Nonlinear Schrödinger equations and generalized Heisenberg uncertainty principle from estimation schemes violating the principle of estimation independence

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One of the advantages of a reconstruction of quantum mechanics based on transparent physical axioms is that it may offer insight to naturally generalize quantum mechanics by relaxing the axioms. Here, we discuss possible extensions of quantum mechanics within a general epistemic framework based on an operational scheme of estimation of momentum given information on the conjugate positions under epistemic restriction. The epistemic restriction is parameterized by a global-nonseparable random variable on the order of Planck constant, an ontic extension to the separable classical phase-space variables. Within the estimation scheme, the canonical quantum laws are reconstructed for a specific estimator and estimation error. In the present work, keeping the Born's quadratic law intact, we construct a class of nonlinear variants of Schrödinger equation and generalized Heisenberg uncertainty principle within the estimation scheme by assuming a more general class of estimation errors. The nonlinearity of the Schrödinger equation and the deviation from the Heisenberg uncertainty principle thus have a common transparent operational origin in terms of generalizations of estimation errors. We then argue that a broad class of nonlinearities and deviations from the Heisenberg uncertainty principle arise from estimation errors violating a plausible inferential-causality principle of estimation independence which is respected by the standard quantum mechanics. This result therefore constrains possible extensions of quantum mechanics and suggests directions to generalize quantum mechanics which comply with the principle of estimation independence.

I. INTRODUCTION

The linearity of the Schrödinger equation with Hermitian quantum Hamiltonian, together with the Born's quadratic law, i.e., the Born's statistical interpretation of wave function, and the Heisenberg uncertainty principle, are the central tenets of quantum mechanics. Hitherto, they have passed all experimental tests with unprecedented accuracy. In spite of their monumental empirical successes, there are nonetheless important reasons to mull over possible generalizations of, and deviations from, these canonical quantum laws: as a guide to conceive stringent precision tests of quantum mechanics which is motivated by the argument that the linearity of the theory might be an approximation to a deeper theory with an extremely weak nonlinearity [1,2], or that the Born's quadratic law applies only in a specific situation of quantum equilibrium [3]; to construct a general framework for a broad class of nonclassical theories which provide a foil to the standard quantum mechanics to better understand conceptually what deeply distinguishes quantum mechanics from the alternative nonclassical theories [4–10]; to study their information-processing capabilities in comparison with those based on quantum mechanics [11–16]; to resolve the infamous measurement problem that the present linear Schrödinger equation may lead to an embarrassing superposition of per-

ceptible macroscopic objects [17–22] (i.e., the well-known Schrödinger's cat [23]); and to develop a general framework which may encompass quantum mechanics and general relativity [24,25].

It has been argued, however, that nonlinear modifications of the Schrödinger equation may violate the relativistic causality principle of no-signaling [26–30]. Moreover, introducing a non-Hermitian quantum Hamiltonian [31] may also be in conflict with no-signaling [32]. In contrast to this, while quantum mechanics allows stronger than classical correlation [33,34], such nonclassical correlations cannot be used to perform faster than light communication; hence, quantum mechanics elegantly respects no-signaling. However, as the Popescu-Rohrlich box shows [35], quantum mechanics is not the only theory which allows stronger than classical correlation and at the same time also complies with no-signaling. Hence, no-signaling is not sufficient to uniquely single out quantum mechanics from among all possible nonclassical theories. These simple but fundamental results yet suggest that the abstract quantum laws may be deeply rooted in some forms of causality principles. This belief is further supported by the theoretical findings that introducing nonlinearity in the Schrödinger equation and a deviation from the Heisenberg uncertainty principle may also lead to violations of the second law of thermodynamics [36,37] (see, however, Ref. [38]). In addition, a deviation from the Heisenberg uncertainty principle may imply stronger than quantum correlation [39], which

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in turn allows implausible computational power [40–47]. Is quantum mechanics the unique nonclassical theory that obeys certain causality principles [35]? This line of inquiry to reconstruct quantum mechanics from deep but simple and transparent axioms [48] may not only lead to a better understanding on the meaning of quantum mechanics but may also offer fresh insight and useful intuition to suggest a logically coherent possible extensions of quantum mechanics by slightly varying the parameters unfixed by the axioms or by relaxing some of the axioms.

On the other hand, previously, we have shown that the abstract formalism of nonrelativistic spinless quantum mechanics can be reconstructed within an epistemic framework based on an operational scheme of estimation of momentum given the information on the conjugate positions [49], under a fundamental epistemic restriction [50] so that the allowed probability distribution of positions that an agent can prepare are irreducibly parametrized by the underlying momentum field. The momentum field is assumed to fluctuate randomly, induced by a global-nonseparable random variable on the order of the Planck constant, an ontic extension to the separable classical phase-space variables [51]. Within this operational scheme of estimation under epistemic restriction, we showed in Refs. [49,51] that the mathematical rules of quantum mechanics in complex Hilbert space formalism, including the linear Schrödinger equation with Hermitian quantum Hamiltonian, the Born's quadratic law, and also the exact form of the Heisenberg uncertainty principle, emerge when the associated estimator and estimation error, take specific forms. Such a specific operational scheme of estimation of momentum based on information on positions has a well-defined implementation in terms of weak momentum value measurement [52–54], which leads to a simple method for the reconstruction of quantum wave function [49,55].

In the present work, we show that the above epistemic framework based on the operational scheme of estimation under epistemic restriction, is flexible to transparently accommodate a broad class of possible extensions of quantum mechanics. Keeping the Born's quadratic law intact, we first construct a broad class of nonlinear variants of the Schrödinger equation and generalized Heisenberg uncertainty principle, by choosing a more general class of estimation errors. Both deviations from the canonical laws of quantum mechanics have thus a common origin from, and a transparent operational meaning in terms of, the generalizations of the estimation errors. They are thus deeply interrelated. In particular, there is no nonlinearity without a deviation from the Heisenberg uncertainty principle, suggesting that it is difficult to modify a part of quantum mechanics without changing the other important parts of the theory. We then show that a broad class of nonlinear Schrödinger equations and deviations from the Heisenberg uncertainty principle arise from estimation errors violating a plausible inferential-causality principle of estimation independence [56]. By contrast, the principle of estimation independence is strictly and pleasingly respected by the specific estimation error leading to the standard quantum mechanics. The result thus constrains possible extensions of quantum mechanics and offers insight to the kinds of generalization of quantum mechanics which comply with the principle of estimation independence.

The rest of the paper is organized as follows. In Sec. II, we give a brief summary on the epistemic reconstruction of nonrelativistic spinless quantum mechanics proposed in Refs. [49,51,56], based on the operational scheme of estimation of momentum given information on positions under epistemic restriction parameterized by a global random variable, with specific estimator and estimation error. In Sec. III A, we consider a generalization of the estimation scheme by employing a class of more general estimation errors, based on which we derive a broad nonlinear variants of Schrödinger equation in Sec. III B and generalized Heisenberg uncertainty principle in Sec. III C. We proceed in Sec. IV to discuss the relation between the resulting nonlinearity in the Schrödinger equation and the deviation from the Heisenberg uncertainty principle, and show that a large class of nonlinearities and deviations from the Heisenberg uncertainty principle arise from estimation schemes with estimation errors violating a physically transparent and plausible inferential-causality principle of estimation independence. We end in Sec. V with conclusions and offer a sketch of several future directions for possible generalizations of quantum mechanics which do not violate the principle of estimation independence.

II. QUANTUM MECHANICS FROM A SPECIFIC SCHEME OF ESTIMATION UNDER EPISTEMIC RESTRICTION PARAMETERIZED BY A GLOBAL RANDOM VARIABLE ON THE ORDER OF PLANCK CONSTANT

Consider a system with a spatial configuration $q = (q_1, \ldots, q_N)$ and the conjugate momentum $p = (p_1, \ldots, p_N)$. First, recall that in classical mechanics, working within the Hamilton-Jacobi formalism [57], the momentum field can be written as

$$\tilde{p}_C(q, t) = \partial_q S_C(q, t), \qquad (1)$$

where $\partial_q = (\partial_{q_1}, \ldots, \partial_{q_N})$, and $S_C(q, t)$ is a real-valued scalar function of the positions q and time t, called as the Hamilton's principal function. (In the paper, we label the momentum field with $\tilde{p}$, whereas p is used to denote the specific value of momentum). It is then clear from Eq. (1) that in classical mechanics, given a momentum field $\tilde{p}_C(q)$ (trivial dependence on time is notationally suppressed) arising in a fixed experimental arrangement, it is in principle possible for an agent, by repeating the experiments many times, to prepare an ensemble of trajectories with arbitrary distribution of positions $\rho(q)$. Namely, each trajectory in the momentum field $\tilde{p}_C(q)$ can be assigned an arbitrary weight $\rho(q)$. Hence, in classical mechanics, the distribution of positions $\rho(q)$ is fundamentally independent of, and thus is not irreducibly parametrized by, the underlying momentum field $\tilde{p}_C(q)$.

We postulate that the above epistemic freedom, namely the freedom to prepare the probability distribution of positions independent of the underlying momentum field, is no longer respected in the microscopic world [51]. Assume first that in the microscopic world, there is a global-nonseparable variable ξ of action dimensional, fluctuating randomly and inducing a random fluctuations of the momentum field $\tilde{p}(q, t; \xi) = (\tilde{p}_1(q, t; \xi), \ldots, \tilde{p}_N(q, t; \xi))$. We then assume that the ensemble of trajectories obtained by identically repeating the experiment suffers a fundamental epistemic restriction [51]:

Namely, unlike in classical mechanics discussed above, it is no longer possible for an agent to assign each trajectory in the momentum field $\tilde{p}(q;\xi)$ an arbitrary weight. The probability distributions of positions therefore fundamentally depend on, and thus are irreducibly parametrized by, the underlying momentum field $\tilde{p}(q;\xi)$. To make explicit this intrinsic dependence, we write the probability distribution of positions as $\rho_{\tilde{p}}(q)$ with a subscript $\tilde{p}$. Furthermore, we assume that in the formal limit of vanishing global fluctuation ξ, the epistemic restriction disappears, i.e., $\lim_{\xi \to 0} \rho_{\tilde{p}}(q) = \rho(q)$, and we regain classical mechanics satisfying Eq. (1) with the epistemic freedom recovered. The fluctuation of ξ thus characterizes the strength of the epistemic restriction and therefore must be practically ignorable in the macroscopic physical regime. We emphasize that the global-nonseparable variable ξ provides an ontic extension to the separable classical phase-space variables.

We have argued in Refs. [49,51] that the abstract mathematical rules of nonrelativistic spinless quantum mechanics can be derived within an operational scheme of estimation of momentum given information on positions, under the above epistemic restriction, combined with the Bayesian reasoning given the experimental settings. A concrete illustration of the reconstruction of quantum laws based on such an estimation scheme in a single- and double-slit experiment is given in Appendix A. First, suppose that the agent has access to q via some position measurement. Note that, as emphasized by Bell [58], any measurement should be reducible to the measurement of position. Since q is sampled from $\rho_{\tilde{p}}(q)$ parametrized by $\tilde{p}(q;\xi)$, then it must somehow contain some information about $\tilde{p}(q;\xi)$. How can the agent use her information about position, in the most reasonable way, to estimate the conjugate momentum? To answer this parameter estimation problem, we need to choose the estimator and the associated estimation error [59].

Let us construct a reasonable estimator for $\tilde{p}(q;\xi)$. First, we select a subensemble of trajectories that are passing $q(t)$ at time t, where different trajectories in the subensemble correspond to different fluctuations of ξ. Then, along each of the trajectory in the subensemble, we make a naive classical momentum measurement via two consecutive position measurements as follows. Just before the system is detected at $q(t)$, we perform a sufficiently weak measurement of the position at time $t - \Delta t$ without appreciably disturbing the trajectory, yielding $q(t - \Delta t)$, where Δt is extremely small. The velocity along the trajectory at $q(t)$ can then be computed in the conventional way by evaluating the difference between $q(t)$ and $q(t - \Delta t)$ and dividing it with Δt, from which we also get the momentum $\tilde{p}(q;\xi)$ along that particular trajectory. Note that because of the fluctuation of ξ, each such single measurement of momentum must yield a random outcome. We then define the estimator $\overline{p}(q) = (\overline{p}_1(q), \ldots, \overline{p}_N(q))$ for $\tilde{p}(q;\xi)$ at time t by taking the average of the above measurement outcomes over all the trajectories in the subensemble. Within the statistical model, such a conditional ensemble average of momentum $\overline{p}(q)$ thus corresponds to the average of $\tilde{p}(q;\xi)$ over ξ, i.e.,

$$\overline{p}(q) \doteq \int d\xi \; \tilde{p}(q;\xi)\chi(\xi), \tag{2}$$

where $\chi(\xi)$ is the probability distribution of ξ. Clearly, by construction, in the absence of ξ, the above scheme for estimating the momentum reduces to the conventional measurement of momentum at q in classical mechanics which must give back Eq. (1).

Next, to have a smooth correspondence with classical mechanics, we assume that the above estimator $\overline{p}(q)$ for $\tilde{p}(q;\xi)$ at q can be written as follows:

$$\overline{p}(q) \doteq \partial_q S(q), \tag{3}$$

where $S(q)$ is a real-valued scalar function, so that in the macroscopic physical regime, the estimator is expected to approach the gradient of the Hamilton's principal function, i.e., $\overline{p}(q) = \partial_q S(q) \to \partial_q S_C(q)$, recovering Eq. (1) of classical mechanics. Of course, since we want to reconstruct quantum mechanics from the above estimation scheme, for consistency, we need to check afterward whether the above operational protocol for estimating the momentum at $q(t)$, by first weakly measuring the position at time $t - \Delta t$ and then followed immediately by a position postselection (strong position measurement) at time t, is consistent with quantum mechanics. That this is indeed the case is shown by Wiseman in Ref. [60] (see also Refs. [49,55]), which has led to the impressive experimental reconstruction of the average trajectory in the double-slit experiment [61]. Namely, implementing the above estimation of the momentum at q with the quantum weak measurement over a preselected wave function $\psi(q)$ and a position postselection at q [52–54], indeed yields Eq. (3), where $S(q)$ is identified as the phase of the quantum wave function $\psi(q)$.

Moreover, given q, let us assume that the error in a single-shot estimation of $\tilde{p}(q;\xi)$ with the estimator $\overline{p}(q) = \partial_q S(q)$ has the following specific form [49]:

$$\epsilon_p(q;\xi) \doteq \tilde{p}(q;\xi) - \partial_q S(q) = \frac{\xi}{2}\partial_q \ln \rho_{\tilde{p}}(q). \tag{4}$$

One can see that in the mathematical limit $\xi \to 0$, the estimation error is vanishing, and we regain the classical relation of Eq. (1), $\lim_{\xi \to 0} \tilde{p} = \overline{p} = \partial_q S$, so that the epistemic restriction disappears, as required. Furthermore, assuming that $\rho_{\tilde{p}}(q)$ is vanishing at the boundary, the above estimation error is on average vanishing for all ξ, i.e., $\int dq \epsilon_p(q;\xi)\rho_{\tilde{p}}(q) = \frac{\xi}{2} \int dq \partial_q \rho_{\tilde{p}}(q) = 0$, $dq = dq_1 \ldots dq_N$; hence, it is desirably (weakly) unbiased.

Let us further assume that the global variable ξ is fluctuating randomly on a microscopic timescale so that its first and second moments are independent of time, given by [51]

$$\overline{\xi} \doteq \int d\xi \; \xi \; \chi(\xi) = 0, \quad \overline{\xi^2} = \hbar^2. \tag{5}$$

The left equation guarantees that the conditional (subensemble) average of p given q is equal to the estimator satisfying Eq. (3); i.e., from Eq. (4), we have $\overline{p}(q) = \int d\xi \tilde{p}(q;\xi)\chi(\xi) = \partial_q S(q)$. On the other hand, the right equation in Eq. (5) shows that the strength of the estimation error is on the order of Planck constant. It therefore ensures that in the macroscopic physical regime, the estimation error is much smaller than the estimator, i.e., $|\partial_q S| \gg |\frac{\xi}{2}\partial_q \ln \rho_{\tilde{p}}|$, so that Eq. (4) effectively reduces back to the classical relation: $\tilde{p} \approx \partial_q S$. Finally, one can also argue that in the above estimation scheme, the

estimator $\overline{p}(q) = \partial_q S(q)$ best estimates $\tilde{p}(q; \xi)$, in the sense that it minimizes the mean-squared (MS) error defined as $\mathcal{E}_p^2 \doteq \int dq d\xi (\epsilon_p(q; \xi))^2 \chi(\xi) \rho_{\tilde{p}}(q)$ [49] (see also Appendix B). This estimation scheme is thus also consistent with the argument advanced in Refs. [62,63] wherein Eq. (3), with $S(q)$ is given by the phase of the wave function, is interpreted as the optimal estimate of momentum based on the measurement of position.

For later comparison, let us write Eq. (4) as

$$\tilde{p}(q; \xi) = \partial_q S(q) + \frac{\xi}{2} \frac{\partial_q \rho_{\tilde{p}}(q)}{\rho_{\tilde{p}}(q)}. \tag{6}$$

Hence, we have a random momentum field which is decomposed into two terms. We emphasize that, by construction, the above decomposition of the random momentum field is not ontic (physical), happening in physical space. Rather, the decomposition is epistemic (i.e., informational); namely, it happens in the agent's mind, artificially devised by the agent to describe her best estimate of the momentum given information on positions [the first term on the right-hand side of Eq. (6)] and the associated single-shot estimation error [the second term] [49]. Equation (6) is just the specific epistemic restriction we postulated in Ref. [51], based on which we derived the mathematical formalism of quantum mechanics.

Within the epistemic reconstruction based on the specific operational scheme of estimation under epistemic restriction, the quantum wave function $\psi(q, t)$ characterizing a preparation is a mathematical object which summarizes the estimator of Eq. (3) and the estimation error of Eq. (4) via $(S(q, t), \rho_{\tilde{p}}(q, t))$ as [49]

$$\psi(q, t) \doteq \sqrt{\rho_{\tilde{p}}(q, t)} \exp[iS(q, t)/\hbar]. \tag{7}$$

As in Refs. [60,62,63], $S(q, t)$ operationally defined in Eq. (3) indeed constitutes the phase of the quantum wave function. In this sense, basically, the estimation of momentum given information on positions described above thus operationally leads to the reconstruction of quantum wave function characterizing the preparation [55]. For example, consider a preparation setting so that quantum mechanically it results in a Gaussian wave function $\psi(q) = (\frac{1}{2\pi\sigma_q^2})^{1/4} e^{-(q-q_o)^2/4\sigma_q^2 + ip_o q/\hbar}$. Within the above epistemic interpretation, noting Eqs. (3) and (4), it means that given information on q, the agent should assign $\overline{p}(q) = p_o$ as her best estimate of the momentum of the system, with the single-shot estimation error $\epsilon_p(q; \xi) = -\frac{\xi}{2\sigma_q^2}(q - q_o)$ so that the MS error reads $\mathcal{E}_p^2 = \hbar^2/4\sigma_q^2$. In particular, a preparation leading to a plane-wave function, $\psi(q) \sim e^{ip_o q/\hbar}$, means that the agent's best estimate of momentum $\overline{p} = p_o$ is sharp with a vanishing MS error, $\mathcal{E}_p^2 = 0$.

Hence, by construction, quantum wave function is not an agent-independent objective physical attribute of the system, but it represents the agent's estimation about the momentum field arising in her preparation based on information on the conjugate positions [49]. Note that from the definition of wave function in Eq. (7), the epistemic decomposition of momentum field in Eq. (6) is invariant under the transformation of wave function $\psi \mapsto Z\psi$, where Z is an arbitrary complex constant. Namely, the estimator and the estimation error of Eqs. (3) and (4) are invariant under such transformation of

wave function. ψ and $Z\psi$ thus represent the same estimation scheme; i.e., the statistical content encoded in ψ and $Z\psi$ are the same, as in standard quantum mechanics. One can also see that, by construction, Eq. (7) leads to the Born's quadratic law,

$$\rho_{\tilde{p}}(q, t) = |\psi(q, t)|^2. \tag{8}$$

Finally, within the above specific estimation scheme, the linear Schrödinger equation can be seen as a Bayesian rule for updating the specific estimator and estimation error represented by the wave function via Eqs. (3) and (4), when she does not make measurement [49]. To see this, first note that measurement is in practice carried out by making a selection of a subensemble of trajectories associated with a particular measurement outcome (see Appendix A for a concrete illustration). No measurement thus corresponds to no selection of trajectories. In the absence of measurement, it is therefore natural for the agent to update her estimation represented by the wave function by imposing the statistical-informational constraints of conservation of trajectories and average energy. It is shown in Ref. [51] that, within the estimation scheme with the specific estimator and estimation error given by Eqs. (3) and (4), the above conservation principles lead to the derivation of the celebrated linear Schrödinger equation. We shall rederive the linear Schrödinger equation as a specific case of a more general dynamical equation in Sec. III B. Moreover, the Heisenberg-Kennard uncertainty relation between momentum and position can be traced back to the trade-off between the MS errors of simultaneous estimations of momentum field and mean position, which in turn is implied by the specific choice of estimation error of Eq. (4). This fundamentally distinctive feature of quantum mechanics will also be rederived in Sec. III C as a specific case of a more general uncertainty relation.

III. GENERALIZED ESTIMATION ERRORS: NONLINEAR SCHRÖDINGER EQUATION, AND GENERALIZED HEISENBERG UNCERTAINTY PRINCIPLE

A. A class of generalized estimation errors

One of the advantages of the epistemic reconstruction of quantum mechanics within the operational scheme of estimation under epistemic restriction is that it provides a flexible operational framework for transparently accommodating a broad class of possible generalizations of quantum mechanics. As summarized above, since the exact forms of the linear Schrödinger equation and the Heisenberg uncertainty principle can be obtained starting from the scheme of estimation of momentum given positions with the help of specific estimator and estimation error respectively given by Eqs. (3) and (4), it is instructive to generalize the above estimation scheme by relaxing Eqs. (3) and/or (4), to search for possible nontrivial extensions of quantum mechanics. To this end, recall that, as discussed in the previous section, the choice of the estimator of Eq. (3) is primarily motivated by a desire to have a smooth macroscopic classicality, requiring the estimator to recover the classical relation of Eq. (1) in the macroscopic physical regime. In this sense, the form of the estimator of Eq. (3) appears to be very natural. By contrast, the form of the

estimation error of Eq. (4) appears to be apparently *ad hoc*. Hence, it is instructive to try various possible alternative forms of estimation error, and work out and analyze the modifications they imply to the canonical laws of standard quantum mechanics such as the linear Schrödinger equation and the Heisenberg uncertainty principle.

Let us therefore consider a generalized scheme of estimation of the momentum based on information on the conjugate positions, with the estimator given by Eq. (3), but with an estimation error which generalizes Eq. (4) having the following general form:

$$\epsilon_{p_f}(q;\xi) \doteq \tilde{p}(q;\xi) - \partial_q S(q)$$
$$= \frac{\xi}{2}\frac{\partial_q \rho_{\tilde{p}}(q)}{\rho_{\tilde{p}}(q)} + \frac{\xi}{2}f(\rho_{\tilde{p}}(q), \partial_q \rho_{\tilde{p}}(q)), \quad (9)$$

where ξ is again assumed to satisfy Eq. (5), and $f = (f_1(\rho_{\tilde{p}}, \partial_q \rho_{\tilde{p}}), \ldots, f_N(\rho_{\tilde{p}}, \partial_q \rho_{\tilde{p}}))$ is a real vector-valued function of $\rho_{\tilde{p}}(q)$ and its spatial gradient $\partial_q \rho_{\tilde{p}}(q)$. Generalization to include higher degrees of spatial derivatives of $\rho_{\tilde{p}}(q)$ are straightforward. Comparing Eq. (9) with Eq. (4), we have thus added a minimal yet general nontrivial correction term given by the last term on the right-hand side of Eq. (9).

Several desirable properties of the specific estimation scheme of Sec. II are shared by the above more general estimation scheme. First, in the limit of vanishing global fluctuation ξ, the estimation error of Eq. (9) is vanishing, and we consistently recover Eq. (1) of classical mechanics, i.e., $\lim_{\xi \to 0} \tilde{p} = \overline{p} = \partial_q S$. Next, in the macroscopic regime where the estimation error of Eq. (9) is much smaller than the estimator of Eq. (3), we again effectively regain the classical relation of Eq. (1), i.e., $\tilde{p} \approx \partial_q S(q)$. Moreover, noting Eq. (5), from Eq. (9), the conditional average of p given q is equal to the estimator of Eq. (3), i.e., $\int d\xi \tilde{p}(q;\xi)\chi(\xi) = \partial_q S(q)$, as required. Finally, as shown in Appendix B, like the specific scheme of estimation in Sec. II, in the estimation scheme with the general estimation error of Eq. (9), the estimator of Eq. (3) also provides the best estimate of momentum based on information on positions, minimizing the MS error.

We show below that the general form of estimation error of Eq. (9) will lead to a broad class of nonlinear variants of Schrödinger equation when the agent does not make measurement (Sec. III B) and a class of generalized Heisenberg uncertainty principle (Sec. III C). We note that in Ref. [56] we have also briefly discussed a specific modification of estimation error of Eq. (4) leading to a specific deviation from the Heisenberg uncertainty principle; this specific modification belongs to the class of estimation errors of Eq. (9) with a specific $f = \Lambda \partial_q \rho_{\tilde{p}}(q)$, where Λ is a dimensionless real constant.

B. A class of nonlinear Schrödinger equations

Let us derive the equation that governs the time evolution of the agent's estimation of momentum given information on positions, namely the time evolution of the estimator and estimation error respectively given by Eqs. (3) and (9), when the agent does not make any selection of trajectories. We thus need to find out how the agent should rationally update the pair of functions $S(q, t)$ and $\rho_{\tilde{p}}(q, t)$ which determine the

estimator and estimation error, provided that she does not make any selection of trajectories. To do this, first, we rewrite Eq. (9) as

$$\tilde{p}(q;\xi) = \partial_q S(q) + \frac{\xi}{2}\frac{\partial_q \rho_{\tilde{p}}(q)}{\rho_{\tilde{p}}(q)} + \frac{\xi}{2}f(\rho_{\tilde{p}}(q), \partial_q \rho_{\tilde{p}}(q)). \quad (10)$$

As for the case of Eq. (6), by construction, the above decomposition of the random momentum field on the left-hand side, into three terms on the right-hand side, is not ontic, happening in physical space; rather, it is epistemic, artificially constructed in the agent's mind to organize her experiences.

Now, for simplicity, we confine our discussion to a system of N one-dimensional (or $N/3$ three-dimensional) particles subjected to a scalar potential $V(q)$ with the classical Hamiltonian taking the following form: $H(p, q) = \sum_{j=1}^{N} p_j^2/2m_j + V(q)$, where m_j is the mass of the jth particle. (Application to more general classical Hamiltonian can be done following the same steps below). In this case, the velocity $\dot{q}_j = dq_j/dt$, $j = 1, \ldots, N$ and the momentum are related as $\dot{q}_j = \partial H/\partial p_j = p_j/m_j$, $j = 1, \ldots, N$, so that inserting Eq. (10), the velocity field is epistemically decomposed as $\tilde{q}_j(q;\xi) = \tilde{p}_j/m_j = \frac{\partial_{q_j} S}{m_j} + \frac{\xi}{2m_j}\frac{\partial_{q_j}\rho_{\tilde{p}}}{\rho_{\tilde{p}}} + \frac{\xi}{2m_j}f_j(\rho_{\tilde{p}}, \partial_q \rho_{\tilde{p}})$, $j = 1, \ldots, N$. The first term on the right-hand side is just the agent's best estimate of the velocity based on information on positions, and the other two terms comprise the estimation error. Hence, averaging over ξ and noting Eq. (5), we obtain the conditional average velocity at q which is equal to the best estimate, i.e.,

$$\overline{\dot{q}_j}(q) = \partial_{q_j} S(q)/m_j, \quad (11)$$

$j = 1, \ldots, N$.

Next, since the agent does not make any selection of trajectories, it is reasonable to require that her estimator and estimation error should be updated in such a way that they respect the conservation of trajectories or probability current. The agent's estimation should therefore satisfy the following continuity equation: $\partial_t \rho_{\tilde{p}} + \sum_{j=1}^{N} \partial_{q_j}(\overline{\dot{q}_j}\rho_{\tilde{p}}) = 0$. Inserting Eq. (11), one thus obtains

$$\partial_t \rho_{\tilde{p}} + \sum_{j=1}^{N} \partial_{q_j}\left(\frac{\partial_{q_j} S}{m_j}\rho_{\tilde{p}}\right) = 0. \quad (12)$$

Moreover, note that since the underlying momentum field is random due to the fluctuation of ξ, each single trajectory does not in general conserve the energy. However, since the agent does not make any selection of trajectories, it is reasonable to assume that her estimation should respect a weaker constraint of conservation of *average* energy, i.e.,

$$\frac{d}{dt}\langle H \rangle_{\{S,\rho_{\tilde{p}}\}} = 0. \quad (13)$$

Here, the average energy $\langle H \rangle_{\{S,\rho_{\tilde{p}}\}}$ is defined as in conventional probability theory, i.e., $\langle H \rangle_{\{S,\rho_{\tilde{p}}\}} = \int dq d\xi dp H(p, q)\text{Pr}(p, q|\xi)\chi(\xi)$, where $\text{Pr}(p, q|\xi) = \prod_{j=1}^{N} \delta(p_j - \tilde{p}_j(q;\xi))\rho_{\tilde{p}}(q)$ is "the epistemically restricted phase-space distribution" induced by the momentum field $\tilde{p}(q;\xi)$ defined in Eq. (10) [51,55].

We show below that the above two reasonable statistical-informational constraints for updating the agent's estimation of the momentum field when she does not make any

selection of trajectories, i.e., the conservation of trajectories and average energy respectively mathematically expressed by Eqs. (12) and (13), are sufficient to deduce the time evolution of $S(q, t)$ and $\rho_{\tilde{p}}(q, t)$, which in turn determines the time evolution of the agent's estimator and estimation error via respectively Eqs. (3) and (9). First, to solve Eq. (13), we must first compute the ensemble average energy, using Eq. (10), to obtain

$$\langle H \rangle_{\{S, \rho_{\tilde{p}}\}}$$

$$\dot{=} \int dq d\xi dp H(p, q) \prod_{j=1}^{N} \delta(p_j - \tilde{p}_j(q; \xi)) \chi(\xi) \rho_{\tilde{p}}(q)$$

$$= \sum_{j=1}^{N} \int dq \rho_{\tilde{p}}(q) \left[\frac{(\partial_{q_j} S)^2}{2m_j} + V + \frac{\hbar^2}{8m_j} \left(\frac{\partial_{q_j} \rho_{\tilde{p}}}{\rho_{\tilde{p}}} \right)^2 \right]$$

$$+ D_f[\rho_{\tilde{p}}], \tag{14}$$

where we have used Eq. (5), and D_f is a functional of $\rho_{\tilde{p}}(q)$ defined as

$$D_f[\rho_{\tilde{p}}] \dot{=} \sum_{j=1}^{N} \int dq \left(\frac{\hbar^2}{4m_j} \frac{\partial_{q_j} \rho_{\tilde{p}}}{\rho_{\tilde{p}}} f_j + \frac{\hbar^2}{8m_j} f_j^2 \right) \rho_{\tilde{p}}(q). \tag{15}$$

Taking the total derivative of Eq. (14) with respect to time, one gets

$$\frac{d}{dt} \langle H \rangle_{\{S, \rho_{\tilde{p}}\}} = \sum_{j=1}^{N} \int dq \, \partial_t \rho_{\tilde{p}}(q) \left[\partial_t S + \frac{(\partial_{q_j} S)^2}{2m_j} + V \right.$$

$$\left. - \frac{\hbar^2}{2m_j} \frac{\partial_{q_j}^2 \sqrt{\rho_{\tilde{p}}}}{\sqrt{\rho_{\tilde{p}}}} + \mathcal{N}_f(\rho_{\tilde{p}}) \right], \tag{16}$$

where we have made use of Eq. (12), and $\mathcal{N}_f$ is defined as the functional derivative of $D_f[\rho_{\tilde{p}}]$ with respect to $\rho_{\tilde{p}}(q)$ as

$$\mathcal{N}_f(\rho_{\tilde{p}}(q)) \dot{=} \frac{\delta D_f}{\delta \rho_{\tilde{p}}(q)}. \tag{17}$$

See Appendix C for the straightforward derivation. Equating the right-hand side of Eq. (16) to zero, i.e., imposing the conservation of average energy of Eq. (13), one thus obtains the following equation:

$$\partial_t S + \sum_{j=1}^{N} \left[\frac{(\partial_{q_j} S)^2}{2m_j} - \frac{\hbar^2}{2m_j} \frac{\partial_{q_j}^2 \sqrt{\rho_{\tilde{p}}}}{\sqrt{\rho_{\tilde{p}}}} \right] + V + \mathcal{N}_f(\rho_{\tilde{p}}) = 0. \tag{18}$$

Hence, to comply with the conservation of trajectories and average energy, the agent's estimation of the momentum based on information on positions with the associated estimator and estimation error determined by $(S(q), \rho_{\tilde{p}}(q))$ via Eqs. (3) and (9), must satisfy a pair of differential equations, i.e., Eqs. (12) and (18). Finally, defining the wave function as in Eq. (7), the two coupled differential equations can be recast in a compact form into the following general nonlinear Schrödinger equation:

$$i\hbar \partial_t \psi(q, t) = -\sum_{j=1}^{N} \frac{\hbar^2}{2m_j} \partial_{q_j}^2 \psi(q, t) + V(q) \psi(q, t)$$

$$+ \mathcal{N}_f(|\psi(q)|^2) \psi(q, t); \tag{19}$$

that is, Eqs. (12) and (18) are respectively the imaginary and the real parts of Eq. (19). In the limit of vanishing $\mathcal{N}_f$, we regain the standard linear Schrödinger equation

$$i\hbar \partial_t \psi(q, t) = -\sum_{j=1}^{N} \frac{\hbar^2}{2m_j} \partial_{q_j}^2 \psi(q, t) + V(q) \psi(q, t). \tag{20}$$

$\mathcal{N}_f$ defined in Eq. (17) thus determines the form and strength of the nonlinearity in the Schrödinger equation of Eq. (19). Moreover, when the estimation error $\epsilon_{p_f}(q; \xi)$ is much smaller than the estimator $\partial_q S$, or the global fluctuation ξ is ignorable, the third and fifth terms in Eq. (18) (i.e., the $\hbar$-dependent terms) are ignorable, so that it reduces smoothly to the classical Hamilton-Jacobi equation: $\partial_t S + \sum_{j=1}^{N} \frac{(\partial_{q_j} S)^2}{2m_j} + V = 0$.

One can see that the above general scheme of estimation under epistemic restriction provides a flexible framework to construct a broad class of nonlinear variants of Schrödinger equation with a transparent operational meaning. As a concrete example, first, consider an estimation scheme so that f that appears in the estimation error of Eq. (9) has the following form:

$$f_j(\rho_{\tilde{p}}(q)) = \Lambda_j \rho_{\tilde{p}}(q)^{\alpha}, \tag{21}$$

$j = 1, \ldots, N$, where Λ_j is a real parameter with the dimension [length]$^{-1}$, and α is a nonvanishing real number. In the limit $\Lambda_j \to 0$, we have $f_j \to 0$, $j = 1, \ldots, N$, so that the estimation error of Eq. (9) reduces back to the specific form assumed in Sec. II given by Eq. (4). Inserting Eq. (21) into Eq. (15), one has

$$D_f[\rho_{\tilde{p}}] = \sum_{j=1}^{N} \int dq \left(\frac{\hbar^2 \Lambda_j}{4m_j} \rho_{\tilde{p}}^{\alpha} \partial_{q_j} \rho_{\tilde{p}} + \frac{\hbar^2 \Lambda_j^2}{8m_j} \rho_{\tilde{p}}^{2\alpha+1} \right). \tag{22}$$

From Eqs. (17) and (22), we therefore obtain

$$\mathcal{N}_f(\rho_{\tilde{p}}(q)) = \Omega \rho_{\tilde{p}}(q)^{2\alpha} = \Omega |\psi(q)|^{4\alpha}, \tag{23}$$

where $\Omega = \sum_{j=1}^{N} \frac{\hbar^2 \Lambda_j^2}{8m_j}(2\alpha + 1)$, and we have used Eq. (8) in the last equality. Inserting into Eq. (19), we finally obtain the following polynomial nonlinear Schrödinger equation:

$$i\hbar \partial_t \psi(q, t) = -\sum_{j=1}^{N} \frac{\hbar^2}{2m_j} \partial_q^2 \psi(q, t) + V(q) \psi(q, t)$$

$$+ \Omega |\psi(q, t)|^{4\alpha} \psi(q, t), \tag{24}$$

which gives the well-known quadratic nonlinear Schrödinger equation when $\alpha = 1/2$.

As another example, and for later comparison, consider an estimation scheme so that f in Eq. (9) has the following form:

$$f_j(\rho_{\tilde{p}}(q), \partial_q \rho_{\tilde{p}}(q)) = \Lambda_j \left(\frac{\partial_{q_j} \rho_{\tilde{p}}(q)}{\rho_{\tilde{p}}(q)} \right)^{\beta}, \tag{25}$$

$j = 1, \ldots, N$, where $\beta > 1$, and Λ_j is a real parameter with the dimension of [length]$^{\beta-1}$. We have thus assumed a higher order error term postulated in Eq. (4). Inserting Eq. (25) into

Eq. (15), we obtain

$$D_f[\rho_{\tilde{p}}] = \sum_{j=1}^{N} \int dq \left(\frac{\hbar^2 \Lambda_j}{4m_j} \partial_{q_j}\rho_{\tilde{p}} \left(\frac{\partial_{q_j}\rho_{\tilde{p}}}{\rho_{\tilde{p}}} \right)^{\beta} + \frac{\hbar^2 \Lambda_j^2}{8m_j} \rho_{\tilde{p}} \left(\frac{\partial_{q_j}\rho_{\tilde{p}}}{\rho_{\tilde{p}}} \right)^{2\beta} \right). \tag{26}$$

Finally, using Eq. (17), the nonlinearity $\mathcal{N}_f$ in the Schrödinger equation of Eq. (19) can be computed to get

$$\mathcal{N}_f(|\psi|^2) = \sum_{j=1}^{N} \frac{\hbar^2 \Lambda_j}{4m_j} \left[-\beta \left(\frac{\partial_{q_j}|\psi|^2}{|\psi|^2} \right)^{\beta+1} - (\beta+1)\partial_{q_j}\left(\frac{\partial_{q_j}|\psi|^2}{|\psi|^2} \right)^{\beta} \right]$$
$$+ \sum_{j=1}^{N} \frac{\hbar^2 \Lambda_j^2}{8m_j} \left[-(2\beta-1)\left(\frac{\partial_{q_j}|\psi|^2}{|\psi|^2} \right)^{2\beta} - 2\beta\partial_{q_j}\left(\frac{\partial_{q_j}|\psi|^2}{|\psi|^2} \right)^{2\beta-1} \right], \tag{27}$$

where we have used Eq. (8).

Let us give a few remarks concerning the derivation of the class of nonlinear variants of Schrödinger equation of Eq. (19). First, we note importantly that defining the wave function as in Eq. (7) amounts to the assumption that the Born's quadratic law of Eq. (8) is kept valid. This is unlike the generalizations of quantum mechanics suggested in Refs. [3,12], wherein the Born's quadratic law is somehow violated.

Notice that the epistemic decomposition of the momentum field of Eq. (10) is invariant under the addition of a global phase to the wave function, i.e., $\psi \mapsto e^{i\alpha}\psi$, where α is an arbitrary real number. But, unlike the specific estimation scheme of Sec. II with the epistemic decomposition of momentum field given by Eq. (6), that in Eq. (10) is in general no longer invariant under the more general transformation of wave function: $\psi \mapsto Z\psi$, where Z is an arbitrary complex number. We note, however, that while the epistemic decomposition of momentum field of Eq. (10) with the specific f given by Eq. (21) is not invariant under the transformation $\psi \mapsto Z\psi$, that with the specific f given by Eq. (25) is invariant under the transformation. As will be argued in Sec. IV, the two different fs in Eqs. (21) and (25), leading to two different variants of nonlinearity in the Schrödinger equations respectively given by Eqs. (23) and (27), are also fundamentally distinguished with respect to certain inferential-causality principle. One can also see that the form of the nonlinearity $\mathcal{N}_f$ determined in Eq. (17) does not depend on $S(q)$, which is due to the assumption that f in Eq. (9) does not depend on $S(q)$ either. Of course, it can be mathematically extended to depend also on $S(q)$. However, in this case, both the estimator of Eq. (3) and the estimation error depend on $S(q)$, so that they are no longer independent of each other, which is undesirable from the information theoretical point of view.

Note further that, using the definition of wave function in Eq. (7), the average energy given in Eq. (14) can be expressed in terms of wave function as

$$\langle H \rangle_{\{S,\rho_{\tilde{p}}\}} = \langle\psi|\hat{H}|\psi\rangle + D_f[|\psi|^2], \tag{28}$$

where $\hat{H} = \sum_{j=1}^{N} \hat{p}_j^2/2m_j + V(\hat{q})$ is the usual Hermitian quantum Hamiltonian. Hence, $D_f[|\psi|^2]$ defined in Eq. (15) provides the correction to the quantum average energy $\langle\psi|\hat{H}|\psi\rangle$. The functional form of this correction term de-

pends on the estimation error of Eq. (9) via $f(\rho_{\tilde{p}}, \partial_q\rho_{\tilde{p}}) = f(|\psi|^2, \partial_q|\psi|^2)$, and is responsible for the appearance of the nonlinearity $\mathcal{N}_f$ in the Schrödinger equation of Eq. (19) via Eq. (17). The correction of average energy $D_f[|\psi|^2]$, and thus the nonlinearity $\mathcal{N}_f(|\psi|^2)$, vanishes for all ψ (i.e., for all estimation schemes characterized by $(S, \rho_{\tilde{p}})$), if and only if $f = 0$, so that the generalized estimation error of Eq. (9) reduces back to the specific estimation error of Eq. (4), leading to the standard quantum mechanics. Moreover, unlike the quantum average energy $\langle\psi|\hat{H}|\psi\rangle$, the correction term $D_f[|\psi|^2]$ is in general not bilinear in ψ.

Next, as in the case of linear Schrödinger equation, the nonlinear Schrödinger equation of Eq. (19) conserves the average energy and probability current. In fact, as demonstrated above, we have upgraded the conservation of average energy and conservation of trajectories (which implies the conservation of probability current) as the principles which single out the dynamical equation when the agent does not make any selection of trajectories, encompassing both the linear and nonlinear variants of the Schrödinger equation [49] (see also Appendix A). We emphasize that the above two constraints, i.e., conservation of trajectories and average energy, are not agent-independent objective physical constraints like the principle of least action. Rather, they are subjective epistemic constraints conditional on the agent's action that she does not make a selection of trajectories manifested in the setting of the experiment. Hence, the nonlinear Schrödinger equation of Eq. (19) should be seen as a Bayesian-inferential rule to update the agent's estimation about her system when she does not make any selection of trajectories.

If the agent instead makes a selection trajectories, she must no longer impose conservation of average energy and trajectories, so that the Bayesian updating of her estimation no longer follows the nonlinear Schrödinger equation of Eq. (19). Such a selection of trajectories is necessary when the agent makes a measurement [49] (see also Appendix A). From this observation, the nonlinearity in the Schrödinger equation of Eq. (19) therefore clearly, by construction, has nothing to do with the problem of Schrödinger's cat, unlike those nonlinearities discussed in Refs. [18–22], which were introduced mathematically to circumvent this central aspect of quantum measurement problem. We note additionally that within our estimation scheme, since we assume that the system has a definite configuration all the time as in classical mechanics, by construction, there is no problem of Schrödinger's cat.

Finally, we mention that some authors have proposed several different frameworks to introduce nonlinearities in the Schrödinger equation [1,2,24,64,65], with the main goal to give a guide for stringent precision test of quantum mechanics. In particular, Weinberg offered an elegant general "Hamiltonian framework" to nonlinearly generalize quantum mechanics [2]. This is done by assuming that, unlike in standard quantum mechanics, the average energy, or, the "Hamiltonian functional," is in general nonbilinear in the wave functions as in our epistemic model. Moreover, the time evolution, i.e., the nonlinear Schrödinger equation, is obtained by imposing the action principle. We emphasize that in Weinberg's approach, the nonlinearities are devised as possible mathematical innovations relative to the original linear theory, rather than motivated by deep conceptual reflection. While mathematically elegant and beautiful, the physical origin and operational meaning of the nonlinearities are not entirely clear so that the physical correspondence with the original linear theory is not conceptually transparent. Moreover, it suffers the same interpretational problem as that of the linear theory.

By contrast, here we work within a general epistemic framework based on an operationally transparent scheme of estimation of momentum given the positions under epistemic restriction. Most importantly, unlike those in Refs. [1,2,24,64,65], within the general scheme of estimation, by construction, the nonlinearity in the Schrödinger equations has a conceptually transparent operational meaning in terms of generalization of estimation errors. This transparent interpretation of the nonlinearities in the Schrödinger equation, as will be discussed in Sec. IV, allows us to impose a physically transparent and reasonable inferential-causality principle which rules out a broad class of nonlinear generalizations of the Schrödinger equation. Another important conceptual advantage of our general epistemic framework based on the scheme of estimation of momentum given information on positions is that, as will be shown in Sec. III C, we can directly derive the associated modifications of the Heisenberg uncertainty principle, and study its transparent relation with the resulting nonlinearities in the Schrödinger equation.

The above observation also suggests an interesting point that the principle of conservation of average energy and trajectories employed in the present paper to derive the (linear and nonlinear) Schrödinger equation are deeply connected with the action principle used in Weinberg's approach. A derivation of the (linear and nonlinear) Schrödinger equation using action principle, following that in Ref. [66], which is closely related to our derivation, is given in the Appendix D. Note, importantly, that unlike the least action principle, which is independent of the agent's action and thus objective, the principle of conservation of average energy and trajectories employed in the present work are epistemic or informational, conditional on the agent's action. Within our model, as discussed above, the Schrödinger equation arises only when the agent does not make measurement (i.e., she does not make a selection of trajectories associated with the measurement outcomes) so that the conservation of average energy and trajectories apply. By contrast, from the principle of least action, it seems to be unclear why (at least the linear) Schrödinger equation only applies when the agent does not make a measurement. Hence, while the two approaches lead

to the same equation, the meaning of the resulting Schrödinger equation are different. Moreover, the principle of conservation of average energy and trajectories are natural, transparent, and intuitive, whereas the principle of least action is somehow *ad hoc*.

C. A class of generalized Heisenberg uncertainty principle

In this section, we derive a broad class of generalized Heisenberg uncertainty principle from the general estimation error of Eq. (9). For notational simplicity, we consider a system with one spatial degree of freedom. Note before proceeding that to derive the uncertainty relations rigorously within the epistemic framework based on the generalized scheme of estimation, we need to develop a detailed mechanism of measurement. However, we shall not pursue this problem, and instead assume that any reliable measurement mechanism within the generalized scheme of estimation must satisfy a reasonable informational requirement as mentioned below.

Consider first the estimation scheme discussed in Sec. II, namely when f in Eq. (9) is vanishing so that the estimation error takes the specific form given by Eq. (4). This specific estimation scheme, as elaborated in Refs. [49,51], reproduces the prediction of standard quantum mechanics. In this case, it was shown in Ref. [51] that, in general, the ensemble average of a physical quantity $O(p, q)$ up to second order in p is equal to the average of the outcomes of the quantum measurement of a Hermitian quantum observable $\hat{O}$ associated with O, i.e.,

$$\langle O \rangle_{\{S, \rho_{\tilde{p}}\}} = \int dq\, dp\, d\xi\, O(p, q) \mathrm{Pr}_{\{S, \rho_{\tilde{p}}\}}(p, q|\xi) \chi(\xi)$$
$$= \langle \psi | \hat{O} | \psi \rangle = \sum_j o_j \mathrm{Pr}(o_j | \psi), \qquad (29)$$

where we have used Eqs. (5) and (7). Here $\mathrm{Pr}_{\{S, \rho_{\tilde{p}}\}}(p, q|\xi) = \sum_{j=1}^{N} \delta(p_j - \tilde{p}_j(q; \xi)) \rho_{\tilde{p}}(q)$ with $\tilde{p}(q; \xi)$ defined in Eq. (6), o_j, $j = 1, 2, \ldots$ is the eigenvalue of $\hat{O}$, and $\mathrm{Pr}(o_j | \psi)$ is the probability to obtain outcome o_j. This equality suggests that, while each single measurement outcome given by one of the eigenvalues of $\hat{O}$ does not in general reveal the objective value of O prior to measurement, each single measurement outcome can be seen as an unbiased estimate of the average value of O, so that the average of the measurement outcomes is equal to the average of O as expressed in Eq. (29). In particular, as a corollary of Eq. (29), we have $\sigma_{\hat{p}}^2 \doteq \langle \psi | (\hat{p} - \langle \psi | \hat{p} | \psi \rangle)^2 | \psi \rangle = \langle (p - \langle p \rangle_{\{S, \rho_{\tilde{p}}\}})^2 \rangle_{\{S, \rho_{\tilde{p}}\}} = \sigma_p^2$, and similarly $\sigma_{\hat{q}}^2 \doteq \langle \psi | (\hat{q} - \langle \psi | \hat{q} | \psi \rangle)^2 | \psi \rangle = \langle (q - \langle q \rangle_{\{S, \rho_{\tilde{p}}\}})^2 \rangle_{\{S, \rho_{\tilde{p}}\}} \doteq \sigma_q^2$. Namely, the variance of the outcomes of momentum (position) measurement, i.e., $\sigma_{\hat{p}}^2(\sigma_{\hat{q}}^2)$, is equal to the variance of the momentum (position) of the statistical model, $\sigma_p^2(\sigma_q^2)$.

We assume below that the above conclusion drawn from the case when $f = 0$ — namely that the statistical mean of measurement outcomes of physical quantities (up to second order in momentum) reproduces the statistical mean of the corresponding (classical) physical quantity of the underlying statistical model — can be carried over to the case when $f \neq 0$. A similar assumption is also postulated by Weinberg in his general Hamiltonian framework for introducing nonlinearity in the Schrödinger equation [2]. Hence, we require that, within

the epistemic framework based on the generalized estimation scheme with the estimation error given by Eq. (9), even when $f \neq 0$, any reliable measurement scheme must be such that the variance of the outcome of the momentum measurement is equal to the variance of the momentum p of the underlying statistical model, denoted by $\sigma_{p_f}^2$ (with a subscript f). Similarly, the variance of the outcome of the position measurement is equal to the variance of q of the statistical model $\sigma_{q_f}^2$. To study the uncertainty relation between the statistics of the outcomes of measurement of momentum and position in this generalized estimation scheme, it is thus sufficient to develop the uncertainty relation between $\sigma_{p_f}^2$ and $\sigma_{q_f}^2$.

First, from Eq. (9), we can compute the MS error for the estimation of momentum field to obtain, noting Eq. (5),

$$\mathcal{E}_{p_f}^2 = \int dq (\epsilon_{p_f}(q;\xi))^2 \chi(\xi) \rho_{\tilde{p}}(q) = \frac{\hbar^2}{4} J_{q_f} + C_f. \quad (30)$$

Here, $J_{q_f} \doteq \int dq (\frac{\partial_q \rho_{\tilde{p}}(q)}{\rho_{\tilde{p}}(q)})^2 \rho_{\tilde{p}}(q)$ is the Fisher information about the mean position contained in $\rho_{\tilde{p}}(q)$, and C_f is a functional of $\rho_{\tilde{p}}(q)$ defined as

$$C_f[\rho_{\tilde{p}}] \doteq \frac{\hbar^2}{4} \int dq \left(2 \frac{\partial_q \rho_{\tilde{p}}}{\rho_{\tilde{p}}} f + f^2 \right) \rho_{\tilde{p}}(q)$$
$$= 2m D_f[\rho_{\tilde{p}}], \quad (31)$$

where we have used Eq. (15) in the second equality which is valid for the specific case of particles in a scalar potential. On the other hand, in the estimation of mean position $q_o \doteq \int dq\, q \rho_{\tilde{p}}(q)$ with the unbiased estimator q, the associated MS error must satisfy the Cramér-Rao inequality [59]:

$$\mathcal{E}_{q_f}^2 = \int dq (q - q_o)^2 \rho_{\tilde{p}}(q) \geqslant \frac{1}{J_{q_f}}. \quad (32)$$

Combining Eq. (30) with Eq. (32), we thus obtain the following uncertainty relation between the MS errors of the simultaneous estimation of momentum field and mean position:

$$\mathcal{E}_{p_f}^2 \mathcal{E}_{q_f}^2 \geqslant \frac{\hbar^2}{4} + \frac{C_f}{J_{q_f}}. \quad (33)$$

On the other hand, from Eq. (10), the variance of the momentum can be computed to obtain

$$\sigma_{p_f}^2 = \Delta_{p_f}^2 + \mathcal{E}_{p_f}^2, \quad (34)$$

where we have used Eq. (5), $\mathcal{E}_{p_f}^2$ is given in (30), and $\Delta_{p_f}^2 \doteq \int dq (\partial_q S(q) - \int dq' \partial_{q'} S(q') \rho_{\tilde{p}}(q'))^2 \rho_{\tilde{p}}(q)$ is the variance of the estimator $\partial_q S(q)$. Hence, the variance of the momentum can be decomposed into the accuracy of the estimation of momentum $\mathcal{E}_{p_f}^2$ of Eq. (30), and the precision of the estimation $\Delta_{p_f}^2$. Moreover, one also straightforwardly has $\sigma_{q_f}^2 = \mathcal{E}_{q_f}^2$. Multiplying this with Eq. (34), and using Eq. (33), one finally obtains the following uncertainty relation between the variances of momentum and position:

$$\sigma_{p_f}^2 \sigma_{q_f}^2 = \Delta_{p_f}^2 \mathcal{E}_{q_f}^2 + \mathcal{E}_{p_f}^2 \mathcal{E}_{q_f}^2$$
$$\geqslant \Delta_{p_f}^2 \mathcal{E}_{q_f}^2 + \frac{\hbar^2}{4} + \frac{C_f}{J_{q_f}}. \quad (35)$$

Furthermore, when $C_f = 0$ we regain the Heisenberg-Kennard uncertainty relation [67,68]

$$\sigma_p^2 \sigma_q^2 \geqslant \Delta_p^2 \mathcal{E}_q^2 + \frac{\hbar^2}{4} \geqslant \frac{\hbar^2}{4}, \quad (36)$$

where, e.g., $\sigma_p^2 \doteq \sigma_{p_f}^2|_{C_f=0}$, etc. This is the case for all estimation schemes characterized by the pairs of $(S, \rho_{\tilde{p}})$, if and only if $f = 0$ so that the estimation error of Eq. (9) reduces back to the specific form given by Eq. (4). The last term on the right-hand side of Eq. (35) thus provides a nontrivial correction to the Heisenberg-Kennard uncertainty relation of Eq. (36). In Ref. [56], we have derived Eq. (35) but for a specific case of Eq. (9) with $f = \Lambda \partial_q \rho_{\tilde{p}}$.

Now, let us consider a specific preparation characterized by a Gaussian wave function, $\psi(q) = (2\pi\sigma_{q_f}^2)^{-1/4} e^{-(q-q_o)^2/4\sigma_{q_f}^2 + i p_o q/\hbar}$. In this case, we have $\rho_{\tilde{p}}(q) = (2\pi\sigma_{q_f}^2)^{-1/2} e^{-(q-q_o)^2/2\sigma_{q_f}^2}$ so that $J_{q_f} = 1/\sigma_{q_f}^2 = 1/\mathcal{E}_{q_f}^2$, and therefore Eq. (32) is saturated. Noting Eq. (30), it follows then that Eq. (33) is also saturated. Moreover, since for Gaussian wave function $S(q) = p_o q$, we have $\Delta_{p_f}^2 = 0$, Eq. (34) becomes $\sigma_{p_f}^2 = \mathcal{E}_{p_f}^2$. Combining all these facts, we thus finally obtain, for Gaussian wave functions,

$$\sigma_{p_f}^2 \sigma_{q_f}^2 = \frac{\hbar^2}{4} + \sigma_{q_f}^2 C_f, \quad (37)$$

which reduces to the usual relation for Gaussian wave function in standard quantum mechanics when $C_f = 0$, i.e., when $f = 0$. Hence, for nonvanishing C_f, unlike in standard quantum mechanics, the product $\sigma_{p_f}^2 \sigma_{q_f}^2$ of the variances of momentum and position depends on the profile of the Gaussian wave function; i.e., it is no longer invariant for all Gaussians. As a concrete example, consider the case when f is given by Eq. (21) with $\alpha = 1/2$, so that C_f in Eq. (37) has the form $C_f = \frac{\hbar^2 \Lambda^2}{4} \int dq \rho_{\tilde{p}}^2 = \frac{\hbar^2 \Lambda^2}{8\pi^{1/2}\sigma_{q_f}} \geqslant 0$, where $\Omega = \frac{\hbar^2 \Lambda^2}{4m}$ characterizes the strength of the nonlinearity in the quadratic nonlinear Schrödinger equation of Eq. (24). Hence, in this case, we have $\sigma_{p_f}^2 \sigma_{q_f}^2 \geqslant \sigma_p^2 \sigma_q^2 = \frac{\hbar^2}{4}$, i.e., the model has a "stronger than quantum uncertainty." Moreover, increasing the strength of the estimation error Λ increases both the uncertainty and nonlinearity. Next, let us consider the case when f is given by Eq. (25) with $\beta = 3$. In this case, we have $C_f = \frac{\hbar^2}{4\sigma_{q_f}^6}(6\Lambda\sigma_{q_f}^2 + 15\Lambda^2)$, so that $C_f < 0$ for $-\frac{6}{15}\sigma_{q_f}^2 < \Lambda < 0$, and $C_f \geqslant 0$ otherwise. When $C_f < 0$, we thus have $\sigma_{p_f}^2 \sigma_{q_f}^2 \leqslant \sigma_p^2 \sigma_q^2 = \frac{\hbar^2}{4}$. Note that such a "weaker than quantum uncertainty" does not necessarily mean that the statistical model is more classical than quantum mechanics. This can be seen from the fact that even in this case f in Eq. (25) could be very large.

IV. DISCUSSION: NONLINEARITY, DEVIATION FROM THE HEISENBERG UNCERTAINTY, AND ESTIMATION INDEPENDENCE

We have shown that within the scheme of estimation of momentum given information on positions, with the estimator of Eq. (3) and the generalized estimation errors of Eq. (9) encapsulated (up to ξ) by the wave function defined in Eq. (7), the agent's estimation when she does not make any

measurement must be updated in time according the nonlinear Schrödinger equation of Eq. (19). Moreover, the variances of the outcomes of momentum and position measurements must satisfy the generalized Heisenberg-Kennard uncertainty relation of Eq. (35). We emphasize that, by construction, both the nonlinearity in the Schrödinger equation and the deviation from the Heisenberg uncertainty principle, respectively characterized by $\mathcal{N}_f$ and C_f defined in Eqs. (17) and (31), arise from the same estimation error of Eq. (9) via f. They therefore should be closely related to each other.

Indeed, in general, by construction, it is easy to see that no nonlinearity in the Schrödinger equation arises without a deviation from the Heisenberg uncertainty principle. In particular, noting Eqs. (17) and (31), for a single one-dimensional particle of mass m, they are directly related as

$$\mathcal{N}_f(|\psi(q)|^2) = \frac{1}{2m} \frac{\delta C_f}{\delta \rho_{\tilde{p}}(q)}\bigg|_{\rho_{\tilde{p}}(q)=|\psi(q)|^2}. \tag{38}$$

The above relation shows that to have nonlinearity in the Schrödinger equation for a single particle, the deviation from the Heisenberg uncertainty principle, i.e., C_f defined in Eq. (31), cannot be a functional linear in $\rho_{\tilde{p}}(q)$ or/and in its spatial derivatives, $\partial_q \rho_{\tilde{p}}(q)$. One may thus conclude that, within the estimation scheme, the nonlinearity is generated by the deviation from the Heisenberg uncertainty principle. Furthermore, from Eq. (38), since it is possible to have $C_f \neq 0$ with $\mathcal{N}_f = 0$, one can still have a nontrivial deviation from the Heisenberg uncertainty relation without inducing nonlinearity in the Schrödinger equation. This is the case, for example, when $f = F(q)$, where F is independent of $\rho_{\tilde{p}}(q)$ and $\partial_q \rho_{\tilde{p}}(q)$, so that from Eq. (31), C_f is linear in $\rho_{\tilde{p}}(q)$ and $\partial_q \rho_{\tilde{p}}(q)$. It suggests that one can still have a superposition principle while the Heisenberg uncertainty relation is to some extent modified.

Remarkably, within the epistemic reconstruction based on scheme of estimation of momentum based on information on positions with the estimation error having the general form of Eq. (9), noting Eqs. (17) and (31), and assuming that the definition of wave function is given by Eq. (7), i.e., assuming that the Born's quadratic law of Eq. (8) stays solid, the linear Schrödinger equation of Eq. (20), and the exact form of Heisenberg-Kennard uncertainty relation of Eq. (36), are regained if and only if $f = 0$, so that the estimation error of Eq. (9) reduces back to the specific form given by Eq. (4). Keeping this observation in mind, we may therefore conclude that not only standard quantum mechanics corresponds to a specific estimation scheme with the specific estimator and estimation error given respectively by Eqs. (3) and (4), but also that it is difficult to nontrivially modify a part of quantum mechanics, e.g., the linearity of the Schrödinger equation, without changing the other fundamental parts of the theory, e.g., the exact form of the Heisenberg uncertainty principle.

Finally, having obtained the various variants of Schrödinger equations given in Eq. (19) and uncertainty relations of Eq. (35), how do we choose among them? To this end, remember first that the standard linear Schrödinger equation of Eq. (20) and the Heisenberg-Kennard uncertainty relation of Eq. (36) have passed all stringent tests conceived to date. Moreover, there are striking theoretical results

which suggest that nonlinearities in the Schrödinger equation and/or deviations from the exact Heisenberg uncertainty principle may imply violations of some forms of causality, e.g., superluminal signaling [26–30] and/or the second law of thermodynamics [36,37]. It is therefore instructive to see, within the general epistemic framework based on the operational scheme of estimation of momentum given information on positions, if the specific estimation error given by Eq. (4), which together with the estimator of Eq. (3) leads to the standard linear Schrödinger equation and the exact form of Heisenberg-Kennard uncertainty relation [49,51], might be justified based on some reasonable premises about causality.

To investigate this last tantalizing question, let us discuss a physically transparent and plausible inferential-causality principle of estimation independence introduced in Ref. [56]. Consider two systems, referred to as system 1 and system 2, with a configuration (q_1, q_2) and the corresponding conjugate momentum (p_1, p_2), prepared independently of each other. First, recall that in classical mechanics, for such independent preparations of two systems, the total Lagrangian is decomposable, so that the associated Hamilton's principal function is also decomposable, i.e., $S_C(q_1, q_2, t) = \int^{(q,t)} dt' (L_1(q_1', \dot{q}_1') + L_2(q_2', \dot{q}_2')) = \int^{(q_1,t)} dt' L_1(q_1', \dot{q}_1') + \int^{(q_2,t)} dt' L_2(q_2', \dot{q}_2') = S_{C_1}(q_1, t) + S_{C_2}(q_2, t)$, where L_j is the (classical) Lagrangian associated with system j, $j = 1, 2$. To have a smooth classical correspondence, it is therefore reasonable to assume that, within the generalized estimation scheme, $S(q)$ defined in Eq. (3) for such pairs of independent preparations should also be decomposable:

$$S(q_1, q_2) = S_1(q_1) + S_2(q_2). \tag{39}$$

Moreover, it is also natural to assume that in such pairs of independent preparations, the probability distribution of positions are factorizable, as in classical mechanics, i.e.,

$$\rho_{\tilde{p}}(q_1, q_2) = \rho_{\tilde{p}_1}(q_1)\rho_{\tilde{p}_2}(q_2). \tag{40}$$

Recalling the definition of wave function given in Eq. (7), the above two assumptions amount to the postulate in standard quantum mechanics that the wave function associated with the independent preparations of the two systems is factorizable (unentangled), i.e., $\psi(q_1, q_2) = \sqrt{\rho_{\tilde{p}}}e^{iS/\hbar} = \sqrt{\rho_{\tilde{p}_1}\rho_{\tilde{p}_2}}e^{i(S_1+S_2)/\hbar} = \psi_1(q_1)\psi_2(q_2)$. The principle of estimation independence then requires that in such independent preparations, the estimation of momentum $\tilde{p}_j$ of system j, i.e., the associated estimator $\overline{p}_j$ and estimation error ϵ_{p_j}, should be reasonably independent of the position q_i of the system i, $i \neq j$, $i, j = 1, 2$ [56]. It thus captures an intuitive form of inferential-causality constraint.

We shall impose the above plausible requirement to scrutinize the various estimation schemes discussed in the previous sections. Let us first consider the estimation scheme discussed in Sec. II, i.e., when the estimator and the estimation error take the specific forms respectively given by Eqs. (3) and (4), leading to the standard quantum mechanics [49,51]. Inserting Eq. (39) into Eq. (3), one has

$$\overline{p}_j = \partial_{q_j}S(q_1, q_2) = \partial_{q_j}S_j(q_j), \tag{41}$$

$j = 1, 2$. Hence, the estimator $\overline{p}_j$ for estimating the momentum field $\tilde{p}_j$ of system j is indeed independent of the position

q_i of system i, $i \neq j$, $i, j = 1, 2$, respecting the principle of estimation independence. Moreover, inserting Eq. (40) into Eq. (4), one obtains

$$\epsilon_{p_j} = \frac{\xi}{2} \partial_{q_j} \ln \rho_{\tilde{p}}(q_1, q_2) = \frac{\xi}{2} \partial_{q_j} \ln \rho_{\tilde{p}_j}(q_j), \qquad (42)$$

$j = 1, 2$. Namely, the error ϵ_{p_j} of estimating $\tilde{p}$ of system j is also independent of q_i of system i, $i \neq j$, $i, j = 1, 2$, satisfying the requirement of estimation independence. In this sense, standard quantum mechanics with the linear Schrödinger equation and the exact form of the Heisenberg uncertainty principle reformulated within the operational scheme of estimation thus elegantly respects the natural inferential-causality principle of estimation independence.

Let us show that the above natural requirement of estimation independence is not fulfilled by a broad class of schemes of estimation of momentum given information on positions discussed in Sec. III with an estimator having the same form as that in Sec. II given by Eq. (3), but with an estimation error of the form given by Eq. (9) which generalizes Eq. (4) via a nonvanishing f. We only need to check whether the estimation error of Eq. (9) passes the requirement of estimation independence. Since the first term on the right-hand side of Eq. (9) is already shown above respecting the principle of estimation independence, we need only to examine the correction term f under the estimation independence.

Consider first the specific scheme of estimation of momentum given information on positions with the estimation error having the form of Eq. (9) where f is given by Eq. (21), leading to the polynomial nonlinear Schrödinger equation of Eq. (24). Inserting Eq. (40) into Eq. (21), one has

$$\begin{aligned} f_j\big(\rho_{\tilde{p}}(q_1, q_2)\big) &= f_j\big(\rho_{\tilde{p}_1}(q_1) \rho_{\tilde{p}_2}(q_2)\big) \\ &= \Lambda_j\big(\rho_{\tilde{p}_1} \rho_{\tilde{p}_2}\big)^\alpha \neq \Lambda_j \rho_{\tilde{p}_j}^\alpha = f_j\big(\rho_{\tilde{p}_j}(q_j)\big), \end{aligned} \quad (43)$$

$j = 1, 2$. Hence, in this case, the error $\epsilon_{p_{f_j}}$ of estimating momentum $\tilde{p}_j$ of system j depends on the position q_i of system i, $i \neq j$, $i, j = 1, 2$, even when the two systems are prepared independently of each other, violating the principle of estimation independence. In fact, one can check that any f which is an analytical function only of $\rho_{\tilde{p}}$ (hence, independent of its spatial gradient) will not pass the reasonable requirement of estimation independence. This shows that, within the operational scheme of estimation of momentum based on information on positions with the estimation error having the general form of Eq. (9), the requirement of estimation independence rules out a broad class of forms of estimation errors, thus excluding a broad class of nonlinear generalizations of Schrödinger equation.

By contrast, one can straightforwardly show that the estimation error of Eq. (9) with f given by Eq. (25) satisfies the plausible requirement of estimation independence; i.e., inserting Eq. (40) into Eq. (25), we have

$$\begin{aligned} f_j\big(\rho_{\tilde{p}_1}(q_1) \rho_{\tilde{p}_2}(q_2)\big) &= \Lambda_j \left(\frac{\partial_{q_j}\big(\rho_{\tilde{p}_1}(q_1) \rho_{\tilde{p}_2}(q_2)\big)}{\rho_{\tilde{p}_1}(q_1) \rho_{\tilde{p}_2}(q_2)} \right)^\beta \\ &= \Lambda_j \left(\frac{\partial_{q_j} \rho_{\tilde{p}_j}(q_j)}{\rho_{\tilde{p}_j}(q_j)} \right)^\beta = f_j\big(\rho_{\tilde{p}_j}(q_j)\big), \quad (44) \end{aligned}$$

$j = 1, 2$. Indeed, all f which have the form $f_j = G(\frac{\partial_{q_j} \rho_{\tilde{p}}}{\rho_{\tilde{p}}})$, where G is some scalar function of $\frac{\partial_{q_j} \rho_{\tilde{p}}}{\rho_{\tilde{p}}}$, satisfy the requirement of estimation independence. Note, however, that while this class of forms of f does fulfill the requirement of estimation independence, it in general does not transform in the same way as the rest of terms in the epistemic decomposition of momentum field of Eq. (10), so that the latter does not transform covariantly.

A different kind of f which satisfies the requirement of estimation independence takes the form $f_j = G(\frac{\partial_{q_j} \rho_{\tilde{p}}}{\rho_{\tilde{p}}}, \partial_{q_j} S)$, $j = 1, \ldots, N$. This can be checked directly for two systems prepared independently of each other so that Eqs. (39) and (40) apply. Namely, we have $f_j(\rho_{\tilde{p}_1} \rho_{\tilde{p}_2}, S_1 + S_2) = G(\frac{\partial_{q_j}(\rho_{\tilde{p}_1} \rho_{\tilde{p}_2})}{\rho_{\tilde{p}_1} \rho_{\tilde{p}_2}}, \partial_{q_j}(S_1 + S_2)) = G(\frac{\partial_{q_j} \rho_{\tilde{p}_j}}{\rho_{\tilde{p}_j}}, \partial_{q_j} S_j) = f_j(\rho_{\tilde{p}_j}, S_j)$, $j = 1, 2$. One can work out directly that such a choice of f will lead to a different class of nonlinear variants of Schrödinger equation and generalized Heisenberg uncertainty principle. Note, however, that in this case, the estimation error becomes correlated with the estimator $\partial_q S$ which is unappealing from the information theoretical view point.

Notice that when f satisfies the requirement of estimation independence, e.g., that given by Eq. (25), the associated correction term $D_f[\rho_{\tilde{p}}]$ to the quantum average energy defined in Eq. (15) for two noninteracting systems is decomposable into that of each system. This can be seen directly by inserting Eq. (40) into Eq. (15) for such fs. Accordingly, in this case, the nonlinearity $\mathcal{N}_f$ defined in Eq. (17) for two noninteracting systems is also decomposable; i.e., one has

$$\mathcal{N}_f\big(\rho_{\tilde{p}_1}(q_1) \rho_{\tilde{p}_2}(q_2)\big) = \mathcal{N}_f\big(\rho_{\tilde{p}_1}(q_1)\big) + \mathcal{N}_f\big(\rho_{\tilde{p}_2}(q_2)\big), \quad (45)$$

as is exemplified by the nonlinearity in Eq. (27). This is not the case when f does not respect the principle of estimation independence, as, e.g., that given by Eq. (21) with the associated nondecomposable nonlinearity in Eq. (23). Within the estimation scheme, the principle of estimation independence thus implies that the product of two wave functions associated with two noninteracting systems will evolve in time independently of each other, as intuitively expected. Such a natural separability condition for the dynamics of noninteracting systems is employed to single out the logarithmic nonlinear Schrödinger equation by Bialynicki-Birula and Mycielski [1]. Moreover, the separability for the dynamics of noninteracting systems are attained in Weinberg's Hamiltonian formalism by imposing the homogeneity condition together with the additivity of the Hamiltonian functional [2]. We emphasize that within our estimation scheme, unlike the latter two approaches, the separability condition for the dynamics of noninteracting systems has a transparent operational interpretation in terms of a natural inferential-causality principle of estimation independence.

Next, it is interesting to note that in the estimation scheme with the generalized estimation error of Eq. (9) and with f given by Eq. (21), which does not comply with the principle of estimation independence, the associated epistemic decomposition of the momentum fields of Eq. (10) is not invariant under the transformation of wave function $\psi \rightarrow Z\psi$, where Z is an arbitrary complex number. In contrast to this,

for f given by Eq. (25), which complies with the estimation independence, the associated epistemic decomposition of the momentum fields of Eq. (10) is invariant under the transformation $\psi \to Z\psi$. It is instructive to ask if this nice relation between the principle of estimation independence and the invariance of epistemic decomposition of the momentum fields of Eq. (10) under $\psi \to Z\psi$ applies for all forms of f. Since the invariance of the nonlinear Schrödinger equation with respect to $\psi \to Z\psi$ is obtained in the Weinberg's Hamiltonian formalism by imposing the homogeneity condition to the Hamiltonian functional [2], this suggests a possible deep connection between the mathematical condition of homogeneity and the physically transparent inferential-causality principle of estimation independence, worth further study in the future.

All the above observations show that the plausible inferential-causality principle of estimation independence puts a tight physical-informational constraint which rules out a significantly large class of mathematically possible modifications of standard quantum mechanics. Indeed, we have argued in Ref. [56] that requiring the estimation error $\epsilon_p(q; \xi)$ to satisfy the following conditions: (i) independent of the estimator $\bar{p} = \partial_q S$, (ii) transforms covariantly with the estimator, and (iii) respecting the principle of estimation independence, will single out the specific form of estimation error given by Eq. (4) up to the statistics of ξ, which has been argued in Refs. [49,51] to imply the standard quantum mechanics. We note that Simon *et al.* [29] argued that the principle of no-signaling can be used to single out the linear quantum dynamics by assuming, at the outset, the quantum kinematics and the quantum trace rule for computing the probability of measurement outcomes. See also Ref. [30] for a similar argument. By contrast, within the above estimation scheme, the principle of estimation independence is used to reconstruct the underlying quantum kinematics by constraining the allowed forms of estimation error, without assuming any quantum structures. While we have assumed the Born's quadratic law of Eq. (8) via the definition of wave function in Eq. (7), it is not the same as, and weaker than, assuming the quantum trace rule as in Ref. [29]. Moreover, within the epistemic reconstruction framework based on the operational scheme of estimation, the linear Schrödinger equation follows from the kinematics via imposing the conservation laws, i.e., the conservation of trajectories and average energy, naturally embodying the assumption that the agent does not make any measurement via a selection of trajectories.

V. CONCLUSIONS AND REMARKS

We have generalized the specific operational scheme of estimation of momentum given information on the conjugate positions under epistemic restriction to reconstruct quantum mechanics proposed in Refs. [49,51], by considering a more general class of estimation errors. We showed that, provided Born's quadratic law is kept intact, it leads to a broad class of nonlinear variants of Schrödinger equation when the agent does not make measurement and a class of generalized Heisenberg uncertainty principle. Within the operational scheme of estimation, both the nonlinearities in the Schrödinger equation and the deviation from the Heisen-

berg uncertainty principle have thus a transparent operational interpretation in terms of generalization of the estimation errors. Hence, they are deeply related to each other; in particular, there is no nonlinearity in the Schrödinger equation without a deviation from the Heisenberg uncertainty principle. With this in mind, it is interesting to further study the connection between the deviation from the Heinseberg uncertainty principle, which allows stronger than quantum correlation [15,39] and in turn may imply implausible computational power [40–47], and the nonlinearity in the Schrödinger equation, which may lead to a violation of no signaling [26–30] and computational schemes fundamentally much faster than quantum computation [11,12]. It is also interesting to investigate the above deep connection between the nonlinearity in the Schrödinger equation and the deviation from the Heisenberg uncertainty principle, with the theoretical results that both may imply violations of the second law of thermodynamics [36,37].

It is remarkable that the linear Schrödinger equation of Eq. (20) and the exact form of Heisenberg uncertainty principle of Eq. (36) are regained for a specific estimation scheme with the estimation error taking the specific form given by Eq. (4), satisfying the principle of estimation independence. On the other hand, other forms of estimation errors violating the principle of estimation independence and/or having unpleasant statistical property from the view of statistical estimation lead to nonlinear corrections to the Schrödinger equation and deviations from the Heisenberg uncertainty principle. Noting that such deviations from the linear Schrödinger equation and Heisenberg uncertainty principle may be in conflict with the principle of no signaling and the second law of thermodynamics, or imply implausible computational power, it is natural to ask if the inferential-causality principle of estimation independence together with other reasonable informational constraints may be upgraded as the axioms to single out uniquely the specific form of estimation error of Eq. (4) leading to the standard quantum mechanics. That this might be so is argued in a different work [56]. Our results also suggest possible deep interlinks between the principle of estimation independence, no-signaling, the second law of thermodynamics, and other principles used to single out quantum correlation such as information causality [45] or data processing inequality [69–71], which are worth further investigation in the future.

The above observation prompts the following question: Beside that mentioned in Sec. III, what kinds of generalizations of the specific estimation scheme of Sec. II comply with the principle of estimation independence, leading to possible nontrivial extensions of the standard quantum mechanics? First, when deriving Eq. (18) by imposing the conservation of average energy of Eq. (13), leading to the derivation of the Schrödinger equation, we have implicitly assumed that the Planck constant $\hbar$, which is the variance of the global random variable ξ, is indeed constant in time. One could thus ponder the possibility that $\hbar$ may, though extremely weakly, depend on time, i.e., $\partial_t \hbar \neq 0$. Such an assumption clearly does not violate the principle of estimation independence and may lead to a weak nontrivial nonlinearity in the Schrödinger equation. We may also study the trade-off between the resulting nonlinearity in the Schrödinger equation

and the possible violations of Born's quadratic law envisioned in Refs. [3,12]. Following Valentini's insight in Ref. [3], it might be interesting to see the implications of such possible weak temporal fluctuation of $\hbar$ in the early universe. One may also impose additional statistical constraints, reflecting some other symmetries of the statistical estimation problems, when exercising the conservation of average energy of Eq. (13). For example, one may assume that some measures of information are (or are not) conserved. Yet another interesting way to generalize quantum mechanics within the operational framework of estimation without violating the principle of estimation independence is to assume that the conservation of trajectories of Eq. (12) is no longer valid as in open systems, or to assume that the conservation of average energy of Eq. (13) is somehow violated as in dynamical collapse models [22], which, for example, might be relevant in the cosmological context [72].

Hence, like other operational approaches to reconstruct quantum mechanics [4–10], the operational scheme of estimation of momentum given the positions under epistemic restriction discussed in the present work provides a general epistemic framework encompassing classical, quantum, and a broad class of possible postquantum theories. Note, however, that unlike those in Refs. [4–10] we have worked directly with the phase-space variables so that the transition to classical mechanics is conceptually less painful. Noting this, it is intriguing to investigate possible hybrid interactions among quantum, postquantum, and classical systems to yet generalize quantum mechanics within the general epistemic framework based on the operational scheme of estimation under epistemic restriction. For example, a hybrid quantum-classical interaction [66,73,74] might find applications in developing approximations in computational physics and chemistry [75], for describing nanomechanical systems in quantum-classical boundary [76], and in the study of quantum gravity [77–79].

APPENDIX A: THE RECONSTRUCTION OF QUANTUM MECHANICS WITHIN THE OPERATIONAL SCHEME OF ESTIMATION UNDER EPISTEMIC RESTRICTION: SINGLE AND DOUBLE SLITS EXPERIMENTS

Consider a beam of particles passing through a screen with a single slit, one by one, followed by the detection of the position of the particles (position measurement), e.g., by a second screen at some time t_0. Suppose that the agent can control the width of the slit (and possibly some other macroscopic setting parameters such as the average kinetic energy of the particles). Within the model, such a set of macroscopic settings determines a random momentum field $\tilde{p}(q; \xi)$ which, unlike in classical mechanics, irreducibly parameterizes the allowed distribution of position $\rho_{\tilde{p}}(q)$. In this preparation setting, the main idea in the epistemic reconstruction of quantum mechanics based on the estimation scheme is that the agent wants to estimate the underlying momentum field associated with the above macroscopic setting, given information on the conjugate positions.

The estimation of the momentum at a given position is carried out in a "naive classical" way as follows [60]. (See also Sec. II of the main text.) Consider a subensemble of the particles that are detected at the screen to be at $q(t)$ at time t, where different trajectories of the particles correspond to different fluctuations of ξ. For each of the particles in the subensemble, we make a sufficiently weak measurement of position at $t - \Delta t$ without appreciably altering the subsequent dynamics of the particle, yielding $q(t - \Delta t)$, where Δt is extremely small. The velocity along the trajectory at $q(t)$ is then computed in the conventional way, i.e., by taking the difference between $q(t)$ and $q(t - \Delta t)$ and dividing it by Δt, from which one also obtains the momentum $\tilde{p}(q; \xi)$. Note that, because of the fluctuation of ξ, each single repetition of such momentum measurement must give random outcome. To overcome this uncertainty, we then define the estimator for the momentum at $q(t)$ by taking the average of the above measurement outcomes over the subensemble of trajectories passing through $q(t)$, i.e., by averaging over ξ as in Eq. (2). Based on this estimate, to have a smooth correspondence with the classical relation of Eq. (1), we then construct a real-valued function $S(q)$ satisfying Eq. (3). Moreover, from the distribution of the position $\rho_{\tilde{p}}(q)$ obtained in measurement, the single-shot estimation error is assumed to take the form given by Eq. (4).

As an example, suppose that the agent's estimate of the momentum field along the direction perpendicular to the direction of the beam obtained operationally by following the above scheme is given by p_o independent of q. Then, following Eq. (3), the agent associates a real-valued function $S(q)$ satisfying $\partial_q S = p_o$ to give $S(q) = p_o q$. Moreover, suppose the distribution of position of the particles is given by a Gaussian distribution $\rho_{\tilde{p}}(q) = \frac{1}{\sqrt{2\pi\sigma_q^2}} e^{-\frac{(q-q_o)^2}{2\sigma_q^2}}$ with a variance σ_q^2 assumed to be determined by the width of the slit. Then, from Eq. (4), the agent should assign a single-shot estimation error for the estimation of momentum as $\epsilon_p(q; \xi) = \frac{\xi}{2} \partial_q \ln \rho_{\tilde{p}}(q) = -\frac{\xi}{2\sigma_q^2}(q - q_o)$ so that the MS estimation error is given by $\mathcal{E}_p^2 \doteq \int dq d\xi (\epsilon_p(q; \xi))^2 \chi(\xi) \rho_{\tilde{p}}(q) = \hbar^2/4\sigma_q^2$, where we have used Eq. (5). The above agent's estimation (knowledge) about the momentum field $\tilde{p}(q; \xi)$ at time t_0, i.e., the estimator and the estimation error, is then recast compactly into a wave function via $(S(q), \rho_{\tilde{p}}(q))$ defined as in Eq. (7), i.e., $\psi_0(q) \doteq \sqrt{\rho_{\tilde{p}}(q)} e^{\frac{i}{\hbar}S(q)} = (\frac{1}{2\pi\sigma_q^2})^{1/4} e^{-\frac{(q-q_o)^2}{4\sigma_q^2} + \frac{i}{\hbar}p_o q}$. Such a reconstruction of wave function is in practice similar to the

reconstruction of wave function via momentum weak value measurement discussed in Ref. [55]. Hence, by decreasing (increasing) the width of the slit, which means decreasing (increasing) σ_q^2, implying sharper (poorer) knowledge of the position, then $\mathcal{E}_p^2$ increases (decreases) so that the agent's estimation about the momentum becomes poorer (sharper), and this leads to a narrower (broader) Gaussian wave function. As a limiting case, suppose that the slit is infinitely wide, so that $\sigma_q^2 \to \infty$, implying an infinitely poor knowledge of the position. In this case, the agent's estimate of the momentum p_o is infinitely sharp with a vanishing MS error, i.e., $\mathcal{E}_p^2 \to 0$, and the agent should assign a plane-wave function $\psi_0(q) \sim e^{ip_o q/\hbar}$ to her preparation.

Now, suppose that the agent postpones the detection of the position of the particle at some later time $t_1 > t_0$. The question is then, given her estimation about the system — i.e., the estimator for the underlying momentum field and the associated estimation error — at time t_0 represented by $\psi_0(q)$, how should she rationally update her estimation at time t_1? Suppose further that during the time $t_0 \leqslant t \leqslant t_1$, the agent does not make any selection of trajectories so that she does not have new information about her system. The only thing that the agent knows is that the system evolves according to some Hamiltonian. In this case, since she does not make a selection of trajectories, her estimation at time t_1 must be updated by respecting the conservation of trajectories and average energy. We have shown in the paper that in this case, the wave function representing the agent's estimation (i.e., the estimator and the estimation error) has to be updated following the Schrödinger equation, either linear or nonlinear, depending on the assumed exact form of the estimation errors. See Sec. III B for the detailed derivation. The linear Schrödinger equation is regained when the estimation error takes the specific from of Eq. (4).

Suppose instead that at some time t_M, $t_0 < t_M < t_1$, the agent makes a measurement on some physical quantities. Such a measurement in practice corresponds to a selection of a subensemble of trajectories associated with the measurement outcome. Namely, in general, a measurement of a physical quantity with an outcome o corresponds to the selection of a subensemble of trajectories leading to the unambiguous assignment of o (see Ref. [49]). As a concrete example, consider the paradigmatic which-way measurement by inserting a screen with a double slits, in the middle between the screen with a single slit and the detecting screen. In this case, the outcome "upper" way ("lower" way) corresponds to the selection of those subensembles of trajectories which pass through the upper (lower) slit. Hence, the measurement is carried out by selecting a particular subset of trajectories, so that the conservation of trajectories and average energy no longer apply. Accordingly, the agent's estimation, represented by the wave function, no longer follows the Schrödinger equation; instead it must follow a wave function collapse reflecting the Bayesian updating in light of the new information associated with the selected subensemble of trajectories [49]. This is the reason why, in standard quantum mechanics, such a which-way measurement demolishes (suppresses) the interference pattern at the detecting screen. That is, since the linear Schrödinger equation is no more valid, the superposition principle no longer applies.

APPENDIX B: PROOF THAT THE ESTIMATOR OF EQ. (3) WITH THE ESTIMATION ERROR OF EQ. (9) MINIMIZES THE MEAN-SQUARED ERROR

First, given information on q, assume a general estimator $T_{p_j}(q)$ for the momentum field $\tilde{p}_j(q; \xi)$, $j = 1, \dots, N$, and compute the associated MS estimation error, to obtain, for each degree of freedom j:

$$
\int dq d\xi \left(\tilde{p}_j(q; \xi) - T_{p_j}(q) \right)^2 \chi(\xi) \rho_{\tilde{p}}(q)
$$

$$
= \int dq d\xi \, \tilde{p}_j(q; \xi)^2 \chi(\xi) \rho_{\tilde{p}}(q) + \int dq \left(-2 T_{p_j}(q) \partial_{q_j} S(q) \right.
$$

$$
\left. + T_{p_j}(q)^2 \right) \rho_{\tilde{p}}(q)
$$

$$
= \int dq d\xi \, \tilde{p}_j(q; \xi)^2 \chi(\xi) \rho_{\tilde{p}}(q)
$$

$$
+ \int dq \left(\left[T_{p_j}(q) - \partial_{q_j} S(q) \right]^2 - \partial_{q_j} S(q)^2 \right) \rho_{\tilde{p}}(q), \quad \text{(B1)}
$$

where we have inserted Eq. (10) and used $\overline{\xi} = 0$ in the first equality to obtain the second term on the right-hand side. It is then clear that the MS error reaches its minimum when

$$
T_{p_j}(q) = \partial_{q_j} S(q),
$$

$j = 1, \dots, N$, as claimed in the main text. In general, one can show that the unbiased estimator for the momentum given information on the positions with minimum MS error is given by the conditional average of momentum given the positions, i.e., $T_{p_j}(q)|_{\{\text{min.MS.error}\}} = \overline{p}_j(q) = \int d\xi \, \tilde{p}_j(q; \xi) \chi(\xi) = \partial_{q_j} S(q)$, $j = 1, \dots, N$.

APPENDIX C: THE DERIVATION OF EQ. (16)

Taking the total derivative of Eq. (14) with respect to time, one first gets

$$
\frac{d}{dt} \langle H \rangle_{\{S, \rho_{\tilde{p}}\}}
$$

$$
= \int dq \left(\frac{\delta \langle H \rangle_{\{S, \rho_{\tilde{p}}\}}}{\delta \rho_{\tilde{p}}(q)} \frac{\partial \rho_{\tilde{p}}(q)}{\partial t} + \frac{\delta \langle H \rangle_{\{S, \rho_{\tilde{p}}\}}}{\delta S(q)} \frac{\partial S(q)}{\partial t} \right)
$$

$$
= \sum_{j=1}^{N} \int dq \left(\left[\frac{(\partial_{q_j} S)^2}{2m_j} - \frac{\hbar^2}{2m_j} \frac{\partial_{q_j}^2 \sqrt{\rho_{\tilde{p}}}}{\sqrt{\rho_{\tilde{p}}}} \right. \right.
$$

$$
\left. \left. + V(q) + \frac{\delta D_f(\rho_{\tilde{p}})}{\delta \rho_{\tilde{p}}} \right] \partial_t \rho_{\tilde{p}}(q) - \partial_{q_j} \left(\rho_{\tilde{p}} \frac{\partial_{q_j} S}{m_j} \right) \partial_t S \right), \quad \text{(C1)}
$$

where we have used the following result for functional derivatives:

$$
\frac{\delta}{\delta \rho_{\tilde{p}}(q)} \int dq' \frac{1}{8} \left[\frac{\partial_{q'} \rho_{\tilde{p}}(q')}{\rho_{\tilde{p}}(q')} \right]^2 \rho_{\tilde{p}}(q')
$$

$$
= -\frac{1}{8} \left(\frac{\partial_q \rho_{\tilde{p}}}{\rho_{\tilde{p}}} \right)^2 - \frac{1}{4} \partial_q \left(\frac{\partial_q \rho_{\tilde{p}}}{\rho_{\tilde{p}}} \right)
$$

$$
= \frac{1}{8} \left(\frac{\partial_q \rho_{\tilde{p}}}{\rho_{\tilde{p}}} \right)^2 - \frac{1}{4} \frac{\partial_q^2 \rho_{\tilde{p}}}{\rho_{\tilde{p}}} = -\frac{1}{2} \frac{\partial_q^2 \sqrt{\rho_{\tilde{p}}}}{\sqrt{\rho_{\tilde{p}}}}, \quad \text{(C2)}
$$

and

$$\frac{\delta}{\delta S(q)} \int dq' \frac{[\partial_q S(q')]^2}{2m} \rho_{\tilde{p}}(q') = -\partial_q \left(\rho_{\tilde{p}} \frac{\partial_q S}{m} \right). \quad \text{(C3)}$$

Noting Eq. (12), the last term in the fourth line of Eq. (C1) becomes $\partial_t \rho_{\tilde{p}} \partial_t S$, so that one obtains

$$\frac{d}{dt} \langle H \rangle_{\{S, \rho_{\tilde{p}}\}} = \sum_{j=1}^{N} \int dq \, \partial_t \rho_{\tilde{p}}(q) \left[\partial_t S + \frac{(\partial_{q_j} S)^2}{2m_j} + V(q) \right.$$
$$\left. - \frac{\hbar^2}{2m_j} \frac{\partial_{q_j}^2 \sqrt{\rho_{\tilde{p}}}}{\sqrt{\rho_{\tilde{p}}}} + \frac{\delta D_f(\rho_{\tilde{p}})}{\delta \rho_{\tilde{p}}} \right], \quad \text{(C4)}$$

as claimed in the main text.

APPENDIX D: NONLINEAR SCHRÖDINGER EQUATION FROM ACTION PRINCIPLE

Here we sketch the derivation of the nonlinear Schrödinger equation using the action principle, following Hall and Reginatto's approach [66]. Assume that $\rho_{\tilde{p}}(q)$ and $S(q)$ constitute a pair of conjugate variables associated with a Hamiltonian functional $\mathcal{H}[\rho_{\tilde{p}}(q), S(q)]$ (it is called as the Hamiltonian ensemble in Ref. [66]). Hence, the time evolution of the above pair of the conjugate variables satisfy the following pair of canonical Hamilton's equations:

$$\frac{\partial \rho_{\tilde{p}}(q)}{\partial t} = \frac{\delta \mathcal{H}}{\delta S(q)} \quad \text{and} \quad \frac{\partial S(q)}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \rho_{\tilde{p}}(q)}. \quad \text{(D1)}$$

Taking the Hamiltonian functional to be equal to the average energy of Eq. (14) of the statistical model,

i.e., $\mathcal{H}[\rho_{\tilde{p}}(q), S(q)] = \langle H \rangle_{\{S, \rho_{\tilde{p}}\}}$, the pair of equations in Eq. (D1) gives respectively the following coupled differential equations:

$$\partial_t \rho_{\tilde{p}} = -\sum_{j=1}^{N} \partial_{q_j} \left(\frac{\partial_{q_j} S}{m_j} \rho_{\tilde{p}} \right),$$

$$\partial_t S = -\sum_{j=1}^{N} \frac{(\partial_{q_j} S)^2}{2m_j} + \frac{\hbar^2}{2m_j} \frac{\partial_{q_j}^2 \sqrt{\rho_{\tilde{p}}}}{\sqrt{\rho_{\tilde{p}}}} - V(q) - \mathcal{N}_f(\rho_{\tilde{p}}),$$

$$\text{(D2)}$$

where $\mathcal{N}_f$ is defined as in Eq. (17). See Appendix C for the detailed calculations.

The above pair of coupled equations are just Eqs. (12) and (18) of the main text, which can be recast into the nonlinear Schrödinger equation of Eq. (19) via the definition of wave function given by Eq. (7). Note crucially that in the above derivation, the pair of equations in Eq. (D2) is obtained via objective least action principle by choosing the correct Hamiltonian functional given by Eq. (14). By contrast, within our epistemic reconstruction based on the estimation under epistemic restriction, the pair of equations in Eq. (D2) are obtained via epistemic-informational constraint of conservation of average energy and trajectories by choosing the correct estimation error of the form in Eq. (9). In this sense, the conservation of average energy and trajectories may provide an epistemic and operational interpretation of the apparently objective and realist principle of least action in terms of estimation of momentum given information on the conjugate positions.

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