


Particle mixing and the emergence of classicality: A spontaneous-collapse-model viewKyrylo Simonov ^{*}*Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria*

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Spontaneous collapse models aim to resolve the measurement problem in quantum mechanics by considering wave-function collapse as a physical process. We analyze how these models affect a decaying flavor-oscillating system whose evolution is governed by a phenomenological non-Hermitian Hamiltonian. In turn, we apply two popular collapse models, the Quantum Mechanics with Universal Position Localization and the Continuous Spontaneous Localization models, to a neutral meson system. By using the equivalence between the approaches to the time evolution of decaying systems with a non-Hermitian Hamiltonian and a dissipator of the Lindblad form in an enlarged Hilbert space, we show that spontaneous collapse can induce the decay dynamics in both quantum state and master equations. Moreover, we show that the decay property of a flavor-oscillating system is intimately connected to the time (a)symmetry of the noise field underlying the collapse mechanism. This (a)symmetry, in turn, is related to the definition of the stochastic integral and can provide a physical intuition behind the Itô-Stratonovich dilemma in stochastic calculus.

DOI: [10.1103/PhysRevA.102.022226](https://doi.org/10.1103/PhysRevA.102.022226)**I. INTRODUCTION**

Despite a distinctly high success of (“standard”) quantum mechanics in describing the microscopic world and covering plenty of phenomena on different energy scales, it meets some conceptual controversies. The linearity of the Schrödinger equation manifests itself as the famous superposition principle, which is one of the cornerstones of quantum mechanics. At high energy scales, it plays a crucial role in the phenomena of particle mixing and oscillations, which are experimentally observed in several systems such as neutral mesons [1,2]. They occur when the energy eigenstates of the particle are not necessarily identical to the interaction eigenstates but rather are their superpositions. For example, a neutral K meson produced in strong interactions as a bound state of a down quark and strange antiquark $d\bar{s}$ can be found to turn into the bound state $\bar{d}s$ via weak-interaction processes. However, the superposition principle does not seem to be relevant in the macroscopic world. We do not observe a table being here and there and a kitten dead and alive at once: our everyday experience shows a break of the dynamics predicted by the linear Schrödinger equation. This observation leads to a question: How does the everyday classical world arise from the quantum world?

Going further, a measurement performed on a quantum system and associated with an interaction between the system and the measurement apparatus (implied to be macroscopic) reveals a definite outcome. In turn, it is intimately connected to the so-called measurement problem, which, in particular, refers to the questions of what selects a certain outcome in a particular experimental run and what makes an actual interaction a measurement (“What makes a measurement a measurement?”) [3,4]. The measurement problem and, in

general, the validity of the superposition principle is a subject of intense experimental verification.

“Standard” quantum mechanics does not explain this quantum-to-classical transition but only postulates an *ad hoc* separation between microscopic (quantum) and macroscopic (classical) worlds. This results in two different types of dynamics of the quantum system:

(i) a stochastic and nonunitary reduction (“collapse”) of the state of the quantum system due to an interaction with a macroscopic system (“measurement apparatus”) in accordance with the Born’s rule;

(ii) a deterministic unitary time evolution governed by the Schrödinger equation before and after the measurement.

A possible approach to the quantum-to-classical transition can be a universal dynamics valid on all scales, which contains both quantum and classical mechanics as approximations. Dynamical reduction models, or so-called collapse models, aim to provide a phenomenological framework to such dynamics. They introduce an ontologically objective mechanism of the wave function collapse, which is implemented by replacing the Schrödinger dynamics with its stochastic (in order to explain why the measurement outcomes occur randomly in accordance with the Born’s rule) and nonlinear (in order to break up a macroscopic superposition and get rid of the Schrödinger’s cat) modification [5–7].

The first and the simplest collapse model is the Ghirardi-Rimini-Weber (GRW) model, also known as the Quantum Mechanics with Spontaneous Localizations (QMSL) model, introduced in 1985 [8]. This model has implemented the collapse mechanism by assuming, with respect to a system of N distinguishable particles, random spontaneous localizations (called “hittings” by Ghirardi [9] and “jumps” by Bell [10]) with a mean rate λ affecting each particle and conserving the usual Schrödinger evolution of the system between the successive localizations. In turn, the GRW model gives rise to the amplification mechanism tuned by the rate λ , which

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sets the collapse strength: spontaneous localizations affect a microscopic object very rarely and can be neglected, while a macroscopic superposition is rapidly reduced [6,7]. Another guiding line implemented by the GRW model is a choice of the preferred basis into which the state of the quantum system is reduced so that a macroscopic object has a definite position in space: for that purpose, the GRW model chooses the position basis. This leads to the definition of the collapse width r_C , which sets a coarse-graining for the wave function of the system. Basically, it sets a scale at which a spatial superposition is effectively reduced and, therefore, is a threshold between macroscopic and microscopic superpositions [11]. For example, a superposition of two localized states separated by a distance $d \ll r_C$ is not significantly affected by the collapse dynamics, whereas one with $d \gg r_C$ will be effectively localized. Taking collapse models seriously, the parameters λ and r_C are two new natural constants introduced by these models.

An important class of dynamical reduction models is represented by those that describe the wave-function collapse as a continuous process induced by an interaction between the system and a (classical) noise field. The benefit of these models is a possibility to govern the universal dynamics by a stochastic differential equation, which is obtained, generally speaking, by adding new nonlinear and stochastic terms to the Schrödinger equation. The typical and, perhaps, most popular models of this class are the Quantum Mechanics with Universal Position Localization (QMUPL) [12] and the Continuous Spontaneous Localization (CSL) [13,14] models. One of the essential differences between these models is the simpler mathematical structure of the QMUPL model: the QMUPL noise field “lives” only in the time dimension, whereas the CSL noise field is spread both in time and space. Therefore, the CSL model introduces two parameters, a collapse rate and a coherence length (just like the GRW model), while, in the QMUPL model, the latter is absent. For many physical systems, both models predict approximately equivalent dynamics. In this paper, however, we provide an example of a setup featuring flavor mixing, for which the QMUPL and the CSL models offer nonequivalent predictions.

Since their appearance on the market, collapse models caught the eye of researchers and were intensively investigated in a plethora of physical systems at different energy scales [15]. In particular, collapse models were analyzed with respect to the spontaneous radiation emission from charged particles [16–18] and put to experimental tests by x-rays [19–23]. Furthermore, spontaneous collapse models were recently studied in the context of cold-atom experiments [24], gravitational waves [25], levitated nanoparticles [26], matter-wave interferometry [27–30], and optomechanical setups [31–33].

Particular attention is attracted to the analysis of the possible effects of a spontaneous collapse in the oscillations at high energies. Mixed systems propose some peculiar features, which have stimulated high interest in these systems as a rich playground for testing the very foundations of quantum mechanics [34–49]. For example, the beauty of the neutral K -meson systems, whose relevant states measured in experiments are superpositions of states with distinct masses, is that they reveal a parallel with the spin- $\frac{1}{2}$ particles and photons.

However, they offer richer properties such as decay and the violation of the \mathcal{CP} discrete symmetry, which leads to surprising results, such as a contradiction between local realism and the \mathcal{CP} violation [50]. In this context, a possible effect of spontaneous collapse on flavor oscillations was studied for neutrinos [51,52] and neutral mesons, in particular, K^0 - \bar{K}^0 correlated pairs produced at the CPLEAR experiment by the Low Energy Antiproton Ring (LEAR) accelerator facility at CERN and at the KLOE and KLOE-2 experiments by the DAΦNE ϕ factory at the Frascati National Laboratories of INFN [51,53]. In turn, the CSL model has predicted exponential damping of the oscillatory behavior of a neutral meson system [51,53].

It has been put under discussion whether the spontaneous collapse can be considered also as a source of particle decay [54–56]. It has been shown that the decay dynamics can be recovered by the mass-proportional CSL model with an asymmetric correlation function of the noise field underlying this collapse model. Mathematically, the asymmetry in the noise field is expressed as a parameter in the quantum state equation of the collapse model, which is intimately related to the choice of the stochastic formalism. Considering this asymmetry free can lead to a nontrivial dependence on the absolute masses of the eigenstates of the time evolution. In Ref. [57], this point of view was criticized by considering a unique fixed master equation in the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) form with a Hermitian Hamiltonian, which, clearly, does not describe the decay property of particles. This argument combined with the fixed asymmetry of the noise field due to the chosen stochastic formalism led to the conclusion that spontaneous collapse models are not able to induce the (exponential) decay dynamics. This is due to the fact that a naïve switch of the stochastic formalism changes both the asymmetry of the noise field and the quantum state equation in such a way that the master equation preserves its form (as it should), and the non-Hermitian part of the Hamiltonian, which covers the decay, clearly, cannot be recovered in such a way.

In this paper, we review the results of Refs. [54–56] in the context of the master equation associated with a collapse model and investigate how the noise field affects the dynamics of neutral mesons. We derive a class of collapse models with a free choice of time-asymmetry of the underlying white-noise field and show that the decay property of a flavor oscillating system (or its absence) can be governed by the corresponding spontaneous collapse dynamics in accordance with Ref. [55]. In turn, the derived collapse models generate a family of master equations parametrized by the decay rates, including the GKLS equation corresponding to the evolution without decay.

The paper is organized as follows: In Sec. II, we discuss the phenomenology of neutral mesons by putting the focus on the nonrelativistic framework via the Wigner-Weisskopf approximation and the corresponding master equation. This approach operates with a two-state non-Hermitian Hamiltonian acting on the flavor Hilbert space and results in exponential decay dynamics in the transition probabilities. Furthermore, we discuss the approach to the neutral meson phenomenology based on the enlarged Hilbert space with included decay product states, which allows us to define a Hermitian Hamiltonian and, in turn, prove the completely positive dynamics of a neutral

meson system. In Sec. III, we discuss the collapse equation in the flavor Hilbert space for a non-Hermitian Hamiltonian and associate it with the master equation introduced in the previous section. After all, we generalize this collapse model to a class of models with a time-asymmetric noise field, where the collapse process turns out to be the source of the decay widths. In Sec. IV, we refer to the QMUPL and the mass-proportional CSL models, generalize them to the time-asymmetric noise field introduced in the previous section, and derive the transition probabilities by using the corresponding master equations.

II. PHENOMENOLOGY OF THE NEUTRAL MESONS

Mixed systems such as the M^0 - \bar{M}^0 systems can be described by a phenomenological Hamiltonian acting on a two-dimensional Hilbert space \mathbf{H}_M called also flavor space. The physical (flavor) states $|M^0\rangle$ and $|\bar{M}^0\rangle$ of a neutral meson are labeled by a flavor quantum number and can decay into the same final states. In particular, the flavor states of neutral K mesons are labeled by the strangeness quantum number ($S = \pm 1$, respectively) and, for the hadronic decays, both can decay via weak interaction into two or three pions. The usual approach to the dynamics of a neutral meson system is based on the Wigner-Weisskopf approximation, which considers an effective non-Hermitian Hamiltonian

$$\hat{H} = \hat{M} - \frac{i}{2}\hat{\Gamma}, \tag{1}$$

with the eigenstates $|M_i\rangle$ (in particular, for neutral mesons, $i = L, H$ corresponding to “light” and “heavy”) with the (distinct) definite masses m_i and known decay widths Γ_i , so that the corresponding eigenvalues read $m_i - \frac{i}{2}\Gamma_i$. In this treatment, $\hat{M} = \hat{M}^\dagger$ is the mass operator, which covers the unitary part of the dynamics, and $\hat{\Gamma} = \hat{\Gamma}^\dagger$ describes the decay. Apart from neutral K mesons, the difference of the decay widths $\Delta\Gamma = \Gamma_L - \Gamma_H$ in the M^0 - \bar{M}^0 systems is tiny. Up to a particular phase convention and neglecting a slight violation of the \mathcal{CP} symmetry in the weak interaction, the mass (lifetime) eigenstates $|M_i\rangle$ of the Hamiltonian (1) can be related to the flavor states as

$$\begin{aligned} |M_H\rangle &= \frac{1}{\sqrt{2}}(|M^0\rangle + |\bar{M}^0\rangle), \\ |M_L\rangle &= \frac{1}{\sqrt{2}}(|M^0\rangle - |\bar{M}^0\rangle). \end{aligned} \tag{2}$$

The Wigner-Weisskopf approximation takes into account only the time evolution of the components of the flavor states, so that the neutral meson system dynamics is described by the effective Schrödinger equation,

$$i\frac{d}{dt}|\psi\rangle_t = \left(\hat{M} - \frac{i}{2}\hat{\Gamma}\right)|\psi\rangle_t, \tag{3}$$

$$|\psi\rangle_t = a(t)|M^0\rangle + b(t)|\bar{M}^0\rangle, \tag{4}$$

where $\hbar = c = 1$ is assumed. Due to the presence of the non-Hermitian part of the Hamiltonian (1), the temporal part of the evolution of the neutral meson system is not normalized.

Namely, considering an arbitrary state $|\psi\rangle_t \in \mathbf{H}_M$, we obtain

$$d\|\psi\rangle_t\|^2 = -\sum_i \Gamma_i |\langle M_i|\psi\rangle_t|^2 dt, \tag{5}$$

so that $|\psi\rangle_t$ is not normalized when $\Gamma_i \neq 0$. In particular, let us consider the quantities actively investigated at the accelerator facilities, the transition probabilities

$$P_{\text{in}\rightarrow\text{out}}(t) = |\langle\psi_{\text{out}}|e^{-i\hat{H}t}|\psi_{\text{in}}\rangle|^2 \tag{6}$$

for the states $|\psi_{\text{in,out}}\rangle \in \mathbf{H}_M$. The most interesting are the transition probabilities for the lifetime eigenstates $|M_i\rangle$,

$$P_{M_i\rightarrow M_j}(t) = e^{-\Gamma_i t} \delta_{ij}, \tag{7}$$

and the flavor states $|M^0\rangle$ and $|\bar{M}^0\rangle$,

$$P_{M^0\rightarrow M^0/\bar{M}^0}(t) = \frac{1}{4} \left\{ \sum_i e^{-\Gamma_i t} \pm 2e^{-\Gamma t} \cos[t\Delta m] \right\}, \tag{8}$$

where $\Gamma = \frac{\Gamma_L + \Gamma_H}{2}$. Since the temporal part of the evolution of the M^0 - \bar{M}^0 system is not normalized due to Eq. (5), the probabilities (7) and (8) are not conserved. A formal renormalization by the decay width would allow us to apply the Born’s rule within a framework of time operator; however, this option is falsified in neutral K -meson systems because of the \mathcal{CP} violation [45].

If the M^0 - \bar{M}^0 system is considered as open (i.e., interacting with the external environment) and affected by some uncontrolled phenomena, the resulting evolution can be described by the following master equation [58]:

$$\begin{aligned} \frac{d\hat{\rho}_t}{dt} &= -i\hat{H}\hat{\rho}_t + i\hat{\rho}_t\hat{H}^\dagger - \frac{1}{2} \sum_i [\hat{L}_i^\dagger \hat{L}_i \hat{\rho}_t + \hat{\rho}_t \hat{L}_i^\dagger \hat{L}_i - 2\hat{L}_i \hat{\rho}_t \hat{L}_i^\dagger] \\ &= i[\hat{\rho}_t, \hat{M}] - \frac{1}{2} \sum_i [\hat{L}_i^\dagger \hat{L}_i \hat{\rho}_t + \hat{\rho}_t \hat{L}_i^\dagger \hat{L}_i - 2\hat{L}_i \hat{\rho}_t \hat{L}_i^\dagger] \\ &\quad - \frac{1}{2} \{\hat{\Gamma}, \hat{\rho}_t\}, \end{aligned} \tag{9}$$

where $\hat{\rho}_t \in \mathcal{D}(\mathbf{H}_M)$ is the density operator¹ which represents the state of the neutral meson system, \hat{L}_i are the Lindblad generators which describe the interaction between the system and environment, and curly brackets denote an anticommutator. In the absence of decay, i.e., when the decay operator $\hat{\Gamma}$ is set to be zero, Eq. (9) possesses the GKLS form [59,60] and, hence, describes a completely positive evolution of the system. This means that the operator $\hat{\rho}_t$ remains a density operator and still represents a state of the quantum system, so that the evolution given by the master equation (9) is physically consistent. In particular, it can govern dissipative and decoherence effects in flavor mixing, which can be signals for new physics. For example, decoherence in neutrino oscillations can reveal the difference between Dirac and Majorana neutrinos and, moreover, break up the \mathcal{CPT} symmetry [44,49].

Although the master equation (9) does not possess the GKLS form if $\hat{\Gamma} \neq 0$, it is possible to prove the completely

¹The set $\mathcal{D}(\mathbf{H}_M)$ of density operators on \mathbf{H}_M is a convex subset of the space $\mathcal{B}(\mathbf{H}_M)$ of bounded linear operators on \mathbf{H}_M formed by the positive self-adjoint trace-class operators.

positive dynamics with respect to the non-Hermitian Hamiltonian (1) in an elegant way by considering particle decay as an open system [58,61]. Namely, the issue of nonconserving probabilities due to Eq. (5) can be resolved by taking into account the decay products. This is done by enlarging the Hilbert space \mathbf{H}_M [spanned by the eigenstates of the Hamiltonian (1)] to the Hilbert space $\mathbf{H} = \mathbf{H}_M \oplus \mathbf{H}_D$, where \mathbf{H}_D is spanned by the orthonormal states $|f_i\rangle$ which represent the decay products [58]. In this new space \mathbf{H} , the decay property of neutral mesons can be incorporated into a GKLS equation as a Lindblad operator. Physically, this means that the decay property is induced by an interaction of the neutral meson system with an environment (analogously to an interaction with the QCD vacuum in quantum field theory), and the resulting time evolution is completely positive [58,62]. We construct the following GKLS equation for $\hat{\rho}_t \in \mathcal{D}(\mathbf{H})$:

$$\begin{aligned} \frac{d\hat{\rho}_t}{dt} = & i[\hat{\rho}_t, \hat{\mathcal{H}}] - \frac{1}{2} \sum_i [\hat{\mathcal{L}}_i^\dagger \hat{\mathcal{L}}_i \hat{\rho}_t + \hat{\rho}_t \hat{\mathcal{L}}_i^\dagger \hat{\mathcal{L}}_i - 2\hat{\mathcal{L}}_i \hat{\rho}_t \hat{\mathcal{L}}_i^\dagger] \\ & - \frac{1}{2} [\hat{\mathcal{L}}_D^\dagger \hat{\mathcal{L}}_D \hat{\rho}_t + \hat{\rho}_t \hat{\mathcal{L}}_D^\dagger \hat{\mathcal{L}}_D - 2\hat{\mathcal{L}}_D \hat{\rho}_t \hat{\mathcal{L}}_D^\dagger], \end{aligned} \quad (10)$$

where

$$\hat{\mathcal{H}} = \begin{pmatrix} \hat{M} & 0 \\ 0 & 0 \end{pmatrix}$$

is the Hamiltonian and

$$\hat{\mathcal{L}}_i = \begin{pmatrix} \hat{L}_i & 0 \\ 0 & 0 \end{pmatrix},$$

in accordance with the notation in Eq. (9). Let us show that the decay property is governed by Eq. (10) if the Lindblad operator $\hat{\mathcal{L}}_D$ has the form

$$\hat{\mathcal{L}}_D = \begin{pmatrix} 0 & 0 \\ \hat{L}_D & 0 \end{pmatrix}. \quad (11)$$

This choice means that $\hat{\mathcal{L}}_D$ represents a transition between the flavor \mathbf{H}_M and the “decay” \mathbf{H}_D subspaces of \mathbf{H} . In this way, if \mathbf{H}_D has the same dimensions as \mathbf{H}_M , then we can decompose $\hat{\mathcal{L}}_D$, which acts in \mathbf{H}_M , as

$$\hat{\mathcal{L}}_D = \sum_i \sqrt{\gamma_i} |f_i\rangle \langle M_i|, \quad (12)$$

where $\gamma_i \geq 0$. By projecting the master equation (10) back to \mathbf{H}_M we obtain a master equation

$$\begin{aligned} \frac{d\hat{\rho}_t}{dt} = & i[\hat{\rho}_t, \hat{M}] - \frac{1}{2} \sum_i [\hat{L}_i^\dagger \hat{L}_i \hat{\rho}_t + \hat{\rho}_t \hat{L}_i^\dagger \hat{L}_i - 2\hat{L}_i \hat{\rho}_t \hat{L}_i^\dagger] \\ & - \frac{1}{2} \{\hat{L}_D^\dagger \hat{L}_D, \hat{\rho}_t\}. \end{aligned} \quad (13)$$

Comparing Eq. (13) with the master equation (9), we see that they are identical if the decay operator is set to be

$$\hat{\Gamma} = \hat{L}_D^\dagger \hat{L}_D = \sum_i \gamma_i |M_i\rangle \langle M_i|, \quad (14)$$

so that $\gamma_i = \Gamma_i$ are simply the decay widths of the corresponding lifetime eigenstates $|M_i\rangle$ of the non-Hermitian Hamiltonian (1). Hence, the choice (14) of the decay operator guarantees the consistency of the evolution of a M^0 - \bar{M}^0 system induced by the master equation (9).

III. COLLAPSE DYNAMICS IN \mathbf{H}_M

In a typical advanced dynamical reduction model, such as the QMUPL and CSL models discussed in Sec. IV, the wave-function collapse is considered as a continuous physical process with respect to the interaction between the quantum system and a randomly fluctuating (noise) field. Mathematically, this is achieved by a nonlinear stochastic modification of the Schrödinger equation,²

$$\begin{aligned} d|\psi\rangle_t = & \left[-i\hat{H}dt + \sqrt{\lambda} \sum_i (\hat{A}_i - R_{\hat{A}_i}) dW_{i,t} \right. \\ & \left. - \frac{\lambda}{2} [\hat{A}_i^\dagger \hat{A}_i - 2R_{\hat{A}_i} \hat{A}_i + R_{\hat{A}_i}^2] dt \right] |\psi\rangle_t, \end{aligned} \quad (15)$$

with

$$R_{\hat{A}_i} = \left\langle \frac{\hat{A}_i^\dagger + \hat{A}_i}{2} \right\rangle_t, \quad (16)$$

where \hat{H} governs the standard Schrödinger part of the evolution, $\lambda \geq 0$ is the coupling constant of the collapse model which represents the localization rate, $W_{i,t}$ is a set of the Wiener processes, \hat{A}_i are the corresponding collapse operators, and $\langle \hat{A}_i \rangle_t := {}_t\langle \psi | \hat{A}_i | \psi \rangle_t$ is the quantum-mechanical expectation value.

The change of the Wiener process $W_{i,t}$ in time is represented by the noise field $dW_{i,t}$ mentioned above, whose correlation function reads

$$\mathbb{E}[dW_{i,t} dW_{j,t'}] = \delta_{ij} \delta(t - t'), \quad (17)$$

where \mathbb{E} denotes the noise average. This noise field is white, i.e., all its frequencies equally contribute to the collapse process. The nature of the noise field still remains an open question: one of the options is a physical field filling the whole space, hence, having presumably a cosmological nature (for example, the relic cosmic neutrino background would be a possible candidate for the noise field of the CSL model [63]). Hence, there is a challenge to study the collapse models with the noise field whose properties differ from ones of a “typical” white noise [64–68]. In turn, the noise field can depend on the direction of time,³ which plays a crucial role in decay dynamics of a M^0 - \bar{M}^0 system, as we show later in this section.

The collapse operators \hat{A}_i define the preferred basis into which the state of the quantum system is reduced. A particular interest has the case of the self-adjoint collapse operators,

²In this section, we use the Itô stochastic formalism. For details see Appendix A.

³Mathematically, this dependence on the time direction is intimately connected to the asymmetry of the correlation function (17) and, in turn, the definition of the stochastic integral. We discuss the mathematical aspects of the asymmetry of Eq. (17) in Appendix A.

which significantly simplifies the collapse equation (15),

$$d|\psi\rangle_t = \left[-i\hat{H}dt + \sqrt{\lambda} \sum_i (\hat{A}_i - \langle \hat{A}_i \rangle_t) dW_{i,t} - \frac{\lambda}{2} \sum_i (\hat{A}_i - \langle \hat{A}_i \rangle_t)^2 dt \right] |\psi\rangle_t. \quad (18)$$

For a neutral meson system, whose standard Schrödinger evolution is governed by the phenomenological Hamiltonian (1), the (flavor) \mathbf{H}_M counterpart of the collapse dynamics can be governed by the self-adjoint collapse operator

$$\hat{A} = \sum_i \tilde{m}_i |M_i\rangle \langle M_i|, \quad (19)$$

where \tilde{m}_i is the mass ratio with respect to the corresponding mass m_i and the reference mass m_0 . The latter can be seen as a free parameter of a collapse model which relates (generally speaking, for ordinary matter) the mass ratio to an average number of constituents of the composite object and tunes the amplification mechanism of the collapse model [69]. In a typical collapse model (in particular, the mass-proportional CSL model discussed in Sec. IV), the amplification mechanism strengthens the collapse effect for a more massive object, suggesting the definition $\tilde{m}_i = \frac{m_i}{m_0}$ for the mass ratio.⁴

Accepting Eq. (18) as the dynamical equation for the state $|\psi\rangle_t \in \mathbf{H}_M$ of a neutral meson system, it is possible to derive the master equation (9) for the corresponding density operator $\hat{\rho}_t = \mathbb{E}[|\psi\rangle_t \langle \psi|_t]$ from it. In this case, the role of the Lindblad generators \hat{L}_i play the collapse operators weighted by the localization rate, so that $\hat{L}_i = \sqrt{\lambda} \hat{A}_i$. However, as we have discussed in Sec. II, Eq. (9) is derived with respect to the non-Hermitian Hamiltonian \hat{H} . Hence, it does not guarantee itself the complete positivity of the resulting dynamics, which could produce no physically consistent state of the neutral meson system in this case. Therefore, we would desire to have a collapse model in \mathbf{H} which induces the GKLS master equation (10). This guarantees that Eq. (18) is associated with the master equation (13), which is a projection of Eq. (10) onto \mathbf{H}_M and, hence, induces a physically consistent dynamics of the $M^0\text{-}\bar{M}^0$ system. Let us build such collapse dynamics in \mathbf{H} . At first, we have to reproduce the self-adjoint collapse operator which induces the reduction in the mass basis due to Eq. (19), namely,

$$\hat{A} = \begin{pmatrix} \hat{A} & 0 \\ 0 & 0 \end{pmatrix}.$$

Another (non-Hermitian) collapse operator

$$\hat{B} = \begin{pmatrix} 0 & 0 \\ \hat{B} & 0 \end{pmatrix}$$

⁴Neutral meson systems reveal opposite behavior: lighter particles decay faster, and the mass ratio $\frac{m_i}{m_0}$ decreases. Thus, as discussed in Ref. [55], an inverted mass ratio $\tilde{m}_i = \frac{m_0}{m_i}$ would be more reasonable for the particles lighter than the constituents of ordinary matter. However, in this paper, we imply that \tilde{m}_i represents the “usual” mass ratio unless otherwise noted.

with

$$\hat{B} = \sum_i \sqrt{\frac{\Gamma_i}{\lambda}} |f_i\rangle \langle M_i| \quad (20)$$

triggers the decay of a neutral meson to the product states $|f_i\rangle$ with the corresponding decay widths Γ_i . With a Hermitian Hamiltonian

$$\hat{\mathcal{H}} = \begin{pmatrix} \hat{M} & 0 \\ 0 & 0 \end{pmatrix},$$

the required dynamics can be governed by the following differential equation for a quantum state $|\Psi\rangle_t \in \mathbf{H}$:

$$d|\Psi\rangle_t = \left\{ -i\hat{\mathcal{H}}dt + \sqrt{\lambda} [(\hat{A} - \langle \hat{A} \rangle_t) dW_t + (\hat{B} - R_{\hat{B}}) dW_t^D] - \frac{\lambda}{2} [(\hat{A} - \langle \hat{A} \rangle_t)^2 + \hat{B}^\dagger \hat{B} - 2R_{\hat{B}} \hat{B} + R_{\hat{B}}^2] dt \right\} |\Psi\rangle_t, \quad (21)$$

with

$$R_{\hat{B}} = \left\langle \frac{\hat{B}^\dagger + \hat{B}}{2} \right\rangle. \quad (22)$$

It can be shown that Eq. (21) is associated with the GKLS master equation (10) for a density operator $\hat{\varrho}_t = \mathbb{E}[|\Psi\rangle_t \langle \Psi|_t]$, where the corresponding Lindblad generators are proportional to the collapse operators, $\hat{L} = \sqrt{\lambda} \hat{A}$ and $\hat{L}_D = \sqrt{\lambda} \hat{B}$.

Going back to the flavor space \mathbf{H}_M , we project the obtained collapse equation (21) onto it. In this way, we find the following collapse equation for $|\psi\rangle_t \in \mathbf{H}_M$:

$$d|\psi\rangle_t = \left\{ -i\hat{M}dt + \sqrt{\lambda} (\hat{A} - \langle \hat{A} \rangle_t) dW_t - \frac{\lambda}{2} [(\hat{A} - \langle \hat{A} \rangle_t)^2 + \hat{B}^\dagger \hat{B}] dt \right\} |\psi\rangle_t, \quad (23)$$

which is associated with the master equation (13). Comparing Eq. (23) with the original collapse equation (18), we see that there appears a drift term proportional to $\lambda \hat{B}^\dagger \hat{B}$ which plays the role of the non-Hermitian part $\hat{\Gamma}$ of the phenomenological Hamiltonian (1) governing the decay property, as we would expect from the master equation (10). However, when looking at Eqs. (21) and (23), the collapse mechanism seems to have no impact on the decay dynamics since its localization rate λ , in fact, does not show up in the terms associated with the decay, so that

$$\begin{aligned} \mathbb{E}[d||\psi\rangle_t|^2] &= -\lambda \mathbb{E}[\langle \psi|_t \hat{B}^\dagger \hat{B} |\psi\rangle_t] dt \\ &= -\sum_i \Gamma_i \mathbb{E}[|\langle M_i | \psi \rangle_t|^2] dt \neq 0, \end{aligned} \quad (24)$$

in accordance with the predictions of the effective Schrödinger equation (3) for the dynamics of neutral mesons. Indeed, the role of “localization rates” for the collapse operators \hat{B} and \hat{B} play the decay widths Γ_i inserted by hand, and we just reproduce the $\hat{\Gamma}$ operator in accordance with Eq. (14). However, it is possible to obtain spontaneous collapse dynamics which does not simply mimic the decay operator $\hat{\Gamma}$ but rather induces additional energy terms which

play the role of the decay widths Γ_i and, hence, recovers the decay dynamics of a $M^0\text{-}\bar{M}^0$ system.

Before proceeding, let us remark on the properties of dynamics governed by Eqs. (13) and (18). Despite the non-linearity of the quantum state equation (18), which makes it difficult to solve, its physical predictions can be analyzed in a simple way by using a peculiar mathematical property of Eqs. (13) and (18). The master equation (13) and, in turn, the physical predictions of the quantum state equation (18) concerning the outcomes of a measurement [in particular, the transition probabilities (7) and (8)] turn out to be invariant⁵ under the phase transformation [64],

$$\begin{aligned} d|\psi\rangle_t &= \left[-i\hat{H}dt + \sqrt{\lambda} \sum_i (e^{i\varphi}\hat{A}_i - \cos(\varphi)\langle\hat{A}_i\rangle_t) dW_{i,t} \right. \\ &\quad \left. - \frac{\lambda}{2} \sum_i (\hat{A}_i^2 - 2e^{i\varphi}\cos(\varphi)\langle\hat{A}_i\rangle_t\hat{A}_i + \cos^2(\varphi)\langle\hat{A}_i\rangle_t^2) dt \right] |\psi\rangle_t, \end{aligned} \quad (25)$$

of Eq. (18). In particular, the original collapse equation (18) can be recovered by choosing $\varphi = 0$, whereas the choice $\varphi = \frac{\pi}{2}$ introduces an imaginary noise field and leads to a more simple Schrödinger-like equation

$$d|\psi\rangle_t = \left[-i\hat{H}dt + i\sqrt{\lambda} \sum_i \hat{A}_i dW_{i,t} - \frac{\lambda}{2} \sum_i \hat{A}_i^2 dt \right] |\psi\rangle_t. \quad (26)$$

Applying the noise field transformation to Eq. (23) we obtain the quantum state equation

$$d|\psi\rangle_t = \left[-i\hat{M}dt + i\sqrt{\lambda}\hat{A}dW_t - \frac{\lambda}{2}(\hat{A}^2 + \hat{B}^\dagger\hat{B})dt \right] |\psi\rangle_t. \quad (27)$$

In the absence of the decay contribution $\hat{B}^\dagger\hat{B}$, the quantum state equation (27) conserves the norm of $|\psi\rangle_t$ and, indeed, there is no built-in direction of time.⁶ As discussed in Appendix A, this can be interpreted as the time-symmetry of the noise field, i.e., the action of the noise field in the bra space (“out” states) and ket space (“in” states) is symmetric. On the other hand, considering a class of more general collapse models with a time-asymmetric noise is more reasonable for the decay dynamics, which does not conserve the norm of $|\psi\rangle_t$. It generates a quantum state equation

$$d|\psi\rangle_t = [-i\hat{M}dt + i\sqrt{\lambda}\hat{A}dW_t - \lambda\beta\hat{A}^2dt] |\psi\rangle_t, \quad (28)$$

⁵This transformation introduced in Ref. [64] can be generalized to the transformation $d|\psi\rangle_t = [-i\hat{H}dt + \sqrt{\lambda} \sum_i (e^{i\varphi}\hat{A}_i - \xi\langle\hat{A}_i\rangle_t) dW_{i,t} - \frac{\lambda}{2} \sum_i (\hat{A}_i^2 - 2e^{i\varphi}\xi^*\langle\hat{A}_i\rangle_t\hat{A}_i + (|\xi|^2 + i\omega)\langle\hat{A}_i\rangle_t^2) dt] |\psi\rangle_t$ with additional parameters $\xi \in \mathbb{C}$ and $\omega \in \mathbb{R}$, which still leaves the master equation (13) invariant.

⁶Notice that the collapse models, generally speaking, do not necessarily need a built-in direction of time for the description of spontaneous collapse, which can seem an essentially time-asymmetric process [70,71].

with the parameter $\beta \in [0, 1]$ which describes this time-asymmetry⁷ and can be seen as tuning of the coupling between the bra and ket spaces with respect to the noise field action [55]. Comparing Eq. (28) with Eq. (27), we see that the decay operator $\hat{\Gamma}$ can be recovered by an operator induced by the collapse dynamics in \mathbf{H}_M ,

$$\hat{\Gamma} \equiv \lambda\hat{B}^\dagger\hat{B} = -\lambda(1 - 2\beta)\hat{A}^2. \quad (29)$$

Decomposing the collapse operators \hat{A} and \hat{B} , we obtain

$$\Gamma_i = -\lambda(1 - 2\beta)\tilde{m}_i^2 \quad (30)$$

in the definition⁸ of the collapse operator \hat{B} . This implies that the spontaneous collapse can be an only source of the two distinct decay widths, and, in turn, the decay dynamics in \mathbf{H}_M is governed by the collapse dynamics.

Summarizing, the non-Hermitian part of the Wigner-Weisskopf effective Hamiltonian (1), which governs the decay property of neutral mesons, can be induced by spontaneous collapse dynamics in \mathbf{H}_M with respect to the quantum state equation (28) or, equivalently, the master equation

$$\begin{aligned} \frac{d\hat{\rho}_t}{dt} &= i[\hat{\rho}_t, \hat{M}] - \frac{\lambda}{2}[\hat{A}^2\hat{\rho}_t + \hat{\rho}_t\hat{A}^2 - 2\hat{A}\hat{\rho}_t\hat{A}] \\ &\quad + \frac{\lambda}{2}(1 - 2\beta)\{\hat{A}^2, \hat{\rho}_t\}. \end{aligned} \quad (31)$$

In other words, a broader class of time-asymmetric collapse models can in principle gain the decay property of neutral mesons. As shown in Refs. [55,56], such a collapse model makes possible to predict the absolute masses m_H and m_L of the lifetime eigenstates as well as the collapse parameters λ and β using the experimentally measured values of the decay constants Γ_H and Γ_L and the difference of masses $\Delta m = m_H - m_L$. We discuss these interesting consequences of the spontaneous collapse dynamics in the following section.

IV. QUANTUM MECHANICS WITH UNIVERSAL POSITION LOCALIZATION AND CONTINUOUS SPONTANEOUS LOCALIZATION MODELS

In the previous section, we obtained a class of collapse models which could explain the full dynamics of a $M^0\text{-}\bar{M}^0$ system in the flavor Hilbert space \mathbf{H}_M , where the oscillations take place. Now, we turn to the collapse models acting in the Hilbert space $L^2(\mathbb{R}^d) \otimes \mathbf{H}_M$ with $d = 1, 2, 3$, which combines the position and flavor spaces, and the collapse is assumed to occur in the spatial part of the state of the system. For the sake of concreteness, we consider the QMUPL and the mass-proportional CSL models, which were widely analyzed in the context of flavor oscillations [51–56], and generalize them to the models with the time-asymmetric noise field. In

⁷The choice $\beta = \frac{1}{2}$ gives a “usual” noise field with no dependence on the direction of time.

⁸The overall minus sign comes from our convention on the direction of time arrow, so that the non-negative decay widths correspond to $\beta \in [\frac{1}{2}, 1]$ in this case.

this way, we analyze the quantum state equation

$$d|\psi\rangle_t = \left[-i\hat{M}dt + i\sqrt{\lambda} \sum_i \hat{\mathbf{A}}_i dW_{i,t} - \lambda\beta \sum_i \hat{\mathbf{A}}_i^2 dt \right] |\psi\rangle_t, \quad (32)$$

with the collapse operators

$$\hat{\mathbf{A}}_i^{\text{QMUPL}} = \hat{q}_i \otimes \hat{A}, \quad (33)$$

$$\hat{\mathbf{A}}_x^{\text{CSL}} = \hat{Q}(\mathbf{x}) \otimes \hat{A}, \quad (34)$$

of the QMUPL and the mass-proportional CSL models, respectively. In the QMUPL model, \hat{q}_i denotes the i th coordinate operator in $L^2(\mathbb{R}^d)$, and the unique introduced collapse parameter is the collapse rate $\lambda \equiv \lambda_Q$ which has the units $(\text{m}^2 \text{s})^{-1}$. In the mass-proportional CSL model,

$$\hat{Q}(\mathbf{x}) = \int \frac{d\mathbf{y}}{(2\pi r_C^2)^{\frac{d}{2}}} e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{2r_C^2}} |\mathbf{y}\rangle\langle\mathbf{y}| \quad (35)$$

is a continuous set of collapse operators acting in $L^2(\mathbb{R}^d)$ which are smeared by a Gaussian function.⁹ In contrast to the QMUPL model, the mass-proportional CSL model introduces two natural constants, the collapse rate being traditionally denoted $\lambda \equiv \gamma$, which has the units $\text{m}^d \text{s}^{-1}$, and the coherence length r_C .

At accelerator facilities, one intensively studies the decay modes of neutral mesons. Observation of a decay mode is a passive measurement procedure, which allows the experimenter to determine the lifetime of a neutral meson or its flavor content (for example, strangeness in the case of K mesons). Hence, in standard quantum mechanics, it leads to the corresponding lifetime and flavor measurement probabilities given by the transition probabilities (7) and (8). With included contribution from spontaneous collapse, these probabilities can be defined through

$$P_{\text{in} \rightarrow \text{out}}(t) = |\langle\psi_{\text{out}}|\hat{U}(t)|\psi_{\text{in}}\rangle|^2 \quad (36)$$

for $|\psi_{\text{in,out}}\rangle \in \mathbf{H}_M$, where $\hat{U}(t)$ is the evolution operator due to the quantum state equation (32). The detailed perturbative calculations of Eq. (36) for the lifetime eigenstates $|M_i\rangle$ and the flavor states $|M^0\rangle$ and $|\bar{M}^0\rangle$ are already known in literature [52,54–56], hence, we choose a simpler procedure and stick to the class of master equations in accordance with Eq. (31), namely,

$$\begin{aligned} \frac{d\hat{\rho}_t}{dt} &= i[\hat{\rho}_t, \hat{M}] - \frac{\lambda}{2} \sum_i [\hat{\mathbf{A}}_i^2 \hat{\rho}_t + \hat{\rho}_t \hat{\mathbf{A}}_i^2 - 2\hat{\mathbf{A}}_i \hat{\rho}_t \hat{\mathbf{A}}_i] \\ &+ \frac{\lambda}{2} (1 - 2\beta) \sum_i \{\hat{\mathbf{A}}_i^2, \hat{\rho}_t\}, \end{aligned} \quad (37)$$

with the collapse operators (33) or (34). Solving Eq. (37), we can derive the required transition probabilities (36), which

now read

$$P_{\text{in} \rightarrow \text{out}}(t) = \text{Tr}[(\mathbb{1} \otimes |\psi_{\text{out}}\rangle\langle\psi_{\text{out}}|)\hat{\rho}_t], \quad (38)$$

where the initial state is chosen as $\hat{\rho}_0 = |\alpha\rangle\langle\alpha| \otimes |\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|$ with $|\alpha\rangle$ representing a Gaussian wave packet of width $\sqrt{\alpha}$.

Following the calculations done in the Appendix B, we find that the QMUPL collapse effect is not exponential, while the CSL model recovers the exponential effect, in accordance with Ref. [55]. In particular, the transition probabilities for the lifetime eigenstates read

$$P_{M_i \rightarrow M_j}^{\text{QMUPL}}(t) = [1 - \lambda_Q \alpha (1 - 2\beta) \tilde{m}_i^2 t]^{-\frac{d}{2}} \delta_{ij}, \quad (39)$$

$$P_{M_i \rightarrow M_j}^{\text{CSL}}(t) = e^{-\frac{\gamma}{(\sqrt{4\pi}r_C)^d} (2\beta - 1) \tilde{m}_i^2 t} \delta_{ij}, \quad (40)$$

whereas the transition probabilities for the flavor states, which reveal the particle oscillations, read

$$\begin{aligned} P_{M^0 \rightarrow M^0 / \bar{M}^0}^{\text{QMUPL}}(t) &= \frac{1}{4} \left\{ \sum_i [1 - \lambda_Q \alpha (1 - 2\beta) \tilde{m}_i^2 t]^{-\frac{d}{2}} \right. \\ &\left. \pm \frac{2 \cos[t \Delta m]}{\left\{ 1 - \frac{\lambda_Q \alpha}{2} [(1 - 2\beta) \sum_i \tilde{m}_i^2 - (\Delta \tilde{m})^2] t \right\}^{\frac{d}{2}}} \right\}, \end{aligned} \quad (41)$$

$$\begin{aligned} P_{M^0 \rightarrow M^0 / \bar{M}^0}^{\text{CSL}}(t) &= \frac{1}{4} \left\{ \sum_i e^{-\frac{\gamma}{(\sqrt{4\pi}r_C)^d} (2\beta - 1) \tilde{m}_i^2 t} \right. \\ &\left. \pm 2e^{-\frac{1}{2} \frac{\gamma}{(\sqrt{4\pi}r_C)^d} [(2\beta - 1) \sum_i \tilde{m}_i^2 + (\Delta \tilde{m})^2] t} \cos[t \Delta m] \right\}. \end{aligned} \quad (42)$$

In particular, the oscillations of K and B mesons were intensively studied through their (semi-)leptonic decays

$$\begin{aligned} K^0 &\rightarrow \pi^- e^+ \nu_e, \\ \bar{K}^0 &\rightarrow \pi^+ e^- \bar{\nu}_e, \\ B^0 &\rightarrow D^- \mu^+ \nu_\mu, \\ \bar{B}^0 &\rightarrow D^+ \mu^- \bar{\nu}_\mu, \end{aligned}$$

at accelerator facilities in the CPLEAR and the BaBar and Belle experiments, respectively. In these experiments, detecting a lepton of a definite charge uniquely identifies the flavor of the decayed meson. In this way, a direct measure of the relative flavor components of the state of a neutral meson is provided by the relative decay rates. The corresponding decay rates can be associated with the transition probabilities for the flavor states up to a normalization factor. Hence, the asymmetry term, which is a combination of the flavor transition probabilities,

$$\mathbb{A}(t) = \frac{P_{M^0 \rightarrow M^0}(t) - P_{M^0 \rightarrow \bar{M}^0}(t)}{P_{M^0 \rightarrow M^0}(t) + P_{M^0 \rightarrow \bar{M}^0}(t)}, \quad (43)$$

plays a significant role by removing the overall normalization factor and, hence, canceling potential systematic biases [72]. For the standard quantum-mechanical transition probabilities

⁹Originally, the CSL model is formulated within the second quantization formalism in order to describe a system of indistinguishable particles. For the sake of simplicity, we formulate it as a first-quantized model following Ref. [57].

(8), it reads

$$\mathbb{A}(t) = \frac{\cos[t\Delta m]}{\cosh\left[\frac{\Delta\Gamma}{2}t\right]}, \quad (44)$$

where $\Delta\Gamma = \Gamma_L - \Gamma_H$. Plugging in the transition probabilities (41) and (42), we include the effects of the QMUPL and the CSL models into the asymmetry term and obtain

$$\begin{aligned} \mathbb{A}^{\text{QMUPL}}(t) &= 2 \cos[t\Delta m] \left[\left(1 - \frac{1}{2} \frac{\lambda_Q \alpha [(1-2\beta)\Delta\tilde{m}^2 - (\Delta\tilde{m})^2] t}{1 - \lambda_Q \alpha (1-2\beta)\tilde{m}_L^2 t}\right)^{\frac{d}{2}} \right. \\ &\quad \left. + \left(1 + \frac{1}{2} \frac{\lambda_Q \alpha [(1-2\beta)\Delta\tilde{m}^2 + (\Delta\tilde{m})^2] t}{1 - \lambda_Q \alpha (1-2\beta)\tilde{m}_H^2 t}\right)^{\frac{d}{2}} \right]^{-1}, \quad (45) \\ \mathbb{A}^{\text{CSL}}(t) &= \frac{\cos[t\Delta m]}{\cosh\left[\frac{\gamma}{(\sqrt{4\pi}r_C)^d}(\beta - \frac{1}{2})\Delta\tilde{m}^2 t\right]} e^{-\frac{\gamma}{(\sqrt{4\pi}r_C)^d} \frac{(\Delta\tilde{m})^2}{2} t}, \quad (46) \end{aligned}$$

where $\Delta\tilde{m}^2 = \tilde{m}_H^2 - \tilde{m}_L^2$.

The asymmetry term (46) with the included CSL collapse dynamics reveals two disentangled effects on the transition probabilities for neutral mesons. At first, there is exponential damping of the interference term, which depends on the mass difference Δm . Second, comparing Eq. (46) with the standard quantum-mechanical asymmetry term (44), which results from the Schrödinger equation with the Wigner-Weisskopf effective Hamiltonian (1), we find the induced exponential decay dynamics with the corresponding decay widths

$$\Gamma_i = -\frac{\gamma}{(\sqrt{4\pi}r_C)^d} (1-2\beta)\tilde{m}_i^2, \quad (47)$$

with respect to the phenomenological decay operator $\hat{\Gamma}$:

$$\hat{\Gamma} = -\frac{\gamma}{(\sqrt{4\pi}r_C)^d} (1-2\beta)\hat{A}^2, \quad (48)$$

as we would expect from Eq. (30). The decay widths appear to depend on the absolute values m_H and m_L of masses of neutral mesons, which play no role in standard quantum mechanics. This fact leads to two important consequences: On the one hand, for $\beta \neq \frac{1}{2}$, it allows us to calculate the absolute masses in terms of experimentally measured quantities [55], namely, decay widths Γ_H and Γ_L and mass difference Δm via the quadratic equation

$$\frac{2\Delta\Gamma}{\Delta\Gamma \pm 2\Gamma} m_L^2 + 2(\Delta m)m_L + (\Delta m)^2 = 0, \quad (49)$$

where the upper sign corresponds to the normal mass ratio while the lower sign fixes the inverted mass ratio, which are discussed in Sec. III. On the other hand, it is possible to estimate the collapse rate

$$\lambda_{\text{CSL}} \equiv \frac{\gamma}{(\sqrt{4\pi}r_C)^d} = \frac{\Gamma_i}{(2\beta - 1)\tilde{m}_i^2} \quad (50)$$

from Eq. (30). In turn, the maximal value of the time-asymmetry $\beta = 1$ establishes a lower bound for the estimated collapse rate, which in terms of experimentally measured

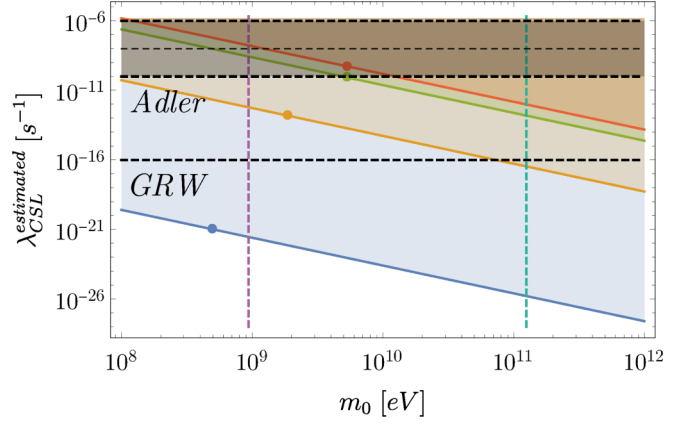


FIG. 1. Lower bounds for the estimated value of the collapse rate λ_{CSL} via Eq. (51) as a function of the reference mass m_0 . The blue, orange, green, and red areas correspond to the values of $\lambda_{\text{CSL}}^{\text{estimated}}$ given by K , D , B , and B_s mesons, respectively. The colored points refer to the rest masses of the corresponding mesons, the purple line is given for the nucleon mass, and the cyan line corresponds to the rest mass of the Higgs boson. Plotted are the theoretical values of the collapse rate $\lambda_{\text{CSL}} = 10^{-16} \text{s}^{-1}$ proposed by Ghirardi, Rimini, and Weber [8] and $\lambda_{\text{CSL}} = 10^{-8 \pm 2} \text{s}^{-1}$ proposed by Adler [73] for $r_C = 10^{-7} \text{m}$. The inverted mass ratio $\tilde{m}_i = \frac{m_0}{m_i}$ is assumed in accordance with Ref. [55].

quantities reads

$$\lambda_{\text{CSL}}^{\text{estimated}} \geq \left(\frac{\Delta m}{m_0 (\sqrt{\Gamma_L^{\pm 1}} - \sqrt{\Gamma_H^{\pm 1}})} \right)^{\mp 2}, \quad (51)$$

where, as in Eq. (49), the upper sign corresponds to the normal mass ratio while the lower sign fixes the inverted mass ratio. In Fig. 1 are plotted the estimated values of λ_{CSL} as a function of the reference mass m_0 .

In the case of the QMUPL model, the neutral meson time evolution is still affected by these two effects, which, however, do not have exponential character (rather algebraic), so that they are, in principle, observable. Hence, the QMUPL and the mass-proportional CSL models offer distinguishable and nonequivalent dynamics for a $M^0 - \bar{M}^0$ system because of the specific link between position and flavor Hilbert spaces. In this way, the mentioned quantities actively studied in the realm of particle physics crucially depend on the coupling between the particles and the noise field introduced by the collapse model and, thus, can give insight into the physics of the collapse mechanism.

V. CONCLUSIONS

Spontaneous collapse models propose a solution to the measurement problem in quantum mechanics by considering the collapse of a wave function as a physical process with respect to the interaction between the system and a (classical) noise field. These models predict interesting effects for the systems at higher energies. In particular, being applied to a flavor-oscillating system, the QMUPL and the mass-proportional CSL models, the two most popular spontaneous-collapse models, predict damping of the oscillations. This

distinguishes them, for example, from the semiclassical gravity approach equipped with the Schrödinger-Newton equation [74–76], which affects a neutral meson system by shifting the mass difference and, thus, changing the frequency of the flavor oscillations [77].

Recently, it was shown that, in the spontaneous collapse models with a white-noise field, its time (a)symmetry can lead to a nontrivial contribution to the quantities, which are usually studied at accelerator facilities [54–56]. Indeed, collapse dynamics induced by the interaction of a neutral meson system with a time-asymmetric noise field results in a dependence on its absolute masses which does not appear in standard quantum mechanics. With respect to this contribution, the QMUPL model provides a nonexponential effect on the time evolution of a M^0 - \bar{M}^0 system, which is in principle observable. In turn, the mass-proportional CSL model can explain the decay property of a M^0 - \bar{M}^0 system and recover its decay dynamics.

In our discussion, we have revisited the effects of the spontaneous collapse on neutral meson systems [54–56] and studied their connection with the properties of the underlying noise field. We have focused on the systems in the meson sector which provide the mixing of particles and antiparticles and are well suited for a nonrelativistic quantum-mechanical treatment since, in this case, the field-theoretical effect is negligible [78]. In this way, we can apply the nonrelativistic collapse models such as the QMUPL and the CSL models in a consistent way. In principle, the results of this paper can be extended to any decaying mixed particles that allow for a nonrelativistic treatment. However, there is a great challenge to construct a spontaneous collapse mechanism in relativistic quantum field theories [79–84], and a closer look into the systems at higher energies would help to move forward the relativistic dynamical reduction program. One of the possible first steps toward it would be to analyze the effect of a nonwhite noise on a flavor-oscillating system, which can be related to a field of a cosmological nature and is more physical than the white noise: the latter with a Lorenz-invariant correlation function produces infinite energy per unit time and unit volume [7,83].

We have found that the decay property incorporated as a non-Hermitian part of a phenomenological Hamiltonian can result from collapse dynamics in an enlarged Hilbert space. Such a space includes the states which correspond to the decay products and is used to construct a Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) equation, which allows us to represent a decaying system as open and to incorporate the particle decay as a Lindblad operator. In the following step, we have questioned whether spontaneous collapse dynamics can be the only source of the decay in the dynamics of a flavor-oscillating system. Our analysis has shown that a broader class of collapse models with an asymmetric white-noise field generates an extra term in the master equation, which induces the decay dynamics and depends on the absolute masses of neutral mesons. Analyzing the obtained master equation, we have confirmed the result of Ref. [55] for the generalized CSL model with an asymmetric noise field, whose collapse dynamics can fully describe the exponential decay behavior of a M^0 - \bar{M}^0 system. In this way, we can conclude that the decay dynamics in flavor-oscillating

systems can be described via the spontaneous collapse mechanism.

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APPENDIX A: STOCHASTIC FORMALISMS AND ASYMMETRY OF THE NOISE FIELD

In stochastic calculus, one of the key objects of study is a stochastic differential equation, which describes the change of a stochastic process X_t in time,

$$dX_t = f(X_t, t)dW_t + g(X_t, t)dt, \quad (\text{A1})$$

where $f(X_t, t)$ is the so-called diffusion term, $g(X_t, t)$ is the so-called drift term, and W_t is a Wiener process with $W_0 = 0$, which represents the Brownian motion (in turn, dW_t represents the white noise) and has the following important properties [85]:

- (i) it is normally distributed with the density $p(W_t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{W_t^2}{2t}}$;
- (ii) it has independent and stationary increments, so that $W_t - W_s$ depends only on $t - s$.

To solve such an equation, we have to define a stochastic integral $\int f dW_t$, which we desire to construct as a Riemann-Stieltjes integral. Focusing on the simplest case $f = W_t$, we define the stochastic integral as the mean-squared limit of the Riemann-Stieltjes sums,

$$\int_0^t W_t dW_t = \text{ms-lim}_{n \rightarrow \infty} \sum_{i=1}^n W(t_i^*) [W(t_i) - W(t_{i-1})], \quad (\text{A2})$$

where ms-lim denotes the mean-squared limit. In contrast with the usual Riemann integral, the value of the constructed stochastic integral (A2) depends on the choice of the intermediate points t_i^* , which lie in the intervals $[t_{i-1}, t_i]$, respectively. A particular choice of t_i^* defines a formalism of stochastic calculus with its specific rules and properties. Generally speaking, there is a continuous family of stochastic formalisms with a corresponding choice of t_i^* . The two formalisms widely discussed in literature are the Itô formalism, which corresponds to the left point choice $t_i^* = t_{i-1}$, and the Stratonovich formalism, which corresponds to the middle point choice $t_i^* = \frac{t_{i-1} + t_i}{2}$ [86,87]. The Itô formalism is widely used in financial mathematics since the integral, in this case, is a martingale, i.e., given only its history up to time t_0 , the expectation value of the Itô integral at any $t > t_0$ is simply its value at t_0 [88]. However, the rules of the Itô calculus, generally speaking, do not coincide with those of the classical calculus: for example, the Itô differentiation rule differs from the classical Leibniz's product rule and is given by

$$d(X_t \cdot Y_t) = dX_t \cdot Y_t + X_t \cdot dY_t + \mathbb{E}[dX_t \cdot dY_t], \quad (\text{A3})$$

where \mathbb{E} denotes the stochastic average. The Stratonovich formalism, which is preferred in physics, operates with the usual rules of calculus.

In turn, the point t_i^* can be connected to the value of the Heaviside function in zero by [86,89]

$$t_i^* = t_{i-1} + \theta(0)(t_i - t_{i-1}), \quad (\text{A4})$$

so that the Itô formalism fixes $\theta(0) = 0$ and the Stratonovich formalism fixes $\theta(0) = \frac{1}{2}$. Typically, the Heaviside function appears in integrals of the correlation function $\mathbb{E}[dW_t dW_{t'}] = \delta(t - t')$ of the white noise since

$$\int_{-\infty}^t \delta(t') dt' = \theta(t), \quad (\text{A5})$$

and its value at zero, $\theta(0)$, generally speaking, belongs to the interval $\theta(0) \in [0, 1]$ and is not well defined. However, once the stochastic formalism is chosen, $\theta(0)$ is fixed due to Eq. (A4). This property can be used for an easy switch between different formalisms, which is governed by the formula

$$f(X_t, t) \circ_{\beta} dW_t = f(X_t, t) \circ_{\beta'} dW_t + (\beta - \beta') \frac{\partial f(X_t, t)}{\partial X_t} f(X_t, t) dt, \quad (\text{A6})$$

where \circ_{β} and $\circ_{\beta'}$ denote the product in the formalisms with $\theta(0) = \beta$ and $\theta(0) = \beta'$, respectively. In particular, the Itô and Stratonovich conventions are related in the following way.¹⁰

$$f(X_t, t) dW_t = f(X_t, t) \circ dW_t - \frac{1}{2} \frac{\partial f(X_t, t)}{\partial X_t} f(X_t, t) dt. \quad (\text{A7})$$

Except for the Stratonovich value $\theta_S(0) = \frac{1}{2}$, the white noise dW_t as well as its correlation function $\delta_{\theta(0)}(t)$ are asymmetric with respect to the chosen value of $\theta(0)$, so that

$$\int_0^t \delta_{\theta(0)}(t') dt' - \int_{-t}^0 \delta_{\theta(0)}(t') dt' = 1 - 2\theta(0). \quad (\text{A8})$$

This asymmetry can be understood better by considering an approximation of $\delta_{\theta(0)}(t)$ by the asymmetric Laplace distribution,

$$\delta_{\varkappa, \varepsilon}(t) = \frac{1}{\varepsilon} \frac{1}{\varkappa + \frac{1}{\varkappa}} e^{-\frac{|t|}{\varepsilon} \varkappa^{\text{sgn}(t)}}, \quad (\text{A9})$$

where $\varepsilon \ll 1$, and $\varkappa \in [0, \infty]$ is the parameter of asymmetry of the distribution. After some algebra, one can derive the identity

$$\theta(0) = \frac{\varkappa^2}{1 + \varkappa^2}, \quad (\text{A10})$$

which illustrates the connection between $\theta(0)$ and the asymmetry of the correlation function of the noise field characterized by the parameter \varkappa .

¹⁰Traditionally, the Stratonovich product is denoted by \circ , whereas the product sign is omitted for the Itô formalism.

In accordance with the discussed rules of stochastic calculus, we can rewrite Eq. (27) in the Stratonovich formalism,

$$d|\psi\rangle_t = \left[-i\hat{M}dt + i\sqrt{\lambda}\hat{A} \circ dW_t - \frac{\lambda}{2}\hat{B}^{\dagger}\hat{B}dt \right] |\psi\rangle_t. \quad (\text{A11})$$

The symmetric choice $\theta_S(0) = \frac{1}{2}$ (hence, time-symmetric noise field) in the Stratonovich formalism makes it clear that Eq. (A11) can be interpreted as a Schrödinger equation with the effective Hamiltonian (1) with the decay operator $\hat{\Gamma} = \lambda\hat{B}^{\dagger}\hat{B}$, and a random potential.

In the Stratonovich formalism, there is no built-in direction of time, which appears for other choices of the stochastic formalism and, hence, asymmetric $\theta(0)$. Let us change the formalism by shifting $\theta(0)$ from its Stratonovich value $\theta_S(0)$ to some another value β . This leads to the equation

$$d|\psi\rangle_t = \left[-i\hat{M}dt + i\sqrt{\lambda}\hat{A} \circ_{\beta} dW_t - \frac{\lambda}{2}(\hat{B}^{\dagger}\hat{B} + (1 - 2\beta)\hat{A}^2)dt \right] |\psi\rangle_t. \quad (\text{A12})$$

Assuming that the collapse dynamics in \mathbf{H}_M is responsible for the decay property of a neutral meson system, i.e., plugging in the decay widths (30), we finish with the stochastic Schrödinger equation

$$d|\psi\rangle_t = [-i\hat{M}dt + i\sqrt{\lambda}\hat{A} \circ_{\beta} dW_t] |\psi\rangle_t, \quad (\text{A13})$$

with, generally speaking, a time-asymmetric noise field dW_t with respect to the chosen value of $\theta(0) = \beta$. In turn, Eq. (A13) reads in Stratonovich formalism

$$d|\psi\rangle_t = \left[-i\hat{M}dt + i\sqrt{\lambda}\hat{A} \circ dW_t + \frac{\lambda}{2}(1 - 2\beta)\hat{A}^2 dt \right] |\psi\rangle_t. \quad (\text{A14})$$

The equivalence of Eqs. (A13) and (A14) means that the decay property can be gained by a collapse model with the underlying time-asymmetric noise field.¹¹ Written in the Itô formalism they correspond to Eq. (28). Hence, the value of the Heaviside function in zero $\theta(0)$ plays the role of a natural constant introduced by such a collapse model (along with λ and r_C) and, in turn, provides physical sense to the choice of the stochastic formalism and the Itô-Stratonovich dilemma. As mentioned in Sec. III, it tunes the coupling between the bra and ket spaces concerning the action of the noise field and, in turn, sets the decay widths of the mixed particles.

¹¹Both Eqs. (A13) and (A14) can be approximated, due to the Wong-Zakai theorem [90], by the same ordinary differential equation

$$i \frac{d}{dt} |\psi\rangle_t = \left[\hat{M} - \sqrt{\lambda}\hat{A}\hat{W}_t^{\varepsilon} + i \frac{\lambda}{2}(1 - 2\beta)\hat{A}^2 \right] |\psi\rangle_t,$$

with a regularized noise $\hat{W}_t^{\varepsilon} = \int_{\mathbb{R}} \delta_{\varepsilon}(t - u) dW_u$ with a mollifier δ_{ε} converging to the δ function, so that its solutions converge to those of Eq. (A13).

**APPENDIX B: COMPUTATIONS OF THE TRANSITION PROBABILITIES FOR THE QMUPL
AND MASS-PROPORTIONAL CSL MODELS**

Plugging the QMUPL and CSL collapse operators (33) and (34) into the master equation (37) we obtain

$$\frac{d\hat{\rho}_t}{dt} = i[\hat{\rho}_t, \hat{M}] - \lambda_Q \sum_i [\beta \{ \hat{q}_i^2 \otimes \hat{A}^2, \hat{\rho}_t \} - (\hat{q}_i \otimes \hat{A}) \hat{\rho}_t (\hat{q}_i \otimes \hat{A})], \quad (\text{B1})$$

$$\frac{d\hat{\rho}_t}{dt} = i[\hat{\rho}_t, \hat{M}] - \gamma \int d\mathbf{x} [\beta \{ \hat{Q}^2(\mathbf{x}) \otimes \hat{A}^2, \hat{\rho}_t \} - (\hat{Q}(\mathbf{x}) \otimes \hat{A}) \hat{\rho}_t (\hat{Q}(\mathbf{x}) \otimes \hat{A})]. \quad (\text{B2})$$

These master equations can be used to calculate the required transition probabilities (38) in a simple way. At first, following Ref. [57], we decompose the density operator $\hat{\rho}_t$ in the position and mass bases,

$$\hat{\rho}_t = \sum_{i,j=H,L} \iint d\mathbf{x} d\mathbf{y} \rho_t^{ij}(\mathbf{x}, \mathbf{y}) |\mathbf{x}\rangle \langle \mathbf{y}| \otimes |M_i\rangle \langle M_j|. \quad (\text{B3})$$

This allows us to rewrite the master equations (B1) and (B2) for the QMUPL and CSL collapse dynamics, respectively, in terms of the matrix elements of the density operator $\hat{\rho}_t$

$$\frac{d}{dt} \rho_t^{ij}(\mathbf{x}, \mathbf{y}) = \left\{ -i(m_i - m_j) - \frac{\lambda_Q}{2} [|\tilde{m}_i \mathbf{x} - \tilde{m}_j \mathbf{y}|^2 - (1 - 2\beta)(\tilde{m}_i^2 \mathbf{x}^2 + \tilde{m}_j^2 \mathbf{y}^2)] \right\} \rho_t^{ij}(\mathbf{x}, \mathbf{y}), \quad (\text{B4})$$

$$\frac{d}{dt} \rho_t^{ij}(\mathbf{x}, \mathbf{y}) = \left\{ -i(m_i - m_j) - \gamma [\beta(\tilde{m}_i^2 + \tilde{m}_j^2)(g * g)(0) - \tilde{m}_i \tilde{m}_j (g * g)(\mathbf{x} - \mathbf{y})] \right\} \rho_t^{ij}(\mathbf{x}, \mathbf{y}), \quad (\text{B5})$$

where

$$(g * g)(\mathbf{x}) = \int d\mathbf{y} g(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) \quad (\text{B6})$$

is the convolution of two Gaussian smearing functions

$$g(\mathbf{x} - \mathbf{y}) = \frac{1}{(2\pi r_C^2)^{\frac{d}{2}}} e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{2r_C^2}} \quad (\text{B7})$$

of the CSL collapse operator (35). Solving these equations, we obtain

$$\rho_t^{ij}(\mathbf{x}, \mathbf{y}) = e^{-i(m_i - m_j)t - \frac{\lambda_Q}{2} [|\tilde{m}_i \mathbf{x} - \tilde{m}_j \mathbf{y}|^2 - (1 - 2\beta)(\tilde{m}_i^2 \mathbf{x}^2 + \tilde{m}_j^2 \mathbf{y}^2)]t} \rho_0^{ij}(\mathbf{x}, \mathbf{y}), \quad (\text{B8})$$

$$\rho_t^{ij}(\mathbf{x}, \mathbf{y}) = e^{-i(m_i - m_j)t - \gamma [\beta(\tilde{m}_i^2 + \tilde{m}_j^2)(g * g)(0) - \tilde{m}_i \tilde{m}_j (g * g)(\mathbf{x} - \mathbf{y})]t} \rho_0^{ij}(\mathbf{x}, \mathbf{y}). \quad (\text{B9})$$

Before plugging these solutions into Eq. (38), we have to calculate the partial trace of $\hat{\rho}_t$ over the position degrees of freedom, i.e., take the following integrals:

$$\int d\mathbf{x} \rho_t^{ij}(\mathbf{x}, \mathbf{x}) = \frac{e^{-i(m_i - m_j)t}}{\left(1 + \frac{\lambda_Q \alpha}{2} [(\tilde{m}_i - \tilde{m}_j)^2 - (1 - 2\beta)(\tilde{m}_i^2 + \tilde{m}_j^2)]t\right)^{\frac{d}{2}}}, \quad (\text{B10})$$

$$\int d\mathbf{x} \rho_t^{ij}(\mathbf{x}, \mathbf{x}) = e^{-i(m_i - m_j)t} e^{-\frac{1}{2} \frac{\gamma}{(\sqrt{4\pi} r_C)^d} [(\tilde{m}_i - \tilde{m}_j)^2 - (1 - 2\beta)(\tilde{m}_i^2 + \tilde{m}_j^2)]t}, \quad (\text{B11})$$

for the QMUPL and the mass-proportional CSL models, respectively, where we have taken into account that the convolution $(g * g)(0) = (\sqrt{4\pi} r_C)^{-d}$. Finally, collecting all the integrals, we find the transition probabilities for the lifetime eigenstates and the flavor states,

$$P_{M_i \rightarrow M_j}^{\text{QMUPL}}(t) = [1 - \lambda_Q \alpha (1 - 2\beta) \tilde{m}_i^2 t]^{-\frac{d}{2}} \delta_{ij}, \quad (\text{B12})$$

$$P_{M^0 \rightarrow M^0 / \bar{M}^0}^{\text{QMUPL}}(t) = \frac{1}{4} \left\{ \sum_i [1 - \lambda_Q \alpha (1 - 2\beta) \tilde{m}_i^2 t]^{-\frac{d}{2}} \pm \frac{2 \cos[t \Delta m]}{\left\{1 - \frac{\lambda_Q \alpha}{2} [(1 - 2\beta) \sum_i \tilde{m}_i^2 - (\Delta \tilde{m})^2] t\right\}^{\frac{d}{2}}} \right\}, \quad (\text{B13})$$

$$P_{M_i \rightarrow M_j}^{\text{CSL}}(t) = e^{-\frac{\gamma}{(\sqrt{4\pi} r_C)^d} (2\beta - 1) \tilde{m}_i^2 t} \delta_{ij}, \quad (\text{B14})$$

$$P_{M^0 \rightarrow M^0 / \bar{M}^0}^{\text{CSL}}(t) = \frac{1}{4} \left\{ \sum_i e^{-\frac{\gamma}{(\sqrt{4\pi} r_C)^d} (2\beta - 1) \tilde{m}_i^2 t} \pm 2e^{-\frac{1}{2} \frac{\gamma}{(\sqrt{4\pi} r_C)^d} [(2\beta - 1) \sum_i \tilde{m}_i^2 + (\Delta \tilde{m})^2] t} \cos[t \Delta m] \right\}. \quad (\text{B15})$$

- [1] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Evidence for the 2π Decay of the K_2^0 Meson, *Phys. Rev. Lett.* **13**, 138 (1964).
- [2] A. Abashian *et al.*, Measurement of the CP Violation Parameter $\sin 2\varphi_1$ in B_d^0 Meson Decays, *Phys. Rev. Lett.* **86**, 2509 (2001).
- [3] Č. Brukner, On the quantum measurement problem, in *Quantum [Un]Speakables II: Half a Century of Bell's Theorem*, edited by R. Bertlmann and A. Zeilinger (Springer, Switzerland, 2017), pp. 95–117.
- [4] V. Baumann and S. Wolf, On Formalisms and Interpretations, *Quantum* **2**, 99 (2018).
- [5] W. Feldmann and R. Tumulka, Parameter diagrams of the GRW and CSL theories of wavefunction collapse, *J. Phys. A: Math. Theor.* **45**, 065304 (2011).
- [6] A. Bassi and G. C. Ghirardi, Dynamical reduction models, *Phys. Rep.* **379**, 257 (2003).
- [7] A. Bassi, K. Lochan, S. Satin, T. P. Singh, and H. Ulbricht, Models of wave-function collapse, underlying theories, and experimental tests, *Rev. Mod. Phys.* **85**, 471 (2013).
- [8] G. C. Ghirardi, A. Rimini, and T. Weber, Unified dynamics for microscopic and macroscopic systems, *Phys. Rev. D* **34**, 470 (1986).
- [9] G. C. Ghirardi, Some lessons from relativistic reduction models in *Open Systems and Measurement in Relativistic Quantum Theory*, edited by H.-P. Breuer and F. Petruccione (Springer, Berlin-Heidelberg, 1999), pp. 117–152.
- [10] J. S. Bell, Are there quantum jumps? in *Schrödinger: Centenary Celebration of a Polymath*, edited by C. Kilmister (Cambridge University Press, Cambridge, 1987), pp. 41–52.
- [11] A. Bassi and H. Ulbricht, Collapse models: From theoretical foundations to experimental verifications, *J. Phys.: Conf. Ser.* **504**, 012023 (2014).
- [12] L. Diósi, Models for universal reduction of macroscopic quantum fluctuations, *Phys. Rev. A* **40**, 1165 (1989).
- [13] P. Pearle, Combining stochastic dynamical state-vector reduction with spontaneous localization, *Phys. Rev. A* **39**, 2277 (1989).
- [14] G. C. Ghirardi, P. Pearle, and A. Rimini, Markov processes in Hilbert space and continuous spontaneous localization of systems of identical particles, *Phys. Rev. A* **42**, 78 (1990).
- [15] M. Carlesso and S. Donadi, Collapse models: Main properties and the state of art of the experimental tests, in *Advances in Open Systems and Fundamental Tests of Quantum Mechanics*, edited by B. Vacchini, H.-P. Breuer, and A. Bassi (Springer, Switzerland, 2019), pp. 1–13.
- [16] A. Bassi and D. Dürr, On the electromagnetic properties of matter in collapse models, *J. Phys. A: Math. Theor.* **42**, 485302 (2009).
- [17] A. Bassi and S. Donadi, Spontaneous photon emission from a non-relativistic free charged particle in collapse models: A case study, *Phys. Lett. A* **378**, 761 (2014).
- [18] S. L. Adler, A. Bassi, and S. Donadi, On spontaneous photon emission in collapse models, *J. Phys. A: Math. Theor.* **46**, 245304 (2013).
- [19] C. Curceanu, B. C. Hiesmayr, and K. Piscicchia, X-rays help to unfuzzy the concept of measurement, *J. Adv. Phys.* **4**, 263 (2015).
- [20] K. Piscicchia *et al.*, Beyond quantum mechanics? Hunting the ‘Impossible’ atoms—Pauli exclusion principle violation and spontaneous collapse of the wave function at test, *Acta Phys. Pol. B* **46**, 147 (2015).
- [21] C. Curceanu *et al.*, Spontaneously emitted X-rays: An experimental signature of the dynamical reduction models, *Found. Phys.* **46**, 263 (2016).
- [22] K. Piscicchia *et al.*, CSL collapse model mapped with the spontaneous radiation, *Entropy* **19**, 319 (2017).
- [23] C. Curceanu, R. Del Grande, M. Laubenstein, and K. Piscicchia, Collapse models tested in the LNGS underground laboratories, *Int. J. Quantum Inform.* **17**, 1941011 (2019).
- [24] M. Bilardello, S. Donadi, A. Vinante, and A. Bassi, Bounds on collapse models from cold-atom experiments, *Phys. A (Amsterdam, Neth.)* **462**, 764 (2016).
- [25] M. Carlesso, A. Bassi, P. Falferi, and A. Vinante, Experimental bounds on collapse models from gravitational wave detectors, *Phys. Rev. D* **94**, 124036 (2016).
- [26] A. Vinante, A. Pontin, M. Rashid, M. Toroš, P. F. Barker, and H. Ulbricht, Testing collapse models with levitated nanoparticles: Detection challenge, *Phys. Rev. A* **100**, 012119 (2019).
- [27] S. Nimmrichter, K. Hornberger, P. Haslinger, and M. Arndt, Testing spontaneous localization theories with matter-wave interferometry, *Phys. Rev. A* **83**, 043621 (2011).
- [28] M. Toroš, G. Gasbarri, and A. Bassi, Colored and dissipative continuous spontaneous localization model and bounds from matter-wave interferometry, *Phys. Lett. A* **381**, 3921 (2017).
- [29] M. Toroš and A. Bassi, Bounds on quantum collapse models from matter-wave interferometry: calculational details, *J. Phys. A: Math. Theor.* **51**, 115302 (2018).
- [30] B. Schirnski, S. Nimmrichter, and K. Hornberger, Quantum-classical hypothesis tests in macroscopic matter-wave interferometry, *Phys. Rev. Research* **2**, 033034 (2020).
- [31] M. Carlesso, M. Paternostro, H. Ulbricht, A. Vinante, and A. Bassi, Non-interferometric test of the continuous spontaneous localization model based on rotational optomechanics, *New J. Phys.* **20**, 083022 (2018).
- [32] J. Nobakht, M. Carlesso, S. Donadi, M. Paternostro, and A. Bassi, Unitary unraveling for the dissipative continuous spontaneous localization model: Application to optomechanical experiments, *Phys. Rev. A* **98**, 042109 (2018).
- [33] M. Carlesso and M. Paternostro, Opto-mechanical tests of collapse models, in *Do Wave Functions Jump?*, edited by V. Allori, A. Bassi, D. Dürr, and N. Zanghì (Springer, Switzerland, 2020).
- [34] M. Genovese, Entanglement properties of kaons and tests of hidden-variable models, *Phys. Rev. A* **69**, 022103 (2004).
- [35] A. Bramon, R. Escribano, and G. Garbarino, Bell's Inequality Tests with Meson–Antimeson Pairs, *Found. Phys.* **26**, 563 (2006).
- [36] J. Bernabéu, J. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and J. Papavassiliou, CPT and Quantum Mechanics Tests with Kaons, [arXiv:hep-ph/0607322](https://arxiv.org/abs/hep-ph/0607322).
- [37] B. C. Hiesmayr and M. Huber, Bohr's complementarity relation and the violation of CP symmetry in the neutral kaon system, *Phys. Lett. A* **372**, 3608 (2008).
- [38] B. D. Yabsley, Quantum entanglement at the psi(3770) and Upsilon(4S), [arXiv:0810.1822](https://arxiv.org/abs/0810.1822).
- [39] M. Blasone, F. Dell'Anno, S. De Siena, M. Di Mauro, and F. Illuminati, Multipartite entangled states in particle mixing, *Phys. Rev. D* **77**, 096002 (2008).

- [40] G. Amelino-Camelia *et al.*, Physics with the KLOE-2 experiment at the upgraded DAΦNE, *Eur. Phys. J. C* **68**, 619 (2010).
- [41] A. Di Domenico *et al.*, Heisenberg's uncertainty relation and Bell inequalities in high energy physics, *Found. Phys.* **42**, 778 (2012).
- [42] B. C. Hiesmayr *et al.*, Revealing Bell's nonlocality for unstable systems in high energy physics, *Eur. Phys. J. C* **72**, 1856 (2012).
- [43] J. Bernabéu, A. Di Domenico, and P. Villanueva-Pérez, Direct test of time reversal symmetry in the entangled neutral kaon system at a ϕ -factory, *Nucl. Phys. B* **868**, 102 (2013).
- [44] A. Capolupo, S. M. Giampaolo, and G. Lambiase, Decoherence in neutrino oscillations, neutrino nature and CPT violation, *Phys. Lett. B* **792**, 298 (2019).
- [45] T. Durt, A. Di Domenico, and B. C. Hiesmayr, Charge-conjugation-parity violation of neutral K-mesons in a temporal wave function model, *Eur. Phys. J. Plus* **134**, 114 (2019).
- [46] K. Simonov, A. Capolupo, and S. M. Giampaolo, Gravity, entanglement and CPT-symmetry violation in particle mixing, *Eur. Phys. J. C* **79**, 902 (2019).
- [47] J. Bernabéu and A. Di Domenico, Can future observation of the living partner influence the past decayed state in entangled neutral K-mesons? [arXiv:1912.04798](https://arxiv.org/abs/1912.04798).
- [48] J. Naikoo *et al.*, A quantum information theoretic quantity sensitive to the neutrino mass-hierarchy, *Nucl. Phys. B* **951**, 114872 (2020).
- [49] L. Buoninfante, A. Capolupo, S. M. Giampaolo, and G. Lambiase, Revealing neutrino nature and CPT violation with decoherence effects, [arXiv:2001.07580](https://arxiv.org/abs/2001.07580).
- [50] R. A. Bertlmann and B. C. Hiesmayr, Bell inequalities for entangled kaons and their unitary time evolution, *Phys. Rev. A* **63**, 062112 (2001).
- [51] M. Bahrami *et al.*, Are collapse models testable with quantum oscillating systems? The case of neutrinos, kaons, chiral molecules, *Sci. Rep.* **3**, 1952 (2013).
- [52] S. Donadi, A. Bassi, C. Curceanu, and L. Ferialdi, The effect of spontaneous collapses on neutrino oscillations, *Found. Phys.* **43**, 1066 (2013).
- [53] S. Donadi, A. Bassi, C. Curceanu, A. Di Domenico, and B. C. Hiesmayr, Are collapse models testable via flavor oscillations? *Found. Phys.* **43**, 813 (2013).
- [54] K. Simonov and B. C. Hiesmayr, Constraints on the noise in dynamical reduction models, *Phys. Lett. A* **380**, 1253 (2016).
- [55] K. Simonov and B. C. Hiesmayr, Spontaneous collapse: A solution to the measurement problem and a source of the decay in mesonic systems, *Phys. Rev. A* **94**, 052128 (2016).
- [56] K. Simonov and B. C. Hiesmayr, Can a spontaneous collapse in flavour oscillations be tested at KLOE? *EPJ Web Conf.* **166**, 00006 (2018).
- [57] A. Tilloy, Comment on "Spontaneous collapse: A solution to the measurement problem and a source of the decay in mesonic systems", *Phys. Rev. A* **96**, 056101 (2017).
- [58] R. A. Bertlmann, W. Grimus, and B. C. Hiesmayr, Open-quantum-system formulation of particle decay, *Phys. Rev. A* **73**, 054101 (2006).
- [59] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Completely positive dynamical semigroups of N -level systems, *J. Math. Phys.* **17**, 821 (1976).
- [60] G. Lindblad, On the generators of quantum dynamical semigroups, *Commun. Math. Phys.* **48**, 119 (1976).
- [61] B. C. Hiesmayr, The GKLS Master equation in high energy physics, *Open Sys. Info. Dyn.* **24**, 1740008 (2017).
- [62] B. C. Hiesmayr, Testing quantum mechanics in high-energy physics, in *Theoretical Foundations of Quantum Information Processing and Communication*, edited by E. Brüning and F. Petruccione (Springer, Berlin-Heidelberg, 2010), pp. 141–185.
- [63] R. Mishra, A. Vinante, and T. P. Singh, Testing spontaneous collapse through bulk heating experiments: An estimate of the background noise, *Phys. Rev. A* **98**, 052121 (2018).
- [64] S. L. Adler and A. Bassi, Collapse models with non-white noises, *J. Phys. A: Math. Theor.* **40**, 15083 (2007).
- [65] S. L. Adler and A. Bassi, Collapse models with non-white noises: II. Particle-density coupled noises, *J. Phys. A: Math. Theor.* **41**, 395308 (2008).
- [66] A. Bassi and L. Ferialdi, Non-Markovian dynamics for a free quantum particle subject to spontaneous collapse in space: General solution and main properties, *Phys. Rev. A* **80**, 012116 (2009).
- [67] A. Bassi and L. Ferialdi, Non-Markovian Quantum Trajectories: An Exact Result, *Phys. Rev. Lett.* **103**, 050403 (2009).
- [68] L. Ferialdi and A. Bassi, Dissipative collapse models with nonwhite noises, *Phys. Rev. A* **86**, 022108 (2012).
- [69] P. Pearle and E. Squires, Bound state excitation, nucleon decay experiments and models of wave function collapse, *Phys. Rev. Lett.* **73**, 1 (1994).
- [70] D. J. Bedingham and O. J. E. Maroney, Time symmetry in wave-function collapse, *Phys. Rev. A* **95**, 042103 (2017).
- [71] D. J. Bedingham, Collapse models and spacetime symmetries, in *Collapse of the Wave Function: Models, Ontology, Origin, and Implications*, edited by S. Gao (Cambridge University Press, Cambridge, 2018), pp. 74–93.
- [72] M. Thomson, *Modern Particle Physics* (Cambridge University Press, Cambridge, 2013), pp. 386–388.
- [73] S. L. Adler, Lower and upper bounds on CSL parameters from latent image formation and IGM heating, *J. Phys. A: Math. Theor.* **40**, 2935 (2007).
- [74] L. Diósi, Gravitation and quantum-mechanical localization of macro-objects, *Phys. Lett. A* **105**, 199 (1984).
- [75] R. Penrose, Quantum computation, entanglement and state reduction, *Philos. Trans. R. Soc., A* **356**, 1927 (1998).
- [76] M. Bahrami, A. Großardt, S. Donadi, and A. Bassi, The Schrödinger–Newton equation and its foundations, *New J. Phys.* **16**, 115007 (2014).
- [77] A. Großardt and B. C. Hiesmayr, Newtonian self-gravitation in the neutral meson system, *Phys. Rev. D* **91**, 064056 (2015).
- [78] A. Capolupo, C.-R. Ji, Y. Mishchenko, and G. Vitiello, Phenomenology of flavor oscillations with non-perturbative effects from quantum field theory, *Phys. Lett. B* **594**, 135 (2004).
- [79] O. Nicrosini and A. Rimini, Relativistic spontaneous localization: A proposal, *Found. Phys.* **33**, 1061 (2003).
- [80] R. Tumulka, A relativistic version of the Ghirardi–Rimini–Weber model, *J. Stat. Phys.* **125**, 821 (2006).
- [81] D. J. Bedingham, Relativistic state reduction dynamics, *Found. Phys.* **41**, 686 (2011).
- [82] D. Bedingham, Relativistic state reduction model, *J. Phys.: Conf. Ser.* **306**, 012034 (2011).

- [83] P. Pearle, Relativistic dynamical collapse model, *Phys. Rev. D* **91**, 105012 (2015).
- [84] R. Tumulka, A relativistic GRW flash process with interaction, in *Do Wave Functions Jump?*, edited by V. Allori, A. Bassi, D. Dürr, and N. Zanghì (Springer, Switzerland, 2020).
- [85] L. Arnold, *Stochastic Differential Equations: Theory and Applications* (Wiley, New York, 1974).
- [86] R. Mannella and P. V. E. McClintock, Itô versus Stratonovich: 30 years later, *Fluct. Noise Lett.* **22**, 1240010 (2012).
- [87] P. Da Pelo, A. Lanconelli, and A. I. Stan, An Itô formula for a family of stochastic integrals and related Wong–Zakai theorems, *Stoch. Process. Appl.* **123**, 3183 (2013).
- [88] A. J. Roberts, *Elementary Calculus of Financial Mathematics* (SIAM, Philadelphia, 2008).
- [89] A. W. C. Lau and T. C. Lubensky, State-dependent diffusion: Thermodynamic consistency and its path integral formulation, *Phys. Rev. E* **76**, 011123 (2007).
- [90] E. Wong and M. Zakai, On the convergence of ordinary integrals to stochastic integrals, *Ann. Math. Stat.* **36**, 1560 (1965).