Stoner-Wohlfarth switching of the condensate magnetization in a dipolar spinor gas and the metrology of excitation damping

Seong-Ho Shinn [1], Daniel Braun, [2] and Uwe R. Fischer [1]

[1] *Seoul National University, Department of Physics and Astronomy, Center for Theoretical Physics, Seoul 08826, Korea*

[2] *Eberhard-Karls-Universität Tübingen, Institut für Theoretische Physik, 72076 Tübingen, Germany*

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We consider quasi-one-dimensional dipolar spinor Bose-Einstein condensates in the homogeneous-local-spin-orientation approximation, that is, with unidirectional local magnetization. By analytically calculating the exact effective dipole-dipole interaction, we derive a Landau-Lifshitz-Gilbert equation for the dissipative condensate magnetization dynamics, and show how it leads to the Stoner-Wohlfarth model of a uniaxial ferromagnetic particle, where the latter model determines the stable magnetization patterns and hysteresis curves for switching between them. For an external magnetic field pointing along the axial, long direction, we analytically solve the Landau-Lifshitz-Gilbert equation. The solution explicitly demonstrates that the magnetic dipole-dipole interaction *accelerates* the dissipative dynamics of the magnetic moment distribution and the associated dephasing of the magnetic moment direction. Under suitable conditions, dephasing of the magnetization direction due to dipole-dipole interactions occurs within time scales up to two orders of magnitude smaller than the lifetime of currently experimentally realized dipolar spinor condensates, e.g., those produced with the large magnetic-dipole-moment atoms ^{166}Er. This enables experimental access to the dissipation parameter Γ in the Gross-Pitaevskii mean-field equation, for a system currently lacking a complete quantum kinetic treatment of dissipative processes and, in particular, an experimental check of the commonly used assumption that Γ is a single scalar independent of spin indices.

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I. INTRODUCTION

Ever since a phenomenological theory to describe the behavior of superfluid helium II near the λ point was developed by Pitaevskii [1], the dynamics of Bose-Einstein condensates (BECs) under dissipation has been intensely studied (see, e.g., Refs. [2–8]). Experimentally, the impact of Bose-Einstein condensation on excitation damping and its temperature dependence has for example been demonstrated in Refs. [9–12].

Dissipation in the form of condensate loss is defined by a dimensionless damping rate Γ entering the left-hand side of the Gross-Pitaevskii equation, replacing the time derivative as $i\partial_t \to (i-\Gamma)\partial_t$. While a microscopic theory of condensate damping is comparatively well established in the contact-interaction case, using various approaches (cf., e.g., Refs. [5,13–15]), we emphasize the absence of a microscopic theory of damping in *dipolar spinor* gases. While for scalar dipolar condensates, partial answers as to the degree and origin of condensate-excitation damping have been found (see, e.g., Refs. [16–19]), in spinor or multicomponent gases the interplay of anisotropic long-range interactions and internal spinor or multicomponent degrees of freedom leads to a highly intricate and difficult-to-disentangle many-body behavior of condensate-excitation damping.

In this paper, we propose a method to experimentally access Γ in a dipolar spinor condensate by using the dynamics of the unidirectional local magnetization in a quasi-one-dimensional (quasi-1D) dipolar spinor BEC in the presence of an external magnetic field. To this end, we first derive an equation of motion for the magnetization of the BEC that has the form of a Landau-Lifshitz-Gilbert (LLG) equation [20–22], with an additional term due to the dipole-dipole interaction between the atoms. The LLG equation is ubiquitous in nanomagnetism, where it describes the creation and dynamics of magnetization. The static limit of this equation is, in the limit of homogeneous local spin orientation, described by the well-known Stoner-Wohlfarth (SW) model [23–25] of a small magnetic particle with an easy axis of magnetization. We then investigate the magnetization switching after flipping the sign of the external magnetic field, and demonstrate the detailed dependence of the switching dynamics on the dissipative parameter Γ.

For a quasi-two-dimensional (quasi-2D) spinor BEC with inhomogeneous local magnetization, Ref. [26] has studied the magnetic domain wall formation process by deriving a LLG type equation. Here, we derive the LLG equation in a quasi-1D spinor BEC with unidirectional local magnetization in order to establish a most direct connection to the original SW model. In distinction to Ref. [27], which studied the effective quasi-1D dipole-dipole interaction resulting from integrating out the two transverse directions within a simple approximation, we employ below an exact analytic form of the dipole-dipole interaction.

In Sec. II, we establish the quasi-1D spinor Gross-Pitaevskii (GP) equation with dissipation, and equations of motion for the magnetization direction (unit vector) $\boldsymbol{M}$. Section V shows how the LLG equation and the SW model result, and Sec. VI derives analytical solutions to the equations of motion for $\boldsymbol{M}$ when the external magnetic field points along the long, z axis. We summarize our results in Sec. VII.

We defer two longer derivations to the Appendixes. The analytical form of the effective dipole-dipole interaction energy is deduced in Appendix A, and the quasi-1D GP mean-field equation with dissipation is described in detail in Appendix B. Finally, in Appendix C, we briefly discuss to what extent relaxing the usual simplifying assumption that dissipation even in the spinor case is described by a single scalar changes the LLG equation, and whether this affects the SW model and its predictions.

II. GENERAL DESCRIPTION OF DAMPING IN BECs

The standard derivation of the quantum kinetics of Bose-Einstein condensate damping [5] starts from the microscopic Heisenberg equation of motion for the quantum field operator $\hat{\psi}(\boldsymbol{r}, t)$, for a scalar (single-component) BEC in the s-wave scattering limit. We use the results of Ref. [28], which obtained a mean-field equation to describe the dissipation of a scalar BEC, whose form is

$$(i - \Gamma)\hbar \frac{\partial \psi}{\partial t} = H\psi, \tag{1}$$

where ψ is (in the large-N limit) the dominant mean-field part upon expanding the full bosonic field operator $\hat{\psi}$.

In Ref. [1], Pitaevskii obtained a similar but slightly different form of the dissipative mean-field equation based on phenomenological considerations, $i\hbar\frac{\partial \psi}{\partial t} = (1 - i\Gamma)H\psi$, by parametrizing the deviation from exact continuity for the condensate fraction while minimizing the energy [1]. The latter deviation is assumed to be small, which is equivalent to assuming that Γ remains small. This provides a clear physical interpretation of the damping mechanism, namely, one based on particle loss from the condensate fraction. The version of Pitaevskii can be written as

$$(i - \Gamma)\hbar \frac{\partial \psi}{\partial t} = (1 + \Gamma^2)H\psi. \tag{2}$$

It can thus be simply obtained by rescaling time with a factor $1 + \Gamma^2$ compared to Eq. (1). Hence, as long as one does not predict precisely Γ, the two dissipative equations (1) and (2) cannot be distinguished experimentally from the dynamics they induce. From the data of Ref. [11], employing the results of Ref. [29], the estimated typical values are $\Gamma \simeq 0.03$ for a *scalar* BEC of ^{23}Na atoms at $T \simeq T_c/10$ [4]. At low temperatures ($T \to 0$), the latter reference derived $\Gamma \propto T^{3/2} e^{(2\mu - \mu_N)/k_B T}$ for a BEC in a harmonic trap, where μ is the chemical potential of the condensate and μ_N its noncondensate counterpart (also compare the discussion in Ref. [12]). Assuming that indeed Γ is of order 10^{-2} demonstrates that to distinguish between Eqs. (1) and (2) experimentally the theoretical predictions of Γ would need to be precise to order 10^{-4}.

How Eqs. (1) and (2) can be generalized to the dipolar spinor gases is comparatively little investigated. Using a symmetry-breaking mean-field approach by writing the quantum field operator $\hat{\psi}(\boldsymbol{r}, t)$ as $\hat{\psi}(\boldsymbol{r}, t) = \psi(\boldsymbol{r}, t) + \delta\hat{\psi}(\boldsymbol{r}, t)$, with $\psi(\boldsymbol{r}, t) = \langle\hat{\psi}(\boldsymbol{r}, t)\rangle$ and $\langle\delta\hat{\psi}(\boldsymbol{r}, t)\rangle = 0$, Refs. [5,28] showed that Γ can be derived from the three-field correlation function $\langle\delta\hat{\psi}^\dagger(\boldsymbol{r}, t)\delta\hat{\psi}(\boldsymbol{r}, t)\delta\hat{\psi}(\boldsymbol{r}, t)\rangle$ in a basis where $\langle\delta\hat{\psi}(\boldsymbol{r}, t)\delta\hat{\psi}(\boldsymbol{r}, t)\rangle = 0$. Then, $\Gamma \propto A(T)/T$ for a homogeneous scalar BEC, where $A(T)$ is of order unity for $T/T_c > 0.5$ and $A(T) \to 0$ when $T \to 0$. From this microscopic origin, based on correlation functions, it is clear that in principle Γ might depend on the spin indices in a spinor BEC and hence become a tensor (see Appendix C for a corresponding phenomenological generalization). Nevertheless, it is commonly assumed (cf., e.g., Refs. [26,30]) that Γ does not depend on spin indices, and the scalar value found specifically in Ref. [4] for a *scalar* BEC of ^{23}Na atoms is commonly used, while a clear justification of this assumption is missing.

Extending the microscopic derivations in Refs. [5,28] to the spinor case would be theoretically interesting, but it is beyond the scope of the present paper. Here, we instead focus on the question whether the standard assumption that the damping of each spinor component can be described by the mean-field equation [28] leads to experimentally falsifiable dynamical signatures. It will turn out that this assumption introduces an additional strong dephasing in the spin degrees of freedom, amplified by the dipolar interaction. Hence, even on time scales on which the decay of the condensate fraction according to Eq. (1) can be neglected, the relaxation of the magnetization of the BEC potentially offers valuable insights whether the scalar-Γ assumption is justified. Indeed, in Ref. [31] it was shown experimentally that on the time scale of the switching dynamics of the magnetization the number of particles in the condensates remains approximately constant. One might wonder, then, which dissipative mechanism is left. However, as we show, by assuming the same GP equation for each component of the spinor as for scalar bosons, additional dephasing occurs that is in fact much more rapid than the decay of condensate density due to dephasing accelerated by the dipole-dipole interaction.

III. MEAN-FIELD DYNAMICS OF DAMPING IN DIPOLAR SPINOR BECs

For a spinor BEC, linear and quadratic Zeeman interactions are commonly included in the Hamiltonian. The quadratic Zeeman interaction is related to a second-order perturbation term in the total energy that can be induced by the interaction with an external magnetic field (q_B) as well as with the interaction with a microwave field ($q_{\rm MW}$) [32]. Specifically, by applying a linearly polarized microwave field, one can change $q_{\rm MW}$ without changing q_B [33,34]. Hence, we assume that the quadratic Zeeman term can be rendered zero by suitably changing $q_{\rm MW}$.

Following Ref. [26], we thus assert that, for a dipolar spinor BEC without quadratic Zeeman term, the mean-field equation can be written as

$$\begin{aligned}(i - \Gamma)\hbar \frac{\partial \psi(\boldsymbol{r}, t)}{\partial t} = &\left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm tr}(\boldsymbol{r}) + c_0|\psi(\boldsymbol{r}, t)|^2 \right.\\ &\left. - \hbar\{\boldsymbol{b} - \boldsymbol{b}_{dd}(\boldsymbol{r}, t)\}\cdot\hat{\boldsymbol{f}}\right]\psi(\boldsymbol{r}, t)\\ &+ \sum_{k=1}^{S} c_{2k} \sum_{\nu_1,\nu_2,\ldots,\nu_k = x,y,z} F_{\nu_1,\nu_2,\ldots,\nu_k}(\boldsymbol{r}, t)\\ &\times \hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}\psi(\boldsymbol{r}, t),\end{aligned} \tag{3}$$

where $\psi(\boldsymbol{r},t)$ is a vector quantity whose αth component in the spinor basis is $\psi_\alpha(\boldsymbol{r},t)$ (spin-space indices from the beginning of the greek alphabet such as $\alpha,\beta,\gamma,\dots$ are integers running from $-S$ to S). In this expression, $\hbar\hat{\boldsymbol{f}}$ is the spin-S operator where the spin ladder is defined by $\hat{f}_z|\alpha\rangle=\alpha|\alpha\rangle$ and $\langle\alpha|\beta\rangle=\delta_{\alpha,\beta}$, while $F_{\nu_1,\nu_2,\dots,\nu_k}(\boldsymbol{r},t):=\psi^\dagger(\boldsymbol{r},t)\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}\psi(\boldsymbol{r},t)$ are the components of the expectation value of $\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}$. The Larmor frequency vector reads $\boldsymbol{b}=g_F\mu_B\boldsymbol{B}/\hbar$ (with Landé g factor g_F, Bohr magneton μ_B, and the external magnetic induction $\boldsymbol{B}$), and $\hbar\boldsymbol{b}_{dd}(\boldsymbol{r},t)\cdot\boldsymbol{e}_\nu=c_{dd}\int d^3r'\sum_{\nu'=x,y,z}Q_{\nu,\nu'}(\boldsymbol{r}-\boldsymbol{r}')F_{\nu'}(\boldsymbol{r}',t)$. Here, $c_{dd}=\mu_0(g_F\mu_B)^2/(4\pi)$ and $\boldsymbol{e}_\nu$ is a unit vector along the ν axis [32] (by convention, indices from the middle of the greek alphabet such as $\kappa,\lambda,\mu,\nu,\dots=x,y,z$ denote spatial indices), and $Q_{\nu,\nu'}$ is the spin-space tensor defined in Eq. (A2) of Appendix A. Finally, m is the boson mass, c_0 the density-density interaction coefficient, and c_{2k} the interaction coefficient parametrizing the spin-spin interactions, where k is a positive integer running from 1 to S [26]. For example, c_2 is the spin-spin interaction coefficient of a spin-1 gas ($S=1$).

To develop a simple and intuitive physical approach, we consider a quasi-1D gas for which one can perform analytical calculations. We set the trap potential as

$$V_{\rm tr}(x,y,z)=\tfrac{1}{2}m\omega_\perp^2(x^2+y^2)+V(z), \tag{4}$$

so that the long axis of our gas is directed along the z axis and the gas is strongly confined perpendicularly.

For a harmonic trap along all directions, i.e., when $V(z)=m\omega_z^2z^2/2$, we set $\omega_\perp\gg\omega_z$. For a box trap along z, i.e., when $V(z)=0$ for $|z|\leqslant L_z$ and $V(z)=\infty$ for $|z|>L_z$, our gas will be strongly confined along z as long as the quasi-1D condition is satisfied; we discuss below whether the condition is satisfied, in Sec. VI A.

Single-domain spinor BECs have been already realized, for example, using spin-1 ^{87}Rb [35]. This single-domain approximation is common in nanomagnetism (see, for example, Ref. [24]), by assuming magnetic particles much smaller than the typical width of a domain wall. The local magnetization is related to the expectation value $\hbar\boldsymbol{F}(\boldsymbol{r},t)\equiv\hbar\psi^\dagger(\boldsymbol{r},t)\hat{\boldsymbol{f}}\psi(\boldsymbol{r},t)$ of the spatial spin density operator by $\boldsymbol{d}(\boldsymbol{r},t)=g_F\mu_B\boldsymbol{F}(\boldsymbol{r},t)$. A unidirectional local magnetization $\boldsymbol{d}(z,t)$ is then given by

$$\begin{aligned}d_x(z,t)&=d(z,t)\sin\theta(t)\cos\phi(t),\\ d_y(z,t)&=d(z,t)\sin\theta(t)\sin\phi(t),\\ d_z(z,t)&=d(z,t)\cos\theta(t),\end{aligned} \tag{5}$$

where $d_\nu(z,t)=\boldsymbol{d}(z,t)\cdot\boldsymbol{e}_\nu$ is the νth component of $\boldsymbol{d}(z,t)$, $d(z,t)=|\boldsymbol{d}(z,t)|$, $\theta(t)$ is polar angle of $\boldsymbol{d}(z,t)$, and $\phi(t)$ is azimuthal angle of $\boldsymbol{d}(z,t)$. For an illustration of the geometry considered, see Fig. 1. For a single-component dipolar BEC, $\boldsymbol{F}(\boldsymbol{r},t)$ has a fixed direction. To study the relation of the Stoner-Wohlfarth model, in which $\boldsymbol{F}(\boldsymbol{r},t)$ changes its direction, with a dipolar BEC, a multicomponent dipolar BEC should therefore be employed.

In the quasi-1D approximation, the order parameter $\psi_\alpha(\boldsymbol{r},t)$ is commonly assumed to be of the form

$$\psi_\alpha(\boldsymbol{r},t)=\frac{e^{-\rho^2/(2l_\perp^2)}}{l_\perp\sqrt{\pi}}\Psi_\alpha(z,t), \tag{6}$$

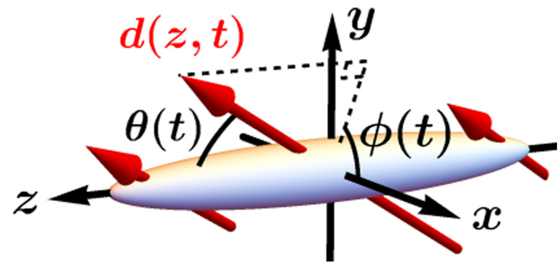

FIG. 1. Schematic of the considered geometry in a quasi-1D gas (shaded ellipsoid). The length of the red magnetization arrows, all pointing in the same direction (homogeneous local-spin-orientation limit), represents $|\boldsymbol{d}(z,t)|$.

where $l_\perp$ is the harmonic oscillator length in the x-y plane and $\rho=\sqrt{x^2+y^2}$. Assuming our gas is in the homogeneous local spin-orientation limit, we may also apply a single mode approximation in space so that $\Psi_\alpha(z,t)=\Psi_{\rm uni}(z,t)\zeta_\alpha(t)$. The time-dependent spinor part is

$$\zeta_\alpha(t)=\langle\alpha|e^{-i\hat{f}_z\phi(t)}e^{-i\hat{f}_y\theta(t)}|S\rangle \tag{7}$$

for spin-S particles [26,32] and the normalization reads $|\zeta(t)|^2:=\zeta^\dagger(t)\zeta(t)=1$. Finally, due to the $(i-\Gamma)$ factor on the left-hand side of Eq. (3), for ease of calculation, we may make the following *Ansatz* for the $\psi_\alpha(\boldsymbol{r},t)$ (cf. Ref. [36]),

$$\psi_\alpha(\boldsymbol{r},t)=\frac{e^{-\rho^2/(2l_\perp^2)}}{l_\perp\sqrt{\pi}}\Psi(z,t)\zeta_\alpha(t)e^{-(i+\Gamma)\omega_\perp t/(1+\Gamma^2)}. \tag{8}$$

From our *Ansätze* in Eqs. (7) and (8), one concludes that the expectation value of the (spatial) spin-density operator is

$$\begin{aligned}\hbar F_x(\boldsymbol{r},t)&=\hbar S\frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}|\Psi(z,t)|^2e^{-2\Gamma\omega_\perp t/(1+\Gamma^2)}\\ &\quad\times\sin\theta(t)\cos\phi(t),\\ \hbar F_y(\boldsymbol{r},t)&=\hbar S\frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}|\Psi(z,t)|^2e^{-2\Gamma\omega_\perp t/(1+\Gamma^2)}\\ &\quad\times\sin\theta(t)\sin\phi(t),\\ \hbar F_z(\boldsymbol{r},t)&=\hbar S\frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}|\Psi(z,t)|^2e^{-2\Gamma\omega_\perp t/(1+\Gamma^2)}\\ &\quad\times\cos\theta(t).\end{aligned} \tag{9}$$

The above equations lead to unidirectional local magnetization, which has been assumed in Eqs. (5), in the quasi-1D limit (after integrating out the strongly confining x and y axes). Note, however, that our *Ansatz* in Eq. (8) is sufficient, but not necessary, for the homogeneous-local-spin-orientation limit, and the homogeneous-local-spin-orientation *Ansatz* is thus designed to render our approach as simple as possible.

Because we are not assuming any specific form of $\Psi(z,t)$ in our *Ansatz* in Eq. (8), we cover every possible time behavior of $|\psi(\boldsymbol{r},t)|^2:=\psi^\dagger(t)\psi(t)$:

$$|\psi(\boldsymbol{r},t)|^2=\frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}|\Psi(z,t)|^2e^{-2\Gamma\omega_\perp t/(1+\Gamma^2)}. \tag{10}$$

Equation (10) explicitly shows that Eq. (8) does not imply an exponentially decaying wave function with time since

$|\Psi(z,t)|^2$ can be any physical function of time t. However, the *Ansatz* (8) simplifies the resulting equation for $\Psi(z,t)$, Eq. (11) below.

By integrating out the x and y directions, the GP equation for a quasi-1D spin-S BEC can be written as (for a detailed derivation see Appendix B)

$$(i-\Gamma)\hbar\frac{\partial\{\Psi(z,t)\zeta_\alpha(t)\}}{\partial t} = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+V(z)+\frac{c_0}{2\pi l_\perp^2}n(z,t)\right\}\Psi(z,t)\zeta_\alpha(t) + \hbar[-\boldsymbol{b}+S\{\boldsymbol{M}(t)-3M_z(t)\boldsymbol{e}_z\}P_{dd}(z,t)]\cdot\left\{\sum_{\beta=-S}^{S}(\hat{\boldsymbol{f}})_{\alpha,\beta}\Psi(z,t)\zeta_\beta(t)\right\} + \sum_{k=1}^{S}\frac{c_{2k}}{2\pi l_\perp^2}n(z,t)\sum_{\nu_1,\nu_2,\dots,\nu_k=x,y,z}SM_{\nu_1,\nu_2,\dots,\nu_k}(t)\left\{\sum_{\beta=-S}^{S}\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}\right)_{\alpha,\beta}\Psi(z,t)\zeta_\beta(t)\right\}, \tag{11}$$

where we defined the two functions

$$M_{\nu_1,\nu_2,\dots,\nu_k}(t) := \frac{1}{S}\sum_{\alpha,\beta=-S}^{S}\zeta_\alpha^\dagger(t)\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}\right)_{\alpha,\beta}\zeta_\beta(t), \tag{12}$$

$$P_{dd}(z,t) := \frac{c_{dd}}{2\hbar l_\perp^3}\int_{-\infty}^{\infty}dz'n(z',t)\left\{G\left(\frac{|z-z'|}{l_\perp}\right)-\frac{4}{3}\delta\left(\frac{z-z'}{l_\perp}\right)\right\}, \tag{13}$$

with the axial density $n(z,t) := \int d^2\rho\,|\psi(\boldsymbol{r},t)|^2 = |\Psi(z,t)|^2e^{-2\Gamma\omega_\perp t/(1+\Gamma^2)}$, where $\int d^2\rho := \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy$. Finally, the function G appearing in P_{dd} is defined as

$$G(\lambda) := \sqrt{\frac{\pi}{2}}(\lambda^2+1)e^{\lambda^2/2}\mathrm{erfc}\left(\frac{\lambda}{\sqrt{2}}\right)-\lambda. \tag{14}$$

We plot $G(\lambda)$ as a function of λ in Fig. 2.

Equation (11) represents our starting point for analyzing the dynamics of magnetization. We now proceed to show how it leads to the LLG equation and the Stoner-Wohlfarth model.

IV. EFFECTIVE LAGRANGIAN DESCRIPTION

To provide a concise phase-space picture of the condensate magnetization dynamics, we discuss in this section a collective coordinate Lagrangian appropriate to our system.

Let $\boldsymbol{M}(t) := \boldsymbol{d}(z,t)/d(z,t)$ where the magnetization $\boldsymbol{d}(z,t)$ is defined in Eq. (5). Explicitly, the local magnetization direction reads $\boldsymbol{M}(t) = (\sin\theta(t)\cos\phi(t), \sin\theta(t)\sin\phi(t), \cos\theta(t))$. Then, from Eqs. (9) and (10), $\boldsymbol{F}(\boldsymbol{r},t) = S\boldsymbol{M}(t)|\psi(\boldsymbol{r},t)|^2$ and one obtains (for a detailed derivation see Appendix B)

$$\frac{\partial\boldsymbol{M}}{\partial t} = \boldsymbol{M}\times\{\boldsymbol{b}+S\Lambda'_{dd}(t)M_z\boldsymbol{e}_z\}-\Gamma\boldsymbol{M}\times\frac{\partial\boldsymbol{M}}{\partial t}, \tag{15}$$

where the renormalized interaction function $\Lambda'_{dd}(t)$ reads

$$\Lambda'_{dd}(t) = \frac{3}{N(t)}\int_{-\infty}^{\infty}dz\,n(z,t)P_{dd}(z,t), \tag{16}$$

and $N(t) := \int d^3r\,|\psi(\boldsymbol{r},t)|^2 = \int_{-\infty}^{\infty}dz\,n(z,t)$. From Eqs. (A9), (A12), and (13), $\Lambda'_{dd}(t)$ is connected to the dipole-dipole interaction contribution $V_{dd}(t)$ by

$$V_{dd}(t) = \frac{3}{2}\hbar S^2\left\{\sin^2\theta(t)-\frac{2}{3}\right\}\int_{-\infty}^{\infty}dz\,n(z,t)P_{dd}(z,t) = \frac{\hbar}{2}S^2N(t)\Lambda'_{dd}(t)\left\{\frac{1}{3}-\cos^2\theta(t)\right\}. \tag{17}$$

We note that in order to obtain the effective quasi-1D dipolar interaction (17), we did not use, in distinction to Ref. [27], any simplifying approximation. A detailed derivation is provided in Appendix A.

Equation (15) is the LLG equation with the external magnetic field in the z direction modified by the magnetization in the z direction due to the dipole-dipole interaction. The corresponding term in units of magnetic field, $\hbar S\Lambda'_{dd}(t)M_z\boldsymbol{e}_z/(g_F\mu_B)$, can be seen as an additional magnetic field that is itself proportional to the magnetization in the z direction, and which leads to an additional nonlinearity in the LLG equation.

From Eqs. (13) and (16), to get how $\Lambda'_{dd}(t)$ depends on time t, one has to calculate the double integral

$$\int dz\int dz'n(z,t)n(z',t)\left\{G\left(\frac{|z-z'|}{l_\perp}\right)-\frac{4}{3}\delta\left(\frac{z-z'}{l_\perp}\right)\right\}. \tag{18}$$

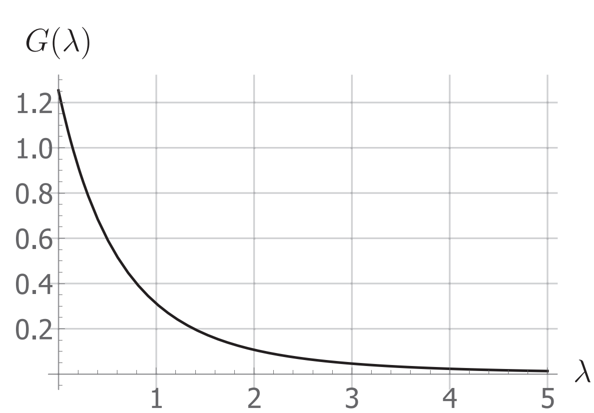

FIG. 2. The function $G(\lambda)$ defined in Eq. (14). Note that $G(\lambda) \simeq 2/\lambda^3 + O(\lambda^{-5})$ for $\lambda \gg 1$, so $G(\lambda)$ is always positive for $\lambda \geqslant 0$.

To achieve a simple physical picture, we assume that $n(z,t)$ does not depend on time t within the time range we are interested in. Then we may write $\Lambda'_{dd}(t) = \Lambda'_{dd}$. The lifetime of a typical dipolar BEC with large atomic magnetic dipole moments such as ^{164}Dy [37], ^{162}Dy and ^{160}Dy [38], or ^{166}Er [31] is of the order of seconds. Since taking into account the time dependence of $n(z,t)$ generally requires a numerical solution of Eq. (11), we here consider the case where $n(z,t)$ is constant in time t as in Ref. [26], to predominantly extract the effect of magnetic dipole-dipole interaction *per se*.

We also neglect the possible effect of magnetostriction. The latter effect, amounting to a distortion of the aspect ratio of the condensate in a harmonic trap as a function of the angle of the external magnetic field with the symmetry axis of the trap, was measured in a condensate of chromium atoms [39] (with a magnetic moment of $6\mu_B$, comparably large to those of Er, $7\mu_B$, and Dy, $10\mu_B$). The magnetostriction effect in that experiment was of the order of 10%. For alkali-metal atoms with spin 1 the effect should be a factor of 6^2 smaller. In addition, theoretical analyses in the Thomas-Fermi limit show that magnetostriction in harmonic traps becomes particularly small for very small or very large asymmetries of the trap [40,41].

More specifically, Ref. [42] has shown that magnetostriction is due to the force induced by the dipole-dipole mean-field potential $\Phi_{dd}(\boldsymbol{r},t)$. In Appendix D, we apply the approach of Ref. [42] to a dipolar spinor BEC. From Eqs. (16), (17), (A1), and (D5), $\Lambda'_{dd}(t)$ contains $\Phi_{dd}(z,t)$ [the quasi-1D form of $\Phi_{dd}(\boldsymbol{r},t)$ defined in Eq. (D5)] by

$$S^2\{1-3M_z^2(t)\}N(t)\hbar\Lambda'_{dd}(t) = 3\int_{-\infty}^{\infty} dz\, n(z,t)\Phi_{dd}(z,t). \tag{19}$$

Hence, our LLG-type equation in Eq. (15) effectively contains the dipole-dipole mean-field potential which causes magnetostriction and the form of Eq. (15) itself will not be changed whether the effect of magnetostriction is large or not. Only the value of $\Lambda'_{dd}(t)$ will be changed because magnetostriction changes the integration domain. Furthermore, we show in Appendix D that, for our quasi-1D system, the effect of magnetostriction is smaller in a box trap than in a harmonic trap. In fact, for the box trap, this effect can be neglected if $L_z/l_\perp$ is sufficiently large. Thus, we may neglect the effect of magnetostriction under suitable limits for both box and harmonic traps.

To get a simple physical idea of the dynamical behavior of our system, let us, for now, assume that there is no damping, $\Gamma = 0$. When the external magnetic field is chosen to lie in the x-z plane, $\boldsymbol{B} = (B_x, 0, B_z)$, Eq. (15) becomes

$$\frac{d\theta}{dt} = b_x \sin\phi,$$
$$\frac{d\phi}{dt} = b_x \cot\theta\cos\phi - b_z - S\Lambda'_{dd}\cos\theta, \tag{20}$$

where we already defined the Larmor frequency vector $\boldsymbol{b} = g_F\mu_B\boldsymbol{B}/\hbar$ below Eq. (3).

By using the Lagrangian formalism introduced in Ref. [43], the Lagrangian L of this system then fulfills

$$\frac{L}{\hbar} = \dot\phi\cos\theta + b_x\sin\theta\cos\phi + b_z\cos\theta + \frac{S}{4}\Lambda'_{dd}\cos(2\theta), \tag{21}$$

where $\dot\phi = d\phi/dt$. The equations of motion are

$$\frac{1}{\hbar}\frac{\partial L}{\partial\theta} = -\dot\phi\sin\theta + b_x\cos\theta\cos\phi - b_z\sin\theta - \frac{S}{2}\Lambda'_{dd}\sin(2\theta),$$
$$\frac{\partial L}{\partial\dot\theta} = 0, \quad \frac{1}{\hbar}\frac{\partial L}{\partial\phi} = -b_x\sin\theta\sin\phi, \quad \frac{1}{\hbar}\frac{\partial L}{\partial\dot\phi} = \cos\theta. \tag{22}$$

One easily verifies that Eq. (21) is indeed the Lagrangian which gives Eqs. (20). Let p_ξ be the conjugate momentum of the coordinate ξ. Since $p_\theta = 0$ and $p_\phi = \hbar\cos\theta$ ($\hbar$ times the z component of $\boldsymbol{M}$), the Hamiltonian H is given by

$$H = -b_x\sqrt{\hbar^2 - p_\phi^2}\cos\phi - b_z p_\phi + \frac{\hbar^2 - 2p_\phi^2}{4\hbar}S\Lambda'_{dd}. \tag{23}$$

Note that the energy $\tilde{E} := H - \hbar S\Lambda'_{dd}/4$ is conserved. Hence, if we put $p_\phi = (p_\phi)_{\rm in}$ and $\phi = \pi/2$ at some time $t = t_0$, $\tilde{E} = -b_z(p_\phi)_{\rm in} - S\Lambda'_{dd}(p_\phi)_{\rm in}^2/2\hbar$. We can then express ϕ as a function of p_ϕ as

$$\cos\phi = -\frac{\tilde{E} + b_z p_\phi + \frac{1}{2\hbar}S\Lambda'_{dd}p_\phi^2}{b_x\sqrt{\hbar^2 - p_\phi^2}}$$
$$= \{(p_\phi)_{\rm in} - p_\phi\}\frac{b_z + S\Lambda'_{dd}\frac{(p_\phi)_{\rm in}+p_\phi}{2\hbar}}{b_x\sqrt{\hbar^2 - p_\phi^2}}. \tag{24}$$

The canonical momentum p_ϕ remains the initial $(p_\phi)_{\rm in}$ when $b_x = 0$, implying that θ does not change when $b_x = 0$, consistent with Eqs. (20). If $|b_x|$ is larger than $|b_z \pm S\Lambda'_{dd}|$, we can have $p_\phi \neq (p_\phi)_{\rm in}$ with $|\cos\phi| \leqslant 1$, which allows for the switching process of the magnetization. Below a threshold value of $|b_x|$ that depends on b_z and $S\Lambda'_{dd}$, p_ϕ has to remain constant for Eq. (24) to be satisfied, which corresponds to simple magnetization precession about the z axis.

When p_ϕ is a function of time, there are two important cases:

$$\text{(a)}\quad |b_z| \gg S\Lambda'_{dd}:\ \cos\phi = \frac{b_z}{b_x}\frac{(p_\phi)_{\rm in} - p_\phi}{\sqrt{\hbar^2 - p_\phi^2}},$$
$$\text{(b)}\quad |b_z| \ll S\Lambda'_{dd}:\ \cos\phi = \frac{S\Lambda'_{dd}}{2b_x}\frac{(p_\phi)_{\rm in}^2 - p_\phi^2}{\hbar\sqrt{\hbar^2 - p_\phi^2}}. \tag{25}$$

We plot the corresponding phase diagrams (θ vs ϕ) in Fig. 3.

Let $(p_\phi)_{\rm in} = \hbar\cos\theta_{\rm in}$, $b_x = b\sin\theta_0$, and $b_z = b\cos\theta_0$. When case (a) holds $|b_z| \gg S\Lambda'_{dd}$, one concludes that $\cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos\phi = \cos\theta_0\cos\theta_{\rm in}$, which is constant. Since $\boldsymbol{d}\cdot\boldsymbol{b} = db(\cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos\phi)$, in case (a) the magnetization $\boldsymbol{d}$ precesses around the external magnetic field $\boldsymbol{B}$, as expected. When case (b) holds, SW switching can occur, to the description of which we proceed in the following.

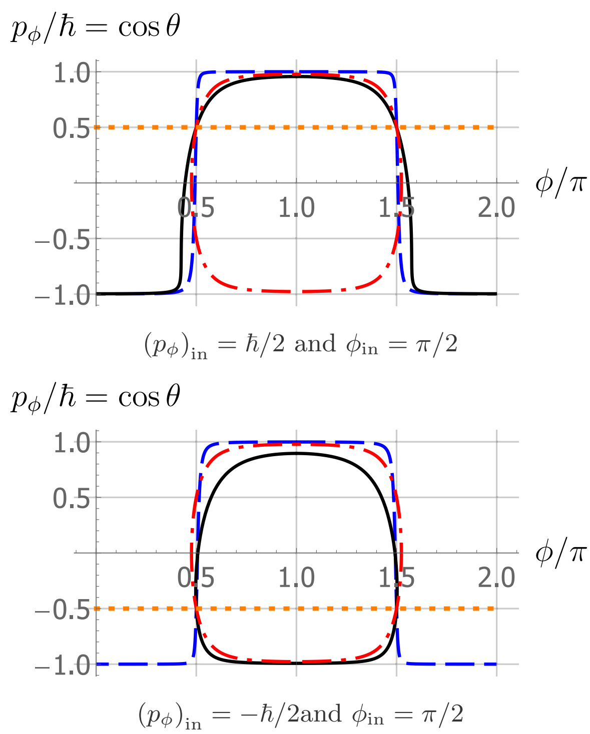

FIG. 3. $p_\phi/\hbar$ vs ϕ/π when $\Gamma = 0$ (no dissipation), with initial values $(p_\phi)_{\rm in} = \hbar/2$ and $\phi_{\rm in} = \pi/2$ (top) and $(p_\phi)_{\rm in} = -\hbar/2$ and $\phi_{\rm in} = \pi/2$ (bottom). Dashed blue line, $b_z/b_x = 0.2$ and $|b_z| \gg S\Lambda'_{dd}$; black line, $b_z/b_x = 0.2$ and $S\Lambda'_{dd}/b_x = 0.6$; dash-dotted red line, $S\Lambda'_{dd}/b_x = 0.6$ and $|b_z| \ll S\Lambda'_{dd}$; dotted orange horizontal line, $b_x = 0$.

V. CONNECTION TO STONER-WOHLFARTH MODEL

The phenomenological SW model can be directly read off from the equations in the preceding section. From Eq. (23), $\tilde{H} := H + \hbar S\Lambda'_{dd}/4$ is given by

$$\frac{\tilde{H}}{\hbar} = -b_x \sin\theta\cos\phi - b_z\cos\theta + \frac{S\Lambda'_{dd}}{2}\sin^2\theta. \quad (26)$$

Let $(b_\nu)_{\rm cr}$ be the value of b_ν at the stability limit where $\partial\tilde{H}/\partial\theta = 0$ and $\partial^2\tilde{H}/\partial\theta^2 = 0$. Then one obtains the critical magnetic fields

$$(b_x)_{\rm cr}\cos\phi = S\Lambda'_{dd}\sin^3\theta, \quad (b_z)_{\rm cr} = -S\Lambda'_{dd}\cos^3\theta, \quad (27)$$

which satisfy the equation

$$\{(b_x)_{\rm cr}\cos\phi\}^{2/3} + (b_z)_{\rm cr}^{2/3} = \{S\Lambda'_{dd}\}^{2/3}. \quad (28)$$

We coin the curve in the (b_x, b_z) plane described by Eq. (28) the switching curve, in accordance with the terminology established in Ref. [44]. Because ϕ changes in time [see Eqs. (20) and Fig. 3], the switching curve depends in general on the timing of the applied external magnetic fields. We note that, for $\phi = 0$, Eqs. (26) and (28) are identical to the SW energy functional

$$\frac{H_{\rm SW}}{\hbar} = -b_x\sin\theta - b_z\cos\theta + K\sin^2\theta \quad (29)$$

and the SW astroid [44], respectively, if we identify $K = S\Lambda'_{dd}/2$.

The LLG equation in Eq. (15) has stationary solutions with $\boldsymbol{M}$ parallel to the effective magnetic field $\hbar\{\boldsymbol{b} + S\Lambda'_{dd}(t)M_z\boldsymbol{e}_z\}/(g_F\mu_B)$. Since we set $\boldsymbol{b}$ to lie in the xz plane, ϕ will go to zero for sufficiently large times. Thus Eq. (26) leads to the SW model (29) due to the damping term in Eq, (15) if $\Gamma > 0$. In Appendix C, we demonstrate that a more general tensorial damping coefficient Γ introduces additional terms on the right-hand side of the LLG equation (15), which involve *time derivatives*. While these will thus not affect the SW phenomenology, which results from the steady states as a function of the applied magnetic fields, and which is thus governed by the vanishing (in the stationary limit) of the first term on the right-hand side of the LLG equation, they affect the detailed relaxation dynamics of the magnetization and its time scales. These deviations can hence can be used to probe deviations from assuming a single scalar Γ.

Before we move on to the next section, we show the characteristic behavior of Λ'_{dd} defined in Eq. (16), for a box-trap scenario defined by $n(z,t) = N/(2L_z)$ for $-L_z \leqslant z \leqslant L_z$ and $n(z,t) = 0$ otherwise (N is the number of particles).

We stress that due to the finite size of the trap along the "long" z direction, in variance with the Hohenberg-Mermin-Wagner theorem holding for infinitely extended systems in the thermodynamic limit, a quasi-1D BEC can exist also at finite temperatures [45]. This remains true up to a ratio of its proper length to the de Broglie wavelength [46], beyond which strong phase fluctuations set in [47]. In fact, such strongly elongated quasi-1D BECs at finite temperature were first realized already long ago (cf., e.g., Ref. [48]).

For the box trap, $\Lambda'_{dd} = \Lambda_{dd}(L_z/l_\perp)$, where

$$\Lambda_{dd}(\lambda) = \frac{3Nc_{dd}}{2\hbar l_\perp^3}\frac{1}{\lambda}\left\{\int_0^{2\lambda} dv\,\left(1 - \frac{v}{2\lambda}\right)G(v) - \frac{2}{3}\right\}. \quad (30)$$

From Eq. (14), $G(v) \simeq 2/v^3 + O(v^{-5})$ for $v \gg 1$, so that

$$\Lambda_{dd}(\lambda) \simeq \frac{Nc_{dd}}{2\hbar l_\perp^3}\frac{1}{\lambda} \qquad \text{for } \lambda = L_z/l_\perp \gg 1. \quad (31)$$

Hence $\Lambda_{dd}(\lambda)$ is a slowly decreasing function of the cigar's aspect ratio λ (keeping everything else fixed). We see below that for the parameters of experiments such as in Ref. [31], the effective magnetic field due to dipolar interactions greatly exceeds the externally applied magnetic fields (in the range relevant for SW switching to be observed) [49].

VI. ANALYTICAL RESULTS FOR AXIALLY DIRECTED EXTERNAL MAGNETIC FIELD

Without dissipation, when $b_x = 0$, $p_\phi = \hbar\cos\theta = \hbar M_z$ is rendered constant [see Eq. (20)]. However, in the presence of dissipation, M_z changes in time even if $b_x = 0$. By employing this change, we propose an experimental method to measure Γ.

For simplicity, we assume that the number density is constant in time (also see Sec. IV) and the external magnetic field points along the z direction, $\boldsymbol{B} = B_z\boldsymbol{e}_z$. Let a critical (see for a detailed discussion below) value of the magnetization be

$$(M_z)_{\rm cr} := -\frac{b_z}{S\Lambda'_{dd}}. \quad (32)$$

Then Eq. (15) can be written as

$$\frac{\partial \boldsymbol{M}}{\partial t} = S\Lambda'_{dd}\boldsymbol{M}\times\boldsymbol{e}_z\{M_z-(M_z)_{\rm cr}\} - \Gamma\boldsymbol{M}\times\frac{\partial \boldsymbol{M}}{\partial t}$$
$$= \boldsymbol{M}\times\boldsymbol{e}_z(b_z+S\Lambda'_{dd}M_z) - \Gamma\boldsymbol{M}\times\frac{\partial \boldsymbol{M}}{\partial t}. \tag{33}$$

Since $\boldsymbol{M}\cdot\frac{\partial \boldsymbol{M}}{\partial t}=0$, by taking the cross product with $\boldsymbol{M}$ on both sides of Eq. (15), one can derive an expression for $\boldsymbol{M}\times\frac{\partial \boldsymbol{M}}{\partial t}$:

$$\frac{\partial M_z}{\partial t} = -\frac{\Gamma S\Lambda'_{dd}}{1+\Gamma^2}\{M_z-(M_z)_{\rm cr}\}\big(M_z^2-1\big)$$
$$= -\frac{\Gamma}{1+\Gamma^2}(b_z+S\Lambda'_{dd}M_z)\big(M_z^2-1\big). \tag{34}$$

Since $\boldsymbol{M}$ is the scaled magnetization, $|\boldsymbol{M}|=1$ with a condensate. Hence, $-1\leqslant M_z\leqslant 1$. Also, according to the discussion below Eq. (26), the generally positive SW coefficient (with units of frequency) K is $S\Lambda'_{dd}/2$.

From Eq. (34), for time-independent Λ'_{dd}, one concludes that there are three time-independent solutions: $M_z=(M_z)_{\rm cr}$ and $M_z=\pm1$. For a box-trapped BEC and constant number density, $\Lambda'_{dd}=\Lambda_{dd}$ which is always positive in the quasi-1D limit [cf. Eq. (30) and the discussion following it]. For some arbitrary physical quasi-1D trap potential, in which the number density is not constant in space, from Eqs. (13) and (16) and Fig. 2 one can infer that $\Lambda'_{dd}>0$, due to the fact that the quasi-1D number density $n(z,t)>0$, $n(z,t)$ has its maximum value near $z=0$ for a symmetric trap centered there, and then $G(\lambda)$ also has its maximum value near $\lambda=0$.

Then, if $|(M_z)_{\rm cr}|<1$, $M_z=(M_z)_{\rm cr}$ is an unstable solution and $M_z=\pm1$ are stable solutions. When $|(M_z)_{\rm cr}|<1$ and $-1<M_z<(M_z)_{\rm cr}$, M_z goes to -1. Likewise, M_z goes to 1 when $(M_z)_{\rm cr}<M_z<1$. This bifurcation does not occur if $|(M_z)_{\rm cr}|>1$. For simplicity, we assume that $|(M_z)_{\rm cr}|<1$. This is the more interesting case due to the possibility of a bifurcation of stable solutions leading to SW switching.

Let $(M_z)_{\rm in}$ be the value of M_z at $t=0$. The analytic solution of Eq. (34) satisfies

$$t = \frac{1+\Gamma^2}{\Gamma S\Lambda'_{dd}}\Bigg[\frac{1}{\{(M_z)_{\rm cr}\}^2-1}\ln\left\{\frac{(M_z)_{\rm in}-(M_z)_{\rm cr}}{M_z-(M_z)_{\rm cr}}\right\} - \frac{1}{2\{1-(M_z)_{\rm cr}\}}\ln\left\{\frac{1-M_z}{1-(M_z)_{\rm in}}\right\} + \frac{1}{2\{1+(M_z)_{\rm cr}\}}\ln\left\{\frac{1+(M_z)_{\rm in}}{1+M_z}\right\}\Bigg]$$
$$= \frac{1+\Gamma^2}{\Gamma}\Bigg[\frac{S\Lambda'_{dd}}{b_z^2-(S\Lambda'_{dd})^2}\ln\left\{\frac{b_z+S\Lambda'_{dd}(M_z)_{\rm in}}{b_z+S\Lambda'_{dd}M_z}\right\} - \frac{1}{2(b_z+S\Lambda'_{dd})}\ln\left\{\frac{1-M_z}{1-(M_z)_{\rm in}}\right\} - \frac{1}{2(b_z-S\Lambda'_{dd})}\ln\left\{\frac{1+(M_z)_{\rm in}}{1+M_z}\right\}\Bigg]. \tag{35}$$

The above equation tells us that, if $(M_z)_{\rm in}\neq(M_z)_{\rm cr}$ and $(M_z)_{\rm in}\neq\pm1$, M_z goes to its stable time-independent solution ($|M_z|=1$) at time $t=\infty$. Thus, we define a *critical switching time*$t_{\rm cr}$ to be the time when $|M_z|=0.99$. Also, note that the form of the LLG equation [Eq. (33)] does not change whether the BEC is confined in a quasi-1D, quasi-2D, or a three-dimensional (3D) geometry. This is because one can find a connection between Λ'_{dd} and the effective dipole-dipole interaction potential $V_{\rm eff}$, so one can measure Γ even if the BEC is effectively confined in a space with dimension higher than one, using Eq. (35).

We point out, in particular, that $t_{\rm cr}$ is inversely proportional to Λ'_{dd}. Hence, for a constant-density quasi-1D BEC confined in $-L_z\leqslant z\leqslant L_z$, $\Lambda'_{dd}=\Lambda_{dd}(L_z/l_\perp)$, and thus $t_{\rm cr}$ is also inversely proportional to the linear number density along z. This follows from the relation between $\Lambda_{dd}(L_z/l_\perp)$ and the linear number density along z displayed in Eq. (30).

For large dipolar interaction, the asymptotic expression for $t_{\rm cr}$ is, assuming $\Gamma\ll1$,

$$t_{\rm cr}\simeq\frac{1}{\Gamma S\Lambda'_{dd}}\ln\left[\frac{5\sqrt{2\big(1-(M_z)_{\rm in}^2\big)}}{|(M_z)_{\rm in}-(M_z)_{\rm cr}|}\right]$$
$$\text{provided}\quad S\Lambda'_{dd}\gg|b_z|\quad\Longleftrightarrow\quad|(M_z)_{\rm cr}|\ll1. \tag{36}$$

The above $t_{\rm cr}$ diverges at $(M_z)_{\rm in}=(M_z)_{\rm cr}$ or ±1, as expected, since $M_z=(M_z)_{\rm cr}$ and $M_z=\pm1$ are time-independent solutions of the LLG equation. We stress that Eq. (36) clearly shows that the magnetic dipole-dipole interaction *accelerates* the decay of M_z. Hence, by using a dipolar spinor BEC with large magnetic dipole moment such as produced from ^{164}Dy or ^{166}Er one may observe the relaxation of M_z to the stable state within the BEC lifetime, enabling the measurement of Γ.

Before we show how the critical switching time $t_{\rm cr}$ depends on $(M_z)_{\rm in}$ and Γ, we qualitatively discuss when our quasi-1D assumption and homogeneous-local-spin-orientation assumption are valid. Typically, spin-spin-interaction couplings are much smaller than their density-density-interaction counterparts, by two orders of magnitude. For a spin-1 ^{23}Na BEC or a spin-1 ^{87}Rb BEC, $c_0\simeq100|c_2|$ [32,35]. Thus we may neglect to a first approximation the S^2 times c_{2k} terms in Eq. (11) (see the discussion at the end of Appendix D). We also require $|(M_z)_{\rm cr}|<1$. Thus, we may additionally neglect the $\boldsymbol{b}$ term compared to the $P_{dd}(z,t)$ term since, for $\boldsymbol{b}=b_z\boldsymbol{e}_z$, $S\Lambda'_{dd}>|\boldsymbol{b}|$ should be satisfied to make $|(M_z)_{\rm cr}|<1$ [see Eq. (32)] and Λ'_{dd} is related to $P_{dd}(z,t)$ by Eq. (16). When $\Gamma=0$, using our *Ansatz* in Eq. (8) and integrating out the x and y directions, Eq. (D4) can be approximated by the expression

$$\mu(t)\Psi(z,t)=\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+V(z)+\frac{c_0}{2\pi l_\perp^2}|\Psi(z,t)|^2+\Phi_{dd}(z,t)\right\}\Psi(z,t), \tag{37}$$

where, from Eqs. (D5), (A1), and (17), the dipole-dipole interaction mean-field potential reads

$$\begin{aligned}\Phi_{dd}(z,t) &= \hbar S^2\{1-3M_z^2(t)\}P_{dd}(z,t)\\ &= \frac{c_{dd}}{2l_\perp^3}S^2\{1-3M_z^2(t)\}\int_{-\infty}^{\infty}dz'\,|\Psi(z',t)|^2\left\{G\left(\frac{|z'-z|}{l_\perp}\right)-\frac{4}{3}\delta\left(\frac{z'-z}{l_\perp}\right)\right\}\\ &= \frac{c_{dd}}{2\pi l_\perp^2}\pi S^2\{1-3M_z^2(t)\}\left\{\int_{-\infty}^{\infty}d\bar{z}\,|\Psi(z+\bar{z}l_\perp,t)|^2G(|\bar{z}|)-\frac{4}{3}|\Psi(z,t)|^2\right\}.\end{aligned} \tag{38}$$

From Fig. 2, the function $G(\lambda)$ is positive and decreases exponentially as λ increases. Thus, if $l_\perp$ is small enough such that $|\Psi(z+\bar{z}l_\perp,t)|^2$ does not change within the range $|\bar{z}|\leqslant 5$, one may conclude that

$$\Phi_{dd}(z,t)\simeq \frac{2\pi}{3}S^2\{1-3M_z^2(t)\}\frac{c_{dd}}{2\pi l_\perp^2}|\Psi(z,t)|^2, \tag{39}$$

due to the property $\int_0^\infty d\lambda\, G(\lambda)=1$.

A spinor ($S=6$) dipolar BEC has been realized using ^{166}Er [31]. For this BEC, $c_0=4\pi\hbar^2 a/m$, where $a\simeq 67a_B$ (a_B is the Bohr radius) and $2\pi S^2 c_{dd}/3=0.4911c_0$. Due to $|M_z(t)|\leqslant 1$ from the definition of $\boldsymbol{M}(t)$, the maximum value of the chemical potential $\mu(t)$ is achieved when $M_z(t)=0$, where

$$\mu(t)\simeq V(z)+\left(c_0+\frac{2\pi}{3}S^2c_{dd}\right)\frac{n(z,t)}{2\pi l_\perp^2}. \tag{40}$$

From above Eq. (40), we may regard the 3D number density as $n(z,t)/(2\pi l_\perp^2)$. In Ref. [31], $N=1.2\times10^5$, $\omega_\perp/(2\pi)=\sqrt{156\times198}=175.75$ Hz, $\omega_z/(2\pi)=17.2$ Hz, $l_\perp=0.589\,\mu$m, and the measured peak number density $\bar{n}_{\rm peak}$ is 6.2×10^{20} m^{-3}. Using Eqs. (37) and (39), by denoting L_z as the Thomas-Fermi radius along z, $(-L_z\leqslant z\leqslant L_z)$ with $V(z)=m\omega_z^2z^2/2$, one derives

$$L_z=\left\{\frac{3(c_0+2\pi S^2c_{dd}/3)N}{4\pi m\omega_z^2 l_\perp^2}\right\}^{1/3}, \tag{41}$$

and the mean number density $\bar{n}=(N/2L_z)/(2\pi l_\perp^2)=6.721\times10^{20}$ m^{-3} as well as chemical potential $\mu/(\hbar\omega_\perp)=m\omega_z^2L_z^2/(2\hbar\omega_\perp)=23.22$. Note that $\bar{n}\simeq 1.1\bar{n}_{\rm peak}$. Because μ is not less than $\hbar\omega_\perp$, the experiment [31] is not conducted within the quasi-1D limit.

The homogeneous-local-spin-orientation approximation is valid when the system size is on the order of the spin healing length ξ_s or less, which has been experimentally verified in Ref. [35]. Using $c_0\simeq 100|c_2|$, $\xi_s\simeq 10\,\xi_d$, where $\xi_d=\sqrt{\hbar^2/(2mc_0\bar{n})}$ is the density healing length and $\xi_s=\sqrt{\hbar^2/(2m|c_2|\bar{n})}$ is the spin healing length. Thus, if L_z is on the order of $10\xi_d$, the homogeneous-local-spin-orientation approximation is justified.

Using the $S=6$ element ^{166}Er, we can provide numerical values which satisfy both the quasi-1D and homogeneous-local-spin-orientation limits, as well as they enable us to explicitly show how $t_{\rm cr}$ depends on $(M_z)_{\rm in}$ in a concretely realizable setup. We consider below two cases: A box trap along z [50] and a harmonic trap along z.

A. Box traps

We set $V(z)=0$ for $|z|<L_z$ and ∞ otherwise. Then $n(z,t)=N/(2L_z)$ and we estimate $\mu\simeq(c_0+2\pi S^2c_{dd}/3)N/(4\pi l_\perp^2L_z)$ from Eq. (40). In this case, $\Lambda'_{dd}=\Lambda_{dd}(L_z/l_\perp)$ as is calculated in Eq. (30). Fixing $B_z=-0.03$ mG and $N=100$, we consider the following two cases:

(1) $\omega_\perp/(2\pi)=2.4\times10^4$ Hz and $L_z=3.125\,\mu$m. Then $L_z/l_\perp=62.03$, $\mu/(\hbar\omega_\perp)=0.1692$, and $L_z/\xi_d=29.55$. Thus, the system is in both the quasi-1D and homogeneous-local-spin-orientation limits. $S\Lambda_{dd}(L_z/l_\perp)=4.074\times10^3$ Hz, $\hbar S\Lambda_{dd}(L_z/l_\perp)/(g_F\mu_B)=0.3969$ mG, and $\theta_{\rm cr}:=\cos^{-1}(M_z)_{\rm cr}$ is 85.67°.

(2) $\omega_\perp/(2\pi)=1.2\times10^4$ Hz and $L_z=6.250\,\mu$m. Then $L_z/l_\perp=87.72$, $\mu/(\hbar\omega_\perp)=0.0846$, and $L_z/\xi_d=29.55$. Thus, again the system is in both the quasi-1D and homogeneous-local-spin-orientation limits. $S\Lambda_{dd}(L_z/l_\perp)=1.028\times10^3$ Hz, $\hbar S\Lambda_{dd}(L_z/l_\perp)/(g_F\mu_B)=0.1002$ mG, and $\theta_{\rm cr}=72.57°$.

Figure 4 shows the relation between $t_{\rm cr}$ and $(M_z)_{\rm in}$.

B. Harmonic traps

We set $V(z)=m\omega_z^2z^2/2$. Using the Thomas-Fermi approximation, from Eq. (40), $\mu=m\omega_z^2L_z^2/2$, where L_z is given by Eq. (41). $(c_0+2\pi S^2c_{dd}/3)n(z,t)/(\pi l_\perp^2)=m\omega_z^2(L_z^2-z^2)$ for $|z|\leqslant L_z$ and $n(z,t)=0$ for $|z|>L_z$. From this $n(z,t)$, we performed a numerical integration to calculate Λ'_{dd} in Eq. (16). Fixing $B_z=-0.03$ mG, we consider the following two cases:

(1) $N=240$, $\omega_\perp/(2\pi)=2000$ Hz, and $\omega_z/(2\pi)=50$ Hz, for which $L_z=5.703\,\mu$m and $L_z/l_\perp=32.68$. We obtain again the quasi-1D and homogeneous-local-spin-orientation limits since $\mu/(\hbar\omega_\perp)=0.3337$ and $L_z/\xi_d=17.85$. Furthermore, $S\Lambda'_{dd}=1.644\times10^3$ Hz, $\hbar S\Lambda'_{dd}/(g_F\mu_B)=1.602\times10^{-1}$ mG, and $\theta_{\rm cr}=79.21°$.

(2) $N=340$, $\omega_\perp/(2\pi)=1000$ Hz, and $\omega_z/(2\pi)=25$ Hz, where $L_z=8.070\,\mu$m and $L_z/l_\perp=32.70$. Again, we have the quasi-1D and homogeneous-local-spin-orientation limits fulfilled due to $\mu/(\hbar\omega_\perp)=0.3341$ and $L_z/\xi_d=17.87$. In addition, $S\Lambda'_{dd}=8.230\times10^2$ Hz, $\hbar S\Lambda'_{dd}/(g_F\mu_B)=8.019\times10^{-2}$ mG, and $\theta_{\rm cr}=68.03°$.

Figure 5 shows for the harmonic traps the relation between $t_{\rm cr}$ and $(M_z)_{\rm in}$.

C. Measurability of critical switching time

Figures 4 and 5 demonstrate that the critical switching time $t_{\rm cr}$ is much smaller than the lifetime of a BEC (several seconds

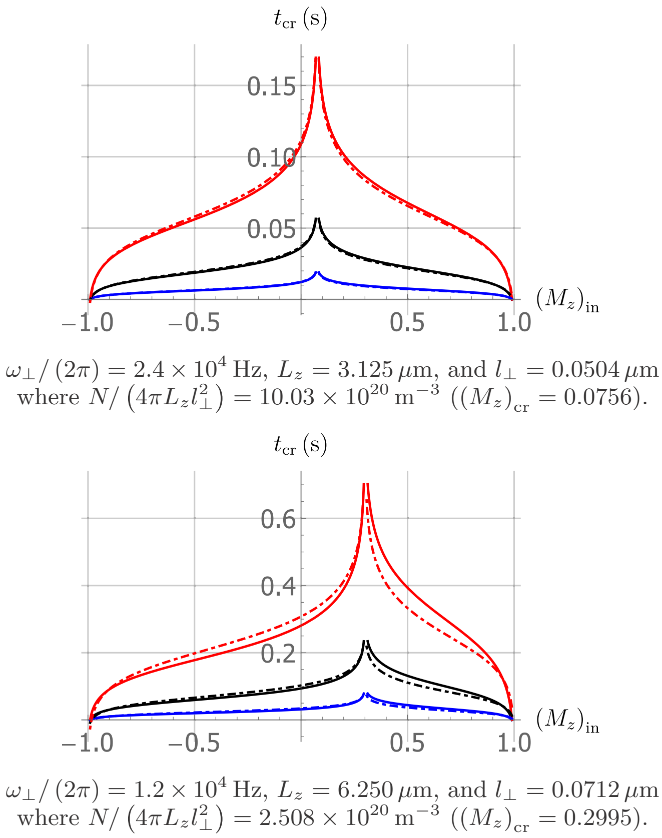

FIG. 4. t_{cr} as a function of $(M_z)_{in}$ when $\boldsymbol{B} = B_z \boldsymbol{e}_z$, where $B_z = -0.03$ mG and particle number $N = 100$. From top to bottom: Red, $\Gamma = 0.01$; black, $\Gamma = 0.03$; and blue, $\Gamma = 0.09$. Solid lines are from the *exact analytic* formula in Eq. (35), and dot-dashed lines are from the asymptotic expression in Eq. (36). Generally, t_{cr} decreases as Γ increases. Also, note that t_{cr} diverges as $(M_z)_{in} \to (M_z)_{cr}$. For larger mean number density $N/(4\pi L_z l_\perp^2)$ (top), the asymptotic expression of t_{cr} is essentially indistinguishable from the exact analytic formula of t_{cr}. Top: $\omega_\perp/(2\pi) = 2.4 \times 10^4$ Hz, $L_z = 3.125\,\mu$m, and $l_\perp = 0.0504\,\mu$m, where $N/(4\pi L_z l_\perp^2) = 10.03 \times 10^{20}$ m^{-3} [$(M_z)_{cr} = 0.0756$]. Bottom: $\omega_\perp/(2\pi) = 1.2 \times 10^4$ Hz, $L_z = 6.250\,\mu$m, and $l_\perp = 0.0712\,\mu$m, where $N/(4\pi L_z l_\perp^2) = 2.508 \times 10^{20}$ m^{-3} [$(M_z)_{cr} = 0.2995$].

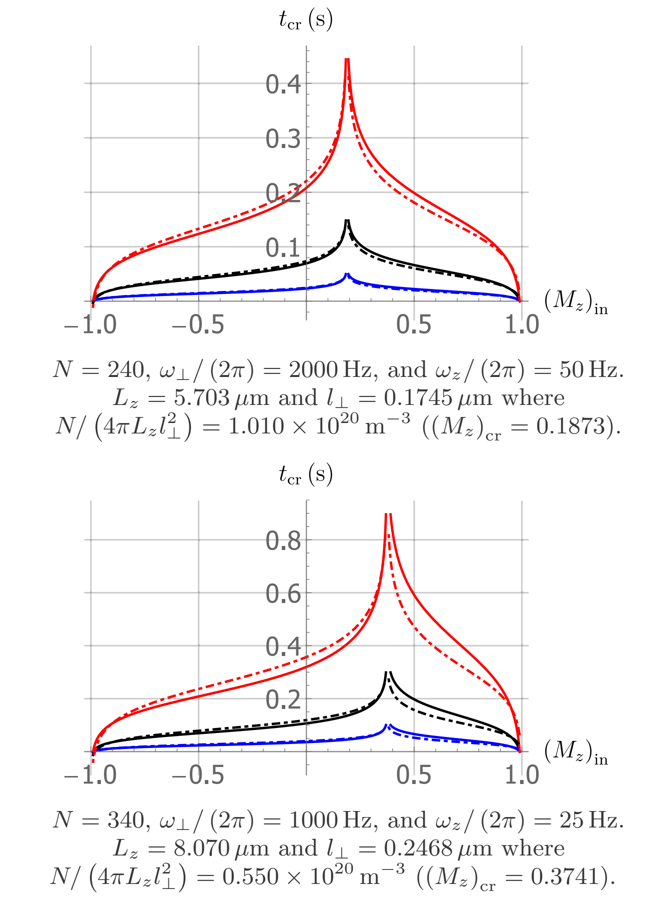

FIG. 5. t_{cr} as a function of $(M_z)_{in}$ when $\boldsymbol{B} = B_z \boldsymbol{e}_z$, where $B_z = -0.03$ mG, for two particle numbers N as shown. From top to bottom: Red, $\Gamma = 0.01$; black, $\Gamma = 0.03$; and blue, $\Gamma = 0.09$. Solid lines are from *exact analytic* formula in Eq. (35), and dot-dashed lines are from the asymptotic expression in Eq. (36). Generally, t_{cr} decreases as Γ increases. Also, note that t_{cr} diverges as $(M_z)_{in} \to (M_z)_{cr}$. For larger mean number density $N/(4\pi L_z l_\perp^2)$ (top), the asymptotic expression of t_{cr} is essentially indistinguishable from the exact analytic formula of t_{cr}. Top: $N = 240$, $\omega_\perp/(2\pi) = 2000$ Hz, and $\omega_z/(2\pi) = 50$ Hz. $L_z = 5.703\,\mu$m and $l_\perp = 0.1745\,\mu$m, where $N/(4\pi L_z l_\perp^2) = 1.010 \times 10^{20}$ m^{-3} [$(M_z)_{cr} = 0.1873$]. Bottom: $N = 340$, $\omega_\perp/(2\pi) = 1000$ Hz, and $\omega_z/(2\pi) = 25$ Hz. $L_z = 8.070\,\mu$m and $l_\perp = 0.2468\,\mu$m, where $N/(4\pi L_z l_\perp^2) = 0.550 \times 10^{20}$ m^{-3} [$(M_z)_{cr} = 0.3741$].

[31]) and thus, by measuring t_{cr} by varying $(M_z)_{in}$, one will be able to obtain the value of Γ, provided Γ indeed does not depend on spin indices as for example Refs. [26,30] have assumed. Conversely, if one obtains from the measurements a different functional relation which does not follow Eq. (35), this implies that Γ may depend on spin indices.

Note that both Figs. 4 and 5 show that t_{cr} is *inversely* proportional to the mean number density $N/(4\pi L_z l_\perp^2)$. Equation (36) states that t_{cr} is inversely proportional to Λ'_{dd}, but except for the box trap case, in which one can *analytically* calculate $\Lambda'_{dd} = \Lambda_{dd}(L_z/l_\perp)$ in Eq. (30), the dependence of Λ'_{dd} and the mean number density $N/(4\pi L_z l_\perp^2)$ is not immediately apparent. Thus, at least for harmonic traps, and in the Thomas-Fermi approximation, one may use the box trap results of Eq. (30) to provide an estimate of the behavior of t_{cr}.

VII. CONCLUSION

For a quasi-1D dipolar spinor condensate with unidirectional local magnetization (that is in the homogeneous-local-spin-orientation limit), we provided an analytical derivation of the Landau-Lifshitz-Gilbert equation and the Stoner-Wohlfarth model. For an external magnetic field along the long axis, we obtained an exact solution of the quasi-1D Landau-Lifshitz-Gilbert equation. Our analytical solution demonstrates that the magnetic dipole-dipole interaction *accelerates* the relaxation of the magnetization to stable states and hence strongly facilitates observation of this process within the lifetime of typical dipolar spinor BECs. Employing this solution, we hence propose a method to experimentally access the dissipative parameter(s) Γ.

We expect, in particular, that our proposal provides a viable tool to verify in experiment whether Γ is indeed independent of spin indices, as commonly assumed, and does not have to be replaced by a tensorial quantity for spinor gases. We hope that this will stimulate further more detailed investigations of the dissipative mechanism in dipolar BECs with internal degrees of freedom.

We considered that the magnetization along z, M_z, has contributions solely from the atoms residing in the condensate, an approximation valid at sufficiently low temperatures. When the magnetization from noncondensed atoms is not negligible, as considered by Ref. [5] for a contact interacting scalar BEC, correlation terms mixing the noncondensed part and the mean field, such as $\sum_{\beta=-S}^{S}\psi_\beta^*(\boldsymbol{r},t)\langle\delta\hat{\psi}_\alpha(\boldsymbol{r},t)\delta\hat{\psi}_\beta(\boldsymbol{r},t)\rangle$, will appear on the right-hand side of Eq. (3). Here, $\delta\hat{\psi}_\alpha(\boldsymbol{r},t)$ is the αth component of quantum field excitations above the mean-field ground state in the spinor basis. Considering the effect of these terms is a subject of future studies.

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APPENDIX A: DERIVATION OF THE EFFECTIVE POTENTIAL V_{eff}

The dipole-dipole interaction term $V_{dd}(t)$ in the total energy is given by [32]

$$V_{dd}(t)=\frac{c_{dd}}{2}\int d^3r\int d^3r'\sum_{\nu,\nu'=x,y,z}F_\nu(\boldsymbol{r},t)Q_{\nu,\nu'}(\boldsymbol{r}-\boldsymbol{r}')F_{\nu'}(\boldsymbol{r}',t), \tag{A1}$$

where c_{dd} is the dipole-dipole interaction coefficient, $F_\nu(\boldsymbol{r},t)=\psi^\dagger(\boldsymbol{r},t)\hat{f}_\nu\psi(\boldsymbol{r},t)$, and $Q_{\nu,\nu'}(\boldsymbol{r})$ is defined as the tensor

$$Q_{\nu,\nu'}(\boldsymbol{r}):=\frac{r^2\delta_{\nu,\nu'}-3r_\nu r_{\nu'}}{r^5} \tag{A2}$$

in spin space, where $r=|\boldsymbol{r}|$ and $r_\nu=\boldsymbol{r}\cdot\boldsymbol{e}_\nu$, with $\boldsymbol{e}_\nu$ being the unit vector along the ν axis. From now on, we define $\boldsymbol{\rho}=(x,y)$ such that $dxdy=d^2\rho=d\varphi d\rho\,\rho$, where $\tan\varphi=y/x$.

Using the convolution theorem, the dipole-dipole interaction term $V_{dd}(t)$ can be expressed by

$$V_{dd}(t)=\frac{c_{dd}}{2}(2\pi)^{D/2}\int d^3k\,\tilde{n}(\boldsymbol{k},t)\tilde{n}(-\boldsymbol{k},t)\tilde{U}_{dd}(\boldsymbol{k},t) \tag{A3}$$

with the Fourier transform

$$U_{dd}(\boldsymbol{\eta},t)=\frac{1}{n(\boldsymbol{r},t)n(\boldsymbol{r}',t)}\sum_{\nu,\nu'=x,y,z}F_\nu(\boldsymbol{r},t)Q_{\nu,\nu'}(\boldsymbol{\eta})F_{\nu'}(\boldsymbol{r}',t), \tag{A4}$$

where $\tilde{g}(\boldsymbol{k},t)=(2\pi)^{-D/2}\int d\boldsymbol{r}\,g(\boldsymbol{r},t)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ is the Fourier transform of the function $g(\boldsymbol{r},t)$ in D-dimensional space $\boldsymbol{r}$ (in our case, $D=3$), $\boldsymbol{\eta}=\boldsymbol{r}-\boldsymbol{r}'$, and $n(\boldsymbol{r},t)=|\psi(\boldsymbol{r},t)|^2$.

By denoting $\boldsymbol{k}=(\boldsymbol{k}_\rho,k_z)$, where $\boldsymbol{k}_\rho=(k_x,k_y)$ with $k_\rho=\sqrt{k_x^2+k_y^2}$ and $\tan\varphi_{k_\rho}=k_y/k_x$, with our mean-field wave function in Eq. (8), one derives

$$\tilde{n}(\boldsymbol{k},t)=\frac{1}{\pi l_\perp^2}\frac{1}{(2\pi)^{3/2}}\int d^2\rho\int_{-\infty}^{\infty}dz\,e^{-(\rho/l_\perp)^2}n(z,t)e^{i\boldsymbol{\rho}\cdot\boldsymbol{k}_\rho}e^{ik_zz}=\frac{1}{2\pi}\tilde{n}(k_z,t)e^{-k_\rho^2l_\perp^2/4}, \tag{A5}$$

where $n(z,t):=|\Psi(z,t)|^2e^{-2\Gamma\omega_\perp t/(1+\Gamma^2)}$. Note the factor of $(2\pi)^{-1}$ appearing, when compared to Eq. (12) in Ref. [27], which is stemming from our definition of Fourier transform.

Denoting $\eta=|\boldsymbol{\eta}|$, by writing $\boldsymbol{e}_\eta$ for the unit vector along $\boldsymbol{\eta}$, we obtain

$$\begin{aligned}U_{dd}(\boldsymbol{\eta},t)=&-\frac{1}{\eta^3}\sqrt{\frac{6\pi}{5}}\left[\left\{Y_2^2(\boldsymbol{e}_\eta)e^{-2i\phi(t)}+Y_2^{-2}(\boldsymbol{e}_\eta)e^{2i\phi(t)}\right\}S^2\sin^2\theta(t)-\left\{Y_2^1(\boldsymbol{e}_\eta)e^{-i\phi(t)}-Y_2^{-1}(\boldsymbol{e}_\eta)e^{i\phi(t)}\right\}S^2\sin\{2\theta(t)\}\right]\\&+\frac{1}{\eta^3}\sqrt{\frac{6\pi}{5}}Y_2^0(\boldsymbol{e}_\eta)\sqrt{\frac{2}{3}}S^2\{3\sin^2\theta(t)-2\},\end{aligned} \tag{A6}$$

where $Y_l^m(\boldsymbol{e}_\eta)$ are the usual spherical harmonics. Its Fourier transform $\tilde{U}_{dd}(\boldsymbol{k},t)$ is

$$\begin{aligned}\tilde{U}_{dd}(\boldsymbol{k},t)=&\frac{1}{(2\pi)^{3/2}}\frac{4\pi}{3}S^2\left\{1-\frac{3}{2}\sin^2\theta(t)\right\}\left(3\frac{k_z^2}{k_\rho^2+k_z^2}-1\right)\\&+\frac{1}{\sqrt{2\pi}}\frac{k_\rho^2}{k_\rho^2+k_z^2}S^2\sin^2\theta(t)\cos\left\{2\varphi_{k_\rho}-2\phi(t)\right\}+\sqrt{\frac{2}{\pi}}\frac{k_\rho k_z}{k_\rho^2+k_z^2}S^2\sin\{2\theta(t)\}\cos\{\varphi_{k_\rho}-\phi(t)\}.\end{aligned} \tag{A7}$$

By plugging Eqs. (A5) and (A7) into Eq. (A3), we finally obtain $V_{dd}(t)$ as

$$V_{dd}(t) = \frac{c_{dd}}{2}\sqrt{2\pi}\int_{-\infty}^{\infty} dk_z\, \tilde{n}(k_z,t)\tilde{n}(-k_z,t)\frac{2S^2}{l_\perp^2\sqrt{2\pi}}\left\{1-\frac{3}{2}\sin^2\theta(t)\right\}\left\{\left(k_z^2l_\perp^2/2\right)e^{k_z^2l_\perp^2/2}E_1\left(k_z^2l_\perp^2/2\right)-\frac{1}{3}\right\}, \tag{A8}$$

where $E_1(x)=\int_x^\infty du\, e^{-u}/u$ is an exponential integral.

Note that Eq. (A8) can be also written as

$$\begin{aligned} V_{dd}(t) &= \frac{c_{dd}}{2}\sqrt{2\pi}\int_{-\infty}^{\infty} dk_z\, \tilde{n}(k_z,t)\tilde{n}(-k_z,t)\tilde{V}_{\rm eff}(k_z,t) \\ &= \frac{c_{dd}}{2}\int_{-\infty}^{\infty} dz\int_{-\infty}^{\infty} dz'\, n(z,t)n(z',t)V_{\rm eff}(z-z',t). \end{aligned} \tag{A9}$$

Due to the fact that $\tilde{V}_{\rm eff}(k_z,t)$ can be obtained by Eq. (A8), we can get $V_{\rm eff}(z,t)$ by inverse Fourier transform. As a preliminary step, we first write some integrals of $E_1(x)$ as follows:

$$\int_{-\infty}^{\infty} dx\, e^{x^2}E_1(x^2)e^{-ikx} = \int_{-\infty}^{\infty} dx\, e^{-ikx}\int_{x^2}^{\infty} dt\,\frac{e^{-(t-x^2)}}{t} = (\pi)^{3/2}e^{k^2/4}\mathrm{erfc}(|k|/2). \tag{A10}$$

Differentiating Eq. (A10) with respect to k two times results in

$$\int_{-\infty}^{\infty} dx\, x^2e^{x^2}E_1(x^2)e^{-ikx} = -(\pi)^{3/2}\left\{\frac{1}{2}\left(\frac{k^2}{2}+1\right)e^{k^2/4}\mathrm{erfc}(|k|/2)-\frac{|k|}{2\sqrt{\pi}}-\frac{2}{\sqrt{\pi}}\delta(k)\right\}. \tag{A11}$$

Therefore, $V_{\rm eff}(z,t)$ can be calculated as

$$\begin{aligned} V_{\rm eff}(z,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk_z\,\frac{2S^2}{l_\perp^2\sqrt{2\pi}}\left\{1-\frac{3}{2}\sin^2\theta(t)\right\}\left\{\left(k_z^2l_\perp^2/2\right)e^{k_z^2l_\perp^2/2}E_1\left(k_z^2l_\perp^2/2\right)-\frac{1}{3}\right\}e^{-ik_zz} \\ &= \frac{S^2}{l_\perp^3}\left\{\frac{3}{2}\sin^2\theta(t)-1\right\}\left\{G(|z|/l_\perp)-\frac{4}{3}\delta(z/l_\perp)\right\}, \end{aligned} \tag{A12}$$

where $G(x)$ is defined in Eq. (14), and $\delta(x)$ is the Dirac delta function.

The Fourier transform of Eq. (A12) acquires the form

$$\begin{aligned} \tilde{V}_{\rm eff}(k_z,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz\, V_{\rm eff}(z,t)e^{ik_zz} = \sqrt{\frac{2}{\pi}}\frac{S^2}{l_\perp^2}\left\{\frac{3}{2}\sin^2\theta(t)-1\right\}\left\{\int_0^\infty dv\, G(v)\cos(k_zl_\perp v)-\frac{2}{3}\right\} \\ &= \sqrt{\frac{2}{\pi}}\frac{S^2}{l_\perp^2}\left\{\frac{3}{2}\sin^2\theta(t)-1\right\}\left[\int_0^\infty du\,\left\{\sqrt{\pi}(2u^2+1)e^{u^2}\mathrm{erfc}(u)-2u\right\}\cos(\sqrt{2}k_zl_\perp u)-\frac{2}{3}\right]. \end{aligned} \tag{A13}$$

From Ref. [51], the following integral involving the complementary error function is

$$\int_0^\infty du\, e^{u^2}\mathrm{erfc}(u)\cos(bu) = \frac{1}{2\sqrt{\pi}}e^{b^2/4}E_1(b^2/4). \tag{A14}$$

By differentiating Eq. (A14) two times with respect to b, we get

$$\int_0^\infty du\, u^2e^{u^2}\mathrm{erfc}(u)\cos(bu) = -\frac{1}{2\sqrt{\pi}}\left\{\frac{1}{2}\left(\frac{b^2}{2}+1\right)e^{b^2/4}E_1(b^2/4)-1+\frac{2}{b^2}\right\}. \tag{A15}$$

Hence, Eq. (A13) becomes

$$\begin{aligned} \tilde{V}_{\rm eff}(k_z,t) &= \sqrt{\frac{2}{\pi}}\frac{S^2}{l_\perp^2}\left\{\frac{3}{2}\sin^2\theta(t)-1\right\}\left[-\left\{\frac{1}{2}\left(k_z^2l_\perp^2+1\right)e^{k_z^2l_\perp^2/2}E_1\left(k_z^2l_\perp^2/2\right)-1+\frac{1}{k_z^2l_\perp^2}\right\}+\frac{1}{2}e^{k_z^2l_\perp^2/2}E_1\left(k_z^2l_\perp^2/2\right)+\frac{1}{k_z^2l_\perp^2}-\frac{2}{3}\right] \\ &= \frac{2S^2}{l_\perp^2\sqrt{2\pi}}\left\{1-\frac{3}{2}\sin^2\theta(t)\right\}\left\{\left(k_z^2l_\perp^2/2\right)e^{k_z^2l_\perp^2/2}E_1\left(k_z^2l_\perp^2/2\right)-\frac{1}{3}\right\}. \end{aligned} \tag{A16}$$

Comparing Eq. (A8) with Eq. (A16), one verifies that Eq. (A12) is the correct result for the effective interaction of the quasi-1D dipolar spinor gas.

APPENDIX B: QUASI-1D GROSS-PITAEVSKII EQUATION WITH DISSIPATION

By introducing an identical damping coefficient for each component of the spinor (cf., e.g., Refs. [26,30]) (i.e., as if each component effectively behaves as a scalar BEC [28]), and neglecting a possible quadratic Zeeman term, the GP equation for a spin-S BEC can be written as [26]

$$(i-\Gamma)\hbar\frac{\partial\psi_\alpha(\boldsymbol{r},t)}{\partial t} = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm tr}(\boldsymbol{r}) + c_0|\psi(\boldsymbol{r},t)|^2\right\}\psi_\alpha(\boldsymbol{r},t) - \hbar\sum_{\beta=-S}^{S}\{\boldsymbol{b}-\boldsymbol{b}_{dd}(\boldsymbol{r},t)\}\cdot(\hat{\boldsymbol{f}})_{\alpha,\beta}\psi_\beta(\boldsymbol{r},t)$$
$$+\sum_{k=1}^{S} c_{2k}\sum_{\nu_1,\nu_2,\ldots,\nu_k=x,y,z} F_{\nu_1,\nu_2,\ldots,\nu_k}(\boldsymbol{r},t)\sum_{\beta=-S}^{S}\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}\right)_{\alpha,\beta}\psi_\beta(\boldsymbol{r},t), \quad \text{(B1)}$$

where $\psi_\alpha(\boldsymbol{r},t)$ is the αth component of the mean-field wave function $\psi(\boldsymbol{r},t)$ (the spin-space index α is an integer taking $2S+1$ values running from $-S$ to S), $F_{\nu_1,\nu_2,\ldots,\nu_k}(\boldsymbol{r},t) := \psi^\dagger(\boldsymbol{r},t)\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}\psi(\boldsymbol{r},t)$, $\hbar\hat{\boldsymbol{f}}$ is the spin-S operator, and $\boldsymbol{b} = g_F\mu_B\boldsymbol{B}/\hbar$ (g_F is the Landé g factor, μ_B is the Bohr magneton, and $\boldsymbol{B}$ the external magnetic field). Finally, $\hbar\boldsymbol{b}_{dd}(\boldsymbol{r},t)\cdot\boldsymbol{e}_\nu = c_{dd}\int d^3r'\sum_{\nu'=x,y,z}Q_{\nu,\nu'}(\boldsymbol{r}-\boldsymbol{r}')F_{\nu'}(\boldsymbol{r}',t)$, where $\boldsymbol{e}_\nu$ is the unit vector along the ν axis ($\nu = x,y,z$) [32]. Applying the formalism of Ref. [1] to a spinor BEC assuming that Γ does not depend on spin indices, one just needs to transform $t\to(1+\Gamma^2)t$ in Eqs. (B1) and (8). We then integrate out the x and y directions in Eq. (B1) to obtain the quasi-1D GP equation.

From Eq. (8) in the main text, we have

$$\int d^2\rho\sum_{\beta=-S}^{S}\frac{e^{-\rho^2/(2l_\perp^2)}}{l_\perp\sqrt{\pi}}\{\hbar\boldsymbol{b}_{dd}(\boldsymbol{r})\cdot(\hat{\boldsymbol{f}})_{\alpha,\beta}\}\psi_\beta(\boldsymbol{r},t)$$
$$=\frac{c_{dd}}{2l_\perp^3}\int_{-\infty}^{\infty}dz'\,n(z',t)\left\{G\left(\frac{|z-z'|}{l_\perp}\right)-\frac{4}{3}\delta\left(\frac{z-z'}{l_\perp}\right)\right\}\Psi(z,t)e^{-\frac{i+\Gamma}{1+\Gamma^2}\omega_\perp t}S\{\boldsymbol{M}(t)-3M_z(t)\boldsymbol{e}_z\}\cdot\sum_{\beta=-S}^{S}(\hat{\boldsymbol{f}})_{\alpha,\beta}\zeta_\beta(t), \quad \text{(B2)}$$

where $\int d^2\rho := \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy$ and $n(z,t) := \int d^2\rho\,|\psi(\boldsymbol{r},t)|^2 = |\Psi(z,t)|^2e^{-2\Gamma\omega_\perp t/(1+\Gamma^2)}$.

For a spin-S BEC, from Eq. (B1), for the trap potential given in Eq. (4) and if we use Eq. (8), by integrating out the x and y directions, one acquires the expression

$$(i-\Gamma)\hbar\frac{\partial\{\Psi(z,t)\zeta_\alpha(t)\}}{\partial t} = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+V(z)+\frac{c_0}{2\pi l_\perp^2}n(z,t)\right\}\Psi(z,t)\zeta_\alpha(t)$$
$$+[-\hbar\boldsymbol{b}+\hbar S\{\boldsymbol{M}(t)-3M_z(t)\boldsymbol{e}_z\}P_{dd}(z,t)]\cdot\left\{\sum_{\beta=-S}^{S}(\hat{\boldsymbol{f}})_{\alpha,\beta}\Psi(z,t)\zeta_\beta(t)\right\}$$
$$+\sum_{k=1}^{S}\frac{c_{2k}}{2\pi l_\perp^2}n(z,t)\sum_{\nu_1,\nu_2,\ldots,\nu_k=x,y,z}SM_{\nu_1,\nu_2,\ldots,\nu_k}(t)\left\{\sum_{\beta=-S}^{S}\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}\right)_{\alpha,\beta}\Psi(z,t)\zeta_\beta(t)\right\}, \quad \text{(B3)}$$

where $M_{\nu_1,\nu_2,\ldots,\nu_k}(t)$ is defined in Eq. (12) and

$$P_{dd}(z,t) = \frac{c_{dd}}{2\hbar l_\perp^3}\int_{-\infty}^{\infty}dz'\,n(z',t)\left\{G\left(\frac{|z-z'|}{l_\perp}\right)-\frac{4}{3}\delta\left(\frac{z-z'}{l_\perp}\right)\right\} = \frac{c_{dd}}{\hbar S^2\{3\sin^2\theta(t)-2\}}\int_{-\infty}^{\infty}dz'\,n(z',t)V_{\rm eff}(z-z',t), \quad \text{(B4)}$$

with $V_{\rm eff}$ defined in Eq. (A12). It is already clear from Eq. (B3) that, besides particle loss from the condensate encoded in a decaying $|\Psi(z,t)|$, dissipation also leads to a *dephasing*, i.e., the decay of $\zeta(t)$ due to the term $-\Gamma\partial\zeta(t)/\partial t$.

From now on, if there is no ambiguity, and for brevity, we drop the arguments such as x,y,z,t from the functions. From Eq. (B3), we then get

$$\hbar\frac{\partial\zeta_\alpha}{\partial t} = -\frac{\hbar}{\Psi}\frac{\partial\Psi}{\partial t}\zeta_\alpha - \frac{\Gamma+i}{1+\Gamma^2}\left(-\frac{\hbar^2}{2m}\frac{1}{\Psi}\frac{\partial^2\Psi}{\partial z^2}+V+\frac{c_0}{2\pi l_\perp^2}n\right)\zeta_\alpha + \frac{\Gamma+i}{1+\Gamma^2}\{\hbar\boldsymbol{b}-S(\boldsymbol{M}-3M_z\boldsymbol{e}_z)\hbar P_{dd}\}\cdot\left\{\sum_{\beta=-S}^{S}(\hat{\boldsymbol{f}})_{\alpha,\beta}\zeta_\beta\right\}$$
$$-\frac{\Gamma+i}{1+\Gamma^2}\sum_{k=1}^{S}\frac{c_{2k}}{2\pi l_\perp^2}n\sum_{\nu_1,\nu_2,\ldots,\nu_k=x,y,z}SM_{\nu_1,\nu_2,\ldots,\nu_k}\left\{\sum_{\beta=-S}^{S}\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}\right)_{\alpha,\beta}\zeta_\beta\right\}. \quad \text{(B5)}$$

Since $\frac{\partial|\zeta|^2}{\partial t} = 0$ due to the normalization $|\zeta|^2 = 1$, we then have

$$0 = 2\mathrm{Re}\left\{-\frac{\hbar}{\Psi}\frac{\partial\Psi}{\partial t}-\frac{\Gamma}{1+\Gamma^2}\left(-\frac{\hbar^2}{2m}\frac{1}{\Psi}\frac{\partial^2\Psi}{\partial z^2}+V+\frac{c_0}{2\pi l_\perp^2}n\right)\right\}+\frac{i}{1+\Gamma^2}\frac{\hbar^2}{2m}\left(\frac{1}{\Psi}\frac{\partial^2\Psi}{\partial z^2}-\frac{1}{\Psi^*}\frac{\partial^2\Psi^*}{\partial z^2}\right)$$
$$+\frac{2\Gamma}{1+\Gamma^2}\{\hbar\boldsymbol{b}-S(\boldsymbol{M}-3M_z\boldsymbol{e}_z)\hbar P_{dd}\}\cdot S\boldsymbol{M}-\frac{2\Gamma}{1+\Gamma^2}\sum_{k=1}^{S}\frac{c_{2k}}{2\pi l_\perp^2}n\sum_{\nu_1,\nu_2,\ldots,\nu_k=x,y,z}S^2M^2_{\nu_1,\nu_2,\ldots,\nu_k}. \quad \text{(B6)}$$

Hence the dynamics of the magnetization direction follows the equation

$$\begin{aligned}\hbar S\frac{\partial M_\nu}{\partial t} &= 2\mathrm{Re}\left\{\sum_{\alpha,\beta=-S}^{S}\zeta_\alpha^\dagger(\hat{f}_\nu)_{\alpha,\beta}\left(\hbar\frac{\partial\zeta_\beta}{\partial t}\right)\right\}\\ &= -\frac{2\Gamma}{1+\Gamma^2}S^2M_\nu\{\hbar\boldsymbol{b}-S(\boldsymbol{M}-3M_z\boldsymbol{e}_z)\hbar P_{dd}\}\cdot\boldsymbol{M}+\frac{2\Gamma}{1+\Gamma^2}M_\nu\sum_{k=1}^{S}\frac{c_{2k}}{2\pi l_\perp^2}n\sum_{\nu_1,\nu_2,\dots,\nu_k=x,y,z}S^3M^2_{\nu_1,\nu_2,\dots,\nu_k}\\ &\quad+\frac{\Gamma}{1+\Gamma^2}\sum_{\mu=x,y,z}\{\hbar b_\mu-S(M_\mu-3M_z\delta_{\mu,z})\hbar P_{dd}\}S\{\delta_{\mu,\nu}+(2S-1)M_\mu M_\nu\}\\ &\quad-\frac{1}{1+\Gamma^2}\sum_{\mu,\kappa=x,y,z}\{\hbar b_\mu-S(M_\mu-3M_z\delta_{\mu,z})\hbar P_{dd}\}\epsilon_{\nu,\mu,\kappa}SM_\kappa\\ &\quad-2\mathrm{Re}\left\{\frac{\Gamma+i}{1+\Gamma^2}\sum_{k=1}^{S}\frac{c_{2k}}{2\pi l_\perp^2}n\sum_{\nu_1,\nu_2,\dots,\nu_k=x,y,z}SM_{\nu_1,\nu_2,\dots,\nu_k}\sum_{\alpha,\beta=-S}^{S}\zeta_\alpha^\dagger(\hat{f}_\nu\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k})_{\alpha,\beta}\zeta_\beta\right\},\end{aligned} \tag{B7}$$

since the scalar product $\zeta^\dagger(\hat{f}_\alpha\hat{f}_\beta+\hat{f}_\beta\hat{f}_\alpha)\zeta=S\{\delta_{\alpha,\beta}+(2S-1)M_\alpha M_\beta\}$ [26].

By direct comparison, we can identify Eq. (B8) below as being identical to Eq. (B21) in Ref. [26], the only difference consisting in the definition of $M_{\nu_1,\nu_2,\dots,\nu_k}$: We employ a scaled version of $M_{\nu_1,\nu_2,\dots,\nu_k}$, which is normalized to S in Ref. [26]. From Eq. (7) in the main text,

$$\sum_{\nu_1,\nu_2,\dots,\nu_k=x,y,z}M_{\nu_1,\nu_2,\dots,\nu_k}\sum_{\alpha,\beta=-S}^{S}\zeta_\alpha^\dagger(\hat{f}_\nu\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k})_{\alpha,\beta}\zeta_\beta=\sum_{\nu_1,\nu_2,\dots,\nu_k=x,y,z}M^2_{\nu_1,\nu_2,\dots,\nu_k}S^2M_\nu, \tag{B8}$$

which is real. Therefore, Eq. (B7) can be written in the form

$$\begin{aligned}\frac{\partial\boldsymbol{M}}{\partial t} &= -\frac{\Gamma}{1+\Gamma^2}\boldsymbol{M}\times[\boldsymbol{M}\times\{\boldsymbol{b}-S(\boldsymbol{M}-3M_z\boldsymbol{e}_z)P_{dd}\}]+\frac{1}{1+\Gamma^2}\boldsymbol{M}\times\{\boldsymbol{b}-S(\boldsymbol{M}-3M_z\boldsymbol{e}_z)P_{dd}\}\\ &= \frac{1}{1+\Gamma^2}\boldsymbol{M}\times(\boldsymbol{b}+3SP_{dd}M_z\boldsymbol{e}_z)-\frac{\Gamma}{1+\Gamma^2}\boldsymbol{M}\times[\boldsymbol{M}\times(\boldsymbol{b}+3SP_{dd}M_z\boldsymbol{e}_z)]\\ &= \boldsymbol{M}\times(\boldsymbol{b}+3SP_{dd}M_z\boldsymbol{e}_z)-\Gamma\boldsymbol{M}\times\frac{\partial\boldsymbol{M}}{\partial t},\end{aligned} \tag{B9}$$

since $\boldsymbol{M}\cdot\frac{\partial\boldsymbol{M}}{\partial t}=0$ holds.

As P is a function of z and t, but $\boldsymbol{M}$ is independent of z [$\boldsymbol{M}$ is the scaled local magnetization and our aim is to study a dipolar spinor BEC with unidirectional local magnetization (the homogeneous-local-spin-orientation limit)], by multiplying with $n(z,t)$ both sides of Eq. (B9) and integrating along z, we finally get the LLG equation

$$\frac{\partial\boldsymbol{M}}{\partial t}=\boldsymbol{M}\times(\boldsymbol{b}+S\Lambda'_{dd}M_z\boldsymbol{e}_z)-\Gamma\boldsymbol{M}\times\frac{\partial\boldsymbol{M}}{\partial t}, \tag{B10}$$

where Λ'_{dd} is defined in Eq. (16). Note here that Λ'_{dd} becomes $\Lambda_{dd}(L_z/l_\perp)$ defined in Eq. (30) when $n(z,t)=N/(2L_z)$ for $-L_z\leqslant z\leqslant L_z$ and $n(z,t)=0$ otherwise.

APPENDIX C: MODIFICATION OF THE LLG EQUATION FOR Γ, A SPIN-SPACE TENSOR

When Γ depends on spin indices, i.e., is a tensor, Eq. (B3) can be generalized to read

$$\sum_{\beta=-S}^{S}(i\delta_{\alpha,\beta}-\Gamma_{\alpha,\beta})\hbar\frac{\partial\{\Psi(z,t)\zeta_\beta(t)\}}{\partial t}=\sum_{\beta=-S}^{S}H_{\alpha,\beta}\Psi(z,t)\zeta_\beta(t). \tag{C1}$$

The spinor part of the wave function is normalized to unity, $|\zeta|^2=1$. Hence, we know that $\frac{\partial|\zeta|^2}{\partial t}=0$. Therefore, from Eq. (C1), we derive the expression

$$\sum_{\alpha,\beta=-S}^{S}\mathrm{Re}\left[-i\zeta_\alpha^*\Gamma_{\alpha,\beta}\frac{\partial\zeta_\beta}{\partial t}-i\zeta_\alpha^*\Gamma_{\alpha,\beta}\zeta_\beta\frac{1}{\Psi}\frac{\partial\Psi}{\partial t}-i\frac{1}{\hbar\Psi}\zeta_\alpha^*H_{\alpha,\beta}\Psi\zeta_\beta\right]-\mathrm{Re}\left[\frac{1}{\Psi}\frac{\partial\Psi}{\partial t}\right]=0. \tag{C2}$$

This then leads us to

$$\frac{\partial M_\nu}{\partial t} = \frac{2}{S}\sum_{\alpha,\beta,\gamma=-S}^{S} \mathrm{Re}\left[-i\zeta_\alpha^*(\hat{f}_\nu)_{\alpha,\beta}\Gamma_{\beta,\gamma}\frac{\partial\zeta_\gamma}{\partial t} - i\zeta_\alpha^*(\hat{f}_\nu)_{\alpha,\beta}\Gamma_{\beta,\gamma}\zeta_\gamma\frac{1}{\Psi}\frac{\partial\Psi}{\partial t} - i\frac{1}{\hbar\Psi}\zeta_\alpha^*(\hat{f}_\nu)_{\alpha,\beta}H_{\beta,\gamma}\Psi\zeta_\gamma\right] - 2\mathrm{Re}\left[M_\nu\frac{1}{\Psi}\frac{\partial\Psi}{\partial t}\right]. \quad \text{(C3)}$$

For scalar Γ, $\Gamma_{\alpha,\beta} \to \Gamma\delta_{\alpha,\beta}$, the equation above becomes Eq. (B7).

From Eqs. (C2) and (C3), one concludes that the stationary solution M_ν of Eq. (C3) is independent of Γ. In other words, whether Γ depends on spin indices or not, the SW model (29) is left unaffected (also see the discussion in Sec. V of the main text).

APPENDIX D: DESCRIPTION OF MAGNETOSTRICTION

For a dipolar spinor BEC without quadratic Zeeman term, when there is no dissipation ($\Gamma = 0$), the mean-field equation in Eq. (3) can be written as

$$\mu_\alpha(t)\psi_\alpha(\boldsymbol{r},t) = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{tr}}(\boldsymbol{r}) + c_0\sum_{\beta=-S}^{S}|\psi_\beta(\boldsymbol{r},t)|^2\right\}\psi_\alpha(\boldsymbol{r},t) - \hbar\{\boldsymbol{b}-\boldsymbol{b}_{dd}(\boldsymbol{r},t)\}\cdot\sum_{\beta=-S}^{S}(\hat{\boldsymbol{f}})_{\alpha,\beta}\psi_\beta(\boldsymbol{r},t)$$
$$+\sum_{k=1}^{S}c_{2k}\sum_{\nu_1,\nu_2,\dots,\nu_k=x,y,z}\;\sum_{\alpha_1,\beta_1,\beta=-S}^{S}\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}\right)_{\alpha_1,\beta_1}\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}\right)_{\alpha,\beta}\psi_{\alpha_1}^*(\boldsymbol{r},t)\psi_{\beta_1}(\boldsymbol{r},t)\psi_\beta(\boldsymbol{r},t), \quad \text{(D1)}$$

where we have substituted $i\hbar\frac{\partial\psi_\alpha(\boldsymbol{r},t)}{\partial t} = \mu_\alpha(t)\psi_\alpha(\boldsymbol{r},t)$.

Since we consider the homogeneous-local-spin-orientation limit, we may write $\psi_\alpha(\boldsymbol{r},t) = \Psi_{\mathrm{uni}}(\boldsymbol{r},t)\zeta_\alpha(t)$. In this limit, we have

$$|\psi(\boldsymbol{r},t)|^2 := \psi^\dagger(\boldsymbol{r},t)\psi(\boldsymbol{r},t) = \sum_{\alpha=-S}^{S}\psi_\alpha^\dagger(\boldsymbol{r},t)\psi_\alpha(\boldsymbol{r},t) = |\Psi_{\mathrm{uni}}(\boldsymbol{r},t)|^2, \quad \text{(D2)}$$

since $\sum_{\alpha=-S}^{S}|\zeta_\alpha(t)|^2 = 1$ from the definition of $\zeta_\alpha(t)$ in Eq. (7). Thus $|\Psi_{\mathrm{uni}}(\boldsymbol{r},t)|^2$ is equal to the number density. Then Eq. (D1) can be written as

$$\mu_\alpha(t)\zeta_\alpha(t)\Psi_{\mathrm{uni}}(\boldsymbol{r},t) = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{tr}}(\boldsymbol{r}) + c_0|\Psi_{\mathrm{uni}}(\boldsymbol{r},t)|^2\right\}\zeta_\alpha(t)\Psi_{\mathrm{uni}}(\boldsymbol{r},t) - \hbar\{\boldsymbol{b}-\boldsymbol{b}_{dd}(\boldsymbol{r},t)\}\cdot\sum_{\beta=-S}^{S}(\hat{\boldsymbol{f}})_{\alpha,\beta}\zeta_\beta(t)\Psi_{\mathrm{uni}}(\boldsymbol{r},t)$$
$$+S\sum_{k=1}^{S}c_{2k}\sum_{\nu_1,\nu_2,\dots,\nu_k=x,y,z}\;\sum_{\beta=-S}^{S}M_{\nu_1,\nu_2,\dots,\nu_k}(t)\left(\hat{f}_{\nu_1}\hat{f}_{\nu_2}\cdots\hat{f}_{\nu_k}\right)_{\alpha,\beta}\zeta_\beta(t)|\Psi_{\mathrm{uni}}(\boldsymbol{r},t)|^2\Psi_{\mathrm{uni}}(\boldsymbol{r},t). \quad \text{(D3)}$$

Now, we decompose the chemical potential $\mu(t)$ as $\mu(t) := \sum_{\alpha=-S}^{S}\mu_\alpha(t)|\zeta_\alpha(t)|^2$. Then one obtains

$$\mu(t)\Psi_{\mathrm{uni}}(\boldsymbol{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{tr}}(\boldsymbol{r}) + \left\{c_0 + S^2\sum_{k=1}^{S}c_{2k}\sum_{\nu_1,\nu_2,\dots,\nu_k=x,y,z}M^2_{\nu_1,\nu_2,\dots,\nu_k}(t)\right\}|\Psi_{\mathrm{uni}}(\boldsymbol{r},t)|^2\right]\Psi_{\mathrm{uni}}(\boldsymbol{r},t)$$
$$+[\Phi_{dd}(\boldsymbol{r},t) - S\hbar\{\boldsymbol{b}\cdot\boldsymbol{M}(t)\}]\Psi_{\mathrm{uni}}(\boldsymbol{r},t), \quad \text{(D4)}$$

where

$$\Phi_{dd}(\boldsymbol{r},t) := S^2c_{dd}\left[\int d^3r'\left\{\sum_{\nu,\nu'=x,y,z}M_\nu(t)Q_{\nu,\nu'}(\boldsymbol{r}-\boldsymbol{r}')M_{\nu'}(t)\right\}\left|\Psi_{\mathrm{uni}}(\boldsymbol{r}',t)\right|^2\right] \quad \text{(D5)}$$

is the dipole-dipole mean-field potential [42] following from the definition of $\boldsymbol{b}_{dd}$ below Eq. (3) in the main text.

Due to $M_x(t) = \sin\theta(t)\cos\phi(t)$, $M_y(t) = \sin\theta(t)\sin\phi(t)$, and $M_z(t) = \cos\theta(t)$, from Eqs. (A4) and (A6), we have

$$\sum_{\nu,\nu'=x,y,z}M_\nu(t)Q_{\nu,\nu'}(\boldsymbol{\eta})M_{\nu'}(t) = -\frac{1}{\eta^3}\sqrt{\frac{6\pi}{5}}\left[\left\{Y_2^2(\boldsymbol{e}_\eta)e^{-2i\phi(t)} + Y_2^{-2}(\boldsymbol{e}_\eta)e^{2i\phi(t)}\right\}\sin^2\theta(t)\right.$$
$$\left.-\left\{Y_2^1(\boldsymbol{e}_\eta)e^{-i\phi(t)} - Y_2^{-1}(\boldsymbol{e}_\eta)e^{i\phi(t)}\right\}\sin\{2\theta(t)\}\right] + \frac{1}{\eta^3}\sqrt{\frac{6\pi}{5}}Y_2^0(\boldsymbol{e}_\eta)\sqrt{\frac{2}{3}}\{3\sin^2\theta(t) - 2\}, \quad \text{(D6)}$$

where $Y_l^m(\boldsymbol{e}_\eta)$ are the usual spherical harmonics.

By using Eq. (A2), an alternative form of Eq. (D6) can be obtained:

$$\sum_{\nu,\nu'=x,y,z} M_\nu(t) Q_{\nu,\nu'}(\boldsymbol{\eta}) M_{\nu'}(t) = \sum_{\nu,\nu'=x,y,z} \frac{\eta^2\delta_{\nu,\nu'} - 3\eta_\nu\eta_{\nu'}}{\eta^5} M_\nu(t) M_{\nu'}(t) = \frac{\eta^2|\boldsymbol{M}(t)|^2 - 3\{\boldsymbol{\eta}\cdot\boldsymbol{M}(t)\}^2}{\eta^5}$$
$$= \frac{\eta^2 - 3\{\boldsymbol{\eta}\cdot\boldsymbol{M}(t)\}^2}{\eta^5}. \tag{D7}$$

Thus, $\Phi_{dd}(\boldsymbol{r}, t)$ can be written as

$$\Phi_{dd}(\boldsymbol{r}, t) = S^2 c_{dd}\left[\int d^3r' \, \frac{|\boldsymbol{r}-\boldsymbol{r}'|^2 - 3\{(\boldsymbol{r}-\boldsymbol{r}')\cdot\boldsymbol{M}(t)\}^2}{|\boldsymbol{r}-\boldsymbol{r}'|^5}|\Psi_{\rm uni}(\boldsymbol{r}', t)|^2\right]$$
$$= S^2 c_{dd}\left[\int d^3\bar{r}' \, \frac{|\bar{\boldsymbol{r}}-\bar{\boldsymbol{r}}'|^2 - 3\{(\bar{\boldsymbol{r}}-\bar{\boldsymbol{r}}')\cdot\boldsymbol{M}(t)\}^2}{|\bar{\boldsymbol{r}}-\bar{\boldsymbol{r}}'|^5}|\Psi_{\rm uni}(\bar{\boldsymbol{r}}', t)|^2\right]$$
$$= -\frac{3}{2}S^2 c_{dd}\sin^2\theta(t)\int d^3\bar{\eta}\, |\Psi_{\rm uni}(\bar{\boldsymbol{\eta}}+\bar{\boldsymbol{r}}, t)|^2\frac{1}{\bar{\eta}^5}\left[\bar{\eta}^2 - \bar{\eta}_z^2 - 2\{\bar{\eta}_x\sin\phi(t) - \bar{\eta}_y\cos\phi(t)\}^2\right]$$
$$- 3S^2 c_{dd}\sin\{2\theta(t)\}\int d^3\bar{\eta}\, |\Psi_{\rm uni}(\bar{\boldsymbol{\eta}}+\bar{\boldsymbol{r}}, t)|^2\frac{\bar{\eta}_z}{\bar{\eta}^5}\{\bar{\eta}_x\cos\phi(t) + \bar{\eta}_y\sin\phi(t)\}$$
$$+ \frac{1}{2}S^2 c_{dd}\{1 - 3\cos^2\theta(t)\}\int d^3\bar{\eta}\, |\Psi_{\rm uni}(\bar{\boldsymbol{\eta}}+\bar{\boldsymbol{r}}, t)|^2\frac{1}{\bar{\eta}^5}\left(3\bar{\eta}_z^2 - \bar{\eta}^2\right), \tag{D8}$$

where $\bar{\boldsymbol{r}} := \boldsymbol{r}/L$ with L a given length scale (so that $\tilde{\boldsymbol{r}}$ is a dimensionless vector). For example, in the quasi-1D setup with trap potential being Eq. (4), $L = l_\perp$. Note that, in the special case where $\boldsymbol{M}(t) = M_z(t)\boldsymbol{e}_z$, the form of Eq. (D8) becomes identical to Eq. (6) in Ref. [41].

Since we concentrate on quasi-1D gases, with trap potential given by Eq. (4) in the main text, we explicitly compute the form of $\Phi_{dd}(\boldsymbol{r}, t)$ for the quasi-1D setup. By writing

$$|\Psi_{\rm uni}(\boldsymbol{r}, t)|^2 = \frac{e^{-\rho^2/l_\perp^2}}{\pi l_\perp^2}|\Psi(z, t)|^2, \tag{D9}$$

and integrating out x and y directions, one can get the quasi-1D dipole-dipole interaction mean-field potential $\Phi_{dd}(z, t)$ as follows [which is in Eq. (38)]:

$$\Phi_{dd}(z, t) = \frac{c_{dd}}{2l_\perp^2}S^2\left\{1 - 3M_z^2(t)\right\}\left\{\int_{-\infty}^{\infty} d\bar{z}\, |\Psi(z + \bar{z}l_\perp, t)|^2 G(|\bar{z}|) - \frac{4}{3}|\Psi(z, t)|^2\right\}. \tag{D10}$$

Now, let us consider a box trap in the quasi-1D case, i.e., $V(z) = 0$ for $|z| \leqslant L_z$ and $V(z) = \infty$ for $|z| > L_z$ where $V(z)$ is in Eq. (4). Then we may write

$$|\Psi(z, t)|^2 = \begin{cases} \frac{N}{2L_z} & \text{for } |z| \leqslant L_z \\ 0 & \text{for } |z| > L_z, \end{cases} \tag{D11}$$

since $V(z) = 0$ for $-L_z \leqslant z \leqslant L_z$. Thus, $\Phi_{dd}(z, t)$ can be written as

$$\Phi_{dd}(z, t) = \begin{cases} \bar{\Phi}_{dd}(t)\left\{\int_{-(L_z+z)/l_\perp}^{(L_z-z)/l_\perp} d\bar{z}\, G(|\bar{z}|) - \frac{4}{3}\right\} & \text{for } |z| \leqslant L_z \\ \bar{\Phi}_{dd}(t)\int_{-(L_z+z)/l_\perp}^{(L_z-z)/l_\perp} d\bar{z}\, G(|\bar{z}|) & \text{for } |z| > L_z, \end{cases} \tag{D12}$$

where $\bar{\Phi}_{dd}(t) := Nc_{dd}S^2\{1 - 3M_z^2(t)\}/(2L_z l_\perp^2)$. $\Phi_{dd}(z, t)$ is discontinuous at $z = \pm L_z$ because of the sudden change of the density at the boundary ($z = \pm L_z$) due to the box-trap potential.

Defining the scaled density-density mean-field potential $\bar{\Phi}_{dd}(z) := \Phi_{dd}(z, t)/\bar{\Phi}_{dd}(t)$, we obtain Fig. 6 for two different axial extensions, $L_z/l_\perp = 10$ and 30. As Fig. 6 clearly illustrates, in a box-trapped quasi-1D gas, $\Phi_{dd}(z, t)$ becomes approximately constant for $|z| < L_c$ and $L_c \to L_z$ for $L_z/l_\perp \gg 1$. Depending on the value of $\boldsymbol{M}(t)$, $\Phi_{dd}(\boldsymbol{r}, t)$ will introduce either a repulsive or an attractive force. This force will, however, exist only near the boundary for a box trap, where it can lead to a slight modification of the density of atoms. Its relative influence decreases with increasing extension of the trapped gas along the z axis, and can therefore be consistently neglected in the approximation of constant particle-density.

However, to assess whether significant magnetostriction occurs, one has to consider, in addition to Φ_{dd}, the trap potential $V_{\rm tr}$ and the "quasi"-density-density interaction mean-field potential Φ_0 defined as

$$\Phi_0(\boldsymbol{r}, t) := \left\{c_0 + S^2\sum_{k=1}^{S} c_{2k}\sum_{\nu_1,\nu_2,\ldots,\nu_k=x,y,z} M_{\nu_1,\nu_2,\ldots,\nu_k}^2(t)\right\}|\Psi_{\rm uni}(\boldsymbol{r}, t)|^2. \tag{D13}$$

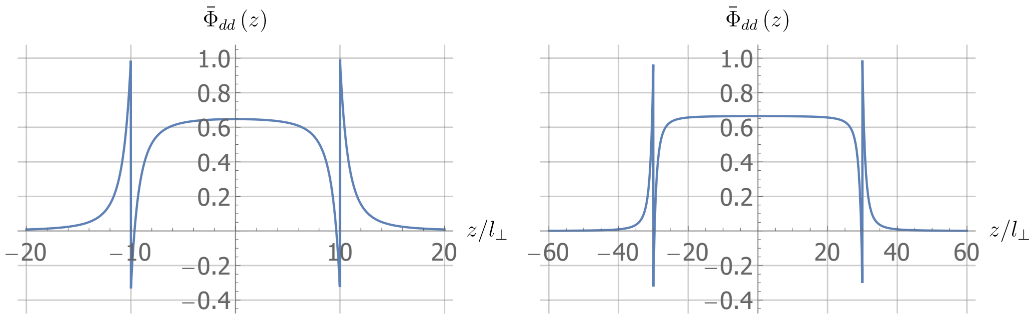

FIG. 6. Scaled dipole-dipole mean-field potential $\bar{\Phi}_{dd}(z)$ as a function of z for a quasi-1D box trap. Left: $L_z/l_\perp = 10$. Right: $L_z/l_\perp = 30$.

We can coin $\Phi_0(\boldsymbol{r}, t)$ a quasi-density-density interaction mean-field potential because only c_0 is a density-density interaction coefficient (c_{2k} are interaction coefficients parametrizing the spin-spin interactions for a spin-S gas where k is an integer with $1 \leqslant k \leqslant S$. For example, c_2 is the spin-spin interaction coefficient of a spin-1 gas). In our quasi-1D case, this $\Phi_0(\boldsymbol{r}, t)$ potential is $\Phi_0(z, t)$, where

$$\Phi_0(z,t) := \left\{\frac{c_0}{2\pi l_\perp^2} + S^2 \sum_{k=1}^{S} \frac{c_{2k}}{2\pi l_\perp^2} \sum_{\nu_1,\nu_2,\ldots,\nu_k=x,y,z} M^2_{\nu_1,\nu_2,\ldots,\nu_k}(t)\right\} |\Psi(z,t)|^2. \tag{D14}$$

In the main text, we assume that $c_0 \gg S^2 \sum_{k=1}^{S} c_{2k} \sum_{\nu_1,\nu_2,\ldots,\nu_k=x,y,z} M^2_{\nu_1,\nu_2,\ldots,\nu_k}(t)$. For spin-1 ^{23}Na or ^{87}Rb, $S = 1$ and $c_0 \simeq 100|c_2|$ [32,35], so this is an appropriate assumption (note that $|\boldsymbol{M}(t)| = 1$). The values of the c_{2k} are not yet established for ^{166}Er. We therefore tacitly assume in the main text, when calculating concrete numerical examples for ^{166}Er, that the above condition also still holds, despite the prefactor S^2 enhancing the importance of spin-spin interactions in $\Phi_0(z, t)$. When this assumption is not applicable, one is required to take into account the time dependence of $\Phi_0(z, t)$ due to $\boldsymbol{M}(t)$ together with magnetostriction due to $\Phi_{dd}(z, t)$, which will change the system size L_z as a function of t. This will in turn change the integration domain and quasi-1D density $n(z, t) = |\Psi(z, t)|^2$ in Eq. (19), and incur also a changed time dependence of $\Lambda'_{dd}(t)$, and the solution of the coupled system of Eqs. (B10) and (D4) needs to be found self-consistently.

For a harmonic trap, due to the resulting inhomogeneity of $|\Psi(z, t)|^2$, $\Phi_{dd}(z, t)$ will have more significant spatial dependence than its box-trap counterpart shown in Fig. 6. Here, we note that Ref. [41] has already shown (for a spin-polarized gas, c_{2k} couplings not included) that magnetostriction occurs in a harmonic trap. The effect of magnetostriction is generally expected to be larger in a harmonic trap when compared to a box trap with similar geometrical and dynamical parameters for large relative system size $L_z/l_\perp \gg 1$, at least under the above condition that the $S^2 c_{2k}/c_0$ are sufficiently small.

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