
$$\beta^{(2)}(t) = \beta(t;\xi_c) = Q(t)\left\{1 - \frac{\xi_c^2\theta\eta}{4\hbar^2}\right\} + P(t)\left(\frac{\xi_c\eta}{2\hbar}\right)^2,$$
$$\gamma^{(2)}(t) = \gamma(t;\xi_c) = \frac{\xi_c}{\hbar}[\eta P(t) + \theta Q(t)],$$
$$\delta^{(2)}(t) = \delta(t;\xi_c) = \left(\frac{\xi_c^2\sqrt{-\theta\eta}}{4\hbar^2}\right)[\eta P(t) - \theta Q(t)]. \tag{42}$$

Apparently the presence and nonvanishing nature of $\delta^{(2)}(t)$ here, however, suggest a nonvanishing Φ_G. So, it will be difficult to demonstrate that $\Phi_G = 0$ in this case by just employing (35) here. However, it turns out that this nonvanishing $\delta^{(2)}(t)$ can be eliminated by using a time-independent unitary transformation $\boldsymbol{U} \in \mathrm{SU}(1,1) \otimes \mathrm{U}(1)$. Indeed, by making use of the time-independent unitary transformation (A5) and the relations in (A6), we readily see from (1) that

$$\mathcal{H}^{(2)}(t) = \boldsymbol{U}\mathcal{H}^{(1)}(t)\boldsymbol{U}^\dagger \tag{43}$$

[1]For example, $\gamma(t) = 0\ \forall t$ implies from (5) that $P(t) \propto Q(t)\ \forall t$. In fact writing this more explicitly, we have $\frac{P}{Q} = -\frac{\theta}{\eta}$.

where the respective parameters are given in (A11) and $\beta = \beta_1$ in (A13). This indicates that the U(1) part of the total dynamical symmetry group $SU(1,1) \otimes U(1)$ (B2) (see Appendix B) is also fixed by this fine tuned value of $\beta = \beta_1$ in (A13); it is not arbitrary. This, in turn, fixes all other parameters of SU(1, 1) in (A11). And this feature makes it difficult to demonstrate the vanishing of Φ_G, just by employing (35), as mentioned above. In any case, we see that the dilatation term in (41) can be eliminated for all times t and thereby ensuring $\Phi_G = 0$. Thus, for this particular value of $\xi = \xi_c$, the phase turns out to be integrable. Perhaps, a more transparent way to understand it would be to consider the following identity:

$$\alpha^{(2)}(t)\beta^{(2)}(t) - [\delta^{(2)}(t)]^2 = \alpha^{(1)}(t)\beta^{(1)}(t) \quad \forall t, \tag{44}$$

which follows trivially from (40) and (42) and can be regarded as a corollary of (43). This demonstrates the invariance under the SU(1, 1) or rather $SO(2,1) = SU(1,1)/\mathbb{Z}_2$ subgroup of $SU(1,1) \otimes U(1)$, of the corresponding frequency (B13) (see Appendix B): $\omega(t;\xi_c) = 2\sqrt{\alpha^{(1)}(t)\beta^{(1)}(t)} = 2\sqrt{\alpha^{(2)}(t)\beta^{(2)}(t) - [\delta^{(2)}(t)]^2}$, written in terms of either set of the parameters. This, in turn, implies that the parameters are indeed connected by SO(2, 1) transformation (B8) and (B13), and as has been elaborated in Appendix B this $\delta^{(2)}(t)$ can be regarded as the time component of a spacelike three-vector. But now it can be eliminated for all times by a global (time-independent) "Lorentz transformation," in (2+1) dimensions. And finally when this SO(2, 1) transformation matrix is lifted to its covering group SU(1, 1) (see Appendix B), it gives the SU(1, 1) part of the transformation matrix $\mathbf{U} \in SU(1,1) \otimes U(1)$ (A5) (see Appendix A).

Since this $\boldsymbol{U}$ is time independent, no connection term of the manner [(B18) in Appendix B] will arise here, and for any state $|\Psi(t)\rangle$ the time evolution of which is governed by $\mathcal{H}^{(2)}(t)$ as $i\hbar\partial_t|\Psi(t)\rangle = \mathcal{H}^{(2)}(t)|\Psi(t)\rangle$ we have a corresponding state $[\boldsymbol{U}^\dagger|\Psi(t)\rangle]$, which time evolves by $\mathcal{H}^{(1)}(t)$, as $i\hbar\frac{\partial(\boldsymbol{U}^\dagger|\Psi(t)\rangle)}{\partial t} = \mathcal{H}^{(1)}(t)[\boldsymbol{U}^\dagger|\Psi(t)\rangle]$. And thus we are back to $\mathcal{H}^{(1)}(t)$ again, where the corresponding $\delta^{(1)}(t) = 0$ and the Berry connection vanishes as a result. In other words, for this critical value ξ_c of the parameter ξ, it is indeed possible to transform away the dilatation term, by subjecting the Hamiltonian $\mathcal{H}^{(2)}(t)$ to a time-independent unitary transformation $\mathcal{H}^{(2)}(t) \to \boldsymbol{U}^\dagger\mathcal{H}^{(2)}(t)\boldsymbol{U}$ (43), resulting in vanishing of the Berry phase.

This analysis of course will not hold for any values of ξ other than ξ_c, i.e., for $\xi \neq \xi_c$. Nevertheless, here also, it is possible to eliminate the dilatation term completely, but only by a unitary transformation $\mathcal{W}(t) \in SU(1,1)$ (B16) which is necessarily time dependent. Here, unlike the the above case, it is not essential, however, to have $\mathcal{W}(t)$ belonging to the entire product group $SU(1,1) \otimes U(1)$; retaining the U(1) part becomes optional. Consequently the disappearance of Berry's phase here is only apparent in nature [68,70] and it gets lodged in the dynamical part, albeit retaining its geometrical characteristics [67]. This has been elaborated in Appendix B.

Now returning back to our main objective, let us try to relate this geometric phase shift obtained in the Heisenberg picture, with the more familiar form of Berry phases acquired by state vectors. For that we revert back to the Schrödinger picture. First let us rewrite (32) and (33) as

$$\boldsymbol{a}_\pm(T) = \boldsymbol{a}_\pm(0)\exp(-i\Theta_{\pm,d} - i\Phi_G) \tag{45}$$

where

$$\Theta_{\pm,d} = \int_0^T (\hbar\omega \mp \gamma\hbar), \qquad \Phi_G = \int_0^T \frac{\beta}{\omega}\frac{d}{d\tau}\left(\frac{\delta}{\beta}\right)d\tau \tag{46}$$

are the dynamical and geometric phases, respectively.

Let $\mathcal{U}(0,t)$ be the Schrödinger evolution operator of our concerned system, generated by the Hamiltonian (4). Then, $\boldsymbol{a}_\pm(t) = \mathcal{U}^\dagger(0,t)\boldsymbol{a}_{S\pm}(t)\mathcal{U}(0,t)$, where $\boldsymbol{a}_{S\pm}(t)$ are the ladder operators in the Schrödinger picture. Note that the time dependence is not entirely frozen here, even in this Schrödinger picture; it creeps in through the time-dependent parameters.

We can therefore write

$$\begin{aligned}
&\frac{(\boldsymbol{a}_+^\dagger(T))^{n_1}(\boldsymbol{a}_-^\dagger(T))^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}}|0,0;t=0\rangle_S \\
&= \mathcal{U}^\dagger(0,T)\frac{(\boldsymbol{a}_{S+}^\dagger(T))^{n_1}(\boldsymbol{a}_{S-}^\dagger(T))^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}}\mathcal{U}(0,T)|0,0;t=0\rangle_S \\
&= \mathcal{U}^\dagger(0,T)\frac{(\boldsymbol{a}_{S+}^\dagger(T))^{n_1}(\boldsymbol{a}_{S-}^\dagger(T))^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}}e^{-i\phi_{0,0}}|0,0;t=T\rangle_S \\
&= \mathcal{U}^\dagger(0,T)|n_1,n_2;t=T\rangle_S e^{-i\phi_{0,0}} \\
&= e^{i(\phi_{n_1,n_2}-\phi_{0,0})}|n_1,n_2;t=0\rangle_S
\end{aligned} \tag{47}$$

where ϕ_{n_1,n_2} represents the total adiabatic phase acquired by $|n_1,n_2;t=0\rangle_S$ after evolving by $\mathcal{H}(t)$ over its complete period T. Further using (45) we also find

$$\begin{aligned}
&\frac{(\boldsymbol{a}_+^\dagger(T))^{n_1}(\boldsymbol{a}_-^\dagger(T))^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}}|0,0;t=0\rangle_S \\
&= e^{in_1(\Theta_{+,d}+\Phi_G)}e^{in_2(\Theta_{-,d}+\Phi_G)}\frac{(\boldsymbol{a}_+^\dagger(0))^{n_1}(\boldsymbol{a}_-^\dagger(0))^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}} \\
&\quad\times|0,0;t=0\rangle_S \\
&= e^{in_1(\Theta_{+,d}+\Phi_G)}e^{in_2(\Theta_{-,d}+\Phi_G)}|n_1,n_2;t=0\rangle_S.
\end{aligned} \tag{48}$$

Note that here we have made use of the fact that $\boldsymbol{a}_\pm(t=0) = \boldsymbol{a}_{S\pm}(t=0)$. Now comparing the above two equations (47) and (48), we get

$$\phi_{n_1,n_2} = \phi_{0,0} + [n_1(\Theta_{+,d} + \Phi_G) + n_2(\Theta_{-,d} + \Phi_G)]. \tag{49}$$

So, the Berry phase acquired by the state $|n_1,n_2;t=0\rangle_S$ is given by

$$\phi_B^{(n_1,n_2)} = \phi_B^{(0,0)} + (n_1+n_2)\Phi_G. \tag{50}$$

This kind of linear nature in the Berry phases of different eigenstates is a general result [71] for any Hamiltonian with an equally spaced discrete spectrum. And in our case the total Hamiltonian (13) is partitioned into two commuting parts corresponding to $\boldsymbol{a}_+$ and $\boldsymbol{a}_-$, where each part produces its own equally spaced spectrum in its respective sub-Hilbert space $\mathbb{H}_\pm$ and the tensor product of which forms the total Hilbert space: $\mathbb{H} = \mathbb{H}_+ \otimes \mathbb{H}_-$.

Importantly, it is the difference of the Berry phases of different eigenstates which contributes to the expectation value of any operator at time t in a state obtained from any initial state and evolving under an adiabatic Hamiltonian, $\langle\hat{\boldsymbol{O}}\rangle(t) = \langle\psi(t)|\hat{\boldsymbol{O}}(t)|\psi(t)\rangle$, where the ground-state contribution $\phi_B^{(0,0)}$ cancels out. And most experiments concerning Berry's phase [72] employ this idea only. Our derivation certainly provides complete information which, in principle, may facilitate the predictions of such cases.

V. CLASSICAL ANALOG: HANNAY ANGLES

We now take up the study of the classical analog of this quantal geometric phase, namely, the Hannay angles [66]. To clinch the correspondence we will exploit the correspondence principle of quantum mechanics with classical mechanics, using coherent states [73] and some suitable chosen quantum operators that represent the classical action and angle variables. For that, it will be convenient to recall the concept of instantaneous Hamiltonians introduced in the previous section to discuss the concept of time-reversal symmetry in our context and to interpret the time-dependent Hamiltonian $\mathcal{H}(t)$ (4), the time dependence of which stems from the presence of a set of time-dependent parameters $\alpha(t)$, $\beta(t)$, $\delta(t)$, and $\gamma(t)$, as a collection of an infinite number of time-independent planar systems labeled by time, say t_0, and the set of parameters takes its values to be fixed by its respective values for time t_0 as $\alpha(t_0)$, $\beta(t_0)$, $\gamma(t_0)$, $\delta(t_0)$. Left on their own, these individual systems evolving by Hamiltonians like $\mathcal{H}(t_0)$, with parametric values frozen at $\alpha(t_0)$, $\beta(t_0)$, $\gamma(t_0)$, $\delta(t_0)$, will give rise to periodic motion in their respective phase spaces at the classical level, as we show below. In fact, each such instantaneous Hamiltonian $\mathcal{H}(t_0)$ can be brought to the standard form of a pair of decoupled planar oscillators by suitable unitary (canonical) transformations of the respective quantum (classical) systems. And this ensures the occurrence of periodic motion in the phase spaces, facilitating the introduction of canonical action and angle variables, for each of these classical systems corresponding to the instantaneous Hamiltonians. As one can easily see, we can accomplish this task by introducing a set of canonically conjugate position and momentum operators, the so-called quadrature variables, from the ladder operators $\boldsymbol{a}_+$ and $\boldsymbol{a}_-$ as

$$\hat{q}_\pm = \sqrt{\frac{\hbar\alpha}{2\omega_\pm}}(\boldsymbol{a}_\pm^\dagger + \boldsymbol{a}_\pm), \quad \hat{p}_\pm = i\sqrt{\frac{\hbar\omega_\pm}{2\alpha}}(\boldsymbol{a}_\pm^\dagger - \boldsymbol{a}_\pm). \tag{51}$$

Utilizing (9) and (11), this shows $\hat{q}_\pm = \hat{q}_\pm(q_1, q_2, p_1, p_2)$, $\hat{p}_\pm = \hat{p}_\pm(q_1, q_2, p_1, p_2)$ are linearly dependent on the old coordinates and momenta. At the quantum level, this can be implemented at each such instant t_0, by suitable unitary transformations $\mathcal{V}(t_0) \in \mathrm{SU}(1,1)\otimes\mathrm{U}(1)$. This is exactly like the case of $\mathcal{W}(t_0)$ as in (B16) (see Appendix B), which helped us to eliminate just the dilatation term, without touching the Zeeman term in $\mathcal{H}(t)$ (4), except that we are now eliminating the Zeeman term also. But the explicit construction of such a $\mathcal{V}(t_0)$ is neither easy nor required in our case, as we are dealing with the instantaneous classical systems here. In fact, as one can easily see, at the classical level (where phase-space variables are just c numbers) an analogous linear canonical transformation with the coefficients determined by the values of $\alpha, \beta, \gamma, \delta$ at $t = t_0$ canonically transforms the instantaneous classical systems from $\{q_1, q_2; p_1, p_2\}$ canonical pairs to $\{q_+, q_-; p_+, p_-\}$ canonical pairs. Correspondingly, we get from the classical Hamiltonian in old phase-space variables, which is the classical counterpart of our quantum Hamiltonian (4) at the instant t_0, the 2D decoupled harmonic-oscillator-like Hamiltonian, written just in terms of new phase-space variables:

$$\mathcal{H}_{\mathrm{Cla}}(t_0) = \frac{\omega_+^2}{2\alpha}q_+^2 + \frac{\alpha}{2}p_+^2 + \frac{\omega_-^2}{2\alpha}q_-^2 + \frac{\alpha}{2}p_-^2. \tag{52}$$

This is the classical counterpart of unitarily transformed Hamiltonian (4): $\mathcal{V}(t_0)\mathcal{H}(t_0)\mathcal{V}^\dagger(t_0)$. Now if we were to consider time evolution in the full time-dependent system governed by $\mathcal{H}(t)$, instead of just the time-independent Hamiltonians $\mathcal{H}(t_0)$ with parameters frozen at fixed values, we would be required to augment the unitary transformed Hamiltonian by a suitable connection term like $i\hbar\mathcal{V}(t)\partial_t\mathcal{V}^\dagger(t)$, so that the total Hamiltonian $\mathcal{H}_{\mathrm{total}}(t)$, as in (B18), can govern the time evolution of the transformed states $[\mathcal{V}(t)|\Psi(t)\rangle]$. And as shown in the case of $\mathcal{W}(t)$ (B16) in Appendix B here too we can show that the geometrical phase will now occur in the dynamical phase obtained through $\mathcal{H}_{\mathrm{total}}(t)$, but will retain its geometrical feature. Similarly the classical counterpart of this $\mathcal{H}_{\mathrm{total}}(t)$ can be obtained by simply adding a term of the form $\frac{\partial F}{\partial t}$, where F is a suitable generating function [74], to (52). In this section, we are, of course, not bothered about this extra time-derivative term in $\mathcal{H}_{\mathrm{total}}(t)$, because the parameters α, β, γ, and δ are held frozen to their respective values corresponding to the instant $t = t_0$. Additionally, each such instantaneous Hamiltonian $\mathcal{H}_{\mathrm{Cla}}(t_0)$ (52) in the classical case gives rise to periodic motion in phase space, if considered as a separate system on its own, and hence allows us to introduce corresponding action and angle variables, as mentioned above.

Now in this classical case [73] let $\{C(I, \boldsymbol{R})\}$ denote a continuous family of periodic trajectories $C(I, \boldsymbol{R})$ in the phase space associated with the classical Hamiltonians $\mathcal{H}_{\mathrm{Cla}}(\boldsymbol{R})$ and let $\omega(I, \boldsymbol{R})$ be the angular velocity on $C(I, \boldsymbol{R})$, where each curve is equipped with a definite origin for angle variable v conjugated to the action variables I. Now, during the adiabatic evolution, a point in phase space follows a trajectory of constant action, and only the angular coordinate $v(t)$ will evolve in time, and its value at time t is given by

$$v(t) = v(0) + \int_0^t \omega(I, \boldsymbol{R}(s))ds + \Delta v_I^{\mathrm{H}}(t). \tag{53}$$

This involves an integration along the curve $C(I, \boldsymbol{R}(t))$, and also contains, apart from the usual dynamical contribution, a geometrical one also, the so-called Hannay angle $\Delta v_I^{\mathrm{H}}(t)$. Note that, since our classical Hamiltonian $\mathcal{H}_{\mathrm{Cla}}(\boldsymbol{R})$ has two degrees of freedom, we expect two sets of action-angle coordinates $\{\{I_i, v_i\} : i = 1, 2\}$.

Now let us consider the coherent states [75–77] of our two-dimensional harmonic oscillator (13), which are supposed to be the best approximations to a classical state. The coherent states analogous to [78] in this case are the tensor product of two independent Glauber-Klauder-Sudarshan coherent states [77], which are simultaneous (normalized) eigenstates of the two mutually commuting annihilation operators $\boldsymbol{a}_+$ and $\boldsymbol{a}_-$:

$$|z_1, z_2; \boldsymbol{R}\rangle = |z_1(\boldsymbol{R})\rangle \otimes |z_2(\boldsymbol{R})\rangle,$$
$$\boldsymbol{a}_+|z_1(\boldsymbol{R})\rangle = z_1|z_1(\boldsymbol{R})\rangle,$$
$$\boldsymbol{a}_-|z_2(\boldsymbol{R})\rangle = z_2|z_2(\boldsymbol{R})\rangle,$$
$$|z_1, z_2; \boldsymbol{R}\rangle = e^{-(|z_1|^2+|z_2|^2)/2} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{z_1^{n_1}}{\sqrt{n_1!}}\frac{z_2^{n_2}}{\sqrt{n_2!}}|n_1, n_2; \boldsymbol{R}\rangle. \tag{54}$$

Further, it has been shown in [73] that a suitable quantum operator for classical action I_i is $\hat{I}_i(\boldsymbol{R}) = \hbar\hat{N}_i(\boldsymbol{R})$, where $\hat{N}_i(\boldsymbol{R})$ is the ith number operator, with $i \in \{+, -\}$ in our case. Now, let $\hat{U}_i(\boldsymbol{R})$'s be the unitary operators defined through their action, $\hat{U}_1(\boldsymbol{R})|n_1, n_2; \boldsymbol{R}\rangle = |n_1 - 1, n_2; \boldsymbol{R}\rangle$, $\hat{U}_1(\boldsymbol{R})|0, n_2; \boldsymbol{R}\rangle = 0$, and similarly for $\hat{U}_2(\boldsymbol{R})$. They essentially correspond to the well-known polar decompositions of the operators like $\boldsymbol{a}$ into the so-called number $\hat{N}$ and phase operators $\hat{\theta}$: $\boldsymbol{a}_\pm = \sqrt{\hat{N}_\pm}e^{i\hat{\theta}_\pm}$, where $\hat{U}_i(\boldsymbol{R})$ can be thought of as the operator corresponding to $e^{-i\hat{\theta}_i}$. It is also shown that the expectation values of these operators in the state $|z_1, z_2; \boldsymbol{R}\rangle$ are given by $I_i = \langle\hat{I}_i(\boldsymbol{R})\rangle = |z_i|^2\hbar$ and $\langle\hat{U}_i(\boldsymbol{R})\rangle = e^{i\times\arg(z_i)}$, so that in the classical limit we can identify $z_j = \sqrt{\frac{I_j}{\hbar}}e^{-iv_j}$.

And this natural relationship between the ladder operators of the quantum system and the corresponding action and angle like operators was the main driving motivation behind our unconventional approach to determine the Berry phases by solving the evolution equations of $\boldsymbol{a}_\pm$ in the Heisenberg picture, which also provides a natural framework for semiclassical correspondence. In fact, the geometrical part of the phases acquired by $\boldsymbol{a}_\pm$ over a complete period of an adiabatic cycle, as found in (32) and (33), is precisely Hannay's angle of the corresponding classical adiabatic evolution, as we identify below. So we see that we could determine at one go the Berry phases as well as Hannay's angle from (32) and (33). Additionally, had we taken the more conventional route we would have been required to determine the exact energy eigenfunctions of the quantum system in order to find the intended geometric phases, which is not a very easy job to do for a generalized two-dimensional Harmonic oscillator such as (4). Hence the overall approach we took, though it was not the generic one, suited our desired goals more appropriately.

Now returning back to the original point, we consider the following wave packet as the initial state, which best approximates the initial conditions for the corresponding classical adiabatic evolution,

$$|z_1, z_2; \boldsymbol{R}(0)\rangle = e^{-(|z_1|^2+|z_2|^2)/2} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{z_1^{n_1}}{\sqrt{n_1!}}\frac{z_2^{n_2}}{\sqrt{n_2!}}|n_1, n_2; \boldsymbol{R}(0)\rangle \tag{55}$$

and evolve it adiabatically over a complete cycle, to get, using (49),

$$\begin{aligned}\mathcal{U}(0,T)|z_1, z_2; \boldsymbol{R}(0)\rangle &= e^{-(|z_1|^2+|z_2|^2)/2} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{z_1^{n_1}}{\sqrt{n_1!}}\frac{z_2^{n_2}}{\sqrt{n_2!}}e^{-i\phi_{0,0}}|n_1, n_2; \boldsymbol{R}(T)\rangle \\ &\quad\times e^{-in_1(\Theta_{+,d}+\Phi_G)}e^{-in_2(\Theta_{-,d}+\Phi_G)} \\ &= e^{-(|z_1|^2+|z_2|^2)/2} \times e^{-i\phi_{0,0}} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{(z_1e^{-i(\Theta_{+,d}+\Phi_G)})^{n_1}}{\sqrt{n_1!}} \\ &\quad\times \frac{(z_2e^{-i(\Theta_{-,d}+\Phi_G)})^{n_1}}{\sqrt{n_2!}}|n_1, n_2; \boldsymbol{R}(T)\rangle \\ &= e^{-i\phi_{0,0}}|z_1e^{-i(\Theta_{+,d}+\Phi_G)}, z_2e^{-i(\Theta_{-,d}+\Phi_G)}; \boldsymbol{R}(T)\rangle.\end{aligned} \tag{56}$$

Therefore, up to a global phase factor, the coherent state associated with the initial Hamiltonian $\mathcal{H}(0)$ evolves to another coherent state $|z_1(T), z_2(T); \boldsymbol{R}(T)\rangle$, now associated with the Hamiltonian $\mathcal{H}(T)$ at time T, where $z_1(T) = z_1e^{-i(\Theta_{+,d}+\Phi_G)}$ and $z_2(T) = z_2e^{-i(\Theta_{-,d}+\Phi_G)}$. Thus, from the evolution of z_1 and z_2 in (56), through a complete period of the adiabatic Hamiltonian, we can identify $\Theta_{\pm,d} = \int_0^T \omega_\pm(t')dt'$, where $\omega_i = \frac{1}{\hbar}\frac{\partial E_{n_1,n_2}}{\partial n_i}$ (like $\frac{\partial \mathcal{H}_{\text{Cla}}}{\partial I_i}$), with the dynamical phases and Φ_G [from (34)], with the angular shift, which was obtained classically by Hannay.

The expectation values of the new set of position-momentum, i.e., the quadrature operators (51), which are now time dependent, are found to be given by [75]

$$\langle\hat{q}_\pm\rangle = \sqrt{2\hbar/\omega_\pm}\text{Re}(z_i),$$
$$\langle\hat{p}_\pm\rangle = \sqrt{2\hbar\omega_\pm}\text{Im}(z_i) \quad (i = 1, 2,\ \text{respectively}). \tag{57}$$

On adopting the parametrization of z_1 and z_2, introduced above, the expectation values of these phase-space operators in the transported state are obtained as

$$\langle\hat{q}_\pm\rangle_T = \sqrt{2I_i/\omega_\pm}\cos[v_i(0) + \Theta_{\pm,d} + \Phi_G],$$
$$\langle\hat{p}_\pm\rangle_T = -\sqrt{2I_i\omega_\pm}\sin[v_i(0) + \Theta_{\pm,d} + \Phi_G], \tag{58}$$

showing that the corresponding classically canonical conjugate phase-space variables, i.e., the classical counterparts of quadrature operators $\hat{q}_\pm$, $\hat{p}_\pm$ (51), undergo oscillatory motion. Thus the geometric phase Φ_G, entering into the nonstationary coherent state through all of its stationary components, i.e., the energy eigenstates, generates in the classical limit ($\hbar \to 0$, $|z_i| \to \infty$, $\sqrt{I_i} = \sqrt{\hbar}|z_i| \to$ finite) the angle variable conjugate to the action I_i and is given by the phase of z_i. Therefore the additional phase of z_i, i.e., one over and above the dynamical phase, can be identified with the Hannay angle, which can clearly be understood from classical arguments.

Before we conclude this section, we would like to draw the attention of the reader to an earlier work [72], where also Berry phase was computed using the Heisenberg picture. However, in contrast to our approach, the authors of that work had incorporated adiabatic approximation right at the level of the time-dependent Schrödinger equation, thereby identifying an effective adiabatic Hamiltonian $\mathcal{H}_{ad}(t)$ as described in [79,80], which then governs the time evolution of adiabatic state vectors in the Schrödinger picture. That also helped them to obtain an effective Heisenberg equation of motion for any general phase-space operator, under adiabatic approximation. On the other hand, in our approach, we worked with the original Hamiltonian itself and only implemented adiabaticity while solving the respective Heisenberg equations of motion explicitly.

Finally, to clinch the correspondence with Hannay's angle, the authors of [72] directly took expectation of the time evolved quantum operator calculated in the way mentioned above, with the initial coherent state of the system best suited for the initial conditions of the dynamics. And finally, they extracted Hannay's angle from the oscillatory sinusoidal terms, which resemble the ones we had in (57), occurring in the time evolved expectation value obtained in the Heisenberg picture. This is again in contrast to our case, as we determined Hannay's angle of our system using the Schrödinger picture. That is, we calculated the time evolved expectation value, using the initial ($t = 0$) quantum operator and time evolved coherent state, after a complete periodic cycle of the Hamiltonian. On the way, of course, we had to make use of our preceding findings, namely, the Berry phases calculated in the Heisenberg picture. Further, while determining Hannay's angle from time evolved expectation, we made use of two special quantum operators [73], which are the authentic quantum versions of the classical action and angle variables, thereby obtaining the genuine classical correspondence with Hannay's angle.

VI. CONCLUSIONS

We have considered the system of a harmonic oscillator in the Moyal plane, but with the additional feature that there is noncommutativity among momentum components, also like the spatial ones, and other parameters are varying slowly with time. Although periodicity, rather that adiabaticity, is more relevant in the computation of geometrical phase, as shown by Aharonov and Anandan [81], we nevertheless find the original adiabatic approach due to Berry convenient to execute. For that we introduce a form of Bopp shift, which is more general in nature and does not involve any effective Plank constant $\hbar_{\rm eff}$, as has been done in the literature. Through this Bopp shift, we can generate a certain dilatation term involving the commutative phase-space variable ($[q_i, q_j] = 0 = [p_i, p_j], [q_i, p_j] = i\hbar\delta_{ij}$), which plays an indispensable role in generating this geometrical, i.e., Berry, phase. We have also provided an unconventional approach to compute this geometrical phase shift initially in the Heisenberg picture and then related it with the conventional Berry phase in the Schrödinger picture. Finally, the classical analog of the Hannay angle was also computed using Glauber-Sudarshan coherent states. We finally observe that the emergent Berry phase (geometrical phase shift) depends on both types of noncommutative parameters (θ and η) and it will vanish in the situation if either one of these parameters were to vanish. Thus we can conclude that the noncommutative phase-space structure induces a suitable geometry on the circuit Γ in parameter space of the 2D time-dependent harmonic oscillator system, which manifests in the appearance of the associated geometrical phase shift, when a circuital adiabatic excursion in the parameter space is considered.

We would also like to mention that the effective commutative Hamiltonian $\mathcal{H}(t)$ (4), obtained by making use of our Bopp shift (3), takes its value in $su(1,1) \oplus u(1)$ Lie algebra (B9), which contains terms responsible for explicit breaking of time-reversal symmetry of the family of instantaneous Hamiltonians $\mathcal{H}(t)$'s, which is a necessary condition to get Berry's phase [38,62]. The instantaneous eigenstates of the Hamiltonians $\mathcal{H}(t)$ can therefore be taken to belong to the representation space of the group $\mathrm{SU}(1,1) \otimes \mathrm{U}(1)$ and the occurrence of geometrical phase becomes inevitable in this case, as has been shown in [82,83]. Our result therefore corroborates this general observation. And as long as the noncommutative parameters θ and η can be regarded as fundamental parameters in some appropriate energy scale the resulting Berry phase can also be regarded as fundamental.

Finally, regarding physical models exhibiting Berry phase with phase-space noncommutative structures, one can perhaps envisage designing a planar system of charged anyons, having fractional spin (related to θ) and trapped in a harmonic potential well, with a normal magnetic field B (related to η) [84–86] and where the mass and frequency parameters are both taken to be slowly varying as periodic functions of time. We feel that, with this, one can have some experimental demonstrations of this phenomenon, which should have some bearings in condensed-matter systems [63,87].

Last but not least, we briefly mention some interesting directions in which our present paper can be extended. The first is to construct coherent-state Euclidean path-integral [88,89] formulation invoking adiabatic iterative prescription [90,91] for calculating nonadiabatic corrections on Berry phase in noncommutative phase space. Apart from the 2D oscillator, one also can think of other exactly integrable systems where the partition function in the coherent-state path-integral method can be computed so that one can eventually obtain Gibb's entropy of the system and also try to make a possible connection with von Neumann entropy [92,93] of the system in the presence of the modified geometric phase.

The analysis presented here can perhaps also be extended to compute the quantum information metric and Berry curvature [94] from the effective action [89] corresponding to the above-mentioned path integral, so that one can try to connect some features in noncommutative quantum mechanics with quantum information science. Of course, any endeavor to detect the effect of noncommutativity is a challenging enterprise. In light of the present paper, there could appear many surprises in this area. We hope to return to some of these issues in our future work.

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APPENDIX A: UNITARY EQUIVALENCE OF DIFFERENT REALIZATIONS OF NC PHASE-SPACE ALGEBRA

As we have already mentioned, there is yet another realization of the whole phase-space noncommutative algebra given in [39,50], unlike the one used by us (3). And Berry phase too was studied using that realization in noncommutative phase space, albeit in a completely different system involving a gravitational potential well [36], but was found to vanish where an equivalent scaled version of the realization (A2) [39], given below, was employed. What we would like to show here is that our realization (A4) is a more general one in the sense that the one parameter (ξ) family of noncommutative algebra (A1) given below and the realization (A2) occurring in [39,50] are unitarily equivalent to our realization (A4), only for a particular value of $\xi = \xi_c$ (A3), and hence will produce the same physical results for this value only. However, our realization (A4) is the only one which persists to be valid for other values of ξ, i.e., for $\xi \neq \xi_c$ also.

To demonstrate this above-mentioned equivalence between the realizations (A2) and (A4) holding only for $\xi = \xi_c$ (A3), let us consider the following structure of noncommutativity among the phase-space variables:

$$[\hat{x}_i, \hat{x}_j] = i\xi\theta\epsilon_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\xi\eta\epsilon_{ij},$$
$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad \theta\eta < 0, \tag{A1}$$

where θ and η are constant parameters, ϵ_{ij} is an antisymmetric constant tensor, and ξ is a scaling parameter. We then introduce the commuting coordinates q_i and momenta p_i, respectively, satisfying the usual Heisenberg algebra: $[q_i, q_j] = 0 = [p_i, p_j]$, $[q_i, p_j] = i\hbar\delta_{ij}$. Note that in our notation these q_i's and p_i's carry no overhead hats, in contrast to their noncommutative counterparts ($\hat{x}_i$'s and $\hat{p}_i$'s).

In [50], the realization in terms of the above q_i's and p_i's is given by the following linear transformation:

$$\hat{x}_i^{(1)} = \sqrt{\xi}\left(q_i - \frac{\theta}{2\hbar}\epsilon_{ij}p_j\right), \quad \hat{p}_i^{(1)} = \sqrt{\xi}\left(p_i + \frac{\eta}{2\hbar}\epsilon_{ij}q_j\right), \tag{A2}$$

which holds only if

$$\xi = \xi_c := \left(1 + \frac{\theta\eta}{4\hbar^2}\right)^{-1}; 4\hbar^2 + \theta\eta > 0. \tag{A3}$$

But this realization of the algebra (A1) is not unique [84]. Indeed, we provide below another possible realization of (A1) in terms of another linear transformation, as

$$\hat{x}_i^{(2)} = q_i - \frac{\xi\theta}{2\hbar}\epsilon_{ij}p_j + \frac{\xi\sqrt{-\theta\eta}}{2\hbar}\epsilon_{ij}q_j,$$
$$\hat{p}_i^{(2)} = p_i + \frac{\xi\eta}{2\hbar}\epsilon_{ij}q_j + \frac{\xi\sqrt{-\theta\eta}}{2\hbar}\epsilon_{ij}p_j. \tag{A4}$$

The merit of this realization is that it is valid for any value of ξ and need not be fixed to the value given in (A3). This is unlike the one in (A2). Clearly, neither of the transformations (A2) or (A4) represents a canonical transformation, as they change the basic commutator algebra. It is, however, quite obvious that for the value of the ξ parameter, fine tuned to value in (A3), the realizations should be unitarily equivalent. We now construct this unitary transformation explicitly, which maps the realization (A2) to the other one (A4). To that end, let us make the following ansatz of the unitary operator,

$$\mathbf{U} = \exp\left[-i\frac{\sigma\mathbf{D}}{\hbar}\right]\exp\left[-i\frac{\beta\mathbf{L}}{\hbar}\right]\exp[-i\alpha_2\vec{p}^2]\exp[-i\alpha_1\vec{q}^2], \tag{A5}$$

where $\mathbf{D} = \vec{q}\cdot\vec{p} + \vec{p}\cdot\vec{q}$ and $\mathbf{L} = \vec{q}\wedge\vec{p}$ are, respectively, the dilatation and angular momentum operators,[2] and relate these two representations as

$$\hat{x}_i^{(2)} = \mathbf{U}\hat{x}_i^{(1)}\mathbf{U}^\dagger, \quad \hat{p}_i^{(2)} = \mathbf{U}\hat{p}_i^{(1)}\mathbf{U}^\dagger. \tag{A6}$$

Note that we have taken the parameters σ and β to be dimensionless, in contrast to the parameters α_1 and α_2, which are dimensionful. Now making use of Hadamard identity we can easily show that

$$\hat{x}_i^{(2)} = Aq_i - B\epsilon_{ij}p_j + C\epsilon_{ij}q_j + Dp_i,$$
$$\hat{p}_i^{(2)} = Ep_i + F\epsilon_{ij}q_j + G\epsilon_{ij}p_j - Hq_i, \tag{A7}$$

where

$$A = \lambda\sqrt{\xi}[\cos(\beta) + \alpha_1\theta\sin(\beta)],$$
$$B = \frac{\sqrt{\xi}}{\lambda}\left[\left(\frac{\theta}{2\hbar} - 2\alpha_1\alpha_2\theta\hbar\right)\cos(\beta) + 2\alpha_2\hbar\sin(\beta)\right],$$
$$C = \sqrt{\xi}\lambda[\sin(\beta) - \alpha_1\theta\cos(\beta)],$$
$$D = \frac{\sqrt{\xi}}{\lambda}\left[\left(\frac{\theta}{2\hbar} - 2\alpha_1\alpha_2\hbar\theta\right)\sin(\beta) - 2\alpha_2\hbar\cos(\beta)\right],$$
$$E = \frac{\sqrt{\xi}}{\lambda}[(1 - 4\alpha_1\alpha_2\hbar^2)\cos(\beta) + \eta\alpha_2\sin(\beta)],$$
$$F = \lambda\sqrt{\xi}\left[\frac{\eta}{2\hbar}\cos(\beta) + 2\alpha_1\hbar\sin(\beta)\right],$$
$$G = \frac{\sqrt{\xi}}{\lambda}[(1 - 4\alpha_1\alpha_2\hbar^2)\sin(\beta) - \eta\alpha_2\cos(\beta)],$$
$$H = \lambda\sqrt{\xi}\left[\frac{\eta}{2\hbar}\sin(\beta) - 2\alpha_1\hbar\cos(\beta)\right].$$
$$\lambda = \exp(-2\sigma). \tag{A8}$$

On the other hand, all these eight coefficients A–H in (A8) can be determined easily by comparing (A7) with (A4) and are provided below in two segregated clusters:

$$H = 0, \quad D = 0, \quad A = 1, \quad C = \xi\frac{\sqrt{-\theta\eta}}{2\hbar} \tag{A9}$$

and

$$B = \xi\frac{\theta}{2\hbar}, \quad E = 1, \quad F = \frac{\xi\eta}{2\hbar}, \quad G = \xi\frac{\sqrt{-\theta\eta}}{2\hbar}. \tag{A10}$$

The reason for this segregation is that a simple inspection suggests that we can make use of the first three equations in

[2]While the former represents the scalar operator, the latter represents a pseudoscalar operator in a commutative plane and generates appropriate transformations. It is also quite well known that the three scalar generators ($\mathbf{D}$, $\vec{p}^2$, $\vec{q}^2$) form a closed SO(1, 2) algebra [70], while $\mathbf{L}$ commutes with all of them: $[\mathbf{L}, \vec{q}^2] = [\mathbf{L}, \vec{p}^2] = [\mathbf{L}, \mathbf{D}] = 0$.

(A9) to solve for α_1, α_2, and λ in terms of the single parameter β as

$$\alpha_1 = \frac{\eta}{4\hbar^2}\tan(\beta), \quad \alpha_2 = \frac{\theta}{4\hbar^2}\left[\frac{\tan(\beta)}{1+\frac{\theta\eta}{4\hbar^2}\tan^2(\beta)}\right],$$
$$\lambda = \{\sqrt{\xi}[\cos(\beta)+\alpha_1\theta\sin(\beta)]\}^{-1}, \tag{A11}$$

and then this β can be determined by first making use of the fourth equation in (A9) to get the following quadratic equation:

$$\frac{\xi}{4\hbar^2}(-\theta\eta)^{\frac{3}{2}}\tan^2(\beta) - \left(\frac{\theta\eta}{2\hbar} - 2\hbar\right)\tan(\beta) - \xi\sqrt{-\theta\eta} = 0, \tag{A12}$$

yielding the following two roots for β:

$$\beta_1 = \tan^{-1}\left(\sqrt{\frac{-\theta\eta}{4\hbar^2}}\right) \quad \beta_2 = -\tan^{-1}\left[\left(\frac{-\theta\eta}{4\hbar^2}\right)^{\frac{3}{2}}\right]. \tag{A13}$$

It is now a matter of a lengthy but straightforward computation to verify that only when β_1 from (A13) along with α_1, α_2, and λ from (A11) are substituted to the set of expressions of B, E, F, and G in (A8) they readily yield the corresponding expression given in (A10). This therefore provides an explicit demonstration of the unitary equivalence of two realizations (A2) and (A4) for specific values of $\xi = \xi_c$ in (A3). For other values of ξ the realization (A2) will not hold, in contrast to the realization (A4), which persists to hold. In this sense the realization (A4) is more general and in this paper we are basically working with the algebra (2) and its realization (3), which are nothing but Eqs. (A1) and (A4) themselves with $\xi = 1$.

APPENDIX B: DYNAMICAL SYMMETRY GROUP SU(1, 1) ⊗ U(1) OF THE HAMILTONIAN (4) AND APPARENT REMOVABILITY OF THE BERRY PHASE

Following Wei-Norman method [95] we can readily identify the Lie algebraic structure [96] of the Hamiltonian operator (4). To that end let us introduce the generators:

$$\boldsymbol{K}_+ = \frac{iq_i^2}{2}, \quad \boldsymbol{K}_- = \frac{ip_i^2}{2}, \quad \boldsymbol{K_0} = \frac{i(p_iq_i+q_ip_i)}{4},$$
$$\boldsymbol{L} = \epsilon_{ij}q_ip_j. \tag{B1}$$

It can now be shown quite easily that $\boldsymbol{K}_\pm$, $\boldsymbol{K_0}$, and $\boldsymbol{L}$ satisfy the $su(1,1) \oplus u(1)$ Lie algebra [97]:

$$[\boldsymbol{K_0}, \boldsymbol{K}_\pm] = \pm\hbar \boldsymbol{K}_\pm, \quad [\boldsymbol{K}_+, \boldsymbol{K}_-] = -2\hbar\boldsymbol{K_0},$$
$$[\boldsymbol{K}_\pm, \boldsymbol{L}] = [\boldsymbol{K_0}, \boldsymbol{L}] = 0 \tag{B2}$$

where $\boldsymbol{L}$ is the $u(1)$ generator commuting with all $su(1,1)$ generators. Upon exponentiating in a suitable manner, like in (A5), they will generate all the elements of the group SU(1, 1) ⊗ U(1).

To be more transparent, let us introduce the dimensionless generators $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, and $\boldsymbol{T}_0$ defined through $\boldsymbol{K}_\pm$ and $\boldsymbol{K}_0$ as

$$\boldsymbol{K}_+ = \theta(\boldsymbol{T}_1 + i\boldsymbol{T}_2), \quad \boldsymbol{K}_- = -\eta(\boldsymbol{T}_1 - i\boldsymbol{T}_2),$$
$$\boldsymbol{T}_0 = \frac{\boldsymbol{K}_0}{\sqrt{-\theta\eta}}, \quad \theta\eta < 0 \tag{B3}$$

where θ and η are the dimensionfull minimal scale factors in noncommutative phase space, which help us to maintain consistent dimension of all SU(1,1) generators in (B1), and in terms of the dimensionless basis $\boldsymbol{T}_\mu$(where $\mu = 0, 1, 2$) the above (B2) commutation relations take a more suggestive form as

$$[\boldsymbol{T_0}, \boldsymbol{T_i}] = i\tilde{\hbar}\epsilon_{ij}\boldsymbol{T_j}, \quad [\boldsymbol{T_i}, \boldsymbol{T_j}] = -i\tilde{\hbar}\epsilon_{ij}\boldsymbol{T_0}, \quad (i, j = 1, 2), \tag{B4}$$

where $\tilde{\hbar} = \frac{\hbar}{\sqrt{-\theta\eta}}$ is the dimensionless reduced Planck's constant. Notice at this stage that $\boldsymbol{T_0}$ and $\boldsymbol{T_1}$ are skew Hermitian like $\boldsymbol{K_\pm}$ and $\boldsymbol{K_0}$, but $\boldsymbol{T_2}$ is Hermitian.

A faithful finite 2D representation [98,99] "Π" of this SU(1,1) is furnished by the Pauli matrices $\vec{\boldsymbol{\sigma}}$'s as[3]

$$\Pi(\boldsymbol{T}_0) = \frac{\tilde{\hbar}}{2}\sigma_3, \quad \Pi(\boldsymbol{T}_i) = -\frac{i\tilde{\hbar}}{2}\sigma_i. \tag{B5}$$

Using this representation, one can easily see that any $su(1,1)$ Lie algebra element $A^\mu\Pi(\boldsymbol{T}_\mu)$, with coefficients A^μ and $\mu = 0, 1, 2$, takes the following traceless form:

$$A^\mu\Pi(\boldsymbol{T}_\mu) = \frac{\tilde{\hbar}}{2}\begin{pmatrix} A^0 & -A \\ A^* & -A^0 \end{pmatrix}, \quad A = A^1 + iA^2. \tag{B6}$$

If this object is now subjected to a SU(1,1) transformation by $\mathcal{U} \in SU(1,1)$ as

$$A^\mu\Pi(\boldsymbol{T}_\mu) \to \mathcal{U}A^\mu\Pi(\boldsymbol{T}_\mu)\mathcal{U}^\dagger := B^\mu\Pi(\boldsymbol{T}_\mu), \tag{B7}$$

where we could easily replace $\mathcal{U} \to \mathcal{V} \in SU(1,1) \otimes U(1)$, as $[\boldsymbol{L}, \boldsymbol{K_\mu}] = 0\ \forall\mu$, the resulting object in (B7) will again be another $su(1,1)$ element with some other coefficient B^μ, where the tracelessness property will be preserved along with the determinant. Particularly, the latter, i.e., the preservation of the determinant, implies that we must have the following identity holding:

$$(A^1)^2 + (A^2)^2 - (A^0)^2 = (B^1)^2 + (B^2)^2 - (B^0)^2. \tag{B8}$$

We immediately conclude that A^μ can be regarded as a three-vector transforming under Lorentz transformation SO(2,1) in (2+1) dimensions where A^i's and A^0 may be thought of as representing spatial and temporal components, respectively. This connection of the Lorentz group SO(2, 1) with its double cover SU(1, 1) or SL(2, ℝ) is well known in the literature: $SO(2,1) = SU(1,1)/\mathbb{Z}_2 = SL(2,\mathbb{R})/\mathbb{Z}_2$; all of them share the same Lie algebra.

We can now apply all of these formalisms to our system Hamiltonian (4), which can be written as a linear combination of the original infinite-dimensional representation of

[3]Observe at this stage that in this finite-dimensional representation it is rather $\Pi(\boldsymbol{T}_0)$ which is only Hermitian and $\Pi(\boldsymbol{T}_i)$'s are skew Hermitian. This is a typical and peculiar feature of finite-dimensional representations of the Lie algebra associated with noncompact unitary groups like SU(1,1).

SU(1, 1) ⊗ U(1) group generators (B1) as

$$\mathcal{H}(t) = -2i[\alpha(t)\boldsymbol{K}_- + \beta(t)\boldsymbol{K}_+ + 2\delta(t)\boldsymbol{K_0}] \\ - \gamma(t)\boldsymbol{L} = \mathcal{H}_{\text{gho}}(t) - \gamma(t)\boldsymbol{L}. \tag{B9}$$

Clearly the part of Hamiltonian $H_{\text{gho}}(t)$ in (7) is an $su(1, 1)$ Lie algebra valued element. Rewriting this in terms of generators $\boldsymbol{T}_\mu := (\boldsymbol{T}_0, \boldsymbol{T}_1, \boldsymbol{T}_2)$ introduced in (B3) we get

$$\mathcal{H}_{\text{gho}}(t) = -2iA^\mu(t)\boldsymbol{T}_\mu, \tag{B10}$$

where

$$A^\mu(t) = \begin{pmatrix} A^1(t) \\ A^2(t) \\ A^0(t) \end{pmatrix} = \begin{pmatrix} [-\eta\alpha(t) + \theta\beta(t)] \\ i[\eta\alpha(t) + \theta\beta(t)] \\ [2\sqrt{-\theta\eta}\delta(t)] \end{pmatrix}. \tag{B11}$$

Note that $A^2(t)$ as occurs here is purely imaginary and this ensures the Hermiticity of $\mathcal{H}_{\text{gho}}(t)$ (B10). Now if the above three-vector $A^\mu(t)$ scaled by $\tilde{\hbar}$ as

$$A^\mu(t) \to \tilde{A}^\mu(t) := \tilde{\hbar}A^\mu(t) \tag{B12}$$

invariance of the SO(2,1) norm of $\tilde{A}^\mu(t)$ readily gives

$$[\tilde{A}^1(t)]^2 + [\tilde{A}^2(t)]^2 - [\tilde{A}^0(t)]^2 \\ = 4\hbar^2[\alpha(t)\beta(t) - \delta^2(t)] = \hbar^2\omega^2(t) > 0, \tag{B13}$$

where $\omega(t) > 0$ in (10) is the frequency of the $\mathcal{H}_{\text{gho}}(t)$ and is invariant under the instantaneous Lorentz transformation.

Further $2\delta(t) \propto A^0(t)$ here in (B11) and (B13) can be identified with the temporal component of the spacelike three-vector A_μ. Consequently, at any instant $t = t_0$, the tip of the three-vector $A^\mu(t_0)$ will lie on a 2D hyperboloid the tangent plane of which is orthogonal to A^μ and as time evolves A^μ will trace out a trajectory (in fact a closed loop here) intersecting this one-parameter family of such hyperboloids. Further, the spacelike nature of A^μ implies that at the instant $t = t_0$ we can choose a suitable SO(2,1) transformation such that $\delta(t_0)$ vanishes in this particular Lorentz frame. Obviously this needs to change from moment to moment and therefore be time dependent. This, on the other hand, can be induced by subjecting the Hamiltonian $\mathcal{H}_{\text{gho}}(t)$ (B9) and (B10) to a time-dependent unitary transformation belonging to the covering group $\mathcal{W}(t) \in \text{SU}(1, 1)$ in the manner of (B7). One can, of course, consider the bigger group SU(1, 1) ⊗ U(1) also here, but the presence of U(1) is quite inconsequential here and therefore optional in nature. This has to be contrasted with (43) in Sec. IV, where we need to choose a specific U(1) element other than the identity element.

In fact it is not very difficult to construct such a unitary operator $\mathcal{W}(t)$. To that end consider an instantaneous SO(2,1) transformation $\Lambda(t_0)$ transforming the triplet

$$(\alpha(t_0), \beta(t_0), \delta(t_0)) \to (\alpha'(t_0), \beta'(t_0), \delta'(t_0)) \\ := (\alpha(t_0), \beta'(t_0), 0) \tag{B14}$$

in such a manner that the coefficient of the dilatation term vanishes. Using (B13), we readily obtain

$$\beta'(t_0) = \frac{\alpha(t_0)\beta(t_0) - \delta^2(t_0)}{\alpha'(t_0)}. \tag{B15}$$

One can easily verify, at this stage, that a corresponding unitary transformation $\mathcal{W}(t)$ at an arbitrary time t can be constructed as

$$\mathcal{W}(t) = \exp\left[\frac{i}{\hbar}\frac{\delta(t)}{2\alpha(t)}\vec{q}^2\right]. \tag{B16}$$

Under this transformation the instantaneous total Hamiltonian $\mathcal{H}(t)$ (6) indeed transforms as

$$\mathcal{H}(t) \to \mathcal{W}(t)\mathcal{H}(t)\mathcal{W}^\dagger(t) \\ = \alpha(t)p_i^2 + \left(\frac{\alpha(t)\beta(t) - \delta^2(t)}{\alpha(t)}\right)q_i^2 - \gamma(t)\epsilon_{ij}q_ip_j \tag{B17}$$

eliminating the dilatation term. However, from the time-dependent Schrödinger equation, one can easily recognize that (B17) should not be identified as the Hamiltonian responsible for the time evolution of the transformed instantaneous state $[\mathcal{W}(t)|\Psi(t)\rangle]$ as $\mathcal{W}(t)$ itself has an explicit time dependence. Indeed, it is not difficult to see that the time evolution of the state $[\mathcal{W}(t)|\Psi(t)\rangle]$ is governed by the effective Hamiltonian $\tilde{\mathcal{H}}(t)$, obtained by augmenting the one in (B17) by a suitable "connection" term as

$$\mathcal{H}(t) \to \tilde{\mathcal{H}}(t) = \mathcal{W}(t)\mathcal{H}(t)\mathcal{W}^\dagger(t) - i\hbar\mathcal{W}(t)\frac{d}{dt}\mathcal{W}^\dagger(t) \tag{B18}$$

so that $i\hbar\partial_t[W(t)|\psi(t)\rangle] = \tilde{\mathcal{H}}(t)[W(t)|\psi(t)\rangle]$ holds. This has to be contrasted with the case involving time-independent unitary transformation $\boldsymbol{U}$ (A5) [see discussion below (44)] connecting $\mathcal{H}^{(1)}(t)$ and $\mathcal{H}^{(2)}(t)$ (43). Expanding this $\tilde{\mathcal{H}}(t)$ we obtain

$$\tilde{\mathcal{H}}(t) = \alpha(t)p_i^2 + \left(\frac{\alpha(t)\beta(t) - \delta^2(t) - \frac{\alpha}{2}\frac{d}{dt}\left(\frac{\delta(t)}{\alpha(t)}\right)}{\alpha(t)}\right)q_i^2 - \gamma(t)\boldsymbol{L} \\ = \mathcal{H}_{\text{sho}}(t) - \gamma(t)\boldsymbol{L}. \tag{B19}$$

This is like a usual 2D harmonic oscillator Hamiltonian with just the Zeeman coupling $\gamma(t)\boldsymbol{L}$. Here we also observe that $\gamma(t)\boldsymbol{L}$, $\mathcal{H}_{\text{sho}}(t)$, and $\tilde{\mathcal{H}}(t)$ commute among each other, even at different times. So, they have simultaneous instantaneous eigenstates.

To find out the eigenstates of the system Hamiltonian (B19) one may proceed and introduce the annihilation (and corresponding creation) operator. This can just be obtained by setting $\delta = 0$ and replacing $\beta \to \tilde{\beta}$ in (9) to get

$$\tilde{\boldsymbol{a}}_j = \left(\frac{\tilde{\beta}}{4\alpha\hbar^2}\right)^{\frac{1}{4}}\left[q_j + i\sqrt{\frac{\alpha}{\tilde{\beta}}}p_j\right], \quad j = 1, 2 \tag{B20}$$

with $\tilde{\beta} = (\frac{\alpha(t)\beta(t) - \delta^2(t) - \frac{\alpha}{2}\frac{d}{dt}(\frac{\delta(t)}{\alpha(t)})}{\alpha(t)})$, satisfying the commutation relation $[\hat{\tilde{\boldsymbol{a}}}_j, \hat{\tilde{\boldsymbol{a}}}_k^\dagger] = \delta_{jk}$. Accordingly, the system Hamiltonian (B19) can be written as

$$\tilde{\mathcal{H}}(t) = \hbar\tilde{\omega}(t)(\tilde{\boldsymbol{a}}_j^\dagger\tilde{\boldsymbol{a}}_j + 1) + i\hbar\gamma(t)\epsilon_{jk}\tilde{\boldsymbol{a}}_j^\dagger\tilde{\boldsymbol{a}}_k \quad (j, k) \in \{1, 2\} \tag{B21}$$

where

$$\tilde{\omega}(t) = 2\sqrt{[\alpha(t)\beta(t) - \delta^2(t)] - \frac{\alpha(t)}{2}\frac{d}{dt}\left(\frac{\delta(t)}{\alpha(t)}\right)}. \tag{B22}$$

Again introducing operators $\boldsymbol{a}_\pm$ through time-independent canonical transformation (11) we get the Hamiltonian (B21) in standard diagonal Hermitian form as

$$\tilde{\mathcal{H}} = \hbar\tilde{\omega}(\tilde{\boldsymbol{a}}_j^\dagger\tilde{\boldsymbol{a}}_j + 1) - \hbar\gamma(t)(\tilde{\boldsymbol{a}}_+^\dagger\tilde{\boldsymbol{a}}_+ - \tilde{\boldsymbol{a}}_-^\dagger\tilde{\boldsymbol{a}}_-), \quad j \in \{+,-\}. \tag{B23}$$

The instantaneous nondegenerate eigenstates are now given by

$$|n_+, n_-;(t)\rangle = \frac{(\tilde{\boldsymbol{a}}_+^\dagger)^{n_+}(\tilde{\boldsymbol{a}}_-^\dagger)^{n_-}}{\sqrt{n_+!}\sqrt{n_-!}}|0,0;(t)\rangle,$$
$$\tilde{\boldsymbol{a}}_\pm(t)|0,0;(t)\rangle = 0. \tag{B24}$$

Note that we have written the time argument t within the parentheses as (t) in order to distinguish these towers of eigenstates from (15), which are clearly not the same; they are built upon different instantaneous ground states.

Correspondingly the instantaneous discrete energy eigenvalues are given by

$$\begin{aligned} E_{n_+,n_-}(t) &= \hbar\tilde{\omega}(n_+ + n_- + 1) - \hbar\gamma(t)(n_+ - n_-) \\ &\simeq \hbar(n_+ + n_- + 1)\omega(t)\left[1 - \frac{\alpha(t)}{\omega(t)^2}\frac{d}{dt}\left(\frac{\delta(t)}{\alpha(t)}\right)\right] \\ &\quad - \hbar\gamma(t)(n_+ - n_-) \end{aligned} \tag{B25}$$

where $\omega = 2\sqrt{\alpha\beta - \delta^2}$. Here it suffices to work in the first order of adiabaticity (manifested through the order of time derivatives of the parameters). This is tantamount to ignoring the higher-order time derivatives of the slowly varying parameters in (B22) and (B25).

Now since $\tilde{\mathcal{H}}(t)$ and $\mathcal{H}(t)_{\text{sho}}$ in (B19) commute with each other they share the same eigenspaces. Consequently, we can express an eigenstate (B24) of $\tilde{\mathcal{H}}(t)$, as a linear combination of eigenstates of $\mathcal{H}(t)_{\text{sho}}$, as

$$|n_+, n_-;(t)\rangle = \sum_{n_1+n_2=n_++n_-} C_{n_1,n_2}^{n_+,n_-}|n_1,n_2;(t)\rangle_{\text{sho}}^{2\text{D}} \tag{B26}$$

where we have denoted the eigenstates of $\mathcal{H}(t)_{\text{sho}}$ as

$$|n_1,n_2;(t)\rangle_{\text{sho}}^{2\text{D}} = \frac{(\tilde{\boldsymbol{a}}_1^\dagger)^{n_1}(\tilde{\boldsymbol{a}}_2^\dagger)^{n_2}}{\sqrt{n_1!}\sqrt{n_2!}}|0,0;(t)\rangle \tag{B27}$$

and $n_1 + n_2 = n_+ + n_-$. This restriction ensures that the eigenstates are taken from a single eigenspace of $\mathcal{H}(t)_{\text{sho}}$ [100]. Note that $C_{n_1,n_2}^{n_+,n_-}$'s are themselves time independent, because the ladder operators diagonalizing $\tilde{\mathcal{H}}(t)$ are derivable from the ladder operators of $\mathcal{H}(t)_{\text{sho}}$, using time-independent invertible linear transformation (11). This also ensures that both sets of lowering operators $\{\tilde{\boldsymbol{a}}_1, \tilde{\boldsymbol{a}}_2\}$ and $\tilde{\boldsymbol{a}}_\pm$ annihilate the same instantaneous ground state $|0,0;(t)\rangle$ at time t.

Since we are working in the regime where the adiabatic theorem works properly, the Berry phase for an eigenstate $|n_+, n_-\rangle$, if it exists, would be given by

$$\begin{aligned} \Phi_{(n_+,n_-)}^{(G)} &= -i\int dt\,\langle n_+,n_-;(t)|\frac{d}{dt}|n_+,n_-;(t)\rangle \\ &= -i\int dt \sum_{\substack{n_1+n_2=m_1+m_2\\=n_++n_-}} C_{m_1,m_2}^{n_+,n_-\star}C_{n_1,n_2}^{n_+n_-}\,{}_{\text{sho}}^{2\text{D}}\langle m_1,m_2(t)| \\ &\quad \times \frac{d}{dt}|n_1,n_2;(t)\rangle_{\text{sho}}^{2\text{D}}. \end{aligned} \tag{B28}$$

Now for a pair of tuples (m_1, m_2) and (n_1, n_2) sandwiching inside the sum there are two possibilities: either (i) $m_1 = n_1, m_2 = n_2$ or (ii) $m_1 \neq n_1, m_2 \neq n_2$.

Let us consider the case of the second possibility first as

$$\begin{aligned} &{}_{\text{sho}}^{2\text{D}}\langle m_1,m_2;(t)|\frac{d}{dt}|n_1,n_2;(t)\rangle_{\text{sho}}^{2\text{D}} \\ &= \left({}_{\text{sho}}^{1\text{D}}\langle m_1(t)|\frac{d}{dt}|n_1(t)\rangle_{\text{sho}}^{1\text{D}}\right) \times \left({}_{\text{sho}}^{1\text{D}}\langle m_2(t)|n_2(t)\rangle_{\text{sho}}^{1\text{D}}\right) \\ &\quad + \left({}_{\text{sho}}^{1\text{D}}\langle m_2(t)|\frac{d}{dt}|n_2(t)\rangle_{\text{sho}}^{1\text{D}}\right) \times \left({}_{\text{sho}}^{1\text{D}}\langle m_1(t)|n_1(t)\rangle_{\text{sho}}^{1\text{D}}\right) = 0 \end{aligned} \tag{B29}$$

whereas for the case of the first possibility $(m_1, m_2) = (n_1, n_2)$ we get

$$\begin{aligned} &\int dt\left({}_{\text{sho}}^{2\text{D}}\langle n_1,n_2;(t)|\frac{d}{dt}|n_1,n_2;(t)\rangle_{\text{sho}}^{2\text{D}}\right) \\ &= \int dt\left({}_{\text{sho}}^{1\text{D}}\langle n_1(t)|\frac{d}{dt}|n_1(t)\rangle_{\text{sho}}^{1\text{D}} + {}_{\text{sho}}^{1\text{D}}\langle n_2(t)|\frac{d}{dt}|n_2(t)\rangle_{\text{sho}}^{1\text{D}}\right) \\ &= 0 \end{aligned} \tag{B30}$$

as this represents the Berry phase of a pair of decoupled 1D simple harmonic oscillators which we know to have vanishing Berry phase. So $\Phi_n^{(G)} = 0$, implying the total Hamiltonian $\tilde{\mathcal{H}}(t)$ does not produce any Berry phase by itself, apparently.

However, the total dynamical phase acquired by $|n_+, n_-;(t)\rangle$ after a complete cycle Γ of time period T by the Hamiltonian $\tilde{\mathcal{H}}(t)$ is obtained by using (B25), to get

$$\begin{aligned} \Phi_{n_+,n_-}(T) &= \int_0^T dt\,\frac{E_{n_+,n_-}(t)}{\hbar} \\ &= \int_0^T dt\left[\left(n_+ + \frac{1}{2}\right)(\omega+\gamma) + \left(n_- + \frac{1}{2}\right)(\omega-\gamma)\right. \\ &\quad \left. -(n_+ + n_- + 1)\frac{\alpha}{\omega}\frac{d}{dt}\left(\frac{\delta}{\alpha}\right)\right], \end{aligned} \tag{B31}$$

agreeing with our result (49) after a gauge transformation (36). Indeed, the last term, although it now occurs in the dynamical phase, nevertheless retains its geometric character as it represents the line integral of the one-form $\boldsymbol{A}$ (36) along the closed loop Γ as $\int_\Gamma \boldsymbol{A}$ and therefore the resulting phase is a functional of Γ and matches exactly with the Berry phase (46). This whole exercise therefore shows how one can eliminate the crucial dilatation term responsible for the Berry phase through a time-dependent unitary transformation to find it to reappear again in disguise within the dynamical part, revealing its geometric origin, when considering the total adiabatic phase as a whole. It should therefore show up in suitably designed interference experiments.

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