Quantum parameter estimation of the frequency and damping of a harmonic oscillator

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We determine the quantum Cramér-Rao bound for the precision with which the oscillator frequency and damping constant of a damped quantum harmonic oscillator in an arbitrary Gaussian state can be estimated. This goes beyond standard quantum parameter estimation of a single-mode Gaussian state for which typically a mode of fixed frequency is assumed. We present a scheme through which the frequency estimation can nevertheless be based on the known results for single-mode quantum parameter estimation with Gaussian states. Based on these results, we investigate the optimal measurement time. For measuring the oscillator frequency, our results unify previously known partial results and constitute an explicit solution for a general single-mode Gaussian state. Furthermore, we show that with existing carbon nanotube resonators see J. Chaste *et al.* [Nat. Nanotechnol. **7**, 301 (2012)] it should be possible to achieve a mass sensitivity of the order of an electron mass $\mathrm{Hz}^{-1/2}$.

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I. INTRODUCTION

The harmonic oscillator is one of the most important model systems in all of physics. It is exactly solvable, both classically and quantum mechanically, and plays a fundamental role in quantum field theories, where its elementary excitations can be identified with, e.g., photons or phonons. The harmonic oscillator arises as the low-amplitude limit of a much wider class of nonharmonic oscillators, and its regular motion is at the basis of time and frequency measurements. Indeed, the most precise measurements of a physical quantity are often achieved when transducing their variations into frequency changes. It is therefore of utmost importance to figure out how precisely the two characteristic quantities of a harmonic oscillator, namely, its frequency and its damping, can be measured in principle. A partial answer was provided in [1], where the quantum Cramér-Rao bound (QCRB) for the frequency measurement of an undamped harmonic oscillator in an arbitrary pure quantum state was calculated. The QCRB is the ultimate lower bound for the uncertainty with which a parameter can be estimated. It is optimized over all possible positive operator-valued measurements (a class of measurements that includes but is more general than the usual projective von Neumann measurements) and over all data analysis procedures (in the sense of unbiased estimator functions of the measurement results alone). It becomes relevant when all technical noise sources are eliminated, and only the noise inherent in the quantum state remains. Importantly, the QCRB can be saturated in the limit of a large number of measurements.

A damped harmonic oscillator leads, however, naturally to mixed quantum states, and for those the calculation of the QCRB is much more difficult than for pure states, owing to the need to diagonalize the density operator in an infinitely dimensional Hilbert space. In [2] an attempt was made to obtain the QCRB for the frequency of a kicked and damped oscillator [3], by using the formulas for Gaussian states. Indeed, in [4] the exact QCRB was found for any of the five parameters that uniquely fix an arbitrary Gaussian state of a harmonic oscillator. However, those formulas were derived for an oscillator of fixed frequency, and they cannot be directly applied for frequency estimation. Doing so would amount to considering the Hamiltonian $H = \hbar\omega a^\dagger a$ as a generator of a phase shift, i.e., the unknown parameter ω multiplies a Hermitian generator, whose variance gives, up to a factor of 4, the pure-state quantum Fisher information (QFI). However, this ignores that the annihilation and creation operators themselves depend on ω. That they do so is most easily seen by writing them in the Fock basis and realizing that the wave functions corresponding to the energy eigenstates depend on ω through the oscillator length. Physically, ignoring the ω dependence of $a, a^\dagger$ hence implies that one neglects the ω dependence of the energy eigenstates, which is particularly important at small times, i.e., much smaller than the period of the oscillator.

One might then think that calculating the QCRB for the damped harmonic oscillator is a hopeless endeavor if the formulas for the Gaussian states cannot be applied, and the state is not already diagonalized. Here we show, however, that there is a well-defined procedure that allows one to use these formulas nevertheless for the large and experimentally most relevant class of initial Gaussian states, by carefully incorporating the consequences of a change of frequency. This allows us to fully solve the problem of parameter estimation of a (weakly) damped harmonic oscillator, described by a Lindblad master equation (ME).

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II. GENERAL FRAMEWORK

We start by briefly describing the dynamics of a damped harmonic oscillator. Afterwards we review the closed-form expression for the general quantum Fisher information for single-mode Gaussian states [4].

A. Dynamics

We consider a quantum harmonic oscillator with bare frequency ω weakly coupled to a Markovian environment. Assuming the validity of the Born-Markov approximation and the rotating-wave approximation, the density matrix ρ of the oscillator evolves according to the quantum optical ME [5,6],

$$\frac{d\rho}{dt} = -i\omega[\hat{a}^\dagger\hat{a},\rho] + \frac{\gamma}{2}\bar{n}(2\hat{a}^\dagger\rho\hat{a} - \hat{a}\hat{a}^\dagger\rho - \rho\hat{a}\hat{a}^\dagger) + \frac{\gamma}{2}(\bar{n}+1)(2\hat{a}\rho\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\rho - \rho\hat{a}^\dagger\hat{a}), \tag{1}$$

where we have introduced the mean thermal photon number of the bath $\bar{n} = (e^x - 1)^{-1}$ at frequency ω, the dimensionless inverse temperature $x \equiv \hbar\omega/k_BT$, and the damping constant γ. Quantum mechanical oscillators are more commonly described in the literature by a quantum Brownian motion ME, expressed in terms of position and momentum operators $\hat{q}$ and $\hat{p}$. However, the two are closely related: it has been shown that the most general Markovian ME bilinear in $\hat{q}$ and $\hat{p}$ turns into the quantum optical ME when requesting a thermal steady state and phase covariance in the creation and annihilation operators $a^\dagger, a$, whereas the quantum Brownian motion master equation is obtained when requesting translational invariance of the dissipative part [7,8]. When the latter is formulated in terms of creation and annihilation operators, it takes a form very similar to Eq. (1) (compare Eqs. (10) and (18) in [8]). For the sake of simplicity we therefore restrict ourselves to Eq. (1).

By introducing the quadrature operator $\mathbf{X} = (\hat{q},\hat{p})^{\mathrm{T}}$, the three-dimensional vector $\mathbf{S}(t) = (M\omega\sigma_{qq}, \sigma_{pp}/M\omega, \sigma_{pq})^{\mathrm{T}}$, where $\sigma_{AB} \equiv 1/2 \times \langle AB + BA\rangle - \langle A\rangle\langle B\rangle$, and by using the ME, (1), one finds the equations of motion [9]

$$\frac{d\langle\mathbf{X}\rangle(t)}{dt} = G\langle\mathbf{X}\rangle(t), \tag{2a}$$

$$\frac{d\mathbf{S}(t)}{dt} = K\mathbf{S}(t) + \mathbf{S}_{\mathrm{inh}}, \tag{2b}$$

where

$$G = \begin{pmatrix} -\gamma/2 & 1/M \\ -M\omega^2 & -\gamma/2 \end{pmatrix}, \quad K = \begin{pmatrix} -\gamma & 0 & 2\omega \\ 0 & -\gamma & -2\omega \\ -\omega & \omega & -\gamma \end{pmatrix}, \tag{2c}$$

and $\mathbf{S}_{\mathrm{inh}} = \gamma\hbar(2\bar{n}+1)/2\ (1,1,0)^{\mathrm{T}}$. The solutions of the time evolution of the first-order moments are given by $\langle\mathbf{X}\rangle(t) = \exp(Gt)\langle\mathbf{X}\rangle(0)$. For the second-order moments we get $\mathbf{S}(t) = \exp(Kt)\mathbf{S}(0) + K^{-1}(\exp(Kt) - \mathcal{I})\mathbf{S}_{\mathrm{inh}}$, where $\mathcal{I}$ denotes the identity operator.

The two phase-space coordinates $\hat{q}$ and $\hat{p}$ are linked to the annihilation and creation operators $\hat{a}_\omega$ and $\hat{a}^\dagger_\omega$ of the mode by

$$\hat{q} = \sqrt{\frac{\hbar}{2M\omega}}(\hat{a}^\dagger_\omega + \hat{a}_\omega), \tag{3a}$$

$$\hat{p} = i\sqrt{\frac{\hbar}{2}M\omega}(\hat{a}^\dagger_\omega - \hat{a}_\omega). \tag{3b}$$

Summing up, ω, γ, and $\bar{n}$ are coded into a state by the dynamics, (1), but in addition, a state specified initially, e.g., in the Fock basis acquires an ω dependence due to the ω dependence of the harmonic oscillator energy eigenstates (oscillator length).

B. QFI of single-mode Gaussian states

1. Gaussian state

The Wigner function for an arbitrary density matrix ρ of a continuous variable system with a single degree of freedom (such as a single harmonic oscillator) is defined by [10]

$$W(q,p) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} e^{-2ipy/\hbar}\langle q-y|\rho|q+y\rangle dy. \tag{4}$$

By definition, a Gaussian state is a state whose Wigner function is Gaussian. Thus, for a Gaussian state of a single harmonic oscillator (such as a single mode of an electromagnetical field) the Wigner function takes the general form [11]

$$W(q,p) = \frac{P}{\pi}\exp\left[-\frac{1}{2}(\mathbf{X}-\langle\mathbf{X}\rangle)^{\mathrm{T}}\Sigma^{-1}(X-\langle\mathbf{X}\rangle)\right], \tag{5}$$

where $\mathbf{X} = (\hat{q},\hat{p})^{\mathrm{T}}$ is the quadrature operator, Σ is the covariance matrix, $\langle\ldots\rangle \equiv \mathrm{tr}(\rho\ldots)$ defines the expectation value, and $P = \mathrm{tr}\rho^2$ is the purity. For single-mode Gaussian states the purity is completely described by the covariance matrix and is given by [12]

$$P = \frac{\hbar}{2\sqrt{\det(\Sigma)}}. \tag{6}$$

Next, we recall that a general single-mode Gaussian state ρ can always be represented as a rotated squeezed displaced thermal state ν [11,13], i.e.,

$$\rho = R(\psi)D(\alpha)S(z)\nu S^\dagger(z)D^\dagger(\alpha)R^\dagger(\psi), \tag{7}$$

where $S(z) = \exp[(1/2)(z\hat{a}^{\dagger 2} - z^*\hat{a}^2)]$ is the squeezing operator, $R(\psi) = \exp(i\psi\hat{a}^\dagger\hat{a})$ denotes the rotation operator, and $D(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ introduces the displacement operator. By introducing $N_{\mathrm{th}} = \mathrm{tr}(\nu\hat{a}^\dagger\hat{a})$, which denotes the number of initial thermal photons, and $z = re^{i\chi}$, the general Gaussian state can be parametrized by five real parameters: α, ψ, r, χ, and $N_{\mathrm{th}} \in \mathbb{R}$. Note that we keep N_{th} and $\bar{n}$ as independent parameters.

2. Quantum Fisher information

We start from a density operator ρ_θ, which depends on an unknown real scalar parameter θ. To estimate this parameter, m independent measurements with the outcome $\xi = (\xi_1,\xi_2,\ldots,\xi_m)^{\mathrm{T}}$ are taken. From the outcome we construct an estimator $\hat{\theta}_{\mathrm{est}}$. For unbiased estimators the sensitivity with which a parameter θ can be measured has a lower bound, the

so-called quantum Cramér-Rao bound, given by [14–17]

$$\mathrm{Var}[\hat{\theta}_{\mathrm{est}}] \geqslant \frac{1}{m\,\mathrm{I}(\rho_\theta;\theta)}, \tag{8}$$

where $\mathrm{I}(\rho_\theta;\theta)$ denotes the QFI. The fidelity, defined by $\mathcal{F}(\rho_1,\rho_2) = \{\mathrm{tr}[(\sqrt{\rho_1}\rho_2\sqrt{\rho_1})^{1/2}]\}^2$, for two arbitrary single-mode Gaussian states ρ_1 and ρ_2 is given by [18]

$$\mathcal{F}(\rho_1,\rho_2) = \frac{2\exp\left[-\frac{1}{2}(\langle \mathbf{X}_1 - \mathbf{X}_2\rangle)^{\mathrm{T}}(\Sigma_1+\Sigma_2)^{-1}\langle \mathbf{X}_1 - \mathbf{X}_2\rangle\right]}{\sqrt{|\Sigma_1+\Sigma_2| + (1-|\Sigma_1|)(1-|\Sigma_2|)} - \sqrt{(1-|\Sigma_1|)(1-|\Sigma_2|)}}. \tag{9}$$

This formula is valid generally for two Gaussian Wigner functions, regardless of the underlying physical system. It therefore remains valid if the two Wigner functions represent states of two different harmonic oscillators, notably harmonic oscillators that can differ in frequency. Using, further, the fact that the fidelity is linked to the QFI through [4]

$$\mathrm{I}(\rho_\theta;\theta) = -2\,\frac{\partial^2 \mathcal{F}(\rho_\theta,\rho_{\theta+\varepsilon})}{\partial\varepsilon^2}\bigg|_{\varepsilon=0}, \tag{10}$$

one obtains the general QFI for Gaussian states of a single harmonic oscillator of fixed frequency [4],

$$\mathrm{I}(\rho_\theta;\theta) = \frac{1}{2}\frac{\mathrm{tr}\left[(\Sigma^{-1}\partial_\theta\Sigma)^2\right]}{1+P^2} + 2\frac{(\partial_\theta P)^2}{1-P^4} + (\partial_\theta\langle\mathbf{X}\rangle)^{\mathrm{T}}\Sigma^{-1}\partial_\theta\langle\mathbf{X}\rangle. \tag{11}$$

By following the approach adopted by Jiang in Ref. [19] the same result can be obtained [20].

III. UNDAMPED CASE

This section provides a scheme for the calculation of and results for the QFI relevant for estimating the frequency ω in the case of no damping. One can distinguish physical situations that differ in the way the parameter ω is imprinted on the state. The situation we focus on is an initial state independent of ω, i.e., a Gaussian state prepared starting from a thermal state of an oscillator with frequency ω_0, and squeezing, shift, and rotation performed with operators of frequency ω_0 as well. This is the typical situation for measuring an adsorbed mass through a frequency shift: the oscillator has an initial frequency ω_0 and a state (thermal, coherent, squeezed, or other) is prepared for that oscillator. An adsorbed mass instantaneously changes the frequency of the oscillator to ω, and the initial state is hence propagated with a new Hamiltonian H_ω corresponding to frequency ω, including creation and annihilation operators at frequency ω. Hence, the full dependence of H_ω on ω must be taken into account, but not of the initial state.

Another physical situation results if also the initial state depends on ω rather than on ω_0. It has as a consequence that an initial thermal state at frequency ω shows no dynamics under the propagation with H_ω. On the other hand, in such a situation quantum Fisher information is finite even for zero measurement time, as the ω dependence is imprinted on the state not via the dynamics, but simply through the ω dependence of all the Fock states (and the Boltzmann weights in the case of a thermal state). We briefly comment on this unusual situation in Sec. III B.

A. Scheme for the estimation of the quantum Fisher information

We first illustrate that by directly using Eq. (11) for a frequency measurement while ignoring the frequency dependence of $\hat{a}^\dagger$, $\hat{a}$, one does not get the full QFI. Then we justify that one can still use Eq. (11) if one treats the squeezing due to frequency change correctly, which leads to the scheme we propose.

To see that by ignoring the ω dependence of $\hat{a}$ and $\hat{a}^\dagger$, one does not get the correct QFI, we use the known and simple results of the QFI for pure states and a phase shift Hamiltonian, i.e., where the parameter to be measured multiplies a Hermitian generator. The Hamiltonian of the harmonic oscillator $\mathcal{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2)$ clearly has this form of a phase shift Hamiltonian if ω is the parameter and $\hat{a}$, $\hat{a}^\dagger$ are taken as independent of ω. The dynamics of the system is described by $\rho_\omega(t) = U(t)\rho(0)U^\dagger(t)$, where $U(t) = \exp(-it\mathcal{H}/\hbar)$ is the time evolution operator. For an initial pure state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ independent of ω and neglecting the ω dependence of a, $a^\dagger$, the QFI is then simply given by (four times) the variance of the generator in the initial state [21,22],

$$\mathrm{I}(\rho_\omega(t);\omega) = 4\,\mathrm{Var}[t(\hat{a}^\dagger\hat{a} + 1/2), |\psi_0\rangle], \tag{12}$$

where $\mathrm{Var}[A,|\psi_0\rangle] \equiv \langle\psi_0|A^2|\psi_0\rangle - \langle\psi_0|A|\psi_0\rangle^2$ denotes the variance. For a general pure Gaussian state in the form of Eq. (7), i.e., $|\psi_0\rangle = R(\psi)D(\alpha)S(re^{i\chi})|0\rangle$, evaluation of (12) leads straightforwardly to the QFI

$$\mathrm{I}(\rho_\omega(t);\omega) = 4\alpha^2t^2[\cosh(2r) + \cos(\chi)\sinh(2r)] + 2t^2\sinh^2(2r). \tag{13}$$

Now we show that exactly the same result for the QFI is found by directly using Eq. (11), i.e., without taking into account the ω dependence of a, $a^\dagger$. For this we first use that we can write the time-evolved density operator as

$$\rho_\omega(t) = R(\zeta)D(\alpha)S(z)|0\rangle\langle 0|S^\dagger(z)D^\dagger(\alpha)R^\dagger(\zeta), \tag{14}$$

where $\zeta = \psi - \omega t$. The general formula, (11), was evaluated in [4] for a $\rho_\omega(t)$ of this form, with the result

$$\mathrm{I}(\rho_\omega(t);\zeta) = 4\alpha^2\left(\sigma^2\cos^2(\chi) + \frac{1}{\sigma^2}\sin^2(\chi)\right) + \frac{(1-\sigma^4)^2}{2\sigma^4}; \tag{15}$$

see Eq. (16) in [4] for purity $P_0 = 1$. With $\sigma = e^{-r}$, Eq. (15) can be rewritten as [23]:

$$\mathrm{I}(\rho_\omega(t);\zeta) = 4\alpha^2[\cosh(2r) + \cos(\chi)\sinh(2r)] + 2\sinh^2(2r). \tag{16}$$

The error propagation law based on $d/d\omega = -td/d\zeta$ leads to multiplication of (16) by t^2 if we calculate the QFI with

respect to ω rather than with respect to ζ, and so we get the same result as obtained with Eq. (13), for which we have demonstrated that it does not contain the ω dependence of the basis. Hence, direct evaluation of (11) without taking into account the ω dependence of $\hat{a}$ and $\hat{a}^\dagger$ gives an incomplete result for the QFI.

Having demonstrated that naively applying Eq. (11), i.e., without taking into account the frequency dependencies of $\hat{a}$ and $\hat{a}^\dagger$, leads to the wrong result, we now introduce the scheme that we have developed for calculating the full QFI:

(1) Start with an initial Gaussian state given in the Fock basis $\{|n\rangle_{\omega_0}\}$.

(2) Perform a sudden change of frequency $\omega_0 \to \omega$, which corresponds to a squeezing, at time $t = 0$.

(3) Evolve the quantum state with respect to the new frequency ω.

(4) Estimate the QFI $\mathrm{I}(\rho_\omega(t);\omega)$ by using Eq. (11).

(5) Take the limit $\omega \to \omega_0$.

The sudden change in frequency $\omega_0 \to \omega$ at time $t = 0$ ensures that also the frequency dependence of the basis is considered. Furthermore, it can be shown that the frequency jump corresponds to squeezing (see Appendix A), i.e.,

$$|n\rangle_{\omega_0} = S_\omega(s)|n\rangle_\omega, \tag{17}$$

where $s = -\tanh^{-1}(y_1)$ and $y_1 = (\omega_0 - \omega)/(\omega_0 + \omega)$.

It should be noted that the introduced scheme is only needed to determine the QFI for a frequency measurement using Eq. (11). For pure states, for example, the QFI can be determined directly from the overlaps of the states propagated with slightly different frequencies [1] or, equivalently, from the variance of the local generator, if the ω dependence of $\hat{a}_\omega$, $\hat{a}_\omega^\dagger$ is taken into account (see Appendix B). In Appendix D we show that the same result as the one from the scheme introduced above is also obtained from (11) when taking into account the ω dependence of all the operators (squeezing, shift, and rotation) through their dependence on $\hat{a}_\omega$ and $\hat{a}_\omega^\dagger$. Furthermore, since the Fock basis does not depend on the damping constant, the introduced scheme is not needed for calculating the QFI for the estimation of γ.

B. ω dependence of the initial state

We here briefly consider the second physical situation where the initial state depends on ω rather than on ω_0. Such a situation arises, e.g., if a harmonic oscillator with variable frequency ω thermalizes due to contact with a thermal heat bath at fixed temperature T. It is clear that in this case the QFI should be different from 0 even without any propagation at all, with the consequence that one could get information about the frequency of the oscillator, e.g., solely observing its noise, without the need to observe any oscillations. Using (11) leads to

$$\mathrm{I}(\nu;\omega) = \frac{1}{2\omega^2}\left[1 + \frac{2N_{\mathrm{th}}(1+N_{\mathrm{th}})}{1+2N_{\mathrm{th}}(1+N_{\mathrm{th}})} + 2N_{\mathrm{th}}(1+N_{\mathrm{th}})\ln^2\left(\frac{1+N_{\mathrm{th}}}{N_{\mathrm{th}}}\right)\right]. \tag{18}$$

In particular, this implies the finite QFI $\mathrm{I}(|0\rangle\langle 0|;\omega) = 1/2\omega^2$ for the ground state. One can infer the frequency of the oscillator from measuring the fluctuations of the quadratures in the ground state. The resulting signal-to-noise ratio is of order 1 and hence cannot compete with a frequency measurement of an excited oscillator [see, e.g., (25)]. However, in a multimode case, where another parameter to be measured, e.g., the length of a cavity, modifies the frequency of all modes at the same time [24], a QFI proportional to the number of modes can be obtained, which makes such a scheme more interesting. For experimentally relevant sensitivities per square of hertz, also the preparation and measurement times have to be taken into account, such that the information gain per unit time remains finite.

C. Result for the QFI for vanishing damping

By using the introduced scheme we now determine the QFI for the estimation of ω for the general Gaussian state given by Eq. (7). For the time evolution of the Gaussian state with the harmonic oscillator $\mathcal{H} = \hbar\omega(\hat{a}_\omega^\dagger\hat{a}_\omega + 1/2)$ the result below follows (see Appendix C)

$$\begin{aligned}\omega^2\mathrm{I}(\rho(\tau);\omega) = {} & C_3 + 2C_1\sin^2(\tau)\{\sinh^2(2r) \\ & \times \cos^2(\chi+2\psi-\tau)+1+2C_2\alpha^2[\cosh(2r) \\ & + \cos(\chi+4\psi-2\tau)\sinh(2r)]\} \\ & + 2C_1\tau\sin(\tau)\{4C_2\alpha^2\cos(2\psi-\tau)\cosh(2r) \\ & + \cos(\chi+2\psi-\tau)[4C_2\alpha^2\sinh(2r) \\ & + \sinh(4r)]\} + 2C_1\tau^2\{2C_2\alpha^2[\cosh(2r) \\ & + \cos\chi\sinh(2r)] + \sinh^2(2r)\},\end{aligned} \tag{19}$$

where $\tau = \omega t$ and

$$C_1 = \frac{(1+2N_{\mathrm{th}})^2}{1+2N_{\mathrm{th}}(1+N_{\mathrm{th}})}, \tag{20a}$$

$$C_2 = \frac{1}{C_1(1+2N_{\mathrm{th}})}, \tag{20b}$$

$$C_3 = N_{\mathrm{th}}(1+N_{\mathrm{th}})\left[\ln\left(\frac{1+N_{\mathrm{th}}}{N_{\mathrm{th}}}\right)\right]^2. \tag{20c}$$

The first term (C_3) in Eq. (19) results from the ω dependence of the initial photon number N_{th}, the second term is due to the ω dependence of the Fock basis, and the term $\propto t^2$ arises from $\hat{a}^\dagger, \hat{a}$ as the generator of the time evolution.

For an initial thermal state $\rho(0) = \nu$, Eq. (19) reduces to

$$\mathrm{I}(\nu(\tau);\omega) = \frac{2C_1\sin^2(\tau)+C_3}{\omega^2}. \tag{21}$$

Thus, a measurement with $t > \pi/2\omega$ does not provide any additional information regarding the frequency and the QFI has an upper bound $(2C_1+C_3)/\omega^2$—where C_1 itself is bounded by $C_1 \in [1,2]\ \forall N_{\mathrm{th}}$ and C_3 is bounded by $C_3 \in [0,1]\ \forall N_{\mathrm{th}}$. Furthermore, the result demonstrates that one can measure the frequency of a mode of an electromagnetic field without any light at all, just from the vacuum fluctuations. The latter have been measured directly in [25].

While our results from Eq. (19) agree with the obtained QFI for a coherent state [1], our result in Eq. (21) contains an extra term, C_3/ω^2, due to the consideration of the ω dependence of N_{th} neglected in [1]. It should also be noted

that our result agrees with the result by calculating the QFI via the variance in the case of a general pure Gaussian state (see Appendix B).

D. Optimal state

The QFI can be drastically increased by displacing and/or squeezing the initial thermal state. In both cases, the QFI acquires a part proportional to t^2 that always dominates at sufficiently large times. For an initial state displaced with $\alpha \in \mathbb{R}$, the part proportional to t^2 has its maximum at $\chi = 0$. We further point out that the long-term behavior of the QFI for a squeezed thermal state also improves due to additional displacing.

The optimal choice of thermal photons N_{th} depends on the initial state. If the QFI is dominated by the terms due to the squeezing, a large number of photons is favorable. If, on the other hand, the terms due to the displacement, which are $\propto(1+2N_{\text{th}})^{-1}$, dominate, the lowest possible number of photons is desirable. The behavior can be well explained by the Wigner function. A larger N_{th} is equivalent to a wider distribution of the state. This means that a small shift in the Wigner function of the displaced state, e.g., due to the time evolution, is less measurable for larger N_{th}. Consequently, the enlargement of the thermal photons counteracts the additional gain of the displacement. The benefits of squeezing, on the other hand, increase with the thermal photon number. This can be seen directly from Eq. (19), since its QFI is proportional to C_1, which is also the only term that increases with N_{th}.

IV. DAMPED CASE

In this section we calculate the QFI for mixed Gaussian states for the damped harmonic oscillator for estimating the oscillator frequency and damping constant. Furthermore, we determine the optimal measuring scheme and the optimal measuring time and we demonstrate that with existing carbon nanotube resonators it should be possible to achieve a mass sensitivity of the order of an electron mass $\text{Hz}^{-1/2}$.

A. Measuring the oscillator frequency

By sticking to the scheme explained in Sec. III A, we obtain the exact expression for the QFI for a general initial Gaussian state by considering the time evolution given by the ME, (1), which can be found in Appendix C, Eqs. (C7) to (C12). However, since the solution is too heavy to report here, we first look at the long-term behavior and then limit ourselves to specific initial states—the coherent state and squeezed state.

1. Long-term behavior

For longer periods, the solution of ME, (1), relaxes to the thermal equilibrium state, i.e., for $t \gg \gamma^{-1}$,

$$\rho \xrightarrow{t\gg\gamma^{-1}} e^{-\hbar\omega/k_BT}/\text{tr}(e^{-\hbar\omega/k_BT}) \equiv \rho_\infty. \tag{22}$$

It should be remembered that the thermal equilibrium state as well as the mean thermal photon number $\bar{n}$ also depends on the oscillator frequency ω itself. It can therefore be expected that the QFI does not vanish due to the dependency of the final state on the frequency. Since both first-order moments vanish, i.e., $\lim_{t\to\infty}\langle \mathbf{X}\rangle = 0$, only the first two terms in Eq. (11) contribute to the QFI and calculation yields

$$\text{I}(\rho_\infty;\omega) = \frac{1}{2\omega^2}\left[2\bar{n}(1+\bar{n})\ln^2\left(\frac{1+\bar{n}}{\bar{n}}\right) + \frac{1+4\bar{n}(1+\bar{n})}{1+2\bar{n}(1+\bar{n})}\right]. \tag{23}$$

This means that for large times, the QFI has an upper bound given by $2/\omega^2$ (see Fig. 1). The upper bound can be reached in the high-temperature limit. As a consequence, a longer measurement does not necessarily yield a better result for the experiment. In other words, there is an optimal measurement time (OMT) in which the frequency can be measured best, which is in accordance with the physical expectations.

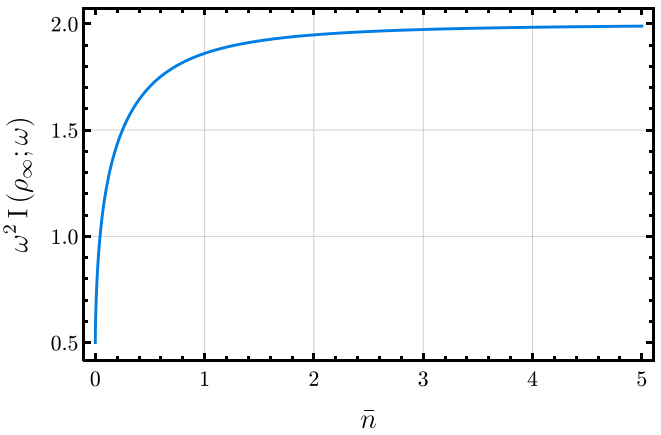

FIG. 1. The long-term behavior of the dimensionless QFI, $\omega^2 \text{I}(\rho_\infty;\omega)$, for a damped Gaussian state for measuring ω is shown as a function of the thermal photon number of the bath. In the limit of validity of (1), the result is independent of the damping constant.

2. Optimal measurement time and maximal quantum Fisher information

Coherent state: We start by considering an initial coherent state $\rho_\alpha(0) = D(\alpha)|0\rangle\langle 0|D^\dagger(\alpha)$. Recall that displacing the initial state is one of the possibilities to strongly increase the QFI in the undamped case. Since displacing the ground state affects only the expectation values of the quadrature operators and not the covariance matrix, the QFI of the coherent state can be written as

$$\text{I}(\rho_\alpha(\tau);\omega) = \text{I}(\rho_0(\tau);\omega) + \text{I}_\alpha(\tau), \tag{24}$$

where $\rho_0(\tau)$ denotes the time-evolved ground state and $\text{I}_\alpha(\tau) = (\partial_\omega\langle \mathbf{X}\rangle)^{\text{T}}\Sigma^{-1}\partial_\omega\langle \mathbf{X}\rangle$. The QFI of the ground state is bounded by $2.135/\omega^2$ (see Appendix C). Thus, the upper bound of the QFI for the ground state is increased by introducing the system-bath coupling, which can be explained by the ω dependence of $\bar{n}$. That is, also in the damped case, the frequency can be measured when the system is initially prepared in the ground state. Straightforward calculation leads to

$$\text{I}_\alpha(\tau) = \frac{4\alpha^2}{\omega^2}\frac{\sin^2(\tau)+\tau\sin(2\tau)+\tau^2}{(2\bar{n}+1)e^{g\tau}-2\bar{n}}, \tag{25}$$

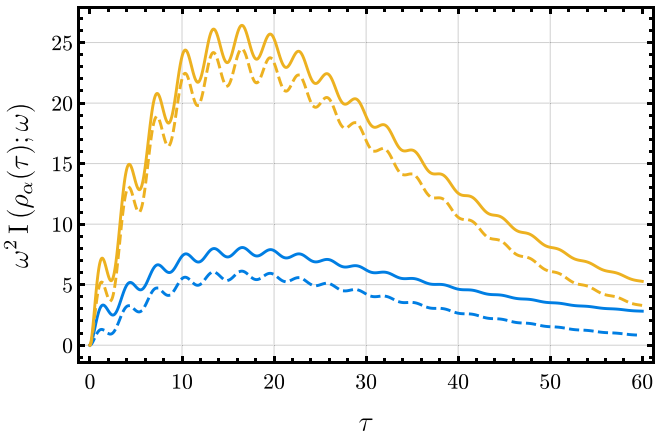

FIG. 2. The dimensionless QFI, $\omega^2 \mathrm{I}(\rho_\alpha;\omega)$, for an initial coherent state (solid curves) is compared with the lower bound $\omega^2 \mathrm{I}_\alpha(\tau)$ (dashed curves) [see Eq. (24)] for measuring ω. Results are depicted for $\bar{n} = 5$ and $g = 0.1$: blue (lower two curves), $\alpha = 1/2$; orange (upper two curves), $\alpha = 1$.

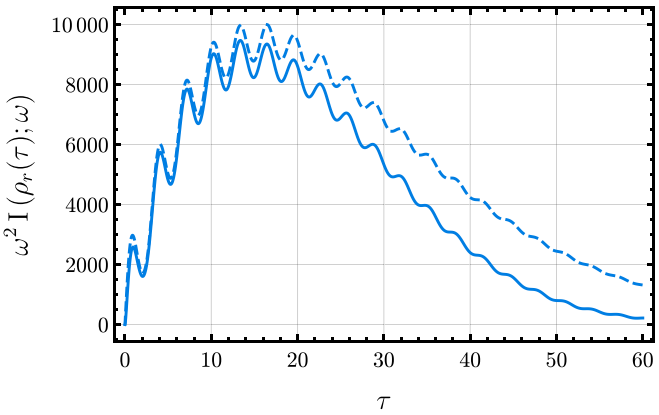

FIG. 3. The dimensionless QFI, $\omega^2 \mathrm{I}(\rho_r;\omega)$, for an initial squeezed state (solid curve) is compared with the approximation from Eq. (32) (dashed curve) for measuring ω. Results are depicted for $\bar{n} = 0.01$, $g = 0.1$, and $r = 2.5$.

where $g = \gamma/\omega$ introduces a dimensionless damping constant. Thus, for frequency measurements a displacement as large as possible is recommended.

Since the QFI of the ground state is bounded (and small), $\mathrm{I}(\rho_\alpha(\tau);\omega) \approx \mathrm{I}_\alpha(\tau)$ applies for $\alpha^2 \gg g^2\bar{n}$ (by assuming $n \gg 1$ and $g \ll 1$). For high enough temperatures, $\bar{n} \gg \alpha^2/g^2$, $\mathrm{I}_\alpha(\tau)$ becomes arbitrarily small and the QFI is then described by the QFI of the ground state. In other words, displacement only improves frequency measurements for resulting mean energies higher than the thermal energy. By neglecting small oscillations, maximization of Eq. (25) provides the maximal QFI $\mathrm{I}_{\max}(\rho;\theta) \equiv \max_\tau \mathrm{I}(\rho;\theta)$ and the OMT $\tau_{\max}$ with $\mathrm{I}(\rho(\tau_{\max});\theta) = \mathrm{I}_{\max}(\rho;\theta)$, i.e.,

$$\mathrm{I}_{\max}(\rho_\alpha(\tau);\omega) = -\frac{2\alpha^2}{\bar{n}g^2\omega^2}\mathcal{W}\left(-\frac{4\bar{n}}{e^2(1+2\bar{n})}\right) \times\left[2+\mathcal{W}\left(-\frac{4\bar{n}}{e^2(1+2\bar{n})}\right)\right], \quad (26)$$

$$\tau_{\max} = \frac{1}{g}\left[2+\mathcal{W}\left(-\frac{4\bar{n}}{e^2(1+2\bar{n})}\right)\right], \quad (27)$$

where $\mathcal{W}(z)$ denotes the Lambert $\mathcal{W}$ function defined by $z = \mathcal{W}(z)e^{\mathcal{W}(z)}$, $z \in \mathbb{C}$.

The Taylor series for $\mathrm{I}(\rho_0(\tau);\omega)$ at $\bar{n} \gg 1$ is

$$\mathrm{I}(\rho_0(\tau);\omega) = \frac{2}{\omega^2}\left[1-\frac{\cos^2(\tau)}{(e^{g\tau}-1)\bar{n}}\right] + O(1/\bar{n}^2). \quad (28)$$

This means that for high temperatures, $\bar{n} \gg 1$, the QFI of the ground state decays more rapidly ($\sim e^{-g\tau}$) than $\mathrm{I}_\alpha(\tau)$ ($\sim \tau^2 e^{-g\tau}$) and varies only slightly close to the time $\tau_{\max}$. Consequently, the use of $\mathrm{I}_\alpha(\tau)$ for estimating the optimal measurement time leads, even in this range, to a good result for the OMT (see Fig. 2).

Furthermore, it should be noted that the smaller g is, the larger $\bar{n}$ can be, so that the OMT is still very well described by $\mathrm{I}_\alpha(\tau)$. By reducing the system-bath coupling, the maximal QFI increases proportionally to $\propto g^{-2}$. However, it should be noted that the OMT also increases proportionally to $\propto g^{-1}$.

Thus, it is natural to consider time as a resource and to introduce the rescaled maximal QFI $\mathrm{I}^{(t)}_{\max}(\rho;\theta) \equiv \max_t \mathrm{I}(\rho;\theta)/t$ and the OMT $\tau^{(t)}_{\max}$ that maximizes it. For an initial coherent state we get

$$\mathrm{I}^{(t)}_{\max}(\rho_\alpha(t);\omega) = -\frac{2\alpha^2}{\bar{n}g\omega}\mathcal{W}\left(-\frac{2\bar{n}}{\mathrm{e}(1+2\bar{n})}\right), \quad (29)$$

$$\tau^{(t)}_{\max} = \frac{1}{g}\left[1+\mathcal{W}\left(-\frac{2\bar{n}}{\mathrm{e}(1+2\bar{n})}\right)\right]. \quad (30)$$

Taking time into account as a resource leads to a reduction in the OMT.

Squeezed state. Besides displacement, squeezing the initial state is another possibility for increasing the QFI in the undamped case. Therefore, we determine the QFI for a squeezed state $\rho_r(0) = S(r)|0\rangle\langle 0|S^\dagger(r)$. For the coherent state we have seen that reducing the temperature leads to an increase in the QFI. This behavior is reasonable, since an increased temperature implies increased damping according to the master equation, (1). A similar behavior can be observed here with the squeezed state. The QFI for a vanishing bath temperature, i.e., $\bar{n} = 0$, reads

$$\begin{aligned}\mathrm{I}(\rho_r(t);\omega) = {}& \{8\omega^2[2e^{g\tau}\sinh^2(r) + e^{2g\tau} - \cosh(2r) + 1]\}^{-1} \\ & \times \{16\tau\sinh(2r)\sin(2\tau)[e^{g\tau} + \cosh(2r) - 1] \\ & - 4(e^{g\tau}-1)\cosh(2r)[2\cos(2\tau) - 3] \\ & + 4e^{g\tau}(e^{g\tau}-1) + (8\tau^2+1)\cosh(4r) \\ & - 8\sinh^2(r)\cosh^2(r)\cos(4\tau) - 8\tau^2 \\ & - 8\cos(2\tau) + 7\}. \end{aligned} \quad (31)$$

Alternatively, for high squeezing and low temperatures, i.e., $r \gg 1$ and $\bar{n} \ll 1$, the QFI can be approximated as (see Fig. 3)

$$\mathrm{I}(\rho_r(\tau);\omega) \approx \frac{e^{2r}[2\tau + \sin(2\tau)]^2}{4\omega^2(e^{g\tau}-1)(1+2\bar{n})}. \quad (32)$$

Thus, the QFI can be significantly increased by squeezing also for an initial thermal state.

Neglecting the oscillations, the OMT can be determined as

$$\tau_{\max} = \frac{1}{g}[2 + \mathcal{W}(-2/e^2)] \approx \frac{1.59}{g}. \tag{33}$$

For sufficiently high squeezing and low temperatures, the OMT no longer depends on the squeezing and temperature.

B. Measuring the damping constant

Next we consider the QFI for estimation of the damping constant. First, the QFI disappears for large times, i.e., $\mathrm{I}(\rho_\infty;\gamma) = 0$. This can be seen directly from the fact that the final thermal state (for the ME approach) itself no longer depends on the damping constant. In other words, there is again an OMT.

After a straightforward but long and tedious calculation we find for the QFI of a general Gaussian state

$$\mathrm{I}(\rho(\tau);\gamma) = \frac{P^2(\tau)g^2\tau^2}{\gamma^2 e^{4g\tau}}\Big\{\alpha^2 e^{2g\tau}\{A_1[\cosh(2r) - \cos(\chi)\sinh(2r)] + a_{1,\tau}\} + \frac{2P^4(\tau)}{1-P^4(\tau)}\big[A_1^2 + A_1(a_{1,\tau} - a_1)\cosh(2r) - a_1 a_{1,\tau}\big]^2$$
$$+ \frac{P^2(\tau)}{1+P^2(\tau)}\big[A_1^4 + A_1^2\big(a_1^2 + a_{1,\tau}^2\big)\cosh(4r) + 2A_1(a_{1,\tau} - a_1)\big(A_1^2 - a_1 a_{1,\tau}\big)\cosh(2r) - 4a_1 a_{1,\tau}A_1^2 + a_1^2 a_{1,\tau}^2\big]\Big\}, \tag{34}$$

where $a_1 = 1 + 2\bar{n}$, $a_{1,\tau} = (e^{g\tau} - 1)a_1$, $A_1 = 1 + 2N_{\mathrm{th}}$, and

$$P(\tau) = e^{g\tau}\big[A_1^2 + a_{1,\tau}^2 + 2a_{1,\tau}A_1\cosh(2r)\big]^{-1/2}. \tag{35}$$

The result does not depend on the rotation angle ψ, but only on the squeezing angle χ. In contrast to the frequency measurement, the QFI for measuring γ is maximized for $\chi = \pi$. This is in agreement with the physical expectation, as the relevant dynamic here is the relaxation of $\langle \mathbf{X} \rangle$. To illustrate the result, we again consider specific initial states—the thermal state, displaced thermal state, and squeezed state.

1. Thermal state

The QFI of a propagated thermal state ν can be written as

$$\mathrm{I}(\nu(\tau);\gamma) = \frac{(\bar{n} - N_{\mathrm{th}})^2 g^2 \tau^2}{\gamma^2[(e^{g\tau} - 1)\bar{n} + N_{\mathrm{th}}][e^{g\tau}(1 + \bar{n}) + N_{\mathrm{th}} - \bar{n}]}. \tag{36}$$

The greater the deviation of the initial temperature from the bath temperature, the better γ can be measured (see Fig. 4). In particular, for a vanishing deviation, i.e., $N_{\mathrm{th}} = \bar{n}$, the QFI vanishes, since in this case the state has no dynamics at all. For $\bar{n} = 0$, the OMT is given by

$$\tau_{\max} = \frac{2 + \mathcal{W}(2N_{\mathrm{th}}e^{-2})}{g}. \tag{37}$$

2. Displaced thermal state

For an initial displaced thermal state $\rho_{\alpha,N_{\mathrm{th}}}(0) = D(\alpha)\nu D^\dagger(\alpha)$ the QFI for measuring γ reads

$$\mathrm{I}(\rho_{\alpha,N_{\mathrm{th}}}(\tau);\gamma) = \mathrm{I}(\nu(\tau);\gamma) + \frac{\alpha^2 g^2 \tau^2}{\gamma^2[2N_{\mathrm{th}} - 2\bar{n} + e^{g\tau}(1 + 2\bar{n})]}. \tag{38}$$

For $N_{\mathrm{th}} = \bar{n}$, the QFI simplifies to

$$\mathrm{I}(\rho_{\alpha,\bar{n}}(\tau);\gamma) = \frac{\alpha^2 g^2 \tau^2}{\gamma^2 e^{g\tau}(1 + 2\bar{n})} \tag{39}$$

and the OMT is given by $\tau_{\max} = 2/g$. For $N_{\mathrm{th}} = \bar{n}$ only the third part of Eq. (11) contributes to the QFI, i.e., the QFI results solely from the relaxation of $\langle p \rangle$, $\langle q \rangle$. Upon considering the rescaled QFI the OMT reduces to $\tau_{\max}^{(t)} = 1/g$.

3. Squeezed state

The low-temperature-limit behavior, i.e., $\bar{n} = 0$, of the QFI for an initial squeezed state ρ_r is given by

$$\mathrm{I}(\rho_r(\tau);\gamma) = \frac{[e^{2g\tau} - 2(e^{g\tau} - 1)]g^2\tau^2\sinh^2(r)}{\gamma^2(e^{g\tau} - 1)[2(e^{g\tau} - 1)\sinh^2(r) + e^{2g\tau}]}. \tag{40}$$

Examples of how the sensitivity with which the damping parameter can be measured improves with squeezing, displacing, and/or increasing the difference $N_{\mathrm{th}} - \bar{n}$ are shown in Fig. 5.

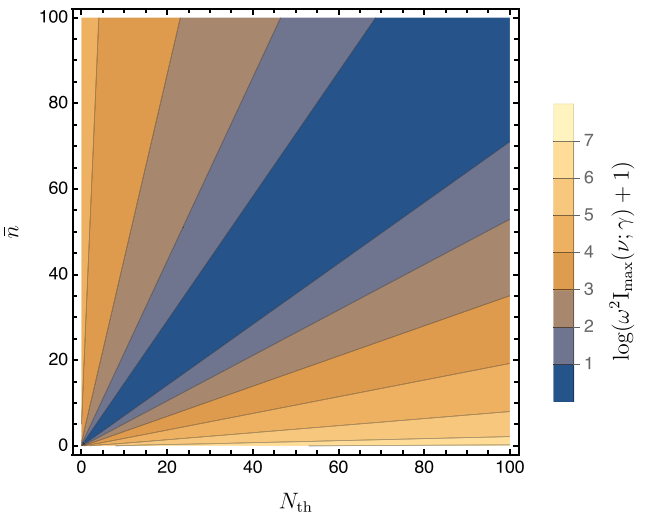

FIG. 4. The dimensionless QFI of a propagated thermal state maximized over time τ for measuring γ. Due to the large variation of the QFI over the parameter range shown the quantity $\log[\omega^2\,\mathrm{I}_{\max}\,(\nu(\tau);\gamma) + 1]$ is shown. Results are depicted for $g = 0.1$.

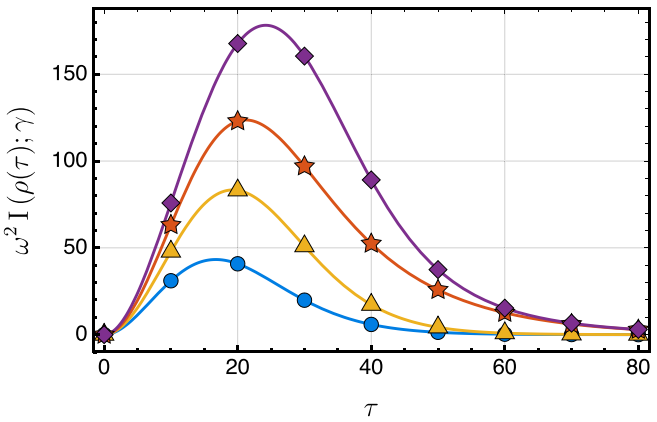

FIG. 5. The dimensionless QFI, $\omega^2 \mathrm{I}(\rho;\gamma)$, for measuring γ. Results are depicted for $\bar{n}=1$, $g=0.1$, $\chi=\pi$ and (blue circles) $N_{\mathrm{th}}=5$, $\alpha=r=0$; (orange triangles) $N_{\mathrm{th}}=10$, $\alpha=r=0$; (red stars) $N_{\mathrm{th}}=10$, $\alpha=1$, $r=0$; and (purple diamonds) $N_{\mathrm{th}}=10$, $\alpha=1$, $r=1/2$.

C. Nanomechanical resonators

In the following we apply the results obtained to nanomechanical resonators, which function as precision mass sensors, as their resonance frequency changes when additional mass is adsorbed. More precisely, we consider carbon nanotube resonators. Using the QCRB, (8), and $\omega=\sqrt{D/M}$, where D is the effective spring constant of the harmonic oscillator, the smallest δM that can be resolved from m measurements of the resonance frequency is given by

$$\delta M_{\mathrm{min}}=\frac{2M}{\omega\sqrt{m\,\mathrm{I}_{\mathrm{max}}(\rho;\omega)}}. \tag{41}$$

Assuming a coherent state with an oscillation amplitude of about 10 nm for the carbon nanotube resonator in [26] ($M=3\times10^{-22}$ kg, $\omega=2\pi\times1.865$ GHz, $T=4$ K, and $Q\sim10^3$), δM_{min} according to (41) is slightly below one proton mass. Using the OMT given by $t_{\mathrm{max}}=270$ ns, the sensitivity corresponds to $\delta M_{\mathrm{min}}\sqrt{t_{\mathrm{max}}}=0.8 m_e\,\mathrm{Hz}^{-1/2}$, which is less than $1/4000$ of the experimentally determined mass sensitivity of slightly more than one proton mass after a 2-s averaging time.

In [1] the theoretically achievable δM_{min} for the carbon nanotube resonator in [27] ($M=10^{-21}$ kg, $\omega=2\pi\times328.5$ MHz, $T=300$ K, and $Q\sim10^3$) was determined to the order of a thousandth of an electron mass. Including the system-bath coupling, δM_{min} increases to about 74 proton masses, where the OMT is given by $t_{\mathrm{max}}=1.5\,\mu$s. This result is equivalent to $0.8\,\mathrm{u}/\sqrt{\mathrm{Hz}}$, which corresponds to approximately one-hundredth of the $78\,\mathrm{u}/\sqrt{\mathrm{Hz}}$ achieved in the experiment.

V. CONCLUSIONS

In summary, we have derived the quantum Cramér-Rao bound for measuring the oscillator frequency and damping constant encoded in the dynamics of a general mixed single-mode Gaussian state of light, including damping through photon loss described by a Lindblad master equation. First, we demonstrated that the known solution for the QFI for Gaussian states of a single harmonic oscillator of fixed frequency cannot be directly applied to frequency measurements. Next, we presented a scheme through which the frequency estimation can nevertheless be based on the results of Pinel *et al.* [4].

Furthermore, we have shown that displacing and/or squeezing the initial state significantly increases the precision with which ω and γ can be estimated. For measuring ω and $r\neq0$, $\chi=0$ is optimal, whereas for measuring γ, $\chi=0$ maximizes the QFI.

Our results can serve as important benchmarks for the precision of frequency measurements of any harmonic oscillator with given damping. In particular, we found optimal measurement times that limit the sensitivity per $\sqrt{\mathrm{Hz}}$ with which frequencies can be measured, in contrast to the undamped case, where, e.g., coherent states lead to an increasing QFI for arbitrarily large times.

APPENDIX A: CHANGE OF BASIS

In presenting the scheme for the estimation of the QFI for measuring ω we made use of the fact that the frequency jump corresponds to squeezing. Next we prove the statement, i.e., the formula

$$|n\rangle_{\omega_0}=S_\omega(s)|n\rangle_\omega, \tag{A1}$$

where $s=-\tanh^{-1}(y_1)$. For the sake of simplicity the two parameters

$$y_1=\frac{\omega_0-\omega}{\omega_0+\omega},\quad y_2=\frac{2\sqrt{\omega_0\omega}}{\omega_0+\omega} \tag{A2}$$

are introduced. A squeezed number state is given by [28]

$$\begin{aligned}{}_\omega\langle m|S_\omega(s)|n\rangle_\omega&=\frac{\sqrt{n!}}{\cosh^{n+1/2}|s|}\sum_{j=0}^{\lfloor n/2\rfloor}\frac{(-d)^j\cosh^{2j}|s|}{(n-2j)!\,j!}\\&\quad\times\sum_{k=0}^{\infty}\frac{d^k\sqrt{(n-2j+2k)!}}{k!}\underbrace{{}_\omega\langle m|n-2j+2k\rangle_\omega}_{=\delta_{m,n-2j+2k}},\end{aligned} \tag{A3}$$

where $d\equiv(s/2|s|)\tanh|s|$ and $\lfloor n/2\rfloor$ denotes the floor function. With $m=n-2j+2k$ and $k\in\mathbb{N}$ we get $k=j+\frac{m-n}{2}\in\mathbb{N}$. This means, in particular, that m and n must be both even or both odd numbers; otherwise, the overlap disappears. If m and n satisfy this condition and by using $\cosh|s|=1/y_2$ and $d=-y_1/2$, the expression can be further simplified as follows:

$${}_\omega\langle m|S_\omega(s)|n\rangle_\omega=\sqrt{y_2 m!n!}\sum_{j=0}^{\lfloor n/2\rfloor}\frac{(-1)^{j+\frac{m-n}{2}}\left(\frac{y_1}{2}\right)^{2j+\frac{m-n}{2}}}{(n-2j)!\,j!\left(j+\frac{m-n}{2}\right)!}y_2^{n-2j}. \tag{A4}$$

By changing the index of summation to $l=n-2j$ we get the new upper bound of $\min(m,n)$, where $\min(m,n)$ denotes the smaller of the two integers m and n. l is also bounded by m, since $k=j-\frac{n-m}{2}=\frac{m-l}{2}\in\mathbb{N}$ and thus $l\leqslant m$. Using the

new index of summation we get [29]

$$
\begin{aligned}
&{}_{\omega}\langle m|S_{\omega}(s)|n\rangle_{\omega} \\
&= \sqrt{\frac{y_2 m!n!}{2^{m+n}}} \sum_{l=0,1}^{\min(m,n)} \frac{(2y_2)^l}{l!} \frac{y_1^{(m+n-2l)/2}(-1)^{(m-l)/2}}{\left(\frac{n-l}{2}\right)!\left(\frac{m-l}{2}\right)!} \\
&= R_{\omega\omega_0}(m,n) \\
&= {}_{\omega}\langle m|n\rangle_{\omega_0},
\end{aligned} \tag{A5}
$$

where $R_{\omega\omega_0}(m,n)$ denotes the overlap matrix element between energy eigenstates of the two oscillators with frequency ω and ω_0. Since this is true for all m, we have proven the formula. Thus, for any density operator

$$
\rho_{\omega_0} = S_{\omega}(s)\tilde{\rho}_{\omega}S_{\omega}^{\dagger}(s), \tag{A6}
$$

where $s = -\tanh^{-1}(y_1)$ and $\tilde{\rho}_{\omega}$ corresponds to the initial state ρ_{ω_0} by replacing the frequency ω_0 of the basis with the new frequency ω. Thus, we have shown that the initial frequency change corresponds to a squeezing. It should be noted that in the case of a vanishing frequency change, i.e., $\omega_0 = \omega, y_1 = 0$, $s = 0$ and $S(s = 0) = \mathcal{I}$ follow and thus $\rho_{\omega_0} = \rho_{\omega}$ is ensured.

APPENDIX B: THE QFI FOR PURE STATES

In the following it is shown that the introduced scheme provides the correct QFI for an undamped pure Gaussian state. Therefore, the QFI is calculated analogously to that in Sec. III A, but this time also the ω dependence of $\hat{a}^{\dagger}, \hat{a}$ is taken into account.

This means that we consider the case where only the dynamics of the state, and not the initial state

$$
\rho_0 = |\psi_0\rangle\langle\psi_0|, \tag{B1}
$$

where $|\psi_0\rangle = R(\psi)D(\alpha)S(re^{i\chi})|0\rangle$, depends on the frequency ω to be measured. For the given Hamiltonian $\mathcal{H}_{\omega} = \hbar\omega(\hat{a}_{\omega}^{\dagger}\hat{a}_{\omega} + 1/2)$, the dynamics of the system is described by $\rho_{\omega} = U_{\omega}\rho_0 U_{\omega}^{\dagger}$, where $U_{\omega} = \exp[-i\omega t(\hat{a}_{\omega}^{\dagger}\hat{a}_{\omega} + 1/2)]$ is the time evolution operator. With the help of the local generator

$$
\mathscr{K} = iU_{\omega}^{\dagger}(t)\frac{\partial U_{\omega}(t)}{\partial\omega} \tag{B2}
$$

the QFI can be rewritten as follows [30]:

$$
\mathrm{I}(\rho_{\omega};\omega) = 4\,\mathrm{Var}\,[\mathscr{K}, |\psi_0\rangle]. \tag{B3}
$$

If $\mathbf{A}$ is a matrix depending on the parameter x, $\mathbf{A} = \mathbf{A}(x)$, then [31]

$$
\frac{\partial}{\partial x}e^{\mathbf{A}(x)} = \left(\int_0^1 e^{\alpha\mathbf{A}(x)}\frac{\partial\mathbf{A}(x)}{\partial x}e^{-\alpha\mathbf{A}(x)}d\alpha\right)e^{\mathbf{A}(x)}. \tag{B4}
$$

Using this formula we can rewrite the local generator $\mathscr{K}$ as [32]

$$
\mathscr{K} = \frac{t}{\hbar}\int_{-1}^{0} V(\alpha)\frac{\partial\mathcal{H}_{\omega}}{\partial\omega}V^{\dagger}(\alpha)d\alpha, \tag{B5}
$$

where $V(\alpha) = \exp(-i\alpha t\mathcal{H}_{\omega}/\hbar)$. The derivative of the Hamiltonian $\mathcal{H}_{\omega}$ with respect to the oscillator frequency ω reads

$$
\frac{\partial\mathcal{H}_{\omega}}{\partial\omega} = \hbar a_{\omega}^{\dagger}\hat{a}_{\omega} + \frac{\hbar}{2}\left[(\hat{a}_{\omega}^{\dagger})^2 + \hat{a}_{\omega}^2 + 1\right], \tag{B6}
$$

where we have made use of $\partial_{\omega}a_{\omega}^{\dagger} = \hat{a}_{\omega}/2\omega$ and $\partial_{\omega}\hat{a}_{\omega} = \hat{a}_{\omega}^{\dagger}/2\omega$, which can be seen from their representation in the Fock-state basis $|n\rangle_{\omega}$. With the help of

$$
e^{-i\psi\hat{a}_{\omega}^{\dagger}\hat{a}_{\omega}}\hat{a}_{\omega}e^{i\psi\hat{a}_{\omega}^{\dagger}\hat{a}_{\omega}} = e^{i\psi}\hat{a}_{\omega} \tag{B7}
$$

we get

$$
\begin{aligned}
&V(\alpha)\frac{\partial\mathcal{H}_{\omega}}{\partial\omega}V^{\dagger}(\alpha) \\
&\quad = \hbar\hat{a}_{\omega}^{\dagger}\hat{a}_{\omega} + \frac{\hbar}{2}\left[e^{-2i\alpha\omega t}(\hat{a}_{\omega}^{\dagger})^2 + e^{2i\alpha\omega t}\hat{a}_{\omega}^2 + 1\right].
\end{aligned} \tag{B8}
$$

Insertion and subsequent integration provide the local generator

$$
\begin{aligned}
\mathscr{K} &= t\left(\hat{a}_{\omega}^{\dagger}\hat{a}_{\omega} + \frac{1}{2}\right) \\
&\quad - \frac{i}{4\omega}\left[(1 - e^{-2i\omega t})\hat{a}_{\omega}^2 + (1 - e^{2i\omega t})\hat{a}_{\omega}^{\dagger 2}\right].
\end{aligned} \tag{B9}
$$

Next, the QFI is calculated. The annihilation and creation operators $\hat{c}_{\omega}$ and $\hat{c}_{\omega}^{\dagger}$, defined by

$$
\begin{aligned}
\hat{c}_{\omega} &= S_{\omega}^{\dagger}(re^{i\chi})D_{\omega}^{\dagger}(\alpha)R_{\omega}^{\dagger}(\psi)\hat{a}_{\omega}R_{\omega}(\psi)D_{\omega}(\alpha)S_{\omega}(re^{i\chi}) \\
&= e^{i\psi}(\cosh(r)\hat{a}_{\omega} + e^{i\chi}\sinh(r)\hat{a}_{\omega}^{\dagger} + \alpha)
\end{aligned} \tag{B10}
$$

and $(\hat{c}_{\omega})^{\dagger} = \hat{c}_{\omega}^{\dagger}$, can be used to rewrite the expectation values of $\mathscr{K}$ as follows:

$$
\langle\psi_0|\mathscr{K}^k|\psi_0\rangle = \langle 0|\mathscr{K}^k|_{\substack{\hat{a}^{\dagger}=\hat{c}^{\dagger} \\ \hat{a}=\hat{c}}}|0\rangle. \tag{B11}
$$

Using the formulas $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle$, we obtain the QFI after a straightforward calculation:

$$
\begin{aligned}
\mathrm{I}(\rho(t);\omega) = \frac{2t}{\omega}&\sin\omega t[4\alpha^2\cos(2\psi - \omega t)\cosh 2r \\
&+ \cos(\chi + 2\psi - \omega t)(4\alpha^2\sinh 2r + \sinh 4r)] \\
&+ \frac{2}{\omega^2}\sin^2\omega t[\sinh^2 2r\cos^2(\chi + 2\psi - \omega t) + 1 \\
&+ 2\alpha^2(\cosh 2r + \cos(\chi + 4\psi - 2\omega t)\sinh 2r)] \\
&+ 2t^2[2\alpha^2(\cosh 2r + \cos\chi\sinh 2r) + \sinh^2 2r].
\end{aligned} \tag{B12}
$$

Comparison with Eq. (19) for a pure Gaussian state, i.e., $N_{\mathrm{th}} = 0$, shows that the results are identical.

APPENDIX C: CALCULATION OF THE QFI

Here we report the calculation of the QFI for measuring ω. First, the dynamics resulting from the ME, (1), is determined. Then the QFI for the undamped case is calculated. Finally, the exact QFI for the damped case is given.

The solutions of the ME, (1), are given by [9]

$$
\langle q\rangle_t = e^{-\frac{\gamma t}{2}}\left[\cos(\omega t)\langle q\rangle_0 + \frac{1}{M\omega}\sin(\omega t)\langle p\rangle_0\right], \tag{C1a}
$$

$$
\langle p\rangle_t = e^{-\gamma t/2}[\cos(\omega t)\langle p\rangle_0 - M\omega\sin(\omega t)\langle q\rangle_0] \tag{C1b}
$$

and

$$\sigma_{qq}(t) = \frac{\hbar}{2M\omega}(1+2\bar{n})(1-e^{-\gamma t}) + e^{-\gamma t}\left[\cos^2(\omega t)\sigma_{qq}(0) + \frac{\sin^2(\omega t)}{M^2\omega^2}\sigma_{pp}(0) + \frac{\sin(2\omega t)}{M\omega}\sigma_{pq}(0)\right], \tag{C2a}$$

$$\sigma_{pp}(t) = \frac{\hbar M\omega}{2}(1+2\bar{n})(1-e^{-\gamma t}) + e^{-\gamma t}[\cos^2(\omega t)\sigma_{pp}(0) + M^2\omega^2\sin^2(\omega t)\sigma_{qq}(0) - M\omega\sin(2\omega t)\sigma_{pq}(0)], \tag{C2b}$$

$$\sigma_{pq}(t) = e^{-\gamma t}\left[\cos(2\omega t)\sigma_{pq}(0) + \frac{1}{M\omega}\sin(\omega t)\cos(\omega t) \times (\sigma_{pp}(0) - M^2\omega^2\sigma_{qq}(0))\right]. \tag{C2c}$$

Indeed, Eq. (C1b) is an immediate consequence of $p = M\partial_t q$. For the general single-mode Gaussian state in Eq. (7) the initial expectation values are given by

$$\langle q\rangle_0 = \alpha\sqrt{\frac{2\hbar}{M\omega_0}}\cos(\psi), \tag{C3a}$$

$$\langle p\rangle_0 = \alpha\sqrt{2\hbar M\omega_0}\sin(\psi), \tag{C3b}$$

$$\sigma_{qq}(0) = \frac{\hbar}{2M\omega_0}(2N_{\rm th}+1) \times [\cosh(2r) + \cos(\chi+2\psi)\sinh(2r)], \tag{C3c}$$

$$\sigma_{pp}(0) = \frac{\hbar M\omega_0}{2}(2N_{\rm th}+1) \times [\cosh(2r) - \cos(\chi+2\psi)\sinh(2r)], \tag{C3d}$$

$$\sigma_{pq}(0) = \frac{\hbar}{2}(2N_{\rm th}+1)\sin(\chi+2\psi)\sinh(2r). \tag{C3e}$$

Here we give the expectation values with respect to the initial frequency ω_0. The time evolution of the ME, (1), on the other hand, is with respect to the new frequency ω, as described in the scheme.

1. Undamped case

We start with the calculation of the QFI for the undamped case of Eq. (19). The undamped dynamic corresponds to the expectation values from Eq. (C1) and Eq. (C2) for $\gamma \to 0$. By using these results we calculated the five parameters of interest: Σ^{-1}, $\partial_\omega\Sigma$, P, $\partial_\omega P$, and $\partial_\omega\langle\mathbf{X}\rangle$. For the sake of clarity we give the results after executing the limit $\omega_0 \to \omega$ and, additionally, use the dimensionless time $\tau = \omega t$. The derivative of the quadrature operator is given by

$$\partial_\omega\langle\mathbf{X}\rangle = \alpha\sqrt{\frac{2M\hbar}{\omega}}\begin{pmatrix}\frac{\tau\sin(\psi-\tau)-\sin(\psi)\sin(\tau)}{M\omega}\\ -\tau\cos(\psi-\tau)-\cos(\psi)\sin(\tau)\end{pmatrix}. \tag{C4}$$

The purity and its derivative read

$$P = \frac{1}{1+2N_{\rm th}}, \tag{C5a}$$

$$\partial_\omega P = \frac{2N_{\rm th}(1+N_{\rm th})\ln(1+1/N_{\rm th})}{\omega(1+2N_{\rm th})^2}, \tag{C5b}$$

whereas the derivative of the covariance matrix is described by the following equations:

$$\frac{2M\omega}{\hbar}\partial_\omega\sigma_{qq}(t) = \frac{1}{\omega P}\{-2\cosh(2r)\sin^2(\tau)+[\cos(\chi+2\psi) - \cos(2\tau-\chi-2\psi)-2\tau \times \sin(2\tau-\chi-2\psi)]\sinh(2r)\} - \frac{\partial_\omega P}{P^2}[\cosh(2r)+\cos(2\tau-\chi-2\psi) \times \sinh(2r)], \tag{C6a}$$

$$\frac{2}{\hbar M\omega}\partial_\omega\sigma_{pp}(t) = \frac{1}{\omega P}\{2\cosh(2r)\sin^2(\tau)+[\cos(\chi+2\psi) - \cos(2\tau-\chi-2\psi)+2\tau \times \sin(2\tau-\chi-2\psi)]\sinh(2r)\} - \frac{\partial_\omega P}{P^2} \times[\cosh(2r)-\cos(2\tau-\chi-2\psi)\sinh(2r)], \tag{C6b}$$

$$\frac{2}{\hbar}\partial_\omega\sigma_{pq}(t) = \frac{\partial_\omega P}{P^2}\sin(2\tau-\chi-2\psi)\sinh(2r) - \frac{1}{\omega P}[\sin(2\tau)\cosh(2r)+2\tau \times \cos(2\tau-\chi-2\psi)\sinh(2r)]. \tag{C6c}$$

By inserting Eqs. (C4)–(C6) into Eq. (11) one obtains Eq. (19).

2. Damped case

By repeating the previous calculations with a nonvanishing γ, we estimate the QFI for damped Gaussian states for measuring ω. Since the solution is too heavy, we specify the three terms in Eq. (11) separately, i.e.,

$$\mathrm{I}(\rho(t);\omega) = \mathrm{I}_{1,\omega} + \mathrm{I}_{2,\omega} + \mathrm{I}_{3,\omega}, \tag{C7}$$

where

$$\mathrm{I}_{1,\omega} = [2(1+P^2)]^{-1}\mathrm{tr}[(\Sigma^{-1}\partial_\theta\Sigma)^2], \tag{C8a}$$

$$\mathrm{I}_{2,\omega} = 2(\partial_\theta P)^2(1-P^4)^{-1}, \tag{C8b}$$

$$\mathrm{I}_{3,\omega} = (\partial_\theta\langle\mathbf{X}\rangle)^{\mathrm{T}}\Sigma^{-1}\partial_\theta\langle\mathbf{X}\rangle. \tag{C8c}$$

Repeating the previous calculation for a nonvanishing damping leads to the following exact result of the QFI:

$$\begin{aligned}
\mathrm{I}_{1,\omega} = \frac{P^4(\tau)}{2\omega^2 e^{4g\tau}[1+P^2(\tau)]}\Big\{ & 8A_1^2\big(A_1^2 + a_{1,\tau}^2 + 2a_{1,\tau}A_1\mathcal{C}_r\big)\mathcal{S}_r^2\tau^2 \\
& + 8A_1\big(A_1^2 + a_{1,\tau}^2 + 2a_{1,\tau}A_1\mathcal{C}_r\big)[A_1\sin(\xi)\mathcal{C}_r + (a_{1,\tau}+A_1\mathcal{C}_r)\sin(2\tau-\xi)]\mathcal{S}_r\tau \\
& + \frac{A_1^2}{2}\big[4A_1^2\big(\mathcal{S}_r^2+2\big) + A_2^2A_3^2 + 2A_2A_3a_{2,\tau}a_3\mathcal{C}_r + a_{2,\tau}^2a_3^2\big(2\mathcal{S}_r^2+1\big) \\
& - 2A_1[4A_1\cos(2\tau) + A_1[\cos(2\xi)+\cos(4\tau-2\xi)+4\sin(\xi)\sin(2\tau-\xi)]\mathcal{S}_r^2 - 4a_{2,\tau}a_3\sin(\tau)\sin(\tau-\xi)\mathcal{S}_r] \\
& + A_1a_{1,\tau}\big[2A_1\big[4A_1\mathcal{C}_r\sin^2(\tau)\big[3+2\cos^2(\tau-\xi)\mathcal{S}_r^2\big] + \mathcal{S}_r\cos(2\tau-\xi)(A_2A_3 - 2a_{2,\tau}a_3\mathcal{C}_r) - 2A_2A_3\mathcal{S}_r\cos(\xi)\big] \\
& + A_2^2A_3^2\mathcal{C}_r + 2A_2A_3a_{2,\tau}a_3 + a_{2,\tau}^2a_3^2\mathcal{C}_r\big] \\
& + \frac{a_{1,\tau}^2}{2}\big(4A_1^2\big(7+6\mathcal{S}_r^2\big) + a_{2,\tau}^2a_3^2 + A_2A_3\big[2a_{2,\tau}a_3\mathcal{C}_r + A_2A_3\big(1+2\mathcal{S}_r^2\big)\big] \\
& + 2A_1\big\{A_1\mathcal{S}_r^2[2[\cos(2\tau-2\xi)-5\cos(2\tau)] - \cos(4\tau-2\xi)+\cos(2\xi)] - 12A_1\cos(2\tau) \\
& + 8A_2A_3\mathcal{C}_r\mathcal{S}_r\sin(\tau)\sin(\tau-\xi) - 4a_{2,\tau}a_3\mathcal{S}_r\cos(\tau)\cos(\tau-\xi)\big\}\big) \\
& + a_{1,\tau}^3[2A_2A_3\mathcal{S}_r\cos(2\tau-\xi) - 4A_1\mathcal{C}_r(\cos(2\tau)-2)] + 2a_{1,\tau}^4\Big\},
\end{aligned} \tag{C9}$$

$$\mathrm{I}_{2,\omega} = \frac{e^{-4g\tau}P^6(\tau)}{2\omega^2[(1-P^4(\tau)]}[A_1A_2A_3 + a_{1,\tau}a_{2,\tau}a_3 + (A_1a_{2,\tau}a_3 + a_{1,\tau}A_2A_3)\mathcal{C}_r - 2a_{1,\tau}A_1\cos(\xi)\mathcal{S}_r]^2, \tag{C10}$$

$$\begin{aligned}
\mathrm{I}_{3,\omega} = \frac{4e^{-2g\tau}\alpha^2P^2(\tau)}{\omega^2}\{ & [a_{1,\tau} + A_1(\mathcal{C}_r + \cos(\chi)\mathcal{S}_r)]\tau^2 + [\cos(\tau-2\psi)(a_{1,\tau}+A_1\mathcal{C}_r) + A_1\cos(\tau-\xi)\mathcal{S}_r]2\tau\sin(\tau) \\
& + [a_{1,\tau} + A_1(\mathcal{C}_r + \cos(2\tau-\xi-2\psi)\mathcal{S}_r)]\sin^2(\tau)\},
\end{aligned} \tag{C11}$$

where we have introduced a new angle $\xi = \chi + 2\psi$, $\mathcal{C}_r = \cosh(2r)$, $\mathcal{S}_r = \sinh(2r)$, and

$$a_1 = 1 + 2\bar{n}, \quad A_1 = 1 + 2N_{\mathrm{th}}, \tag{C12a}$$

$$a_2 = 4\bar{n}(1+\bar{n}), \quad A_2 = 4N_{\mathrm{th}}(1+N_{\mathrm{th}}), \tag{C12b}$$

$$a_3 = \ln(1+1/\bar{n}), \quad A_3 = \ln(1+1/N_{\mathrm{th}}), \tag{C12c}$$

$$a_{1,\tau} = (e^{g\tau}-1)a_1, \quad a_{2,\tau} = (e^{g\tau}-1)a_2. \tag{C12d}$$

Finally, the maximum of the QFI of an initial ground state, which was used to approximate the QFI of the coherent state, is determined. For an initial ground state, the QFI simplifies to

$$\omega^2\,\mathrm{I}(\rho;\omega) = \frac{1+[e^{g\tau}(1+2\bar{n})-2\bar{n}]^2 - 2[e^{g\tau}(1+2\bar{n})-2\bar{n}]\cos(2\tau)}{2[2\bar{n}^2 - 2e^{g\tau}\bar{n}(1+2\bar{n}) + e^{2g\tau}(1+2\bar{n}+2\bar{n}^2)]} + \frac{(e^{g\tau}-1)\bar{n}(1+\bar{n})^2\ln^2(1+1/\bar{n})}{e^{g\tau}(1+\bar{n})-\bar{n}}. \tag{C13}$$

Numerical maximization of $\mathrm{I}(\rho;\omega)$ with respect to the three parameters τ, g, and $\bar{n}$ returns the value $2.135/\omega^2$.

APPENDIX D: ALTERNATIVE WAY OF CALCULATING THE QFI

Next, we show that by considering the full ω dependence of the time evolution operator and assuming that the initial state does not depend on ω, we obtain the same result as we did using our scheme introduced in Sec. III A. This means that we redo the calculation from Eq. (16), but this time we consider the ω dependence correctly.

Let $\rho(0)$ be the initial Gaussian state according to Eq. (7), i.e.,

$$\rho_{\omega_0}(0) = R_{\omega_0}(\psi)D_{\omega_0}(\alpha)S_{\omega_0}(z)v_{\omega_0}S_{\omega_0}^\dagger(z)D_{\omega_0}^\dagger(\alpha)R_{\omega_0}^\dagger(\psi)\,. \tag{D1}$$

For the given Hamiltonian $\mathcal{H}_\omega = \hbar\omega(\hat{a}_\omega^\dagger\hat{a}_\omega + 1/2)$, the dynamics of the system is described by

$$\rho(t) = U_\omega(t)\rho_{\omega_0}(0)U_\omega^\dagger(t) = R_\omega(-\omega t)\rho_{\omega_0}(0)R_\omega^\dagger(-\omega t)\,, \tag{D2}$$

where $U_\omega(t)$ is the time evolution operator. Keeping in mind that we want to use Eq. (11) for estimating the QFI, we need to calculate the expectation values of p, q, p^2, q^2, and $\frac{1}{2}(qp+pq)$. The expectation value of an arbitrary operator A is given by

$$\langle A\rangle_t = \mathrm{tr}[\rho(t)A] = \mathrm{tr}[\rho_{\omega_0}(0)R_\omega^\dagger(-\omega t)AR_\omega(-\omega t)]. \tag{D3}$$

It is therefore useful to express $R^\dagger_\omega(-\omega t)q_\omega R_\omega(-\omega t)$ and $R^\dagger_\omega(-\omega t)p_\omega R_\omega(-\omega t)$ in terms of q_{ω_0} and p_{ω_0}. By using the formulas

$$a_\omega = \frac{1}{2\sqrt{\omega_0\omega}}[(\omega_0+\omega)a_{\omega_0} - (\omega_0-\omega)a^\dagger_{\omega_0}], \tag{D4a}$$

$$R^\dagger_\omega(\psi)a_\omega R_\omega(\psi) = e^{i\psi}a_\omega, \tag{D4b}$$

we obtain

$$R^\dagger_\omega(-\omega t)q_\omega R_\omega(-\omega t) = \cos(\omega t)q_{\omega_0} + \frac{\sin(\omega t)}{M\omega}p_{\omega_0}, \tag{D5a}$$

$$R^\dagger_\omega(-\omega t)p_\omega R_\omega(-\omega t) = \cos(\omega t)p_{\omega_0} - M\omega\sin(\omega t)q_{\omega_0}. \tag{D5b}$$

Inserting Eq. (D5) into the expectation value yields

$$\langle q\rangle_t = \cos(\omega t)\langle q\rangle_0 + \frac{1}{M\omega}\sin(\omega t)\langle p\rangle_0, \tag{D6a}$$

$$\langle p\rangle_t = \cos(\omega t)\langle p\rangle_0 - M\omega\sin(\omega t)\langle q\rangle_0 \tag{D6b}$$

and

$$\sigma_{qq}(t) = \cos^2(\omega t)\sigma_{qq}(0) + \frac{\sin^2(\omega t)}{M^2\omega^2}\sigma_{pp}(0) + \frac{\sin(2\omega t)}{M\omega}\sigma_{pq}(0), \tag{D7a}$$

$$\sigma_{pp}(t) = \cos^2(\omega t)\sigma_{pp}(0) + M^2\omega^2\sin^2(\omega t)\sigma_{qq}(0) - M\omega\sin(2\omega t)\sigma_{pq}(0), \tag{D7b}$$

$$\sigma_{pq}(t) = \cos(2\omega t)\sigma_{pq}(0) + \frac{1}{M\omega}\sin(\omega t)\cos(\omega t)[\sigma_{pp}(0) - M^2\omega^2\sigma_{qq}(0)]. \tag{D7c}$$

These equations are the same as Eqs. (C1) and (C2) (for $\gamma = 0$) and therefore the following calculations are the same. Thus, considering all ω dependencies correctly results in the same QFI as using the scheme introduced in Sec. III A.

[1] D. Braun, Ultimate quantum bounds on mass measurements with a nano-mechanical resonator, Europhys. Lett. **94**, 68007 (2011).

[2] Q. Zheng, Y. Yao, and Y. Li, Optimal quantum parameter estimation in a pulsed quantum optomechanical system, Phys. Rev. A **93**, 013848 (2016).

[3] M. Asjad, G. S. Agarwal, M. S. Kim, P. Tombesi, G. Di Giuseppe, and D. Vitali, Robust stationary mechanical squeezing in a kicked quadratic optomechanical system, Phys. Rev. A **89**, 023849 (2014).

[4] O. Pinel, P. Jian, N. Treps, C. Fabre, and D. Braun, Quantum parameter estimation using general single-mode Gaussian states, Phys. Rev. A **88**, 040102(R) (2013).

[5] G. S. Agarwal, Brownian motion of a quantum oscillator, Phys. Rev. A **4**, 739 (1971).

[6] S. Dattagupta, Brownian motion of a quantum system, Phys. Rev. A **30**, 1525 (1984).

[7] G. Lindblad, Brownian motion of a quantum harmonic oscillator, Rep. Math. Phys. **10**, 393 (1976).

[8] B. Vacchini, Quantum optical versus quantum Brownian motion master-equation in terms of covariance and equilibrium properties, J. Math. Phys. **43**, 5446 (2002).

[9] A. Isar and A. Săndulescu, Damped quantum harmonic oscillator, Rom. J. Phys. **37**, 643 (1992).

[10] E. Wigner, On the quantum correction for thermodynamic equilibrium, Phys. Rev. **40**, 749 (1932).

[11] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. **84**, 621 (2012).

[12] M. G. A. Paris, F. Illuminati, A. Serafini, and S. De Siena, Purity of Gaussian states: Measurement schemes and time evolution in noisy channels, Phys. Rev. A **68**, 012314 (2003).

[13] G. Adam, Density matrix elements and moments for generalized Gaussian state fields, J. Mod. Opt. **42**, 1311 (1995).

[14] S. L. Braunstein and C. M. Caves, Statistical Distance and the Geometry of Quantum States, Phys. Rev. Lett. **72**, 3439 (1994).

[15] A. S. Holevo, *Statistical Structure of Quantum Theory* (Springer Science & Business Media, Nairobi, Kenya, 2003), Vol. 67.

[16] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Springer Science & Business Media, Nairobi, Kenya, 2011), Vol. 1.

[17] M. Hayashi, *Quantum Information* (Springer, Berlin, 2006).

[18] H. Scutaru, Fidelity for displaced squeezed thermal states and the oscillator semigroup, J. Phys. A: Math. Gen. **31**, 3659 (1998).

[19] Z. Jiang, Quantum Fisher information for states in exponential form, Phys. Rev. A **89**, 032128 (2014).

[20] M. Bina, F. Grasselli, and M. G. A. Paris, Continuous-variable quantum probes for structured environments, Phys. Rev. A **97**, 012125 (2018).

[21] S. L. Braunstein, C. M. Caves, and G. J. Milburn, Generalized uncertainty relations: Theory, examples, and Lorentz invariance, Ann. Phys. **247**, 135 (1996).

[22] M. G. Paris, Quantum estimation for quantum technology, Int. J. Quantum. Inf. **7**, 125 (2009).

[23] After correcting $r \to -r$ and $\chi \to 2\chi$ in the definitions of [4].

[24] D. Braun and J. Martin, Heisenberg-limited sensitivity with decoherence-enhanced measurements, Nat. Commun. **2**, 223 (2011).

[25] C. Riek, D. V. Seletskiy, A. S. Moskalenko, J. F. Schmidt, P. Krauspe, S. Eckart, S. Eggert, G. Burkard, and A. Leitenstorfer, Direct sampling of electric-field vacuum fluctuations, Science **350**, 420 (2015).

[26] J. Chaste, A. Eichler, J. Moser, G. Ceballos, R. Rurali, and A. Bachtold, A nanomechanical mass sensor with yoctogram resolution, Nat. Nanotechnol. **7**, 301 (2012).

[27] K. Jensen, K. Kim, and A. Zettl, An atomic-resolution nanomechanical mass sensor, Nat. Nanotechnol. **3**, 533 (2008).

[28] M. M. Nieto, Displaced and squeezed number states, Phys. Lett. A **229**, 135 (1997).

[29] W. Smith, The overlap integral of two harmonic-oscillator wave functions, J. Phys. B **2**, 1 (1969).

[30] S. Boixo, S. T. Flammia, C. M. Caves, and J. M. Geremia, Generalized Limits for Single-Parameter Quantum Estimation, Phys. Rev. Lett. **98**, 090401 (2007).

[31] R. Snider, Perturbation variation methods for a quantum Boltzmann equation, J. Math. Phys. **5**, 1580 (1964).

[32] J. M. E. Fraïsse and D. Braun, Enhancing sensitivity in quantum metrology by Hamiltonian extensions, Phys. Rev. A **95**, 062342 (2017).