Coherence measures with respect to general quantum measurements

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Quantum coherence with respect to orthonormal bases has been studied extensively in the past few years. Recently, Bischof, Kampermann, and Bruß [Phys. Rev. Lett. **123**, 110402 (2019)] generalized it to the case of general positive operator-valued measure (POVM) measurements. Such POVM-based coherence, including the block coherence as a special case, have significant operational interpretations in quantifying the advantage of quantum states in quantum information processing. In this work we first establish an alternative framework for quantifying the block coherence and provide several block coherence measures. We then present several coherence measures with respect to POVM measurements, and prove a conjecture on the l_1-norm related POVM coherence measure.

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I. INTRODUCTION

Quantum coherence is a characteristic feature of quantum mechanics, with wide applications in superconductivity, quantum thermodynamics, and biological processes. From a resource-theoretic perspective the quantification of quantum coherence has attracted much attention and various kinds of coherence measures have been proposed [1–15]. Let ρ be a density operator in d-dimensional complex Hilbert space H. Under a fixed orthonormal basis $\{|i\rangle\}_{i=1}^d$ of H, the state ρ is called incoherent if $\langle i|\rho|j\rangle = 0$ for any $i \neq j$ [1]. Otherwise ρ is called coherent. The coherence theory has achieved fruitful results in the past few years (for recent reviews see, e.g., [16,17]).

From another perspective, the orthonormal basis $\{|i\rangle\}_{i=1}^d$ corresponds to a rank-1 projective measurement (von Neumann measurement) $\{|i\rangle\langle i|\}_{i=1}^d$, and $\langle i|\rho|j\rangle = 0$ is equivalent to $|i\rangle\langle i|\rho|j\rangle\langle j| = 0$. This observation leads one to view the coherence with respect to the orthonormal basis $\{|i\rangle\}_{i=1}^d$ as the coherence with respect to the rank-1 projective measurement $\{|i\rangle\langle i|\}_{i=1}^d$. Along this idea, the concept of coherence can be generalized to the cases of general measurements. Recently, Bischof *et al.* [18] have generalized the concept of coherence to the case of general quantum measurements, i.e., positive operator-valued measures (POVMs), and established the resource theory of coherence based on POVMs. One motivation of this generalization is due to the fact that POVMs may be more advantageous compared to von Neumann measurement in many applications [19]. Moreover, the coherence of a state with respect to a POVM can be interpreted as the cryptographic randomness generated by measuring the POVM on the state [20]. Namely, the amount of POVM coherence in a state is equal to the unpredictability of measurement outcomes relative to an eavesdropper with maximal information about the state, which generalizes the results from [2]. It has been shown that the relative entropy of POVM coherence is equal to the cryptographic randomness gain [20]. It provides an important operational meaning to the concept of coherence with respect to a general measurement. Generalizing the usual coherence theory from an orthonormal basis to a generic POVM had been also the efforts made in Refs. [21,22].

After establishing a framework for quantifying the POVM coherence [18,20], Bischof *et al.* developed [18,20] a scheme by employing the Naimark extension to embed the POVM coherence into the block coherence proposed in Ref. [23] in a lager Hilbert space. The Naimark extension [24,25] states that any POVM can be extended to a projective measurement in a larger Hilbert space. The block coherence was defined with respect to projective measurements, not necessarily rank 1. With this scheme, the relative entropy of POVM coherence C_{rel} and the robustness POVM coherence C_{rot} were proposed. Recently, the structures of different incoherent operations for POVM coherence were investigated [26]. For simplicity, we call the coherence theory with respect to fixed orthonormal bases the standard coherence theory. As the generalizations of the standard coherence, both the block coherence and the POVM coherence reduce to the standard coherence in the case of the von Neumann measurement.

In the present work, we establish an alternative framework for quantifying the block coherence and provide several block coherence measures. We then present several POVM coherence measures. Meanwhile, we also prove a conjecture raised recently.

II. ALTERNATIVE FRAMEWORK FOR QUANTIFYING BLOCK COHERENCE

A. Block incoherent states and block incoherent channels

The block coherence theory was introduced in Ref. [23]. We adopt the framework proposed in Ref. [20] for

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quantifying the block coherence. Consider a quantum system A associated with an m-dimensional complex Hilbert space H. One has partition $H=\oplus_{i=1}^{n}\pi_i$ into orthogonal subspaces π_i of dimension $\dim\pi_i=m_i$, $\sum_{i=1}^{n}m_i=m$. Correspondingly, one gets a projective measurement $P=\{P_i\}_{i=1}^{n}$, with each projector satisfying $P_i(H)=\pi_i$. A state ρ on H is called block incoherent (BI) with respect to P if

$$P_i\rho P_j=0,\quad \forall i\neq j, \tag{1}$$

or

$$\rho=\sum_{i=1}^{n}P_i\rho P_i. \tag{2}$$

We denote the set of all quantum states in H by $\mathcal{S}(H)$, and the set of all block incoherent quantum states by $\mathcal{I}_{\rm B}(H)$. It is easy to check that

$$\mathcal{I}_{\rm B}(H)=\left\{\sum_{i=1}^{n}P_i\rho P_i|\rho\in\mathcal{S}(H)\right\}. \tag{3}$$

A quantum channel is a completely positive and trace preserving (CPTP) linear map of quantum states [27]. A quantum channel ϕ is often expressed by the Kraus operators $\{K_l\}_l$ satisfying $\sum_l K_l^\dagger K_l=I_m$, where I_m is the identity operator on H and $\dagger$ stands for the adjoint. A quantum channel ϕ is called block incoherent if it admits an expression of Kraus operators $\phi=\{K_l\}_l$ such that

$$P_iK_l\rho K_l^\dagger P_j=0,\quad \forall l,\ \forall i\neq j \tag{4}$$

for any $\rho\in\mathcal{I}_{\rm B}(H)$. Such an expression $\phi=\{K_l\}_l$ is called a block incoherent decomposition of ϕ. We denote the set of all quantum channels on H by $\mathcal{C}(H)$, and the set of all block incoherent quantum channels by $\mathcal{C}_{\rm BI}(H)$.

The concept of block coherence can be properly extended to the multipartite systems via the tensor product of the Hilbert spaces of the subsystems, similar to the case of standard coherence theory [16]. For bipartite systems, let A' be another quantum system associating with the m'-dimensional complex Hilbert space H'. Partitioning $H'=\oplus_{i=1}^{n'}\pi_i'$ into orthogonal subspaces π_i' of dimension $\dim\pi_i'=m_i'$, $m'=\sum_{i=1}^{n'}m_i'$, one gets a projective measurement $P'=\{P_i'\}_{i=1}^{n'}$ with each projector P_i' satisfying $P_i'(H')=\pi_i'$. Correspondingly one has concepts such as $\mathcal{S}(H')$, $\mathcal{I}_{\rm B}(H')$, $\mathcal{C}(H')$, and $\mathcal{C}_{\rm BI}(H')$. For the composite Hilbert space $H^{AA'}=H^A\otimes H^{A'}$ associating to the bipartite system AA', we have the projective measurement $P\otimes P'=\{P_i\otimes P_{i'}'\}_{ii'}$. A state $\rho^{AA'}$ on $H^{AA'}$ is called block incoherent with respect to the projective measurement $P\otimes P'$ if

$$(P_i\otimes P_{i'}')\rho^{AA'}(P_j\otimes P_{j'}')=0,\quad \forall(i,i')\neq(j,j'), \tag{5}$$

where $(i,i')\neq(j,j')$ means that $i\neq j$ or $i'\neq j'$.

We denote the set of all states on $H^{AA'}$ by $\mathcal{S}(H^{AA'})$ and the set of all channels on $\mathcal{S}(H^{AA'})$ by $\mathcal{C}(H^{AA'})$. A quantum channel $\phi^{AA'}$ on $\mathcal{C}(H^{AA'})$ is called block incoherent if it admits an expression of Kraus operators $\phi^{AA'}=\{K_l^{AA'}\}_l$ such that

$$(P_i\otimes P_{i'}')K_l^{AA'}\rho^{AA'}\left(K_l^{AA'}\right)^\dagger(P_j\otimes P_{j'}')=0 \tag{6}$$

for all l and $(i,i')\neq(j,j')$. We denote the set of all block incoherent channels on $\mathcal{C}(H^{AA'})$ by $\mathcal{C}_{\rm BI}(H^{AA'})$ and call such an expression $\phi^{AA'}=\{K_l^{AA'}\}_l$ a block incoherent decomposition of $\phi^{AA'}$.

B. An alternative framework for quantifying the block coherence

A framework for quantifying the block coherence has been established in Ref. [20]: any valid block coherence measure $C(\rho;P)$ with respect to the projective measurement P should satisfy the conditions (B1)–(B4) here:

(B1) Faithfulness: $C(\rho;P)\geqslant 0$ with equality if $\rho\in\mathcal{I}_{\rm B}(H)$.

(B2) Monotonicity: $C(\phi_{\rm BI}(\rho);P)\leqslant C(\rho;P)$ for any $\phi_{\rm BI}\in\mathcal{C}_{\rm BI}(H)$.

(B3) Strong monotonicity: $\sum_l p_lC(\rho_l;P)\leqslant C(\rho;P)$ for any block incoherent decomposition $\phi_{\rm BI}=\{K_l\}_l\in\mathcal{C}_{\rm BI}(H)$ of $\phi_{\rm BI}$, $p_l={\rm tr}(K_l\rho K_l^\dagger)$, $\rho_l=K_l\rho K_l^\dagger/p_l$.

(B4) Convexity: $C(\sum_j p_j\rho_j;P)\leqslant\sum_j p_jC(\rho_j;P)$ for any states $\{\rho_j\}_j$ and any probability distribution $\{p_j\}_j$.

This framework coincides with the one in the standard coherence theory [1] if all $\{P_i\}_{i=1}^{n}$ are rank 1. Note that (B3) and (B4) together imply (B2).

The framework of the standard coherence theory [1] had been modified by adding an additivity condition in Ref. [28]. For the block coherence theory, we add the following condition:

(B5) Block additivity:

$$C(p_1\rho_1\oplus p_2\rho_2;P)=p_1C(\rho_1;P)+p_2C(\rho_2;P), \tag{7}$$

where $p_1>0$, $p_2>0$, $p_1+p_2=1$, $\rho_1,\rho_2\in\mathcal{S}(H)$, and for any partition $P=\{P_{k_1}\}_{k_1}\cup\{P_{k_2}\}_{k_2}$ such that $\{k_1\}_{k_1}\cup\{k_2\}_{k_2}=\{k\}_{k=1}^{n}$, $\{k_1\}_{k_1}\cap\{k_2\}_{k_2}=\varnothing$, and $\rho_1P_{k_2}=\rho_2P_{k_1}=0$ for any k_1 and k_2.

With condition (B5), we have the following theorem, which establishes an alternative framework for quantifying the block coherence.

Theorem 1. The framework given by conditions (B1) to (B4) is equivalent to the one given by the conditions (B1), (B2), and (B5).

Proof. We first prove that conditions (B1) to (B4) imply (B1), (B2), and (B5). Suppose that (B1) to (B4) are fulfilled. For the state $p_1\rho_1\oplus p_2\rho_2$ as given in (B5), we construct the BI channel $\phi_{\rm BI}=\{K_1,K_2\}$ with $K_1=\sum_{k_1}P_{k_1}$, $K_2=\sum_{k_2}P_{k_2}$. We have $K_1(p_1\rho_1\oplus p_2\rho_2)K_1^\dagger=p_1\rho_1$ and $K_2(p_1\rho_1\oplus p_2\rho_2)K_2^\dagger=p_2\rho_2$. Then from (B3) we get

$$C(p_1\rho_1\oplus p_2\rho_2;P)\geqslant p_1C(\rho_1;P)+p_2C(\rho_2;P). \tag{8}$$

On the other hand, since $p_1\rho_1\oplus p_2\rho_2=p_1\rho_1+p_2\rho_2$, from (B4) we get

$$C(p_1\rho_1\oplus p_2\rho_2;P)\leqslant p_1C(\rho_1;P)+p_2C(\rho_2;P). \tag{9}$$

Combining (8) and (9) we get the condition (B5).

Next we prove that (B1), (B2), and (B5) imply (B1) to (B4). Suppose conditions (B1), (B2), and (B5) are satisfied. Let $\{K_l\}_{l=1}^{n'}\in\mathcal{C}_{\rm BI}(H)$ be a BI decomposition associated to the system A. Consider the bipartite system AA' with the aforementioned notation and $\rho\in\mathcal{S}(H)$. Let the state $\rho^{AA'}=\rho\otimes|1\rangle\langle 1|$ undergo a BI channel such that

$$\begin{aligned}\phi_{BI}^{AA'}(\rho^{AA'})&=\sum_l(K_l\otimes U_l)(\rho\otimes|1\rangle\langle 1|)(K_l^\dagger\otimes U_l^\dagger)\\&=\sum_l K_l\rho K_l^\dagger\otimes|l\rangle\langle l|,\end{aligned} \tag{10}$$

where

$$U_l = \sum_{k=1}^{n'} |k+l-1\rangle\langle k|$$

are the unitary operators on A'. From (B5), Eq. (10) gives rise to

$$C\left(\sum_l K_l \rho K_l^\dagger \otimes |l\rangle\langle l|; P\otimes P'\right) = \sum_l p_l C(\rho_l; P), \quad (11)$$

where P and P' are rank-1 projective measurements, $p_l = \mathrm{tr}(K_l\rho K_l^\dagger)$, $\rho_l = K_l\rho K_l^\dagger/p_l$, and we have used

$$C(\rho_l \otimes |l\rangle\langle l|; P\otimes P') = C(\rho_l; P). \quad (12)$$

According to (B2), Eqs. (10) and (12) together imply (B3).

Now consider the state

$$\rho^{AA'} = \sum_{l=1}^{n'} p_l \rho_l \otimes |l\rangle\langle l|, \quad (13)$$

with $\{p_l\}_{l=1}^{n'}$ a probability distribution and $\{\rho_l\}_{l=1}^{n'} \subset \mathcal{S}(H)$, $\{|l\rangle\}_{l=1}^{n'}$ orthonormal basis of H'. According to (B5), we have

$$C\left(\sum_l p_l \rho_l \otimes |l\rangle\langle l|; P\otimes P'\right) = \sum_l p_l C(\rho_l; P). \quad (14)$$

Let $\rho^{AA'}$ undergo a BI channel as

$$\begin{aligned}\phi_{\mathrm{BI}}^{AA'}(\rho^{AA'}) &= \sum_{k=1}^{n'} (I^A \otimes |1\rangle\langle k|)\rho^{AA'}(I^A\otimes |k\rangle\langle 1|)\\ &= \sum_j p_j \rho_j \otimes |1\rangle\langle 1|. \quad (15)\end{aligned}$$

Similarly, (B2), (B5), and Eqs. (15) and (16) together imply (B4). ∎

We have provided an alternative framework for block coherence by proving that the conditions (B1) to (B4) are equivalent to the conditions (B1), (B2), and (B5). The similar condition (B5) in the standard coherence has particular advantages in calculating coherence of block diagonal states [29]. The condition (B5) in the block coherence may also simplify the calculations of the block coherence for certain block diagonal states.

C. Several block coherence measures

Under the framework of block coherence above, we now provide several block coherence measures. Denote $P = \{P_i\}_{i=1}^n$ a projective measurement on the Hilbert space H. Propositions 1–5 provide block coherence measures; see the detailed proofs in the Appendix.

Proposition 1. l_1 norm of coherence

$$C_{l_1}(\rho, P) = \sum_{i\neq j} \|P_i\rho P_j\|_{\mathrm{tr}} \quad (16)$$

is a block coherence measure, where $\|M\|_{\mathrm{tr}} = \mathrm{tr}\sqrt{M^\dagger M}$ denotes the trace norm of the matrix M.

Proposition 2. For $\alpha \in (0,1)\cup(1,2]$, coherence based on Tsallis relative entropy

$$C_{T,\alpha}(\rho, P) = \frac{1}{\alpha-1}\left\{\sum_i \mathrm{tr}[(P_i\rho^\alpha P_i)^{1/\alpha}] - 1\right\} \quad (17)$$

is a block coherence measure.

In particular, we have

Corollary 1.

$$\lim_{\alpha\to 1} C_{T,\alpha}(\rho, P) = (\ln 2) C_{\mathrm{rel}}(\rho, P), \quad (18)$$

where

$$C_{\mathrm{rel}}(\rho, P) = \mathrm{tr}(\rho\log_2\rho) - \sum_i \mathrm{tr}[(P_i\rho P_i)\log_2(P_i\rho P_i)], \quad (19)$$

and ln is the natural logarithm.

Proposition 3. Modified trace norm of coherence

$$C_{\mathrm{tr}}(\rho, P) = \min_{\lambda>0, \sigma\in\mathcal{I}_B(H)} \|\rho - \lambda\sigma\|_{\mathrm{tr}} \quad (20)$$

is a block coherence measure.

Proposition 4. Coherence weight

$$\begin{aligned} C_{\mathrm{w}}(\rho, P) &= \min_{\sigma,\tau}\{s \geqslant 0 | \rho = (1-s)\sigma + s\tau, \sigma\in\mathcal{I}_{\mathrm{B}}(H), \tau\in\mathcal{S}(H)\}\\ &= \min_{\sigma}\{s\geqslant 0 | \rho \geqslant (1-s)\sigma, \sigma\in\mathcal{I}_{\mathrm{B}}(H)\} \quad (21)\end{aligned}$$

is a block coherence measure.

Proposition 5. For $\alpha\in[\frac{1}{2}, 1)$, coherence based on sandwiched Rényi relative entropy

$$C_{R,\alpha}(\rho, P) = 1 - \max_{\sigma\in\mathcal{I}_{\mathrm{B}}(H)} \left(\left\{\mathrm{tr}\left[\left(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right]\right\}^{\frac{1}{1-\alpha}}\right) \quad (22)$$

is a block coherence measure.

When P is a rank-1 projective measurement, $C_{l_1}(\rho, P)$ recovers the standard coherence measure $C_{l_1}(\rho)$ proposed in Ref. [1], $C_{T,\alpha}(\rho, P)$ recovers the standard coherence measure proposed in Refs. [9,13,30], $C_{\mathrm{tr}}(\rho, P)$ recovers the standard coherence measure proposed in Ref. [28], $C_{\mathrm{w}}(\rho, P)$ recovers the standard coherence measure $C_{\mathrm{w}}(\rho)$ proposed in Ref. [31], and $C_{R,\alpha}(\rho, P)$ recovers the standard coherence measure proposed in Ref. [14]. In particular, when $\alpha = \frac{1}{2}$,

$$C_{R,\frac{1}{2}}(\rho, P) = 1 - \max_{\sigma\in\mathcal{I}_{\mathrm{B}}(H)} (\mathrm{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2 \quad (23)$$

recovers the standard coherence measure proposed in Ref. [32] when P is a rank-1 projective measurement.

III. COHERENCE MEASURES WITH RESPECT TO GENERAL QUANTUM MEASUREMENTS

We study now the coherence measures with respect to general quantum measurements [20]. A general measurement or a POVM on d-dimensional Hilbert space H is given by a set of positive-semidefinite operators $E = \{E_i\}_{i=1}^n$ with $\sum_{i=1}^n E_i = I_d$ the identity on H. Projective measurement and rank-1 projective measurement are the special cases of POVM. Suppose $E_i = A_i^\dagger A_i$ for any i. We also denote $E = \{A_i\}_{i=1}^n$ with $\sum_{i=1}^n A_i^\dagger A_i = I_d$. Note that $E_i = (U_iA_i)^\dagger(U_iA_i)$ for any unitary $\{U_i\}_{i=1}^n$.

A state ρ is called an incoherent state with respect to E if [18]

$$E_i\rho E_j = 0, \quad \forall i \neq j. \tag{24}$$

Note that this is equivalent to [18]

$$A_i\rho A_j^\dagger = 0, \quad \forall i \neq j. \tag{25}$$

The POVM incoherent channel is defined via the canonical Naimark extension [20]. For POVM $E = \{E_i = A_i^\dagger A_i\}_{i=1}^n$ on d-dimensional Hilbert space H, introduce an n-dimensional Hilbert space H_R with $\{|i\rangle\}_{i=1}^n$ an orthonormal basis of H_R. A canonical Naimark extension $P = \{P_i\}_{i=1}^n$ of $E = \{E_i\}_{i=1}^n$ is described by a unitary matrix V on $H_\varepsilon = H \otimes H_R$ as [20]

$$V = \sum_{ij=1}^n A_{ij} \otimes |i\rangle\langle j|, \tag{26}$$

$$\overline{P} = \{\overline{P}_i = I_d \otimes |i\rangle\langle i|\}_{i=1}^n, \tag{27}$$

$$P_i = V^\dagger \overline{P}_i V, \tag{28}$$

with $\{A_{ij}\}_{ij=1}^n$ satisfying

$$\sum_{i=1}^n A_{ij}^\dagger A_{ik} = \delta_{jk} I_d,$$

$$\sum_{k=1}^n A_{ik} A_{jk}^\dagger = \delta_{ij} I_d,$$

$$A_{i1} = A_i.$$

A channel $\phi \in \mathcal{C}(H)$ is called a POVM incoherent (PI) channel if [20] ϕ allows a Kraus operator decomposition $\phi = \{K_l\}_l$ with $\sum_l K_l^\dagger K_l = I_d$ and there exists a BI channel $\phi' = \{K_l'\}_l \in \mathcal{C}_{\rm BI}(H_\varepsilon)$ with respect to a canonical Naimark extension $P = \{P_i\}_{i=1}^n$ such that

$$K_l\rho K_l^\dagger \otimes |1\rangle\langle 1| = K_l'(\rho \otimes |1\rangle\langle 1|)K_l'^\dagger, \quad \forall l, \tag{29}$$

where $\{K_l'\}_l$ is a BI decomposition of ϕ'. For such case we call $\{K_l\}_l$ a PI decomposition of ϕ.

We denote the set of all PI states as $\mathcal{I}_{\rm P}(H)$, and the set of all PI channels as $\mathcal{C}_{\rm PI}(H)$. Note that $\mathcal{I}_{\rm P}(H)$ may be empty for some POVMs. Note also that such definition of PI operation does not depend on the choice of Naimark extension [20].

A coherence measure for states in Hilbert space H with respect to a general quantum measurement $E = \{E_i\}_{i=1}^n$ should satisfy conditions (P1)–(P4) [20]:

(P1) Faithfulness: $C(\rho, E) \geqslant 0$, with equality if $\rho \in \mathcal{I}_{\rm P}(H)$.

(P2) Monotonicity: $C(\phi_{\rm PI}(\rho), E) \leqslant C(\rho, E)$, $\forall \phi_{\rm PI} \in \mathcal{C}_{\rm PI}(H)$.

(P3) Strong monotonicity: $\sum_l p_l C(\rho_l, P) \leqslant C(\rho, P)$, where $\{K_l\}_l$ is a PI decomposition of a PI channel, $p_l = {\rm tr}(K_l\rho K_l^\dagger)$, $\rho_l = K_l\rho K_l^\dagger/p_l$.

(P4) Convexity: $C(\sum_j p_j\rho_j, E) \leqslant \sum_j p_j C(\rho_j, E)$, $\{\rho_j\}_j \subset \mathcal{S}(H)$, $\{p_j\}_j$ a probability distribution.

Note that the definitions of PI states and PI channels and the conditions (P1)–(P4) all include the projective measurements and the rank-1 projective measurements as special cases [20]. We emphasize that the framework of POVM coherence measure is about POVM $E = \{E_i\}_{i=1}^n$. Hence, any valid coherence measure in terms of $\{A_i\}_i$ should be invariant under the unitary transformation $\{A_i\}_i \to \{U_i A_i\}_i$ for any unitary $\{U_i\}_{i=1}^n$ [20].

An efficient scheme for constructing POVM coherence measures is as follows [18,20]:

$$C(\rho, E) = C(\varepsilon(\rho), \overline{P}), \tag{30}$$

where

$$\varepsilon(\rho) = \sum_{ij=1}^n A_i\rho A_j^\dagger \otimes |i\rangle\langle j|. \tag{31}$$

It can be checked that if $C(\rho_\varepsilon, \overline{P})$ is a unitarily invariant block coherence measure satisfying conditions (B1) to (B4), then $C(\rho, E)$ defined above is a POVM coherence measure satisfying conditions (P1) to (P4) [20]. Here ρ_ε is any state on $H_\varepsilon = H \otimes H_R$. The unitary invariance means that

$$C(\rho_\varepsilon, \overline{P}) = C(U\rho_\varepsilon U^\dagger, U\overline{P}U^\dagger) \tag{32}$$

for any unitary transformation U on H_ε. Employing this scheme and using Propositions 1 to 5, we obtain the following theorem:

Theorem 2. Let $E = \{E_i = A_i^\dagger A_i\}_{i=1}^n$ be a POVM on the Hilbert space H. The following quantities given in (1)–(5) are all POVM coherence measures with respect to E.

(1) l_1 norm of coherence

$$C_{l_1}(\rho, E) = \sum_{i\neq j} \|A_i\rho A_j^\dagger\|_{\rm tr}. \tag{33}$$

(2) For $\alpha \in (0, 1) \cup (1, 2]$, coherence based on Tsallis relative entropy

$$C_{T,\alpha}(\rho, E) = \frac{1}{\alpha - 1}\left\{\sum_i {\rm tr}[(A_i\rho^\alpha A_i^\dagger)^{1/\alpha}] - 1\right\} \tag{34}$$

and

$$\lim_{\alpha\to 1} C_{T,\alpha}(\rho, E) = (\ln 2)C_{\rm rel}(\rho, E), \tag{35}$$

where

$$C_{\rm rel}(\rho, E) = {\rm tr}(\rho \log_2 \rho) - \sum_i {\rm tr}[(A_i\rho A_i^\dagger)\log_2(A_i\rho A_i^\dagger)]. \tag{36}$$

(3) Modified trace norm of coherence

$$C_{\rm tr}(\rho, E) = \min_{\lambda>0, \sigma\in\mathcal{I}_{\rm B}(H_\varepsilon)} \|\varepsilon(\rho) - \lambda\sigma\|_{\rm tr}. \tag{37}$$

(4) Coherence weight

$$C_{\rm w}(\rho, E) = \min_{\sigma\in\mathcal{I}_{\rm B}(H_\varepsilon)} \{s \geqslant 0 | \varepsilon(\rho) \geqslant (1 - s)\sigma\}. \tag{38}$$

(5) For $\alpha \in [\frac{1}{2}, 1)$, coherence based on sandwiched Rényi relative entropy

$$C_{R,\alpha}(\rho, E) = 1 - \max_{\sigma\in\mathcal{I}_{\rm B}(H_\varepsilon)} \left\{{\rm tr}\left[\left(\varepsilon\left(\rho^{\frac{1-\alpha}{2\alpha}}\right)\sigma\varepsilon\left(\rho^{\frac{1-\alpha}{2\alpha}}\right)\right)^\alpha\right]\right\}^{\frac{1}{1-\alpha}}. \tag{39}$$

Proof. To prove the results of Theorem 2, we need to use the results of Propositions 1 to 5. Let $\{|i\rangle\}_{i=1}^n$ be an orthonormal basis for the Hilbert space H_R, and $\overline{P}$ and $\varepsilon(\rho)$ be defined in Eqs. (27) and (31), respectively. Since $C(\rho, E)$

is a POVM coherence measure satisfying conditions (P1) to (P4) if $C(\rho_\varepsilon, \overline{P})$ is a unitarily invariant block coherence measure satisfying conditions (B1) to (B4), we only need to prove the unitary invariance Eq. (32) and show that $C_{l_1}(\rho, E)$, $C_{T,\alpha}(\rho, E)$, $C_{\rm tr}(\rho, E)$, $C_{\rm w}(\rho, E)$, and $C_{R,\alpha}(\rho, E)$ take the forms of Eqs. (33), (34), and (37)–(39) under Eq. (30), respectively.

(1) We prove that $C_{l_1}(\rho_\varepsilon, \overline{P})$ is unitarily invariant. For any unitary U on H_ε, we have

$$\begin{aligned}C_{l_1}(U\rho_\varepsilon U^\dagger, U\overline{P}U^\dagger) &= \sum_{i\neq j}\|U\overline{P}_iU^\dagger U\rho_\varepsilon U^\dagger U\overline{P}_jU^\dagger\|_{\rm tr}\\ &= \sum_{i\neq j}\|\overline{P}_i\rho_\varepsilon\overline{P}_j\|_{\rm tr} = C_{l_1}(\rho_\varepsilon, \overline{P}),\end{aligned}$$

where we have used the fact that the trace norm is unitarily invariant. It is easy to see that $C_{l_1}(\rho, E)$ have the form of Eq. (33).

(2) It is easy to see that $C_{T,\alpha}(\rho_\varepsilon, \overline{P})$ is unitarily invariant. Now we prove that $C_{T,\alpha}(\rho, E)$ has the form of Eq. (34) under Eq. (30).

For the unitary transformation V defined in Eq. (26),

$$\varepsilon_V(\rho) = V(\rho\otimes|1\rangle\langle 1|)V^\dagger = \sum_{ij}A_i\rho A_j^\dagger\otimes|i\rangle\langle j| = \varepsilon(\rho).$$

As a result,

$$\begin{aligned}&{\rm tr}[(\overline{P}_i(\varepsilon_V(\rho))^\alpha\overline{P}_i)^{1/\alpha}]\\ &= {\rm tr}[(\overline{P}_iV(\rho^\alpha\otimes|1\rangle\langle 1|)V_i^\dagger\overline{P})^{1/\alpha}]\\ &= {\rm tr}\left[\left(\overline{P}_i\left(\sum_{jk}A_j\rho^\alpha A_k^\dagger\otimes|j\rangle\langle k|\right)\overline{P}_i\right)^{1/\alpha}\right]\\ &= {\rm tr}[(A_i\rho A_i^\dagger\otimes|i\rangle\langle i|)^{1/\alpha}]\\ &= {\rm tr}[(A_i\rho A_i^\dagger)^{1/\alpha}].\end{aligned}$$

Hence, $C_{T,\alpha}(\rho, E)$ has the form of Eq. (34). Equation (35) can be proved as Corollary 1.

(3) It is easy to see that $C_{\rm tr}(\rho, E)$ has the form of Eq. (37). Now we show that $C_{\rm tr}(\rho_\varepsilon, \overline{P})$ is unitarily invariant. Note that

$$C_{\rm tr}(\rho_\varepsilon, \overline{P}) = \min_{\lambda>0,\sigma}\left\|\rho_\varepsilon - \lambda\sum_{i=1}^n\overline{P}_i\sigma\overline{P}_i\right\|_{\rm tr},$$

where σ is any density operator on H_ε.

For any unitary U on H_ε, we have

$$\begin{aligned}&C_{\rm tr}(U\rho_\varepsilon U^\dagger, U\overline{P}U^\dagger)\\ &= \min_{\lambda>0,\sigma}\left\|U\rho_\varepsilon U^\dagger - \lambda\sum_{i=1}^nU\overline{P}_iU^\dagger\sigma U\overline{P}_iU^\dagger\right\|_{\rm tr}\\ &= \min_{\lambda>0,\sigma}\left\|\rho_\varepsilon - \lambda\sum_{i=1}^n\overline{P}_iU^\dagger\sigma U\overline{P}_i\right\|_{\rm tr}\\ &= \min_{\lambda>0,\sigma}\left\|\rho_\varepsilon - \lambda\sum_{i=1}^n\overline{P}_i\sigma\overline{P}_i\right\|_{\rm tr}\\ &= C_{\rm tr}(\rho_\varepsilon, \overline{P}),\end{aligned}$$

where we have used the facts that the trace norm is unitarily invariant and $\{\sigma : \sigma\in\mathcal{S}(H)\} = \{U^\dagger\sigma U : \sigma\in\mathcal{S}(H)\}$.

(4) It is easy to see that $C_{\rm w}(\rho, E)$ has the form of Eq. (38). Next we show that $C_{\rm w}(\rho_\varepsilon, \overline{P})$ is unitarily invariant. Note that

$$C_{\rm w}(\rho_\varepsilon, \overline{P}) = \min_\sigma\left\{s\geqslant 0|\rho_\varepsilon\geqslant(1-s)\sum_{i=1}^n\overline{P}_i\sigma\overline{P}_i\right\},$$

where σ is any density operator on H_ε.

For any unitary U on H_ε, we have

$$\begin{aligned}&C_{\rm w}(U\rho_\varepsilon U^\dagger, U\overline{P}U^\dagger)\\ &= \min_\sigma\left\{s\geqslant 0|U\rho_\varepsilon U^\dagger\geqslant(1-s)\sum_{i=1}^nU\overline{P}_iU^\dagger\sigma U\overline{P}_iU^\dagger\right\}\\ &= \min_\sigma\left\{s\geqslant 0|\rho_\varepsilon\geqslant(1-s)\sum_{i=1}^n\overline{P}U^\dagger\sigma U\overline{P}_i\right\}\\ &= \min_\sigma\left\{s\geqslant 0|\rho_\varepsilon\geqslant(1-s)\sum_{i=1}^n\overline{P}_i\sigma\overline{P}_i\right\}\\ &= C_{\rm w}(\rho_\varepsilon, \overline{P}),\end{aligned}$$

which completes the proof.

(5) It is easy to see that $C_{R,\alpha}(\rho, E)$ has the form of Eq. (39). Similarly to the proof of (3), one can show that $C_{\rm w}(\rho_\varepsilon, \overline{P})$ is unitarily invariant. ■

We remark that the coherence measure $C_{l_1}(\rho, P)$ was proposed in Ref. [23]. in Ref. [20] the authors conjectured that $C_{l_1}(\rho, E)$ is a well-defined POVM coherence measure satisfying the conditions (P1)–(P4). Combining with our result of Proposition 1, we have strictly proved in Theorem 2 that $C_{l_1}(\rho, E)$ is indeed a well-defined POVM coherence measure.

IV. SUMMARY

We have established an alternative framework for quantifying the coherence with respect to projective measurements, and provided several coherence measures with respect to projective measurements. We then obtained several coherence measures with respect to general POVM measurements, from which a conjecture has been verified concerning the coherence measure $C_{l_1}(\rho, E)$. The coherence with respect to POVM measurements has operational significance. Similarly to the robustness of coherence and the maximum relative entropy of coherence which characterize the success probability of subchannel discrimination [4,8], it would be also an interesting issue to explore the operational meanings of the POVM coherence measures given in Theorem 2. Our results may highlight further investigations on the coherence of quantum states and the applications in quantum information processing.

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APPENDIX

1. Proof of Proposition 1

From the definition of the BI state and the properties of the trace norm, $C_{l_1}(\rho, P)$ satisfies the condition (B1). It satisfies the conditions (B4) and (B5) due to the properties of the trace norm. Since (B3) and (B4) imply (B2), we only need to prove that $C_{l_1}(\rho, P)$ fulfills (B3).

For any BI channel ϕ with BI decomposition $\phi_{\rm BI} = \{K_l\}_l$, $\sum_l K_l^\dagger K_l = I_d$, each K_l has the form [20]

$$K_l = \sum_{i=1}^{n} P_{f_l(i)} M_l P_i, \tag{A1}$$

where $f_l(i)$ is a function on $\{i\}_{i=1}^n$, and M_l is a matrix on H. Denote $p_l = {\rm tr}(K_l \rho K_l^\dagger)$, $\rho_l = K_l \rho K_l^\dagger / p_l$. We have

$$\sum_l p_l C_{l_1}(\rho_l, P) = \sum_{l,i\neq j} \|P_i K_l \rho K_l^\dagger P_j\|_{\rm tr} = \sum_{l,i\neq j} \left\|P_i K_l \sum_{i'\neq j'} P_{i'} \rho P_{j'} K_l^\dagger P_j\right\|_{\rm tr} \tag{A2}$$

$$\leqslant \sum_{l,ij,i'\neq j'} \|P_i K_l P_{i'} \rho P_{j'} K_l^\dagger P_j\|_{\rm tr} = \sum_{l,i'\neq j'} \left\|P_{f_l(i')} K_l P_{i'} \rho P_{j'} K_l^\dagger P_{f_l(j')}\right\|_{\rm tr} \tag{A3}$$

$$= \sum_{l,i'\neq j'} \left\|P_{f_l(i')} K_l \sum_k s_{i'j'k} |\psi_{i'j'k}\rangle\langle\overline{\psi}_{i'j'k}| K_l^\dagger P_{f_l(j')}\right\|_{\rm tr} \tag{A4}$$

$$\leqslant \sum_{lk,i'\neq j'} s_{i'j'k} \left\|P_{f_l(i')} K_l |\psi_{i'j'k}\rangle\langle\overline{\psi}_{i'j'k}| K_l^\dagger P_{f_l(j')}\right\|_{\rm tr}$$

$$= \sum_{k,i'\neq j'} s_{i'j'k} \sum_l \sqrt{\langle\psi_{i'j'k}|K_l^\dagger P_{f_l(i')} K_l|\psi_{i'j'k}\rangle\langle\overline{\psi}_{i'j'k}|K_l^\dagger P_{f_l(j')} K_l|\overline{\psi}_{i'j'k}\rangle} \tag{A5}$$

$$\leqslant \sum_{k,i'\neq j'} s_{i'j'k} \sqrt{\sum_l \langle\psi_{i'j'k}|K_l^\dagger P_{f_l(i')} K_l|\psi_{i'j'k}\rangle} \sqrt{\sum_{l'} \langle\overline{\psi}_{i'j'k}|K_{l'}^\dagger P_{f_{l'}(j')} K_{l'}|\overline{\psi}_{i'j'k}\rangle} \tag{A6}$$

$$= \sum_{k,i'\neq j'} s_{i'j'k} \sqrt{\langle\psi_{i'j'k}| \sum_l K_l^\dagger P_{f_l(i')} K_l|\psi_{i'j'k}\rangle} \sqrt{\langle\overline{\psi}_{i'j'k}| \sum_{l'} K_{l'}^\dagger P_{f_{l'}(j')} K_{l'}|\overline{\psi}_{i'j'k}\rangle}$$

$$\leqslant \sum_{k,i'\neq j'} s_{i'j'k} \sqrt{\langle\psi_{i'j'k}|I_m|\psi_{i'j'k}\rangle} \sqrt{\langle\overline{\psi}_{i'j'k}|I_m|\overline{\psi}_{i'j'k}\rangle} = \sum_{k,i'\neq j'} s_{i'j'k} = \sum_{i'\neq j'} \|P_{i'} \rho P_{j'}\|_{\rm tr} = C_{l_1}(\rho, P). \tag{A7}$$

In Eq. (A2) we have used the property that $\{K_l\}_l$ is a BI decomposition, that is, $P_i K_l (\sum_{i'} P_{i'} \rho P_{i'}) K_l^\dagger P_j = 0$ for any $i \neq j$. In Eq. (A3) we have used $P_i K_l P_{i'} = P_i P_{f_l(i')} K_l P_{i'} = \delta_{i,f_l(i')} P_{f_l(i')} K_l P_{i'}$ which is a result of Eq. (A1). In Eq. (A4) we have used the singular value decomposition, $P_{i'} \rho P_{j'} = \sum_k s_{i'j'k} |\psi_{i'j'k}\rangle\langle\overline{\psi}_{i'j'k}|$ with $\{s_{i'j'k}\}_k$ the singular values, $\{|\psi_{i'j'k}\rangle\}_k$ ($\{|\overline{\psi}_{i'j'k}\rangle\}_k$) a set of orthonormal vectors. In Eq. (A5) we have taken into account the fact that $\||\psi\rangle\langle\varphi|\|_{\rm tr} = \sqrt{\langle\psi|\psi\rangle\langle\varphi|\varphi\rangle}$ for any pure states $|\psi\rangle$ and $|\varphi\rangle$. In Eq. (A6) we have used the Cauchy-Schwarz inequality $\sum_l \sqrt{a_l b_l} \leqslant \sqrt{\sum_l a_l} \sqrt{\sum_{l'} b_{l'}}$ with $a_l \geqslant 0$ and $b_l \geqslant 0$. In Eq. (A7) we have used the fact that $\sum_l K_l^\dagger P_{f_l(i')} K_l \leqslant I_m$ since $P_{f_l(i')} \leqslant I_m$ and $\sum_l K_l^\dagger K_l = I_m$.

2. Proof of Proposition 2

For $\alpha > 0$, the quantum Tsallis relative entropy is defined as

$$D_{T,\alpha}(\rho\|\sigma) = \frac{{\rm tr}(\rho^\alpha \sigma^{1-\alpha}) - 1}{\alpha - 1}, \quad \rho, \sigma \in \mathcal{S}(H),$$
$${\rm supp}(\rho) \subset {\rm supp}(\sigma) \quad \text{when } \alpha \geqslant 1, \tag{A8}$$

where ${\rm supp}(\rho) = \{|\psi\rangle | \rho|\psi\rangle \neq 0\}$ is the support of ρ.

It is shown that for $\alpha > 0$ [33],

$$D_{T,\alpha}(\rho\|\sigma) \geqslant 0, \quad D_{T,\alpha}(\rho\|\sigma) = 0 \Leftrightarrow \rho = \sigma. \tag{A9}$$

Also, $D_\alpha(\rho\|\sigma)$ is monotonic under CPTP maps for $\alpha \in (0; 2]$ [33],

$$D_{T,\alpha}(\phi(\rho)\|\phi(\sigma)) \leqslant D_{T,\alpha}(\rho\|\sigma). \tag{A10}$$

Define

$$D_{T,\alpha}(\rho) = \min_{\sigma \in \mathcal{I}_{\rm B}(H)} D_{T,\alpha}(\rho\|\sigma). \tag{A11}$$

We now prove that

$$D_{T,\alpha}(\rho) = \frac{\left\{\sum_i {\rm tr}\left[(P_i \rho^\alpha P_i)^{\frac{1}{\alpha}}\right]\right\}^\alpha - 1}{\alpha - 1}. \tag{A12}$$

To go ahead, we need the lemmas below.

Lemma 1. Hölder inequality. Suppose $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$, are all positive real numbers, then

(1) when $\alpha \in (0, 1)$,

$$\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^{\frac{1}{\alpha}}\right)^{\alpha} \left(\sum_{i=1}^{n} b_i^{\frac{1}{1-\alpha}}\right)^{1-\alpha}, \tag{A13}$$

and the equality holds if and only if $a_i^{\frac{1}{\alpha}}/b_i^{\frac{1}{1-\alpha}} = a_j^{\frac{1}{\alpha}}/b_j^{\frac{1}{1-\alpha}}$ for any i, j;

(2) when $\alpha > 1$,

$$\sum_{i=1}^{n} a_i b_i \geqslant \left(\sum_{i=1}^{n} a_i^{\frac{1}{\alpha}}\right)^{\alpha} \left(\sum_{i=1}^{n} b_i^{\frac{1}{1-\alpha}}\right)^{1-\alpha}, \tag{A14}$$

and the equality holds if and only if $a_i^{\frac{1}{\alpha}}/b_i^{\frac{1}{1-\alpha}} = a_j^{\frac{1}{\alpha}}/b_j^{\frac{1}{1-\alpha}}$ for any i, j.

Lemma 2 (Ref. [34]). For $r \times r$ positive-semidefinite matrices M and N, it holds that

$$\sum_{j=1}^{r} \lambda_{r+1-j}^{\downarrow}(M)\lambda_j^{\downarrow}(N) \leqslant \mathrm{tr}(MN) \leqslant \sum_{j=1}^{r} \lambda_j^{\downarrow}(M)\lambda_j^{\downarrow}(N), \tag{A15}$$

where $\{\lambda_j^{\downarrow}(M)\}_j$ are the eigenvalues of M in decreasing order.

Now for $\alpha \in (0, 1)$ and $\sigma \in \mathcal{I}_{\mathrm{B}}(H)$, we have

$$\begin{aligned} \mathrm{tr}(\rho^\alpha \sigma^{1-\alpha}) &= \mathrm{tr}\left[\rho^\alpha \sum_{i=1}^{n} (P_i \sigma P_i)^{1-\alpha}\right] \\ &= \sum_{i=1}^{n} q_i^{1-\alpha} \mathrm{tr}\big(\rho^\alpha \sigma_i^{1-\alpha}\big) \\ &\leqslant \left\{\sum_{i=1}^{n} \left[\mathrm{tr}\big(\rho^\alpha \sigma_i^{1-\alpha}\big)\right]^{\frac{1}{\alpha}}\right\}^{\alpha}, \end{aligned} \tag{A16}$$

where $q_i = \mathrm{tr}(P_i \sigma P_i)$, $\sigma_i = P_i \sigma P_i / q_i$, the Hölder inequality has been used, and the equality holds if and only if there exists constant $C \geqslant 0$ such that $q_i = C[\mathrm{tr}(\rho^\alpha \sigma_i^{1-\alpha})]^{\frac{1}{\alpha}}$ for any i. Furthermore,

$$\begin{aligned} \mathrm{tr}\big(\rho^\alpha \sigma_i^{1-\alpha}\big) &= \mathrm{tr}\big(\rho^\alpha P_i \sigma_i^{1-\alpha} P_i\big) \\ &\leqslant \sum_{j=1}^{m_i} \lambda_j^{\downarrow}(P_i \rho^\alpha P_i)\lambda_j^{\downarrow}\big(\sigma_i^{1-\alpha}\big) \\ &= \sum_{j=1}^{m_i} \lambda_j^{\downarrow}(P_i \rho^\alpha P_i)[\lambda_j^{\downarrow}(\sigma_i)]^{1-\alpha} \\ &\leqslant \left\{\sum_{j=1}^{m_i} [\lambda_j^{\downarrow}(P_i \rho^\alpha P_i)]^{\frac{1}{\alpha}}\right\}^{\alpha} \left\{\sum_{j=1}^{m_i} [(\lambda_j^{\downarrow}(\sigma_i))^{1-\alpha}]^{\frac{1}{1-\alpha}}\right\}^{1-\alpha} \\ &= \left\{\mathrm{tr}\left[(P_i \rho^\alpha P_i)^{\frac{1}{\alpha}}\right]\right\}^{\alpha}, \end{aligned} \tag{A17}$$

where Lemma 1 and Lemma 2 have been used. It is easy to check that when

$$\sigma = \frac{\sum_{i=1}^{n} (P_i \rho^\alpha P_i)^{\frac{1}{\alpha}}}{\sum_{i=1}^{n} \mathrm{tr}\left[(P_i \rho^\alpha P_i)^{\frac{1}{\alpha}}\right]}, \tag{A18}$$

Eq. (A11) achieves Eq. (A12). As a result we get Eq. (A12).

For $\alpha > 1$, we have

$$\begin{aligned} \mathrm{tr}(\rho^\alpha \sigma^{1-\alpha}) &= \mathrm{tr}\left[\rho^\alpha \sum_{i} (P_i \sigma P_i)^{1-\alpha}\right] \\ &= \sum_{i} q_i^{1-\alpha} \mathrm{tr}\big(\rho^\alpha \sigma_i^{1-\alpha}\big) \\ &\geqslant \left\{\sum_{i} \left[\mathrm{tr}\big(\rho^\alpha \sigma_i^{1-\alpha}\big)\right]^{\frac{1}{\alpha}}\right\}^{\alpha}, \end{aligned} \tag{A19}$$

and the equality holds if and only if there exists a constant $C \geqslant 0$ such that $q_i = C[\mathrm{tr}(\rho^\alpha \sigma_i^{1-\alpha})]^{\frac{1}{\alpha}}$ for any i. Moreover,

$$\begin{aligned} &\mathrm{tr}\big(\rho^\alpha \sigma_i^{1-\alpha}\big) \\ &\quad= \mathrm{tr}\big(\rho^\alpha P_i \sigma_i^{1-\alpha} P_i\big) \\ &\quad\geqslant \sum_{j=1}^{m_i} \lambda_j^{\downarrow}(P_i \rho^\alpha P_i)\lambda_{m_i+1-j}^{\downarrow}\big(\sigma_i^{1-\alpha}\big) \\ &\quad= \sum_{j=1}^{m_i} \lambda_j^{\downarrow}(P_i \rho^\alpha P_i)\big[\lambda_{m_i+1-j}^{\downarrow}(\sigma_i)\big]^{1-\alpha} \\ &\quad\geqslant \left\{\sum_{j=1}^{m_i} [\lambda_j^{\downarrow}(P_i \rho^\alpha P_i)]^{\frac{1}{\alpha}}\right\}^{\alpha} \left\{\sum_{j=1}^{m_i} \left[\big(\lambda_{m_i+1-j}^{\downarrow}(\sigma_i)\big)^{1-\alpha}\right]^{\frac{1}{1-\alpha}}\right\}^{1-\alpha} \\ &\quad= \left\{\mathrm{tr}\left[(P_i \rho^\alpha P_i)^{\frac{1}{\alpha}}\right]\right\}^{\alpha}. \end{aligned} \tag{A20}$$

In the above derivation, we have used Lemma 1 and Lemma 2. Again, when σ takes the value in Eq. (A18), Eq. (A11) achieves Eq. (A12). As a result we get Eq. (A12).

From Eqs. (A11) and (A16) we see that $D_{T,\alpha}(\rho) \geqslant 0$ and $D_{T,\alpha}(\rho) = 0$ if and only if $\rho \in \mathcal{I}_{\mathrm{B}}(H)$. Then from Eq. (A12) we have

$$\frac{\left\{\sum_i \mathrm{tr}\left[(P_i \rho^\alpha P_i)^{\frac{1}{\alpha}}\right]\right\}^{\alpha} - 1}{\alpha - 1} \geqslant 0,$$

namely,

$$\frac{\sum_i \mathrm{tr}\left[(P_i \rho^\alpha P_i)^{\frac{1}{\alpha}}\right] - 1}{\alpha - 1} \geqslant 0,$$

with the equality holding if and only if $\rho \in \mathcal{I}_{\mathrm{B}}(H)$, which proves that $C_{T,\alpha}(\rho, P)$ satisfies (B1).

For any $\phi_{\mathrm{BI}} \in \mathcal{C}_{\mathrm{BI}}(H)$, from Eqs. (A15) and (A16) we have

$$\begin{aligned} D_{T,\alpha}(\rho) &= \min_{\sigma \in \mathcal{I}_{\mathrm{B}}(H)} D_{T,\alpha}(\rho\|\sigma) = D_{T,\alpha}(\rho\|\sigma^*) \\ &\geqslant D_{T,\alpha}(\phi_{\mathrm{BI}}(\rho)\|\phi_{\mathrm{BI}}(\sigma^*)) \\ &\geqslant \min_{\sigma \in \mathcal{I}_{\mathrm{B}}(H)} D_{T,\alpha}(\phi_{\mathrm{BI}}(\rho)\|\sigma) \\ &= D_{T,\alpha}(\phi_{\mathrm{BI}}(\rho)), \end{aligned} \tag{A21}$$

where $\sigma^* \in \mathcal{I}_{\mathrm{B}}(H)$ such that $\min_{\sigma \in \mathcal{I}_{\mathrm{B}}(H)} D_{T,\alpha}(\rho\|\sigma) = D_{T,\alpha}(\rho\|\sigma^*)$.

From Eq. (A12), Eq. (A21) is equivalent to

$$\begin{aligned} &\frac{\left\{\sum_i \mathrm{tr}\left[(P_i \rho^\alpha P_i)^{\frac{1}{\alpha}}\right]\right\}^{\alpha} - 1}{\alpha - 1} \\ &\quad\leqslant \frac{\left\{\sum_i \mathrm{tr}\left[\big(P_i^\alpha (\phi_{\mathrm{BI}}(\rho))^\alpha P_i\big)^{\frac{1}{\alpha}}\right]\right\}^{\alpha} - 1}{\alpha - 1}, \end{aligned}$$

which is further equivalent to

$$\frac{\sum_i \mathrm{tr}\big[(P_i\rho^\alpha P_i)^{\frac{1}{\alpha}}\big]-1}{\alpha-1} \leqslant \frac{\sum_i \mathrm{tr}\big[\big(P_i^\alpha(\phi_{\mathrm{BI}}(\rho))^\alpha P_i\big)^{\frac{1}{\alpha}}\big]-1}{\alpha-1}.$$

We then proved that $C_{T,\alpha}(\rho,P)$ satisfies (B2).

Now we prove that $C_{T,\alpha}(\rho,P)$ also satisfies (B5). Suppose $\rho=p_1\rho_1\oplus p_2\rho_2$ as described in (B5). Then

$$\begin{aligned}\sum_{i=1}^n \mathrm{tr}\big[(P_i\rho^\alpha P_i)^{\frac{1}{\alpha}}\big] &= p_1\sum_{k_1}\mathrm{tr}\big[\big(P_{k_1}\rho_1^\alpha P_{k_1}\big)^{\frac{1}{\alpha}}\big]+p_2\sum_{k_2}\mathrm{tr}\big[\big(P_{k_2}\rho_2^\alpha P_{k_2}\big)^{\frac{1}{\alpha}}\big]\\ &= p_1\sum_{i=1}^n\mathrm{tr}\big[\big(P_i\rho_1^\alpha P_i\big)^{\frac{1}{\alpha}}\big]+p_2\sum_{i=1}^n\mathrm{tr}\big[\big(P_i\rho_2^\alpha P_i\big)^{\frac{1}{\alpha}}\big]. \quad (A22)\end{aligned}$$

Substituting (A22) into Eq. (17), we then proved that $C_{T,\alpha}(\rho,P)$ satisfies (B5).

3. Proof of Corollary 1

Set $\alpha=1+\varepsilon$. Consider the Taylor expansions around $\varepsilon=0$,

$$\begin{aligned}M^{1+\varepsilon}&=M+\varepsilon M\ln M+o(\varepsilon^2),\\ \ln(M+\varepsilon N)&=\ln M+o(\varepsilon),\\ \frac{1}{1+\varepsilon}&=1-\varepsilon+o(\varepsilon^2),\end{aligned}$$

where M, N are Hermitian matrices, and $o(\varepsilon)$ denotes the infinitesimal term with the order ε or higher around $\varepsilon=0$. We have $P_i\rho^\alpha P_i=P_i(\rho+\varepsilon\rho\ln\rho+o(\varepsilon^2))P_i$. Therefore,

$$\begin{aligned}&\mathrm{tr}\big[(P_i\rho^\alpha P_i)^{\frac{1}{\alpha}}\big]\\ &=\mathrm{tr}\big[(P_i\rho^\alpha P_i)^{1-\varepsilon+o(\varepsilon^2)}\big]\\ &=\mathrm{tr}[(P_i\rho^\alpha P_i)-\varepsilon(P_i\rho^\alpha P_i)\ln(P_i\rho^\alpha P_i)+o(\varepsilon^2)]\\ &=\mathrm{tr}[P_i\rho P_i+\varepsilon P_i(\rho\ln\rho)P_i-\varepsilon(P_i\rho P_i)\ln(P_i\rho P_i)\\ &\quad+o(\varepsilon^2)].\end{aligned}$$

Applying L'Hospital's rule to Eq. (17), we have

$$\begin{aligned}&\lim_{\alpha\to1}C_{T,\alpha}(\rho,P)\\ &=\lim_{\alpha\to1}\frac{d}{d\alpha}\sum_i\mathrm{tr}[(P_i\rho^\alpha P_i)^{1/\alpha}]\\ &=\sum_i\mathrm{tr}[P_i(\rho\ln\rho)P_i-(P_i\rho P_i)\ln(P_i\rho P_i)]\\ &=\mathrm{tr}(\rho\ln\rho)-\sum_i\mathrm{tr}[(P_i\rho P_i)\ln(P_i\rho P_i)]\\ &=(\ln 2)C_{\mathrm{rel}}(\rho,P).\end{aligned}$$

4. Proof of Proposition 3

Obviously, the condition (B1) is satisfied. (B2) is also satisfied as a consequence of the fact that $\|M\|_{\mathrm{tr}}\geqslant\|\phi(M)\|_{\mathrm{tr}}$ for any CPTP map ϕ and any Hermitian matrix M [35]. Concerning (B5), we consider $\rho=p_1\rho_1\oplus p_2\rho_2$ as described in (B5). Any $\sigma\in\mathcal{I}_{\mathrm{BI}}(H)$ can be written as

$$\sigma=q_1\sigma_1\oplus q_2\sigma_2, \quad (A23)$$

with $q_1\geqslant0$, $q_2\geqslant0$, $q_1+q_2=1$, and $\sigma_1,\sigma_2\in\mathcal{S}(H)$, $\sigma_1P_{k_2}=\sigma_2P_{k_1}=0$ for any k_1 and k_2. It follows that

$$\begin{aligned}&C(p_1\rho_1\oplus p_2\rho_2,P)\\ &=\min_{\lambda>0,q_1,\sigma_1,\sigma_2}\|p_1\rho_1\oplus p_2\rho_2-\lambda(q_1\sigma_1\oplus q_2\sigma_2)\|_{\mathrm{tr}}\\ &=\min_{\lambda>0,q_1,\sigma_1,\sigma_2}\left(p_1\left\|\rho_1-\frac{\lambda q_1}{p_1}\sigma_1\right\|_{\mathrm{tr}}+p_2\left\|\rho_2-\frac{\lambda q_2}{p_2}\sigma_2\right\|_{\mathrm{tr}}\right)\\ &=p_1\min_{\lambda_1>0,\sigma_1}\left\|\rho_1-\frac{\lambda q_1}{p_1}\sigma_1\right\|_{\mathrm{tr}}\\ &\quad+p_2\min_{\lambda_2>0,\sigma_2}\left\|\rho_2-\frac{\lambda q_2}{p_2}\sigma_2\right\|_{\mathrm{tr}}\\ &=p_1C(\rho_1)+p_2C(\rho_2,P),\end{aligned}$$

where we have used the facts that $\sigma_1,\sigma_2\in\mathcal{S}(H)$, $\{q_1,q_2\}$ is a probability distribution, $\lambda_1=\frac{\lambda q_1}{p_1}$ and $\lambda_2=\frac{\lambda q_2}{p_2}$.

5. Proof of Proposition 4

It can be proved that $C_{\mathrm{w}}(\rho,P)$ fulfills the conditions (B1), (B3), and (B4) by using a way similar to that adopted in Ref. [31]. Here we equivalently prove that $C_{\mathrm{w}}(\rho,P)$ fulfills (B1), (B2), and (B5). (B1) is evidently satisfied. To prove (B2), suppose $\{K_l\}_l\in\mathcal{C}_{\mathrm{BI}}(H)$ with $\{K_l\}_l$ a BI decomposition. Then there exists $\sigma\in\mathcal{I}_{\mathrm{B}}(H)$ such that

$$\begin{aligned}\rho&\geqslant[1-C_{\mathrm{w}}(\rho,P)]\sigma,\\ \sum_l K_l\rho K_l^\dagger&\geqslant[1-C_{\mathrm{w}}(\rho,P)]\sum_l K_l\sigma K_l^\dagger.\end{aligned}$$

Since $\sum_l K_l\sigma K_l^\dagger\in\mathcal{I}_{\mathrm{B}}(H)$, we obtain $C_{\mathrm{w}}(\sum_l K_l\rho K_l^\dagger,P)\leqslant C_{\mathrm{w}}(\rho,P)$, which proves that (B2) is satisfied.

To prove (B5), let us consider again $\rho=p_1\rho_1\oplus p_2\rho_2$ as described in (B5). Then there exists $\sigma\in\mathcal{I}_{\mathrm{B}}(H)$ such that

$$\begin{aligned}\rho&\geqslant[1-C_{\mathrm{w}}(\rho,P)]\sigma,\\ \sum_{k_1}P_{k_1}\rho P_{k_1}&\geqslant[1-C_{\mathrm{w}}(\rho,P)]\sum_{k_1}P_{k_1}\sigma P_{k_1},\\ \sum_{k_2}P_{k_2}\rho P_{k_2}&\geqslant[1-C_{\mathrm{w}}(\rho,P)]\sum_{k_2}P_{k_2}\sigma P_{k_2}.\end{aligned}$$

Denote $\sum_{k_1}P_{k_1}\sigma P_{k_1}=q_1\sigma_1$, $\sum_{k_1}P_{k_1}\sigma P_{k_1}=q_2\sigma_2$, with $\{q_1,q_2\}$ a probability distribution, $\sigma_1,\sigma_2\in\mathcal{I}_{\mathrm{B}}(H)$. Since

$\sum_{k_1} P_{k_1}\rho P_{k_1} = p_1\rho_1$, $\sum_{k_2} P_{k_2}\rho P_{k_2} = p_2\rho_2$, we have

$$\begin{aligned} \rho_1 &\geqslant \frac{[1-C_w(\rho,P)]q_1}{p_1}\sigma_1, \\ \rho_2 &\geqslant \frac{[1-C_w(\rho,P)]q_2}{p_2}\sigma_2, \\ C_w(\rho_1,P) &\leqslant 1-\frac{[1-C_w(\rho,P)]q_1}{p_1}, \\ C_w(\rho_2,P) &\leqslant 1-\frac{[1-C_w(\rho,P)]q_2}{p_2}, \\ p_1C_w(\rho_1,P) &+ p_2C_w(\rho_2,P) \leqslant C_w(\rho,P). \end{aligned} \tag{A24}$$

Conversely, there exist $\sigma_1', \sigma_2' \in \mathcal{I}_B(H)$ such that

$$\begin{aligned} \rho_1 &\geqslant [1-C_w(\rho_1,P)]\sigma_1', \\ \rho_2 &\geqslant [1-C_w(\rho_2,P)]\sigma_2'. \end{aligned}$$

It follows that

$$\begin{aligned} &p_1\rho_1 \oplus p_2\rho_2 \\ &\quad\geqslant p_1[1-C_w(\rho_1,P)]\sigma_1' + p_2[1-C_w(\rho_2,P)]\sigma_2', \\ &\quad C_w(\rho,P) \leqslant p_1C_w(\rho_1,P) + p_2C_w(\rho_2,P). \end{aligned} \tag{A25}$$

Equations (A26) and (A27) imply (B5), which completes the proof.

6. Proof of Proposition 5

This proof is a generalization of the proof for Theorem 1 in Ref. [14]. For $\alpha \in [\frac{1}{2}, 1)$, $\sigma, \rho \in \mathcal{S}(H)$, the sandwiched Rényi relative entropy is defined as [36,37]

$$F_\alpha(\sigma\|\rho) = \frac{\ln \mathrm{tr}\big[\big(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]}{\alpha-1}.$$

It is shown that [37,38] for $\alpha \in [\frac{1}{2}, 1)$, $F_\alpha(\sigma\|\rho) \geqslant 0$, where the equality holds if and only if $\sigma = \rho$. This is equivalent to that

$$\mathrm{tr}\big[\big(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big] \leqslant 1,$$

and to that

$$\big\{\mathrm{tr}\big[\big(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]\big\}^{\frac{1}{1-\alpha}} \leqslant 1,$$

with the equality holding if and only if $\sigma = \rho$. This says that $C_{R,\alpha}(\rho,P)$ satisfies (B1).

For $\alpha \in [\frac{1}{2}, 1)$, it has been shown that [37,39] for $\sigma, \rho \in \mathcal{S}(H)$, and any CPTP map ϕ,

$$F_\alpha(\phi(\sigma)\|\phi(\rho)) \leqslant F_\alpha(\sigma\|\rho).$$

This implies

$$\begin{aligned} &\mathrm{tr}\big[\big((\phi(\rho))^{\frac{1-\alpha}{2\alpha}}\phi(\sigma)(\phi(\rho))^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big] \\ &\quad\geqslant \mathrm{tr}\big[\big(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big], \\ &\big\{\mathrm{tr}\big[\big((\phi(\rho))^{\frac{1-\alpha}{2\alpha}}\phi(\sigma)(\phi(\rho))^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]\big\}^{\frac{1}{1-\alpha}} \\ &\quad\geqslant \big\{\mathrm{tr}\big[\big(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]\big\}^{\frac{1}{1-\alpha}}. \end{aligned}$$

For any BI map ϕ_{BI}, there exists $\sigma^* \in \mathcal{I}_B(H)$ such that

$$\begin{aligned} &\max_{\sigma\in\mathcal{I}_B(H)} \big\{\mathrm{tr}\big[\big(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]\big\}^{\frac{1}{1-\alpha}} \\ &\quad= \big\{\mathrm{tr}\big[\big(\rho^{\frac{1-\alpha}{2\alpha}}\sigma^*\rho^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]\big\}^{\frac{1}{1-\alpha}} \\ &\quad\leqslant \big\{\mathrm{tr}\big[\big((\phi_{BI}(\rho))^{\frac{1-\alpha}{2\alpha}}\phi_{BI}(\sigma^*)(\phi_{BI}(\rho))^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]\big\}^{\frac{1}{1-\alpha}} \\ &\quad\leqslant \max_{\sigma\in\mathcal{I}_B(H)} \big\{\mathrm{tr}\big[\big((\phi_{BI}(\rho))^{\frac{1-\alpha}{2\alpha}}\sigma(\phi_{BI}(\rho))^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]\big\}^{\frac{1}{1-\alpha}}. \end{aligned}$$

This proves that $C_{R,\alpha}(\rho,P)$ satisfies (B2).

Next we prove $C_{R,\alpha}(\rho,P)$ satisfies (B5). Consider $\rho = p_1\rho_1 \oplus p_2\rho_2$ as described in (B5). As any $\sigma \in \mathcal{I}_{BI}(H)$ can be written as Eq. (A23), it follows that

$$\begin{aligned} &\max_{\sigma\in\mathcal{I}_B(H)} \mathrm{tr}\big[\big(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big] \\ &\quad= \max_{q_1,q_2}\Big\{\big(p_1^{1-\alpha}q_1^\alpha\big)\max_{\sigma_1}\mathrm{tr}\big[\big(\rho_1^{\frac{1-\alpha}{2\alpha}}\sigma_1\rho_1^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big] \\ &\qquad+\big(p_2^{1-\alpha}q_2^\alpha\big)\max_{\sigma_2}\mathrm{tr}\big[\big(\rho_2^{\frac{1-\alpha}{2\alpha}}\sigma_2\rho_2^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]\Big\} \\ &\quad= \max_{q_1,q_2}\big\{p_1^{1-\alpha}q_1^\alpha t_1 + p_2^{1-\alpha}q_2^\alpha t_2\big\} \\ &\quad= p_1^{1-\alpha}p_2^{1-\alpha}t_1t_2\big(p_1^{-1}t_1^{\frac{1}{\alpha-1}} + p_2^{-1}t_2^{\frac{1}{\alpha-1}}\big)^{1-\alpha}, \end{aligned}$$

where

$$\begin{aligned} t_1 &= \max_{\sigma_1}\mathrm{tr}\big[\big(\rho_1^{\frac{1-\alpha}{2\alpha}}\sigma_1\rho_1^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big], \\ t_2 &= \max_{\sigma_2}\mathrm{tr}\big[\big(\rho_2^{\frac{1-\alpha}{2\alpha}}\sigma_2\rho_2^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big], \end{aligned}$$

and Lemma 1 (note here $t_1 > 0$ and $t_2 > 0$) has been taken into account.

Consequently,

$$\begin{aligned} &\max_{\sigma\in\mathcal{I}_B(H)}\Big(\big\{\mathrm{tr}\big[\big(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]\big\}^{\frac{1}{1-\alpha}}\Big) \\ &\quad= \Big\{\max_{\sigma\in\mathcal{I}_B(H)}\mathrm{tr}\big[\big(\rho^{\frac{1-\alpha}{2\alpha}}\sigma\rho^{\frac{1-\alpha}{2\alpha}}\big)^\alpha\big]\Big\}^{\frac{1}{1-\alpha}} \\ &\quad= p_1p_2t_1^{\frac{1}{1-\alpha}}t_2^{\frac{1}{1-\alpha}}\big(p_1^{-1}t_1^{\frac{1}{\alpha-1}} + p_2^{-1}t_2^{\frac{1}{\alpha-1}}\big) \\ &\quad= p_1t_1^{\frac{1}{1-\alpha}} + p_2t_2^{\frac{1}{1-\alpha}}. \end{aligned}$$

This shows that $C_{R,\alpha}(\rho,P)$ satisfies (B5).

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