Resonant generation of a *p*-wave Cooper pair in a non-Hermitian Kitaev chain at the exceptional point

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We investigate a non-Hermitian extension of a Kitaev chain by considering imaginary *p*-wave pairing amplitudes. The exact solution shows that the phase diagram consists of two phases with real and complex Bogoliubov–de Gennes spectra, associated with the $\mathcal{PT}$-symmetry breaking, which is separated by a hyperbolic exceptional line. The exceptional points (EPs) correspond to a specific Cooper pair state $(1 + c_k^\dagger c_{-k}^\dagger)|0\rangle$ with movable k when the parameters vary along the exceptional line. The non-Hermiticity around EP supports resonant generation of such a pair state from the vacuum state $|0\rangle$ of fermions via the critical dynamic process. In addition, we propose a scheme to generate a superconducting state through a dynamic method.

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I. INTRODUCTION

The Kitaev model is a lattice model of a *p*-wave superconducting wire, which realizes Majorana zero modes at the ends of the chain [1]. This has been demonstrated by unpaired Majorana modes exponentially localized at the ends of open Kitaev chains [2–4]. The main feature of this model originates from the pairing term, which violates the conservation of the fermion number but preserves its parity, leading to the superconducting phase. The amplitudes for pair creation and annihilation play an important role in the existence of the gapped superconducting phase. In general, most of the investigations on this model have focused on the case with a Hermitian pairing term. A non-Hermitian term is no longer forbidden both in theory and experiment since the discovery that a certain class of non-Hermitian Hamiltonians could exhibit entirely real spectra [5,6]. The origin of the reality of the spectrum of a non-Hermitian Hamiltonian is the pseudo-Hermiticity of the Hamiltonian operator [7]. It motivates a non-Hermitian extension of the Kitaev model. Many contributions have been devoted to non-Hermitian Kitaev models [8–13] and Ising models [14,15] within the pseudo-Hermitian framework. Also, the experimental schemes for realizing the Kitaev model and related non-Hermitian systems have been presented in Refs. [16] and [17], respectively. In addition, the peculiar features of a non-Hermitian system do not only manifest in statics but also dynamics. From the perspective of non-Hermitian quantum mechanics, it is also a challenge to deal with many-particle dynamics.

In this paper, we investigate a non-Hermitian extension of the Kitaev chain by considering imaginary *p*-wave pairing amplitudes. Theoretically, an open system is regarded as a subsystem of an infinite Hermitian system, while a non-Hermitian Hamiltonian is introduced to describe the physics of the subsystem in a phenomenological way [18]. Non-Hermitian *p*-wave pairing amplitudes may arise from the case, in which the subsystem and the surrounding system are in the superconducting phase. When the whole system is in some nonequilibrium superconducting states, the subsystem should be effectively described by a non-Hermitian pair creation and annihilation. As a concrete step toward this, the quantum tunneling of particle pairs has been studied for two weakly interacting systems as a superconducting tunnel junction [19]. Technically speaking, a non-Hermitian pairing term is not so surprising, because it is equivalent to an imaginary on-site potential (see Appendix A). Non-Hermitian systems exhibit many peculiar dynamic behaviors that never occurred in Hermitian systems. One of the remarkable features is the dynamics at the exceptional point (EP) [20–26] or the spectral singularity (SS) [27–31], where the system has a coalescence state. Recently, there are some works for non-Hermitian many-body EPs [32,33], many-body dynamics [34], and the many-body EP-related dynamics for a noninteracting system [35] and the quantum spin system [36]. In this work, we focus on the EP-related dynamic behavior for the many-body interacting fermion system.

Based on the exact solution, we find that the exceptional line is a hyperbolic line in the parameter space, which separates two regions with real and complex Bogoliubov–de Gennes spectra, associated with the unbroken and the broken $\mathcal{PT}$-symmetric phase, respectively. The EPs move in k space when the parameters vary along the exceptional line. In addition, the critical dynamics supports resonant generation of a *p*-wave Cooper pair: a specific pair state $c_{k_c}^\dagger c_{-k_c}^\dagger|0\rangle$ with selected opposite momentum k_c can be generated from the vacuum state $|0\rangle$ of fermions by the natural time evolution. The selected k_c ranges over the Brillouin zone, determined by the parameters. The underlying mechanism stems from the critical dynamics around the EP that projects an initial state onto the coalescing state. Our work also exemplifies the dynamic nature of a non-Hermitian interacting many-particle system. As an application, it provides an alternative way to generate a superconducting state from an empty state via a

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critical dynamic process rather than cooling down the temperature, and the key feature of this model is that the theoretical results can be demonstrated and tested experimentally via a two-site system, which can exhibit similar EP dynamics as the original model (see Appendix B).

This paper is organized as follows. In Sec. II, we describe the model Hamiltonian. In Sec. III, based on the solutions, we present the phase diagram. In Sec. IV, we study the dynamics in the unbroken-symmetry region, including the time evolution at exceptional lines. In Sec. V, we focus on the critical dynamics for the vacuum state as the initial state. In Sec. VI, we propose a scheme to generate a superconducting state. Finally, we give a summary and discussion in Sec. VII.

II. NON-HERMITIAN KITAEV MODEL

We consider the following fermionic Hamiltonian on a lattice of length N:

$$\mathcal{H} = \sum_{j=1}^{N}[-Jc_j^\dagger c_{j+1} + \text{H.c.} - i\Delta c_j^\dagger c_{j+1}^\dagger - i\Delta c_{j+1}c_j + \mu(2n_j - 1)], \tag{1}$$

where $c_j^\dagger$ (c_j) is a fermionic creation (annihilation) operator on site j, $n_j = c_j^\dagger c_j$, J the tunneling rate, μ the chemical potential, and $i\Delta$ the strength of the p-wave pair creation (annihilation). We define $c_{N+1} = c_1$ for the periodic boundary condition. The Hamiltonian (1) has a rich phase diagram in its Hermitian version, i.e., $i\Delta \to \Delta$, which is a spin-polarized p-wave superconductor in one dimension. This system is known for having topological phases in which there is a zero energy Majorana mode at each end of a long chain. It is also the fermionized version of the familiar one-dimensional transverse-field Ising model [37], which is one of the simplest solvable models exhibiting quantum criticality and demonstrating a quantum phase transition with spontaneous symmetry breaking [38]. In this work, we consider a non-Hermitian extension by imaginary pairing amplitude $i\Delta$. Comparing with the non-Hermitian Kitaev model in previous works [8–14], the present model has parity-time-reversal ($\mathcal{PT}$) symmetry (proved below) and its non-Hermiticity arises from the imaginary pairing term rather than from the on-site potential term. We will show that the quasiparticle spectrum can have two movable EPs, resulting in some exclusive features different from its Hermitian version.

Before solving the Hamiltonian, it is profitable to investigate the symmetry of the system. By the direct derivation, we have $[\mathcal{PT}, \mathcal{H}] = 0$, where the antilinear time-reversal operator $\mathcal{T}$ has the function $\mathcal{T}i\mathcal{T} = -i$, and $(\mathcal{P})^{-1}c_l\mathcal{P} = c_{N-l+1}$. As a usual pseudo-Hermitian system [39], the $\mathcal{PT}$ symmetry in the present model plays the same role to the phase diagram. The spectrum of $\mathcal{H}$ can be real if all the eigenstates can be written as a $\mathcal{PT}$-symmetric form, while complex when the corresponding eigenstates break the $\mathcal{PT}$ symmetry. The concept of EPs in this paper specifies the locations in the parameter space, at which the complex spectrum starts to appear (in general, an EP is any point with a coalescing state). We concentrate our work on the real-spectrum (or unbroken symmetry) region, avoiding the exponentially increased Dirac probability.

In this work, we focus on the dynamics of such a superconducting system, which motivates a more systematic study. Taking the Fourier transformation

$$c_j = \frac{1}{\sqrt{N}}\sum_k e^{ikj}c_k, \tag{2}$$

for the Hamiltonian (1), with wave vector $k \in (-\pi, \pi]$, we have

$$\mathcal{H} = -\sum_k [2(J\cos k - \mu)c_k^\dagger c_k + \Delta \sin k(c_{-k}c_k + c_{-k}^\dagger c_k^\dagger) + \mu]. \tag{3}$$

For the convenience of further analysis, we express the Hamiltonian by using the Nambu representation

$$\mathcal{H} = \sum_{\pi>k>0} \mathcal{H}_k, \tag{4}$$

$$\mathcal{H}_k = 2(c_k^\dagger\ c_{-k})\begin{pmatrix} \mu - J\cos k & \Delta \sin k \\ -\Delta \sin k & J\cos k - \mu \end{pmatrix}\begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}, \tag{5}$$

where the Hamiltonian $\mathcal{H}_k$ in each invariant subspace satisfies the commutation relation

$$[\mathcal{H}_k, \mathcal{H}_{k'}] = 0. \tag{6}$$

This allows us to treat the diagonalization and the dynamics governed by $\mathcal{H}_k$ individually. So far the procedure is the same as those for solving the Hermitian version of $\mathcal{H}$. To diagonalize a non-Hermitian Hamiltonian, we should introduce the Bogoliubov transformation in the complex version:

$$\gamma_k = \cos\theta_k c_k - i\sin\theta_k c_{-k}^\dagger,$$
$$\overline{\gamma}_k = \cos\theta_k c_k^\dagger + i\sin\theta_k c_{-k}, \tag{7}$$

where the complex angle θ_k is determined by

$$\tan(2\theta_k) = \frac{i\Delta \sin k}{\mu - J\cos k}. \tag{8}$$

It is a crucial step to diagonalize a non-Hermitian Hamiltonian, which essentially establishes the biorthogonal modes. It is easy to check that the complex Bogoliubov modes $(\gamma_k, \overline{\gamma}_k)$ satisfy the anticommutation relations

$$\{\gamma_k, \overline{\gamma}_{k'}\} = \delta_{k,k'},$$
$$\{\gamma_k, \gamma_{k'}\} = \{\overline{\gamma}_k, \overline{\gamma}_{k'}\} = 0, \tag{9}$$

which result in the diagonal form of the Hamiltonian

$$\mathcal{H} = \sum_k \varepsilon_k\left(\overline{\gamma}_k\gamma_k - \frac{1}{2}\right). \tag{10}$$

Here

$$\varepsilon_k = 2\sqrt{(\mu - J\cos k)^2 - \Delta^2\sin^2 k} \tag{11}$$

is the dispersion relation of the quasiparticle. Note that the Hamiltonian $\mathcal{H}$ is still non-Hermitian due to the fact that $\overline{\gamma}_k \neq \gamma_k^\dagger$. In addition, quasispectrum ε_k can be real or imaginary, but not zero since the complex Bogoliubov modes $(\gamma_k, \overline{\gamma}_k)$ are not well defined if $\varepsilon_k = 0$, which will be discussed in the next section.

III. PHASE DIAGRAM

According to the theory for a pseudo-Hermitian system [39], the whole parameter space consists of two kinds of regions: a symmetry-unbroken one with a fully real spectrum and a symmetry-broken one with a complex spectrum, which originates from the overthreshold imaginary pairing amplitudes. The reason can be seen from the following derivation. For a given k, the Hamiltonian $\mathcal{H}_k$ in the basis $(|0\rangle_k|0\rangle_{-k}, |1\rangle_k|1\rangle_{-k})$ is expressed as a 2×2 matrix

$$\mathcal{H}_k = 2\begin{pmatrix} J\cos k - \mu & -\Delta\sin k \\ \Delta\sin k & \mu - J\cos k \end{pmatrix}. \quad (12)$$

The eigenstate $|\psi_k^{\pm}\rangle$ with even parity of the particle number is

$$\left|\psi_k^{\pm}\right\rangle = \frac{1}{\sqrt{\Omega_{\rm nh}^{\pm}}}(|0\rangle_k|0\rangle_{-k} + \beta_k^{\pm}|1\rangle_k|1\rangle_{-k}), \quad (13)$$

where $\Omega_{\rm nh}^{\pm} = 1 + |\beta_k^{\pm}|^2$ is the normalization coefficient in the context of a Dirac inner product with

$$\beta_k^{\pm} = \frac{\Delta\sin k}{J\cos k - \mu \pm \varepsilon_k/2}. \quad (14)$$

We note that

$$\mathcal{PT}|\psi_k^{\pm}\rangle = |\psi_k^{\pm}\rangle, \quad \text{for } (\varepsilon_k)^2 > 0,$$
$$\mathcal{PT}|\psi_k^{\pm}\rangle = |\psi_k^{\mp}\rangle, \quad \text{for } (\varepsilon_k)^2 < 0, \quad (15)$$

for the unbroken and the broken $\mathcal{PT}$-symmetric phase, where we used the relation

$$(\mathcal{PT})^{-1}c_k^{\dagger}\mathcal{PT} = e^{-ik}c_k^{\dagger}. \quad (16)$$

As expected, the symmetry of the eigenstates is associated with the reality of the energy level. An eigenstate of $\mathcal{H}$ is constructed as the form

$$|\Psi\rangle = \prod_{\pi>k>0}|\varphi_k^{\lambda}\rangle, \quad (17)$$

where the index $\lambda = 1, 2$ labels the eigenstate in each k sector, $|\varphi_k^{1,2}\rangle = |\psi_k^{+,-}\rangle$, with the eigenenergy

$$E = \sum_{\pi>k>0}\epsilon_k^{\lambda}, \quad (18)$$

with $\epsilon_k^{1,2} = \varepsilon_k^{+,-}$. Therefore, the reality of ε_k determines the reality of the spectrum of $\mathcal{H}$, since a single imaginary ε_k can result in the complex spectrum of $\mathcal{H}$. A quantum phase transition occurs when the complex spectrum appears. Then the phase boundary of $\mathcal{H}$ locates at the touching point of curve ε_k at k axis. The phase boundary (or EP line) in parameter space ($\mu - \Delta$ plane) is determined by the equations [40]

$$\varepsilon_{k_c} = \left[\frac{\partial\varepsilon_k}{\partial k}\right]_{k=k_c} = 0. \quad (19)$$

The boundary is obtained as

$$\mu_c^2 - \Delta_c^2 = J^2, \quad (20)$$

with

$$k_c = \arccos\frac{J}{\mu_c}. \quad (21)$$

In this situation, $\mathcal{H}_{k_c}$ cannot be expressed as the complex Bogoliubov modes $(\gamma_{k_c}, \overline{\gamma}_{k_c})$ since the matrix of $\mathcal{H}_{k_c}$ in an even particle number sector has a Jordan block form,

$$M_c = \frac{-2\Delta_c^2}{\mu_c}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad (22)$$

which is no longer diagonalizable. Two eigenvectors of M_c coalesce to a single one $(1, 1)^{\rm T}$, leading to a set of coalescing eigenstates of $\mathcal{H}$, including the coalescing ground state. Remarkably, $\mathcal{H}_{k_c}$ governs a peculiar dynamics, which is the focus of this work. The phase diagram on parameter $\mu - \Delta$ plane is plotted in Fig. 1(a). The real part of quasiparticle spectra for several typical points in symmetry-unbroken, broken phases, and on the EP lines are plotted in Figs. 1(b) and 1(c). It shows that the pair of EPs are movable and meet at a fixed point. Such a gapless phase is different from its Hermitian version, where the band touching point is the degenerate point

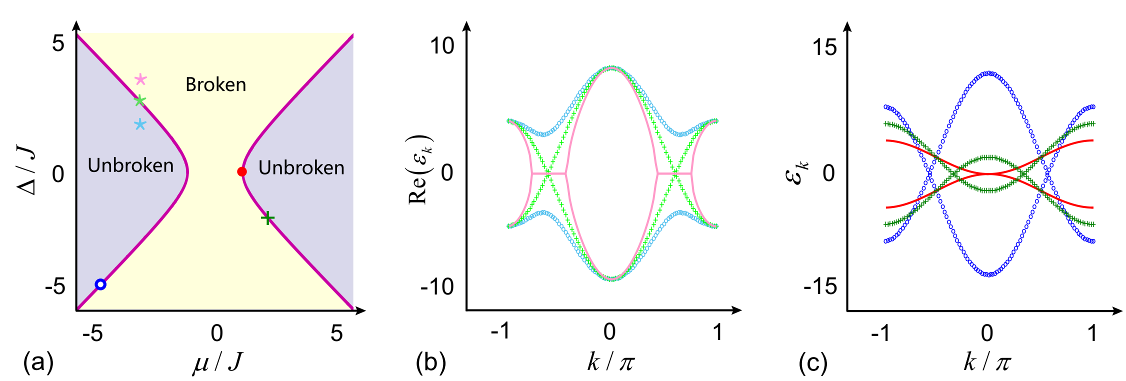

FIG. 1. (a) Phase diagram of the non-Hermitian Kitaev model with imaginary p-wave pairing amplitudes. The phase boundary is hyperbolic exceptional lines (dark magenta), which separate two regions with real (purple) and complex (yellow) Bogoliubov–de Gennes spectra, associated with the unbroken $\mathcal{PT}$-symmetric phase and the broken $\mathcal{PT}$-symmetric phase, respectively. (b) Real part of quasiparticle spectra for three typical points indicated in (a) with "$*$", representing unbroken phase (light blue line with "o"), EP line (light green line with "+"), and broken phase (light pink line), respectively. (c) Quasiparticle spectra for three typical points at the phase boundary line indicated in (a) with filled circle, empty circle, and "+", respectively. The corresponding EPs are movable and merge at fixed $k = 0$ or $k = \pi$.

and it will result in different dynamical behavior in the non-Hermitian Kitaev model, especially near the phase boundary.

IV. DYNAMICS

We study the dynamics in the unbroken-symmetry region, in which ε_k is always real, including the time evolution at exceptional lines. Based on the above analysis, the dynamics is governed by the time evolution operator

$$U(t) = \exp(-i\mathcal{H}t) = \prod_{\pi > k > 0} U_k(t), \tag{23}$$

where

$$U_k(t) = \exp(-i\mathcal{H}_k t). \tag{24}$$

The explicit form of $U_k(t)$ is determined by the diagonal form of $\mathcal{H}_k$, i.e.,

$$\mathcal{H}_k = \varepsilon_k(\overline{\gamma}_k\gamma_k + \overline{\gamma}_{-k}\gamma_{-k} - 1). \tag{25}$$

However, one of the exclusive features of a non-Hermitian system is that $\mathcal{H}_k$ is nondiagonalizable when $k = k_c$. Therefore, we will deal with $U_k(t)$ in two aspects.

(i) In the case of $k \neq k_c$, we have

$$\begin{aligned} U_k(t) = {} & 2[\cos(\varepsilon_k t) - 1]\bar{\gamma}_k\gamma_k\bar{\gamma}_{-k}\gamma_{-k} \\ & + (1 - e^{i\varepsilon_k t})(\bar{\gamma}_{-k}\gamma_{-k} + \bar{\gamma}_k\gamma_k) + e^{i\varepsilon_k t}, \end{aligned} \tag{26}$$

where we have used the identity $(\overline{\gamma}_k\gamma_k)^2 = \overline{\gamma}_k\gamma_k$. This result is also valid for imaginary ε_k. The vacuum state of γ_k is constructed as $|\text{Vac}\rangle_k = \gamma_k|0\rangle$, where $|0\rangle$ is the vacuum state of c_k. Two states $(|00\rangle_k, |11\rangle_k) = (|\text{Vac}\rangle_k|\text{Vac}\rangle_{-k}, \bar{\gamma}_k\bar{\gamma}_{-k}|\text{Vac}\rangle_k|\text{Vac}\rangle_{-k})$ are both the eigenstates of $\mathcal{H}_k$. The time evolution of such two states are

$$U_k(t)\begin{pmatrix} |00\rangle_k \\ |11\rangle_k \end{pmatrix} = \begin{pmatrix} \exp(i\varepsilon_k t)|00\rangle_k \\ \exp(-i\varepsilon_k t)|11\rangle_k \end{pmatrix}, \tag{27}$$

which indicates that it looks like the one in a Hermitian system if ε_k is real. The corresponding Dirac probability, $|U_k(t)|mn\rangle_k|^2$ $(m, n = 1, 0)$, is conservative. However, the Dirac probability of a superposition of two such eigenstates in the even particle number subspace is a periodic function of time with period π/ε_k. It is noted that when k tends to k_c, this period goes to infinity (or nonperiod), which is one of the properties of the critical dynamics.

(ii) In the case of $k = k_c$, $\mathcal{H}_{k_c}$ cannot be expressed as the complex Bogoliubov modes $(\gamma_{k_c}, \overline{\gamma}_{k_c})$. Nevertheless, we can rewrite $\mathcal{H}_{k_c}$ in the form

$$\mathcal{H}_{k_c} = -\frac{2\Delta_c^2}{\mu_c}\left(s_{k_c}^z + is_{k_c}^y\right) \tag{28}$$

by introducing pseudospin operators [41]

$$\begin{aligned} s_k^x &= \frac{1}{2}(c_{-k}^\dagger c_k^\dagger + c_k c_{-k}), \\ s_k^y &= \frac{1}{2i}(c_{-k}^\dagger c_k^\dagger - c_k c_{-k}), \\ s_k^z &= \frac{1}{2}(c_k^\dagger c_k + c_{-k}^\dagger c_{-k} - 1), \end{aligned} \tag{29}$$

which satisfy Lie algebra $[s_k^\alpha, s_k^\beta] = i\varepsilon_{\alpha\beta\gamma}s_k^\gamma$, with $\varepsilon_{\alpha\beta\gamma}$ being the Levi-Civita symbol. The corresponding time evolution operator has the form

$$U_{k_c}(t) = \exp(-i\mathcal{H}_{k_c}t) = 1 - i\mathcal{H}_{k_c}t, \tag{30}$$

based on the identity $(s_k^z + is_k^y)^2 = 0$, or $(\mathcal{H}_{k_c})^2 = 0$.

Obviously, the coalescing eigenstate of $\mathcal{H}_{k_c}$ is the spin-up state in x direction, $s_{k_c}^x|x\rangle = \frac{1}{2}|x\rangle$, and the corresponding eigenstates are

$$|\pm x\rangle = |0\rangle_k|0\rangle_{-k} \pm |1\rangle_k|1\rangle_{-k}. \tag{31}$$

Then the dynamics of the Jordan block is very clear, i.e.,

$$U_{k_c}(t)|x\rangle = |x\rangle, \tag{32}$$

$$U_{k_c}(t)|-x\rangle = |-x\rangle + i4\left(\Delta_c^2/\mu_c\right)t|x\rangle. \tag{33}$$

Any initial states with component $|x\rangle$ obey a nonperiodic (or infinite period) dynamics, which accords with the dynamics of $\mathcal{H}_k$ with $k \to k_c$. In addition, the evolved state $U_{k_c}(t)|-x\rangle$ converges to $|x\rangle$ as time increases. This property also appears in the dynamics of $\mathcal{H}_k$ with $k \to k_c$. Therefore, the system around EPs should exhibit some peculiar critical dynamics. The dynamics of $\mathcal{H}_{k_c}$ alone cannot induce any macroscopic phenomenon, while a set of $\mathcal{H}_k$ near EPs may result in a many-particle effect.

V. *p*-WAVE PAIR GENERATION

In this section, we investigate the critical dynamical behavior by applying the obtained $U(t)$ on a simple initial state. We start with the time evolution of the vacuum state of operators $c_{\pm k}$ as an initial state, i.e.,

$$|\psi_k(0)\rangle = |0\rangle_k|0\rangle_{-k}. \tag{34}$$

In the case of $k \neq k_c$, we have

$$\begin{aligned} |\psi_k(t)\rangle &= U_k(t)|\psi_k(0)\rangle \\ &= [-2i\sin(\varepsilon_k t)\sin^2\theta_k + \exp(i\varepsilon_k t)]|0\rangle_k|0\rangle_{-k} \\ &\quad - \sin(2\theta_k)\sin(\varepsilon_k t)|1\rangle_k|1\rangle_{-k}, \end{aligned} \tag{35}$$

while, for $k = k_c$, we have

$$\left|\psi_{k_c}(t)\right\rangle = (1 + it)|0\rangle_{k_c}|0\rangle_{-k_c} - it|1\rangle_{k_c}|1\rangle_{-k_c}. \tag{36}$$

Accordingly, considering a vacuum state of all fermions (empty state) as an initial state

$$|\Psi(0)\rangle = \prod_{k>0}|\psi_k(0)\rangle = \prod_{k>0}|0\rangle_k|0\rangle_{-k}, \tag{37}$$

we have

$$|\Psi(t)\rangle = \prod_{k>0}U_k(t)|\psi_k(0)\rangle. \tag{38}$$

It is expected that p-wave pairs are generated from the empty state. We are interested in the normalized population of the p-wave pair

$$N(t) = \frac{\langle\Psi(t)|\widehat{N}|\Psi(t)\rangle}{\langle\Psi(t)|\Psi(t)\rangle} = \sum_{k>0}\frac{\langle\psi_k(t)|\widehat{N}_k|\psi_k(t)\rangle}{\langle\psi_k(t)|\psi_k(t)\rangle}, \tag{39}$$

where the total p-wave pair number operator is

$$\widehat{N} = \sum_{k>0}\widehat{N}_k = \sum_{k>0}n_k n_{-k}. \tag{40}$$

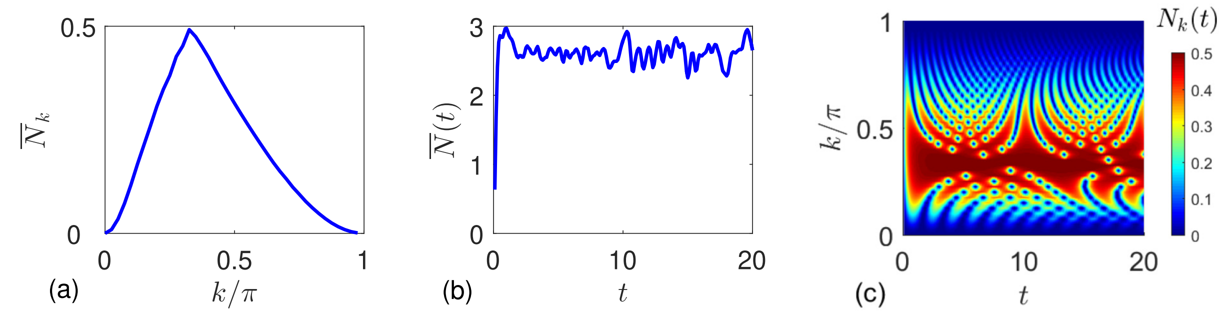

FIG. 2. Plots of (a) $\overline{N}_k$, (b) $\overline{N}(t)$, and (c) $N_k(t)$, which are defined in Eqs. (45), (46), and (43). The parameters are $N = 40$ and $(J, \Delta, \mu) = (1, \sqrt{3}, 2)$.

Then $N(t)$ can be evaluated from $N_k(t)$,

$$N(t) = \sum_{k>0} N_k(t) = \sum_{k>0} \frac{\langle \psi_k(t)|\widehat{N}_k|\psi_k(t)\rangle}{\langle \psi_k(t)|\psi_k(t)\rangle}, \tag{41}$$

and the distribution of $N_k(t)$ determines the property of the nonequilibrium state. For the case of $k \neq k_c$, we have

$$\langle \Psi_k(t)|\Psi_k(t)\rangle = 2|\sin 2\theta_k|^2 \sin^2(\varepsilon_k t) + 1 \tag{42}$$

and

$$N_k(t) = \frac{[\Delta \sin k \sin(t\varepsilon_k)]^2}{(\varepsilon_k/2)^2 + 2[\Delta \sin k \sin(t\varepsilon_k)]^2}, \tag{43}$$

which is a periodic function of time with period $T_k = \pi/\varepsilon_k$. We note that we have $\varepsilon_k \approx 0$ in the vicinity of $k \approx k_c$, and the period become very long. It indicates that we always have $N_k(t) \approx 1/2$ except for some short intervals. For the case of $k = k_c$, Eq. (36) shows that the normalized pair number is

$$N_{k_c}(t) = \frac{t^2}{1 + 2t^2}, \tag{44}$$

which obeys $\lim_{t\to\infty} N_{k_c}(t) = 1/2$, which is in accord with the case with $k \neq k_c$ but an infinitely long period. In order to demonstrate the property of the evolved state, we define the average normalized pair number distribution,

$$\overline{N}_k = \frac{1}{T_k}\int_0^{T_k} N_k(t)dt, \tag{45}$$

and the total average normalized pair number,

$$\overline{N}(t) = \frac{1}{\pi}\int_0^{\pi} N_k(t)dk. \tag{46}$$

We plot quantities $\overline{N}_k$, $\overline{N}(t)$, and $N_k(t)$ for concrete cases in Fig. 2. It indicates that the majority of modes become quasistable after a period of time. Accordingly, the evolved many-body state $|\Psi(t)\rangle$ should exhibit as a macroscopic equilibrium state. To demonstrate this point, we calculate the time evolution of expectation values of the other two physical observables, kinetic energy $\hat{E}$ and order parameter $\hat{P}$, in the present non-Hermitian Kitaev model.

(i) The operator of kinetic energy $\hat{E}$ is defined as

$$\hat{E} = \sum_{j=1}^{N}(c_j^\dagger c_{j+1} + \text{H.c.}) = \sum_{k>0}\hat{E}_k, \tag{47}$$

and the time evolution of the expectation value of $\hat{E}$ is

$$E(t) = \frac{\langle \Psi(t)|\widehat{E}|\Psi(t)\rangle}{\langle \Psi(t)|\Psi(t)\rangle} = \sum_{k>0}\frac{\langle \psi_k(t)|\widehat{E}_k|\psi_k(t)\rangle}{\langle \psi_k(t)|\psi_k(t)\rangle}. \tag{48}$$

In the case of $k \neq k_c$, the evolved state is represented in Eq. (35) and then we can get

$$\begin{aligned} E_k(t) &= \frac{\langle \psi_k(t)|\widehat{E}_k|\psi_k(t)\rangle}{\langle \psi_k(t)|\psi_k(t)\rangle} \\ &= \frac{4\cos k[\Delta \sin k \sin(t\varepsilon_k)]^2}{2[\Delta \sin k \sin(t\varepsilon_k)]^2 + (\varepsilon_k/2)^2}. \end{aligned} \tag{49}$$

In the case of $k = k_c$, the evolved state is represented in Eq. (36) and corresponding $E_{k_c}(t)$ can be expressed as

$$E_{k_c}(t) = \frac{\langle \psi_{k_c}(t)|\widehat{E}_k|\psi_{k_c}(t)\rangle}{\langle \psi_{k_c}(t)|\psi_{k_c}(t)\rangle} = \frac{4t^2\cos k_c}{1 + 2t^2}. \tag{50}$$

(ii) The operator of order parameter $\hat{P}$ is defined as

$$\hat{P} = \sum_{j=1}^{N}(c_j^\dagger c_{j+1}^\dagger + \text{H.c.}) = \sum_{k>0}\hat{P}_k \tag{51}$$

and the time evolution of the expectation value of $\hat{P}$ is

$$P(t) = \frac{\langle \Psi(t)|\widehat{P}|\Psi(t)\rangle}{\langle \Psi(t)|\Psi(t)\rangle} = \sum_{k>0}\frac{\langle \psi_k(t)|\widehat{P}_k|\psi_k(t)\rangle}{\langle \psi_k(t)|\psi_k(t)\rangle}. \tag{52}$$

In the case of $k \neq k_c$, we have

$$\begin{aligned} P_k(t) &= \frac{\langle \psi_k(t)|\widehat{P}_k|\psi_k(t)\rangle}{\langle \psi_k(t)|\psi_k(t)\rangle} \\ &= \frac{-\Delta\varepsilon_k \sin^2 k \sin(2\varepsilon_k t)}{2(\Delta \sin k)^2 \sin^2(\varepsilon_k t) + (\varepsilon_k/2)^2}, \end{aligned} \tag{53}$$

while, for $k = k_c$, we have

$$P_{k_c}(t) = \frac{\langle \psi_{k_c}(t)|\widehat{P}_k|\psi_{k_c}(t)\rangle}{\langle \psi_{k_c}(t)|\psi_{k_c}(t)\rangle} = \frac{-4t \sin k_c}{1 + 2t^2}. \tag{54}$$

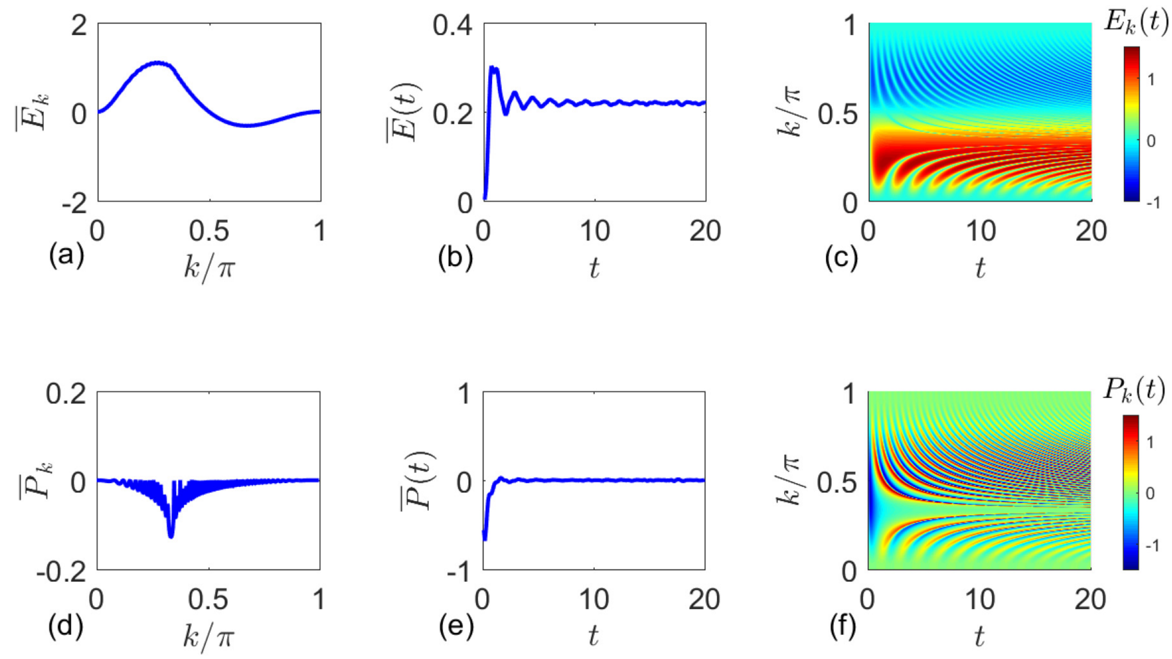

FIG. 3. Plots of (a) $\overline{E}_k$, (b) $\overline{E}(t)$, (c) $E_k(t)$, (d) $\overline{P}_k$, (e) $\overline{P}(t)$, and (f) $P_k(t)$, which are defined in Eqs. (55), (57), (49), (56), (58), and (53). The parameters are $N = 400$ and $(J, \Delta, \mu) = (1, \sqrt{3}, 2)$. It shows that the average of observables become steady as time increases.

We define the average normalized kinetic-energy and order-parameter distributions in the k space as

$$\overline{E}_k = \frac{1}{T_k}\int_0^{T_k} E_k(t)dt \tag{55}$$

and

$$\overline{P}_k = \frac{1}{T_k}\int_0^{T_k} P_k(t)dt. \tag{56}$$

The total average normalized kinetic energy and order parameter are

$$\overline{E}(t) = \frac{1}{\pi}\int_0^{\pi} E_k(t)dk \tag{57}$$

and

$$\overline{P}(t) = \frac{1}{\pi}\int_0^{\pi} P_k(t)dk. \tag{58}$$

We plot quantities $\overline{E}_k$, $\overline{E}(t)$, $E_k(t)$, $\overline{P}_k$, $\overline{P}(t)$, and $P_k(t)$ for concrete cases in Fig. 3. It indicates that our conclusion still holds for physical observables $\hat{E}$ and $\hat{P}$. In the following section, we will investigate the possible property of such a state.

VI. DYNAMICAL GENERATION OF A SUPERCONDUCTING STATE

In this section, as an application of the above result, we investigate the possibility of dynamical generation of the superconducting state via a non-Hermitian Kitaev model. The scheme is that, taking the empty state $\prod_{k>0}|0\rangle_k|0\rangle_{-k}$ as an initial state, the final state, which approaches the ground state of a Hermitian Kitaev Hamiltonian H, is achieved by a driven non-Hermitian Kitaev Hamiltonian $\mathcal{H}$ at EP. Before proceeding, we briefly review the properties of a Hermitian Kitaev model with the Hamiltonian

$$H = \sum_{j=1}^{N}[-Jc_j^{\dagger}c_{j+1} + \text{H.c.} - i\Delta_{\text{h}}c_j^{\dagger}c_{j+1}^{\dagger} + i\Delta_{\text{h}}c_{j+1}c_j + \mu_{\text{h}}(2n_j - 1)]. \tag{59}$$

It has been shown to have a topologically nontrivial (trivial) ground state, when $|\mu_{\text{h}}| < |J|$ ($|\mu_{\text{h}}| > |J|$) in Ref. [1]. The phase diagram is plotted in Fig. 4, with an H-shape boundary separating topologically nontrivial and trivial phases, charac-

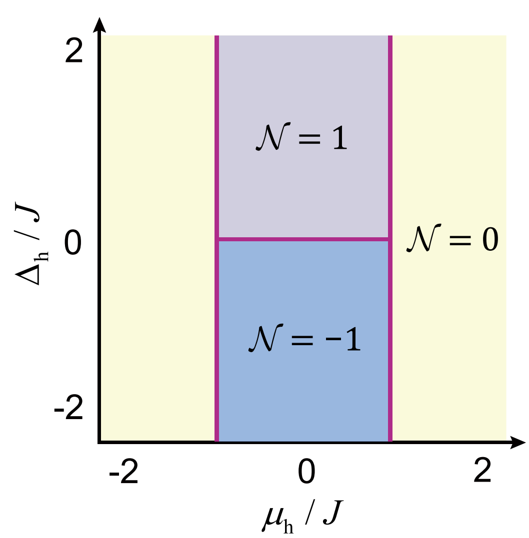

FIG. 4. Phase diagram of Hermitian Kitaev model on the parameter $\mu_h - \Delta_h$ plane. The sky blue, yellow, and purple regions correspond to the winding number -1, 0, and 1, respectively. Dark magenta lines indicate the phase-transition lines.

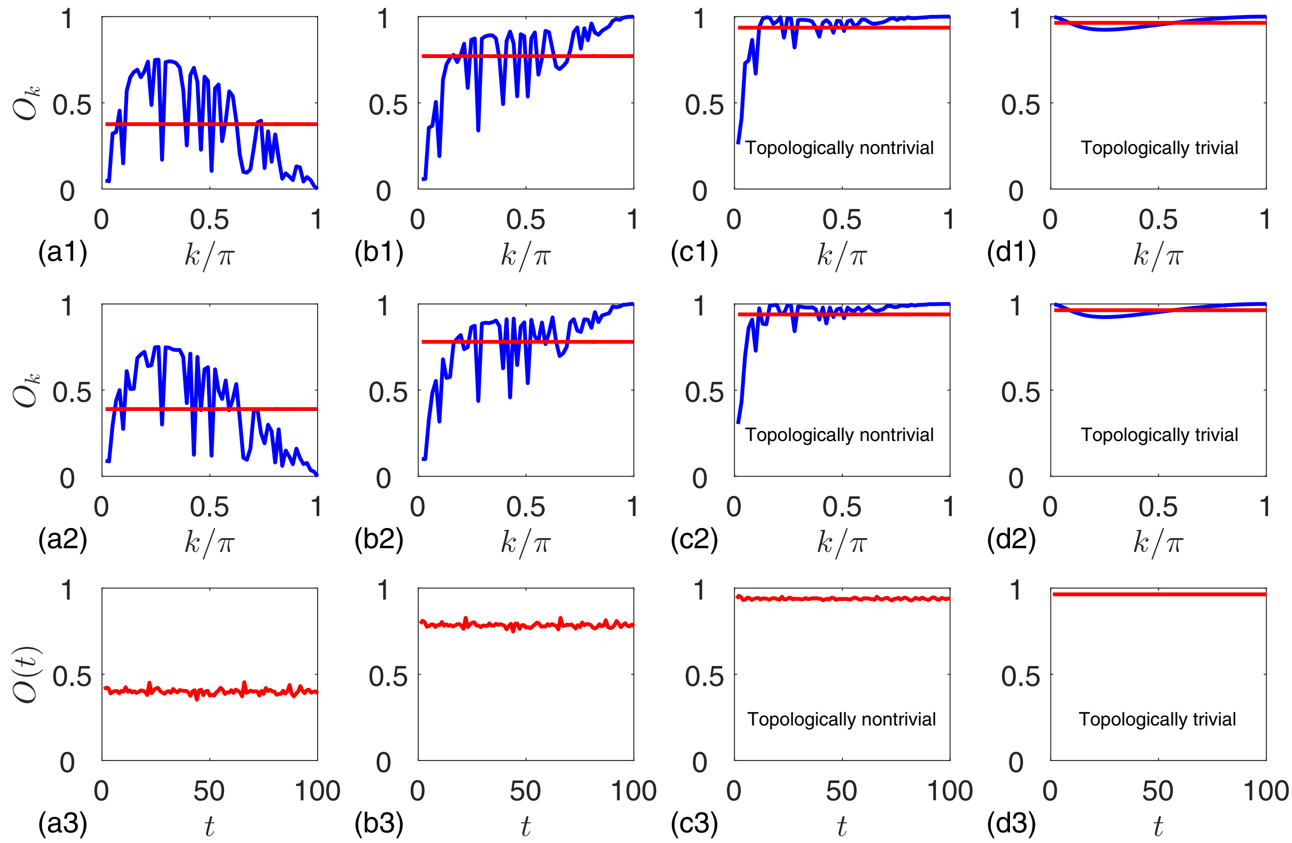

FIG. 5. Numerical simulations of O_k defined in Eq. (71) and $O(t)$ defined in Eq. (70). The four panels in the first row and the second row are the plots of O_k at time $t = 50J^{-1}$ and $100J^{-1}$, respectively. The red lines represent the corresponding O, which are the values (a1) 0.376, (a2) 0.390, (b1) 0.771, (b2) 0.780, (c1) 0.936, (c2) 0.939, (d1) 0.964, and (d2) 0.964. The four panels in the third row are the plots of $O(t)$ with the same parameters in the two above rows. The parameters are $N = 61$, $J = 1$, $\Delta = \Delta_h = 1$, and $\mu = \sqrt{J^2 + \Delta^2}$. Each column of the graph has the same set of parameters, i.e., (a1)–(a3) $\mu_h = -5$, (b1)–(b3) $\mu_h = -0.5$, (c1)–(c3) $\mu_h = 0.9$, and (d1)–(d3) $\mu_h = \mu$. The parameters of the Hermitian Kitaev model in (c1)–(c3) and (d1)–(d3) support topologically nontrivial and trivial superconducting ground states, respectively.

terized by winding number $\mathcal{N}$. By the similar procedure as above, we have

$$H = \sum_{\pi > k > 0} H_k, \tag{60}$$

$$H_k = 2(c_k^\dagger \;\; c_{-k}) \begin{pmatrix} \mu_h - J\cos k & \Delta_h \sin k \\ \Delta_h \sin k & J\cos k - \mu_h \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}, \tag{61}$$

and the Hamiltonian H_k in each invariant subspace satisfies the commutation relation

$$[H_k, H_{k'}] = 0. \tag{62}$$

For a given k, the Hamiltonian H_k in the basis $(|0\rangle_k |0\rangle_{-k}, |1\rangle_k |1\rangle_{-k})$ is expressed as 2×2 matrix

$$h_k = 2\begin{pmatrix} J\cos k - \mu_h & \Delta_h \sin k \\ \Delta_h \sin k & \mu_h - J\cos k \end{pmatrix}. \tag{63}$$

The eigenstate $|\varphi_k^\pm\rangle$ with even parity of the particle number is

$$|\varphi_k^\pm\rangle = \frac{1}{\sqrt{\Omega_h^\pm}} (|0\rangle_k |0\rangle_{-k} + b_k^\pm |1\rangle_k |1\rangle_{-k}), \tag{64}$$

where $\Omega_h^\pm = 1 + |b_k^\pm|^2$ is the normalization coefficient in the context of a Dirac inner product with

$$b_k^\pm = \frac{\Delta_h \sin k}{J\cos k - \mu_h + \epsilon_k^\pm/2} \tag{65}$$

and corresponding energies are

$$\epsilon_k^\pm = \pm 2\sqrt{(\mu_h - J\cos k)^2 + \Delta_h^2 \sin^2 k}. \tag{66}$$

Accordingly, the ground-state wave function can be expressed as

$$|G\rangle = \prod_{\pi > k > 0} |\varphi_k^-\rangle. \tag{67}$$

We note that, for a topological nontrivial ground state, we have

$$\lim_{k\to 0} |\varphi_k^-\rangle = |1\rangle_k |1\rangle_{-k}, \quad \lim_{k\to \pi} |\varphi_k^-\rangle = |0\rangle_k |0\rangle_{-k}, \tag{68}$$

while

$$\lim_{k\to 0} |\varphi_k^-\rangle = |0\rangle_k |0\rangle_{-k}, \quad \lim_{k\to \pi} |\varphi_k^-\rangle = |0\rangle_k |0\rangle_{-k}, \tag{69}$$

for a topological trivial ground state. On the other hand, for the non-Hermitian system, we know that there is a stable final state $\lim_{t\to\infty} |\psi_{k_c}(t)\rangle \propto (|0\rangle_{k_c} |0\rangle_{-k_c} - |1\rangle_{k_c} |1\rangle_{-k_c})$, according to Eq. (36). If we take a matching set of parameters, the stable final state can be an eigenmode of $|\mathrm{G}\rangle$, i.e., $|\psi_{k_c}(t)\rangle = |\varphi_{k_c}^-\rangle$ after normalization. It is probable to obtain a state dynamically under the Hamiltonian $\mathcal{H}$, which is similar to a ground state of H. To characterize how close of an evolved state to a superconducting state we introduce a quantity

$$O(t) = \frac{1}{N}\sum_k O_k(t), \tag{70}$$

where

$$O_k(t) = \langle \varphi_k^- | \psi_k(t)\rangle \tag{71}$$

is the overlap of a specific topological superconducting mode $|\varphi_k^-\rangle$ and a dynamically generated state $|\psi_k(t)\rangle$ via the non-Hermitian system.

We compute the quantity $O(t)$ for various sets of parameters (J, Δ, μ) and $(J, \Delta_\mathrm{h}, \mu_\mathrm{h})$ to search optimal cases with large $O(t)$. We find that there are many cases with large $O(t)$. Here we take four typical cases to demonstrate our results. We plot $O(t)$ and O_k at certain instants in Fig. 5, which show that $O(t)$ oscillates with a very small amplitude. It also indicates that through such a dynamical method, a quasisuperconducting state involving topological trivial and nontrivial can be generated from a simple initial state.

VII. SUMMARY

In summary, we have studied the non-Hermitian extension of a Kitaev chain by considering imaginary p-wave pairing amplitudes. Based on the analysis of the exact solution we find that exceptional lines are hyperbolic lines, which separate two regions with real and complex Bogoliubov–de Gennes spectra, associated with $\mathcal{PT}$-symmetry breaking. The EPs are movable in k space as the parameters vary along the exceptional lines. The non-Hermiticity around EP supports resonant generation of the p-wave Cooper pair state via the critical dynamic process. A specific pair state $(1 + c_k^\dagger c_{-k}^\dagger)|0\rangle$ with selecting momentum k can be generated from the vacuum state $|0\rangle$ of fermions and be frozen forever. The remarkable result obtained by analytical approaches and numerical simulations are that the dynamically generated state via the non-Hermitian system is very close to a specific superconducting ground state, which can be topologically nontrivial or not. This finding provides an alternative way to generate a superconducting state via a critical dynamic process rather than cooling down the temperature. In general, a Hermitian superconductor is an isolated system without particle exchange to the environment. The superconducting state appears after the temperature decreases below the transition point. During the cooling process, the system keeps in an equilibrium thermal state. In contrast, a superconducting state in the present non-Hermitian system is obtained by a dynamic process. Roughly speaking, it can be regarded as the result of Cooper-pair flowing in from the environment. Actually, the key feature of the theoretical results is due to the EP dynamics. This can be demonstrated and tested experimentally via a two-site system, which can exhibit similar EP dynamics as the original model. In Appendix B, we present a two-site system to address this point.

ACKNOWLEDGMENT

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APPENDIX A: POSSIBLE ORIGIN OF THE NON-HERMITIAN PAIRING

Here, we provide a way to understand the physical origin of the pairing term with imaginary amplitudes. We will do it by presenting an equivalent Hamiltonian which is not surprising in the non-Hermitian quantum mechanics. We start our examination by introducing a particle-hole transformation

$$c_j^\dagger \to i c_j, \quad c_j \to -i c_j^\dagger \tag{A1}$$

only for a site with even j (one of the sublattices of a bipartite lattice), at which the relation $\{c_j, c_j^\dagger\} = 1$ still holds, but $n_j \to 1 - n_j$. Then the original Hamiltonian Eq. (1) becomes its equivalent Hamiltonian

$$H_\mathrm{eq} = \sum_{j=1}^{N}[(-1)^{j+1} iJ(c_j^\dagger c_{j+1}^\dagger - c_{j+1}c_j) + \Delta(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j) + (-1)^{j+1}\mu(2c_j^\dagger c_j - 1)]. \tag{A2}$$

It is still a non-Hermitian Kitaev model, with staggered chemical potentials and pairing amplitudes. In contrast to the original one, the non-Hermiticity arises from the hopping term

$$H_\mathrm{hop} = \Delta \sum_{j=1}^{N}(c_j^\dagger c_{j+1} - c_{j+1}^\dagger c_j) \tag{A3}$$

with non-Hermitian hopping strength. The non-Hermitian hopping term is not a new concept. In Ref. [42], it is proposed that an imaginary hopping term can be obtained by the combination of on-site imaginary potentials and magnetic flux. In Ref. [17], the realization of a non-Hermitian magnon hopping term in an ultracold atomic system is also proposed. In addition, such a term is also mentioned in Ref. [34].

Nevertheless, here we would like to understand the physical meaning of the imaginary hopping term in an alternative way. Taking Fourier transformation

$$c_k = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{-ikj}c_j, \tag{A4}$$

we have

$$H_\mathrm{hop} = i2\Delta\sum_k(\sin k)n_k. \tag{A5}$$

It is clear that H_hop can be regarded as the pure imaginary on-site potential in k space. In principle, the coordinate space and the momentum space are considered as equal terms. It turns out that the imaginary potential at the lth site can originate from the coupling to the environment, $c_l^\dagger b_l + b_l^\dagger c_l$, where b_l denotes the particle operator in the environment. In parallel, the imaginary potential at the kth site in momentum space can originate from the coupling to the environment, $c_k^\dagger b_k + b_k^\dagger c_k$, where b_k denotes the particle operator in the environment. The

Fourier transformation

$$c_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-ikj} c_j,$$

$$b_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{-ikj} b_j \tag{A6}$$

tells us that the term $c_k^\dagger b_k + b_k^\dagger c_k$ corresponds to a collective tunneling of particles between the central system and the environment. For a simple case, in which the whole system is multiple coupled wires, the interwire coupling can be responsible for the non-Hermitian term.

APPENDIX B: TWO-SITE KITAEV MODEL

Consider the original Hamiltonian on a two-site system, which has the form

$$H = -[Jc_1^\dagger c_2 + \text{H.c.} + i\Delta(c_1^\dagger c_2^\dagger + c_2 c_1) - \mu(2c_1^\dagger c_1 - 1) - \mu(2c_2^\dagger c_2 - 1)]. \tag{B1}$$

By introducing a particle-hole transformation

$$c_2^\dagger \to -c_2, \quad c_2 \to -c_2^\dagger, \tag{B2}$$

we get its equivalent Hamiltonian

$$H_{\text{eq}} = J(c_1^\dagger c_2^\dagger + c_2 c_1) + i\Delta(c_1^\dagger c_2 + c_2^\dagger c_1) + 2\mu(c_1^\dagger c_1 - c_2^\dagger c_2). \tag{B3}$$

Taking the linear transformation

$$c_{\text{L}} = \frac{c_1 + c_2}{\sqrt{2}}, \quad c_{\text{R}} = \frac{c_1 - c_2}{\sqrt{2}}, \tag{B4}$$

we have

$$H_{\text{eq}} = 2\mu(c_{\text{L}}^\dagger c_{\text{R}} + c_{\text{R}}^\dagger c_{\text{L}}) - J(c_{\text{L}}^\dagger c_{\text{R}}^\dagger + c_{\text{R}} c_{\text{L}}) + i\Delta(c_{\text{L}}^\dagger c_{\text{L}} - c_{\text{R}}^\dagger c_{\text{R}}), \tag{B5}$$

which represents a Kiteav model with $\mathcal{PT}$ (double well) imaginary potential.

Furthermore, taking Jordan-Wigner transformation

$$c_{\text{L}} = -\sigma_{\text{L}}^z \sigma_{\text{L}}^-, \quad c_{\text{R}} = -\sigma_{\text{R}}^z \sigma_{\text{R}}^-, \tag{B6}$$

we have

$$H_{\text{eq}} = -2\mu(\sigma_{\text{L}}^+ \sigma_{\text{R}}^- + \sigma_{\text{L}}^- \sigma_{\text{R}}^+) - J(\sigma_{\text{L}}^+ \sigma_{\text{R}}^+ + \sigma_{\text{L}}^- \sigma_{\text{R}}^-) - 2i\Delta(\sigma_{\text{L}}^z - \sigma_{\text{R}}^z), \tag{B7}$$

or the anisotropic XY model

$$H_{\text{eq}} = -2(J + 2\mu)\sigma_{\text{L}}^x \sigma_{\text{R}}^x + 2(J - 2\mu)\sigma_{\text{L}}^y \sigma_{\text{R}}^y - 2i\Delta(\sigma_{\text{L}}^z - \sigma_{\text{R}}^z). \tag{B8}$$

Importantly, the complex magnetic field is no longer a component of a toy model, and it is claimed that the non-Hermitian XY model can be implemented using three-level atoms with spontaneous decay [43]. Accordingly, our results can be seen experimentally with trapped ions, cavity QED, and atoms in optical lattices.

[1] A. Y. Kitaev, Unpaired Majorana fermions in quantum wires, Phys. Usp. **44**, 131 (2001).

[2] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. D. Sarma, Non-Abelian anyons and topological quantum computation, Rev. Mod. Phys. **80**, 1083 (2008).

[3] A. Stern, Non-Abelian states of matter, Nature (London) **464**, 187 (2010).

[4] J. Alicea, New directions in the pursuit of Majorana fermions in solid state systems, Rep. Prog. Phys. **75**, 076501 (2012).

[5] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry, Phys. Rev. Lett. **80**, 5243 (1998).

[6] C. M. Bender, D. C. Brody, and H. F. Jones, Complex Extension of Quantum Mechanics, Phys. Rev. Lett. **89**, 270401 (2002).

[7] A. Mostafazadeh, Pseudo-Hermiticity versus PT-symmetry: The necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian, J. Math. Phys. **43**, 205 (2002); **43**, 2814 (2002); Pseudo-Hermiticity versus PT-symmetry. II. A complete characterization of non-Hermitian Hamiltonians with a real spectrum, **43**, 3944 (2002); Pseudo-supersymmetric quantum mechanics and isospectral pseudo-Hermitian Hamiltonians, Nucl. Phys. B **640**, 419 (2002); Pseudo-Hermiticity and generalized PT- and CPT-symmetries, J. Math. Phys. **44**, 974 (2003).

[8] X. J. Liu, C. L. M. Wong, and K. T. Law, Non-Abelian Majorana Doublets in Time-Reversal-Invariant Topological Superconductors, Phys. Rev. X **4**, 021018 (2014).

[9] X. H. Wang, T. T. Liu, Y. Xiong, and P. Q. Tong, Spontaneous PT-symmetry breaking in non-Hermitian Kitaev and extended Kitaev models, Phys. Rev. A **92**, 012116 (2015).

[10] C. Yuce, Majorana edge modes with gain and loss, Phys. Rev. A **93**, 062130 (2016).

[11] Q. B. Zeng, B. G. Zhu, S. Chen, L. You, and R. Lü, Non-Hermitian Kitaev chain with complex on-site potentials, Phys. Rev. A **94**, 022119 (2016).

[12] M. Klett, H. Cartarius, D. Dast, J. Main, and G. Wunner, Relation between PT-symmetry breaking and topologically nontrivial phases in the Su-Schrieffer-Heeger and Kitaev models, Phys. Rev. A **95**, 053626 (2017).

[13] H. Menke and M. M. Hirschmann, Topological quantum wires with balanced gain and loss, Phys. Rev. B **95**, 174506 (2017).

[14] C. Li, X. Z. Zhang, G. Zhang, and Z. Song, Topological phases in a Kitaev chain with imbalanced pairing, Phys. Rev. B **97**, 115436 (2018); C. Li, G. Zhang, X. Z. Zhang, and Z. Song, Conventional quantum phase transition driven by a complex parameter in a non-Hermitian PT-symmetric Ising model, Phys. Rev. A **90**, 012103 (2014); C. Li and Z. Song, Finite-temperature quantum criticality in a complex-parameter plane, *ibid.* **92**, 062103 (2015); C. Li, G. Zhang, and Z. Song,

Chern number in Ising models with spatially modulated real and complex fields, *ibid.* **94**, 052113 (2016).
[15] X. Z. Zhang and Z. Song, Non-Hermitian anisotropic XY model with intrinsic rotation-time-reversal symmetry, Phys. Rev. A **87**, 012114 (2013).
[16] M. Franz, Majorana's wires, Nat. Nanotechnol. **8**, 149 (2013).
[17] Y. Ashida, S. Furukawa, and M. Ueda, Parity-time-symmetric quantum critical phenomena, Nat. Commun. **8**, 15791 (2017).
[18] I. Rotter and J. P. Bird, A review of progress in the physics of open quantum systems: Theory and experiment, Rep. Prog. Phys. **78**, 114001 (2015).
[19] P. Matthews, P. Ribeiro, and A. M. García-García, Dissipation in a Simple Model of a Topological Josephson Junction, Phys. Rev. Lett. **112**, 247001 (2014).
[20] W. D. Heiss, The physics of exceptional points, J. Phys. A: Math. Theor. **45**, 444016 (2012).
[21] I. Rotter, A non-Hermitian Hamilton operator and the physics of open quantum systems, J. Phys. A: Math. Theor. **42**, 153001 (2009); I. Rotter and A. F. Sadreev, Avoided level crossings, diabolic points, and branch points in the complex plane in an open double quantum dot, Phys. Rev. E **71**, 036227 (2005).
[22] H. Xu, D. Mason, L. Jiang, and J. G. E. Harris, Topological energy transfer in an optomechanical system with exceptional points, Nature (London) **537**, 80 (2016).
[23] K. L. Zhang, L. Jin, and Z. Song, Helical resonant transport and purified amplification at an exceptional point, Phys. Rev. B **100**, 144301 (2019).
[24] K. L. Zhang, P. Wang, and Z. Song, Exceptional-point-induced lasing dynamics in a non-Hermitian Su-Schrieffer-Heeger model, Phys. Rev. A **99**, 042111 (2019).
[25] K. L. Zhang, X. M. Yang, and Z. Song, Quantum transport in non-Hermitian impurity arrays, Phys. Rev. B **100**, 024305 (2019).
[26] K. L. Zhang and Z. Song, Coherent confinement and unidirectional dynamics of a wave packet induced by a non-Hermitian Su-Schrieffer-Heeger segment, Phys. Rev. A **101**, 032111 (2020).
[27] B. F. Samsonov, Spectral singularities of non-Hermitian Hamiltonians and SUSY transformations, J. Phys. A: Math. Gen. **38**, L571 (2005).
[28] A. A. Andrianov, F. Cannata, and A. V. Sokolov, Spectral singularities for non-Hermitian one-dimensional Hamiltonians: puzzles with resolution of identity, J. Math. Phys. **51**, 052104 (2010).
[29] A. Mostafazadeh, Spectral Singularities of Complex Scattering Potentials and Infinite Reflection and Transmission Coefficients at Real Energies, Phys. Rev. Lett. **102**, 220402 (2009); Optical spectral singularities as threshold resonances, Phys. Rev. A **83**, 045801 (2011).
[30] S. Longhi, Spectral singularities in a non-Hermitian Friedrichs-Fano-Anderson model, Phys. Rev. B **80**, 165125 (2009); Spectral singularities and Bragg scattering in complex crystals, Phys. Rev. A **81**, 022102 (2010).
[31] X. Z. Zhang, L. Jin, and Z. Song, Perfect state transfer in PT-symmetric non-Hermitian networks, Phys. Rev. A **85**, 012106 (2012).
[32] M. Aharonyan and E. G. D. Torre, Many-body exceptional points in colliding condensates, Mol. Phys. **117**, 1971 (2019).
[33] D. J. Luitz and F. Piazza, Exceptional points and the topology of quantum many-body spectra, Phys. Rev. Research **1**, 033051 (2019).
[34] B. Dóra and C. P. Moca, Quantum Quench in $\mathcal{PT}$-Symmetric Luttinger Liquid, Phys. Rev. Lett. **124**, 136802 (2020).
[35] Y. Ashida and M. Ueda, Full-Counting Many-Particle Dynamics: Nonlocal and Chiral Propagation of Correlations, Phys. Rev. Lett. **120**, 185301 (2018).
[36] X. Z. Zhang, L. Jin, and Z. Song, Dynamic magnetization in non-Hermitian quantum spin systems, Phys. Rev. B **101**, 224301 (2020).
[37] P. Pfeuty, The one-dimensional Ising model with a transverse field, Ann. Phys. (NY) **57**, 79 (1970).
[38] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).
[39] A. Mostafazadeh, PT-symmetric cubic anharmonic oscillator as a physical model, J. Phys. A: Math. Gen. **38**, 6557 (2005); Metric operator in pseudo-Hermitian quantum mechanics and the imaginary cubic potential, **39**, 10171 (2006); Delta-function potential with a complex coupling, **39**, 13495 (2006).
[40] L. Jin and Z. Song, Solutions of PT-symmetric tight-binding chain and its equivalent Hermitian counterpart, Phys. Rev. A **80**, 052107 (2009).
[41] G. Zhang and Z. Song, Topological Characterization of Extended Quantum Ising Models, Phys. Rev. Lett. **115**, 177204 (2015).
[42] G. Zhang, X. Q. Li, X. Z. Zhang, and Z. Song, Transmission phase lapse in the non-Hermitian Aharonov-Bohm interferometer near the spectral singularity, Phys. Rev. A **91**, 012116 (2015).
[43] T. E. Lee and C.-K. Chan, Heralded Magnetism in Non-Hermitian Atomic Systems, Phys. Rev. X **4**, 041001 (2014).