Spectral collapse in the two-photon quantum Rabi model

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Spectral collapse, the transition from a discrete to a continuous spectrum, is a characteristic in quantum Rabi models. We explore this phenomenon in the two-photon quantum Rabi model using optical phase space, and we find that, in the so-called degenerate qubit regime, the collapse is similar to the transition from a harmonic to an inverted oscillator with the free-particle potential as a critical transition point. In the degenerate qubit regime, we construct Dirac-normalizable eigenfunctions with well-defined parity for the model. In the general model, we use parity to diagonalize the system in the qubit basis and numerically find that the qubit frequency does not change the critical point where spectral collapse occurs. We numerically confirm the existence of an exceptional state at the critical coupling, and we argue its analytic provenance from both a Born-Oppenheimer and a variational approximation.

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I. INTRODUCTION

The quantum Rabi model [1] describes the minimal coupling between a two-level system and a boson field. Experimental realizations of the model exist in a range of quantum platforms [2–4]. Experimentalists can tailor their quantum systems to explore extensions of the model [5] that has fueled theoretical extensions as well [6–11]. We are interested in the process of two-photon exchange reported with atoms [12] and solid-state devices [13]. In this so-called two-photon quantum Rabi model,

$$H = \frac{\omega_0}{2}\hat{\sigma}_z + \omega a^\dagger \hat{a} + g_2(\hat{a}^{\dagger 2} + \hat{a}^2)\hat{\sigma}_x, \quad (1)$$

the qubit and boson field are described by the two-level energy gap related to the frequency ω_0 and the frequency ω. The coupling between these is given by the parameter g_2. The qubit is described by Pauli matrices $\hat{\sigma}_i$ and the field by annihilation (creation) operators $\hat{a}$ ($\hat{a}^\dagger$). In the two-photon quantum Rabi model, qubit flop accompanies the creation or destruction of two photons. This process resembles parametric up (down) conversion. The latter has varied theoretical and experimental applications [14–20], which might point to the importance of a better understanding of the former.

An interesting characteristic of the single-photon quantum Rabi model is the spectral collapse that occurs in the so-called relativistic regime [21]. There, we can follow the transition from a discrete spectrum in the so-called degenerate qubit regime, where the model is equivalent to a driven cavity, and the so-called relativistic regime, where the model is equivalent to the Dirac equation in (1 + 1)D with a continuous spectrum. The eigenstates of the model interpolating between these two regimes transition from the superposition of even and odd displaced number states to that of infinitely squeezed coherent states, respectively. Here, we want to show that the spectral collapse in the two-photon quantum Rabi model, existing at a critical coupling and providing a spectrum with a discrete and a continuous part [22], has a different nature from the one in the single-photon quantum Rabi model.

In the following, we discuss the spectral collapse mechanism in the two-photon quantum Rabi model. Our approach is twofold.

First, we rewrite the Hamiltonian for the model using optical phase space. In this representation, we show that the mechanism behind the spectral collapse in the degenerate qubit regime is equivalent to the transition from a harmonic to an inverted oscillator. The discrete spectrum of the harmonic oscillator becomes continuous at the critical point where the potential becomes null and the effective Hamiltonian is equivalent to that of a free particle. Our result matches the critical points found in the literature with the addition of a physical understanding of the mechanism behind the spectral collapse. In the degenerate qubit regime, we construct the analytic eigenfunctions of the system in terms of the confluent hypergeometric functions that yield both discrete and continuous solutions. These solutions show well-defined parity.

Second, we provide a diagonalization of the model in the qubit basis that allows us to study the spectral collapse in the general model using numerical methods. We conduct a numerical survey in both the full two-photon quantum Rabi

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Hamiltonian and its diagonalization in the qubit basis to confirm the null effect of the qubit frequency on the critical coupling. We present results for on-resonance and off-resonance surveys that confirm the existence of an exceptional state with a finite norm at the critical coupling of every parameter set. To explore this exceptional eigenstate, we use semiclassical methods in the form of Born-Oppenheimer and variational approaches. Finally, we close with a conclusion.

II. OPTICAL PHASE-SPACE MODEL

We are interested in the mechanism behind spectral collapse reported in the two-photon quantum Rabi model [22,23]. For reasons that will become clear in the following, we move into optical phase space [24],

$$\hat{q} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}), \quad \hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}), \tag{2}$$

and we perform a $\pi/2$ rotation around the axis defined by $\hat{\sigma}_y$ such that we arrive at the Hamiltonian, up to a constant factor,

$$\hat{H}_y = \tfrac{1}{2}[\omega + 2g_2\hat{\sigma}_z]\hat{p}^2 + \tfrac{1}{2}[\omega - 2g_2\hat{\sigma}_z]\hat{q}^2 + \tfrac{1}{2}\omega_0\hat{\sigma}_x. \tag{3}$$

Moving into a frame defined by a unitary rotation in terms of a Fourier-like rotation,

$$\hat{U}(\theta) = [\hat{R}(\theta)]^{-\frac{1}{2}(\hat{\sigma}_z - 1)}, \quad \hat{R}(\theta) = e^{-\frac{i\theta}{2}(\hat{p}^2 + \hat{q}^2)}, \tag{4}$$

and choosing a rotation angle $\theta = \pi/2$, we recover a rotated two-photon quantum Rabi model Hamiltonian,

$$\hat{H}_R = \tfrac{1}{2}(\alpha_+\hat{p}^2 + \alpha_-\hat{q}^2)\hat{\sigma}_0 + \tfrac{1}{2}\omega_0[\hat{R}_{\frac{\pi}{4}}\hat{\sigma}_+ + \hat{R}^\dagger_{\frac{\pi}{4}}\hat{\sigma}_-], \tag{5}$$

where we define the dimensionless auxiliary parameters $\alpha_\pm = \omega \pm 2g_2$ and use the shorthand notation $\hat{\sigma}_0$ for the identity matrix and $\hat{R}_\theta \equiv \hat{R}(\theta)$. This analogy immediately brings to our mind the idea of spectral collapse as the diagonal element of this Hamiltonian transitions from harmonic to free-particle form when the auxiliary parameter takes the value $\alpha_- = 0$ at the critical coupling $g_c = \omega/2$. For values larger than the critical coupling, the diagonal element takes the form of an inverted oscillator, $\alpha_- < 0$.

In the past, we showed that a competition between Hamiltonians with components showing a discrete and a continuous spectrum produces the spectral collapse in the single-photon quantum Rabi model [21]. There, the collapse occurs from the transition from driven-cavity-like in the degenerate-qubit regime, into a relativistic (1 + 1)D Dirac-like Hamiltonian in the relativistic regime. We have a different mechanism in the two-photon quantum Rabi model. Here, the diagonal term shows spectral collapse in the degenerate-qubit regime equivalent to a transition from harmonic oscillator to free-particle and then to inverted harmonic oscillator.

III. DEGENERATE QUBIT REGIME

Let us focus on the boson component attached to the identity element in the qubit basis, an analogy to the so-called degenerate qubit regime where $\omega_0 \to 0$,

$$\hat{H}_0 = \tfrac{1}{2}(\alpha_+\hat{p}^2 + \alpha_-\hat{q}^2). \tag{6}$$

We can think of the second term on the right-hand side as a potential $V(\hat{q}) = \alpha_-\hat{q}^2$ that takes the form of a harmonic oscillator showing a discrete spectrum for $\alpha_- > 0$ with $\omega > 2g_2$, Fig. 1(a), and two regimes with a continuous spectrum in the form of a free particle for $\alpha_- = 0$ with $\omega = 2g_2$, Fig. 1(b), or an inverted oscillator for $\alpha_- < 0$ with $\omega < 2g_2$, Fig. 1(c). Thus, the spectral collapse in the degenerate-qubit regime is related to the transition from harmonic to free-particle potential.

Suppose that the vector $|\lambda_0\rangle$ is an eigenvector of the diagonal element $\hat{H}_0$ with eigenvalue λ_0. We can use a quadrature representation to find linearly independent solutions to the diagonal element,

$$\langle q|\lambda_0\rangle = c_1\, e^{-\frac{1}{4}\alpha q^2}\, {}_1F_1\left(-\nu - \frac{1}{4}; \frac{1}{2}; \frac{1}{2}\alpha q^2\right) + c_2\, q\, e^{-\frac{1}{4}\alpha q^2}\, {}_1F_1\left(-\nu + \frac{1}{4}; \frac{3}{2}; \frac{1}{2}\alpha q^2\right), \tag{7}$$

in terms of the confluent hypergeometric function ${}_1F_1(a;b;z)$ [25]. We introduce the auxiliary parameters $\alpha^2 = (\omega - 2g_2)/(\omega + 2g_2)$ and $\Omega = \sqrt{\omega^2 - 4g_2^2}$, modal amplitudes c_1 and c_2 that include normalization constants, and the scaled eigenvalue

$$\nu = \frac{\omega}{\Omega}\lambda - \frac{1}{2}, \tag{8}$$

that is real for parameters $\omega > 2g_2$, diverges for $\omega = 2g_2$, and becomes complex for $\omega < 2g_2$. In the first case, $\omega > 2g_2$, we recover the discrete, equidistant, harmonic oscillator spectrum,

$$\lambda_n^{(I)} = \frac{\Omega}{\omega}\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots \tag{9}$$

with corresponding Hermite-Gauss eigenfunctions,

$$\langle q|\lambda_n^{(I)}\rangle = \frac{1}{\sqrt{2^n n!}}\left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\alpha q^2} H_n(\alpha q). \tag{10}$$

Then, for the free-particle-like case, $\omega = 2g_2$, the spectrum collapses and becomes continuous with Dirac-δ normalizable forward- and backward-propagating monochromatic plane-wave eigenfunctions,

$$\langle q| \pm \lambda^{(II)}\rangle = \frac{1}{\sqrt{2\pi}} e^{\pm i\sqrt{\lambda^{(II)}}q}, \tag{11}$$

with real positive eigenvalues $\lambda^{(II)} \in [0, \infty)$. In the inverted oscillator case, $\omega < 2g_2$, the spectrum remains continuous and the eigenfunctions are Dirac-δ normalizable [26],

$$\langle q|\lambda^{(III)}\rangle = \frac{e^{-i\frac{\pi}{4}(\eta+1)}\Gamma(-\eta)}{2^{\frac{3}{4}}\pi} D_\eta\left(\pm\left|\frac{\omega - 2g_2}{\omega + 2g_2}\right|^{\frac{1}{4}} e^{3i\frac{\pi}{4}}\sqrt{2}q\right), \tag{12}$$

with real eigenvalue $\lambda^{(III)} \in \mathbb{R}$, where we define the auxiliary real parameter $\eta = i\lambda^{(III)}\Omega^{-1} - 1/2$, and we use the parabolic cylinder function $D_\eta(x)$ [25]. We want to stress that both Dirac-normalizable solutions with continuous spectra allow the construction of states with well-defined parity.

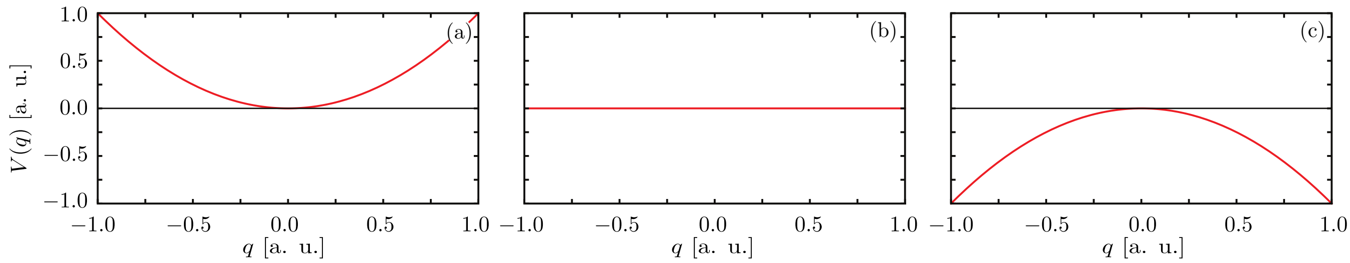

FIG. 1. Effective pseudopotential $V(\hat{q}) = (\omega - 2g_2)\hat{q}^2$ in the diagonal terms of the rotated two-photon quantum Rabi model with (a) $\omega - 2g_2 = 1$, (b) $\omega - 2g_2 = 0$, and (c) $\omega - 2g_2 = -1$.

IV. GENERAL MODEL

Now, let us try to address the process behind the spectral collapse in the general model by considering its symmetries [22]. For starters, we can partition the boson Hilbert space into even and odd sectors and rewrite the rotated two-photon quantum Rabi Hamiltonian, up to a constant,

$$H_q = \frac{\omega_0}{2}\hat{\sigma}_x + 2\omega\hat{K}_z - 2g_2(\hat{K}_+ + \hat{K}_-)\hat{\sigma}_z, \tag{13}$$

in terms of the elements of the su(1, 1) algebra: $K_- = a^2/2$, $K_+ = a^{\dagger}/2$, and $K_z = (a^{\dagger}a + aa^{\dagger})/4$ such that $[\hat{K}_z, \hat{K}_{\pm}] = \pm\hat{K}_{\pm}$ and $[\hat{K}_+, \hat{K}_-] = -2\hat{K}_z$. The parameter q is known as the Bargmann index and takes the value of $q = 1/4$ ($q = 3/4$) in the even (odd) boson subspace defined as $\mathcal{H}_{1/4} = \{|1/4; m\rangle\}$ ($\mathcal{H}_{3/4} = \{|3/4; m\rangle\}$). Each subspace has a parity operator $\hat{\Pi}_q = e^{i\pi(\hat{K}_z - q)}$, and the action of these operators in the subspaces can be found in [27,28]. We use a Fulton-Gouterman (FG) transformation [6] to diagonalize the rotated two-photon quantum Rabi Hamiltonian in the qubit basis,

$$\hat{H}_{\rm FG} = \hat{H}_{q,+}|+\rangle\langle+| + \hat{H}_{q,-}|-\rangle\langle-|. \tag{14}$$

We use the parity $\hat{\Pi}_q$ as an auxiliary operator for this. The four Hamiltonians, one for each boson sector and up to a constant,

$$\hat{H}_{q,\pm} = \pm\frac{\omega_0}{2}\hat{\Pi}_q + 2\omega\hat{K}_z - 2g_2(\hat{K}_+ + \hat{K}_-), \tag{15}$$

have a form in which the first two terms on the right-hand side have a discrete spectrum and the last term has a continuous spectrum. The competition between these terms defines the spectral collapse in the full rotated two-photon quantum Rabi model.

We can numerically explore this transition in each boson subspace, but we will focus on the even excited subspace $\mathcal{H}_{1/4,+}$ for the sake of brevity. Figure 2 shows the first 25 eigenvalues for this subspace associated with the upper diagonal term $\hat{H}_{1/4,+}$ using a truncated boson Hilbert subspace of 2^{13}. We can compare these results with those from the full rotated two-photon quantum Rabi model using a truncated space of 2^{12} photons to good agreement. In both cases, we accept an eigenvalue and eigenfunction pair if the norm of the last 20% components of the boson sector of the eigenvector is less than 10^{-6}. Once we reach the critical value for the coupling constant $g_c = 2\omega$, there is but a single converged eigenfunction as the spectrum becomes continuous and the truncation method is no longer viable to solve the eigenvalue problem. Analytic and numeric results are in good agreement. A key characteristic arises in these results, in that there is always an exceptional solution at the critical coupling $g_c = \omega/2$ in each boson subspace as reported in Ref. [22].

We explore different parameter space to numerically verify the dependence of the critical coupling on just the field frequency. In particular, we surveyed two off-resonance models. For the sake of simplicity, frequencies are given in units of $\tilde{\omega}$, one with fixed qubit frequency $\omega_0 = \tilde{\omega}$ and the other with variable boson frequency $\omega \in [0.45, 0.5]\tilde{\omega}$. Figure 3(a) shows a result of this survey for $\omega = 0.45\tilde{\omega}$ that yields a critical coupling of $g_c = 0.225\tilde{\omega}$, and another with qubit frequency $\omega_0 \in [0.95, 1.05]\tilde{\omega}$ with boson frequency $\omega = 0.5\tilde{\omega}$ that yields $g_c = 0.25\tilde{\omega}$. Figure 3(b) shows a result of this

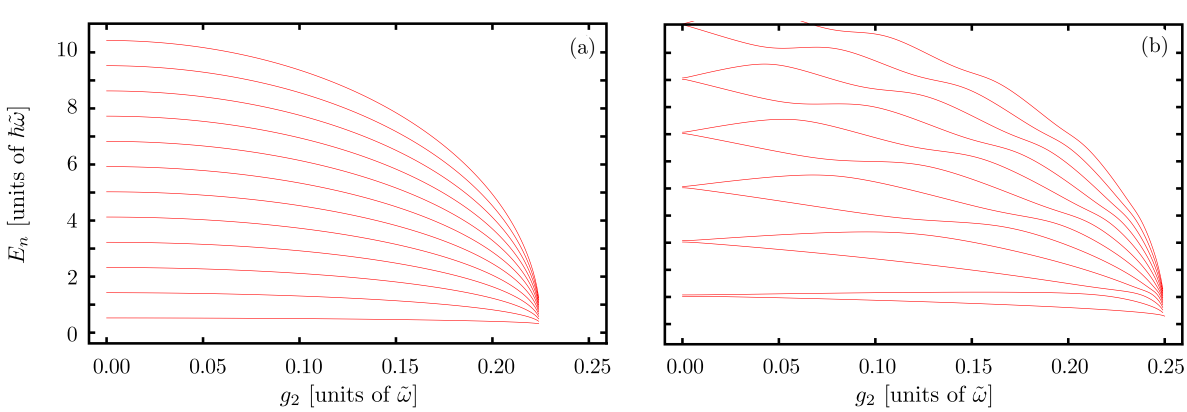

FIG. 2. Spectral collapse in the boson sector $\hat{H}_{1/4,+}$ of the rotated two-photon quantum Rabi model diagonalized in the qubit basis in (a) the degenerate qubit regime $\omega_0 = 0$, $\omega = 0.45\tilde{\omega}$, and $g_{2c} = 0.225\tilde{\omega}$; (b) on-resonance $\omega_0 = \tilde{\omega}$, $\omega = 0.5\tilde{\omega}$, and $g_{2c} = 0.25\tilde{\omega}$.

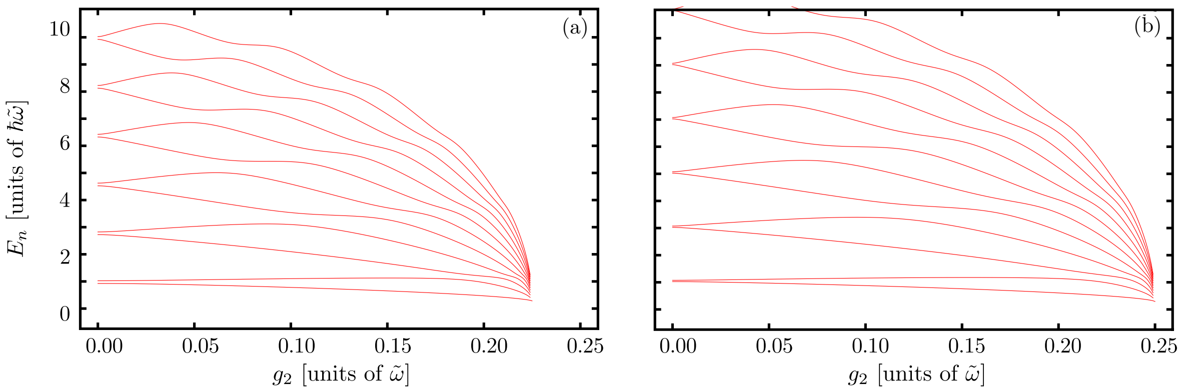

FIG. 3. Spectral collapse in the boson sector $\hat{H}_{1/4,+}$ of the rotated two-photon quantum Rabi model diagonalized in the qubit basis in the off-resonance case with (a) $\omega_0 = \tilde{\omega}$, $\omega = 0.45\tilde{\omega}$, and $g_{2c} = 0.225\tilde{\omega}$; (b) $\omega_0 = 0.95\tilde{\omega}$, $\omega = 0.5\tilde{\omega}$, and $g_{2c} = 0.25\tilde{\omega}$.

survey for $\omega_0 = 0.95\tilde{\omega}$. The surveys explored homogeneous 20-step distributions in the variable frequencies and 200 steps in the coupling parameter. Finer combs in the coupling parameter were implemented centered on the critical coupling $g_2 \in [0.98, 1.02]g_c$ and covering 200 steps to verify the results. It seems that the addition of the parity has no effect on the critical coupling nor on the exceptional solution at the critical coupling.

V. EXCEPTIONAL STATE

Our focus is on the origin of the spectral collapse related to the transition from a harmonic to an inverted oscillator potential in optical phase-space variables for the degenerate qubit regime. However, our numerical simulations confirm the existence of an exceptional solution [23,29], a converged numerical eigenstate, at the transition point given by the critical coupling strength $g_{2c} = \omega/2$. In the degenerate qubit regime, the energy of this exceptional state seems to tend toward an asymptotic limit. To obtain some information about this state, we use the Born-Oppenheimer approximation [30] to decouple the fast frequency modes of the total system and approximate its energy. We treat the optical phase-space variables q and p as a commuting object. This facilitates the diagonalization of the approximated Hamiltonian in Eq. (3) yielding proper energies,

$$E_{\pm}(p,q) = \frac{1}{2}\left[\omega(p^2+q^2) \pm \sqrt{\omega_0^2 + 4g_2^2(p^2-q^2)}\right], \tag{16}$$

in terms of the system parameter set $\{\omega, \omega_0, g_2\}$. Each parameter set provides a different energy landscape. In the uncoupled case, Fig. 4(a), the energy landscape for high and low energies is a parabola. The lowest-energy state is the qubit in the ground and the boson field in the vacuum state. As the coupling increases, moving from left to right in Fig. 4, the lowest Born-Oppenheimer energy landscape becomes unstable but shows a valley related to the exceptional state in the numerics, Fig. 4(d). The second row in Fig. 4 shows the

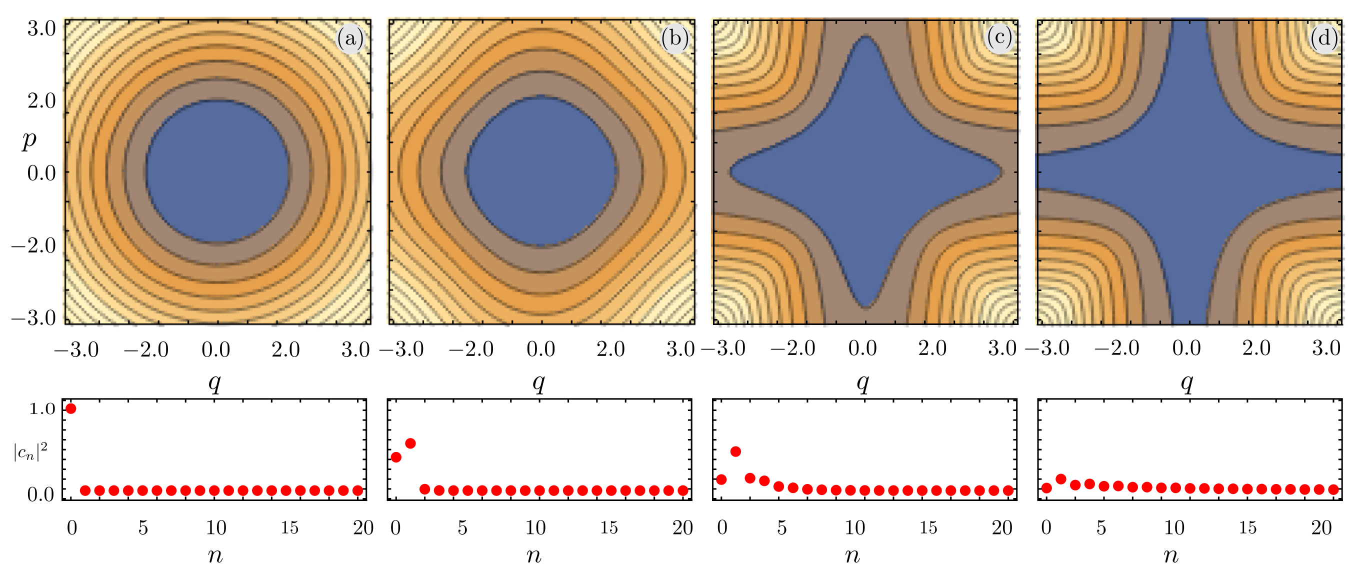

FIG. 4. The upper row shows the lowest-energy $E_-(p,q)$ landscape in optical phase space under the Born-Oppenheimer approximation. The lower row shows the Fock state distribution $|c_n|^2 = |\langle n|\psi\rangle|^2$ of the corresponding exact numerical eigenstate for on-resonance frequency values $\omega_0 = \tilde{\omega}$, $\omega = 0.5\,\tilde{\omega}$, and variable coupling strength $g_2 = \{0.0, 0.1, 0.24, 0.25\}\,\tilde{\omega}$, left to right columns.

TABLE I. Exceptional state eigenvalue comparison between variational $\epsilon_0^{(\mathrm{var})}$ and numerical $\epsilon_0^{(\mathrm{num})}$ approaches for the even, positive parity state.

Model parameters			Variational parameters		Ground energies	
$\frac{g_2}{\tilde{\omega}}$	$\frac{\omega}{\tilde{\omega}}$	$\frac{\omega_0}{\tilde{\omega}}$	α	ζ	$\frac{\epsilon_0^{(\mathrm{var})}}{\hbar\tilde{\omega}}$	$\frac{\epsilon_0^{(\mathrm{num})}}{\hbar\tilde{\omega}}$
0.240	0.480	1	−0.00000003114	0.316478	−0.327647	−0.335529
0.245	0.490	1	−0.00013949500	0.321824	−0.325006	−0.333182
0.245	0.499	1	−0.00000004176	0.315722	−0.319561	−0.331095
0.250	0.500	1	−0.00000003683	0.327155	−0.322394	−0.330865

photon number distribution of the corresponding exceptional state from the numerical diagonalization of Eq. (15). The Born-Oppenheimer approximation provides us with information about the origin of the exceptional state and its energy, but it does not facilitate calculating the form of the state. To figure out the latter, we use a variational approach ansatz to compare with the numerical data. Our previous result for the standard quantum Rabi model spectrum points to eigenstates related to squeezed displaced entangled states [21]. Thus, we make an educated guess for the exceptional state in the form of a squeezed coherent cat state,

$$|\zeta,\alpha,\pi\rangle = \frac{1}{\sqrt{\mathcal{N}_\pi}} S(\zeta)(|\alpha\rangle + \pi|-\alpha\rangle), \tag{17}$$

where the squeezing and coherent parameters ζ and α must minimize the energy of the Hamiltonian in Eq. (15). We introduce a normalization constant $\mathcal{N}_\pi$ for even (odd) states with $\pi = 1$ ($\pi = -1$). The corresponding energy for this state is given by the following expression:

$$\begin{aligned}
&\langle\zeta,\alpha,\pi|H_\pm|\zeta,\alpha,\pi\rangle \\
&= \mp\omega_0\pi \frac{e^{-\alpha^2(1+\tanh 2\zeta)}}{(1+\pi e^{-2\alpha^2})\sqrt{\cosh 2\zeta}} f_\pi(\chi) \\
&\quad + 2\omega\left\{\left[\frac{1}{4} + \frac{\alpha^2}{2} g_\pi(\alpha^2)\right]\cosh 2\zeta + \frac{\alpha^2}{2}\sinh 2\zeta\right\} \\
&\quad - 2g_2\left\{\alpha^2\cosh 2\zeta + 2\left[\frac{1}{4} + \frac{\alpha^2}{2} g_\pi(\alpha^2)\right]\sinh 2\zeta\right\},
\end{aligned} \tag{18}$$

where we use the shorthand notation $\chi = \alpha^2/\cosh 2\zeta$ and define the auxiliary functions

$$f_\pi(x) = \begin{cases} \cos x & \text{if } \pi = 1, \\ \sin x & \text{if } \pi = -1 \end{cases} \tag{19}$$

and

$$g_\pi(x) = \begin{cases} \tanh x & \text{if } \pi = 1, \\ \coth x & \text{if } \pi = -1. \end{cases} \tag{20}$$

There exists a nontrivial dependence of the functional in terms of the parameter set and parity. Thus, we numerically minimize the functional in Eq. (18) and compare it with the value obtained from the numerical diagonalization of Eq. (15). Table I shows a summary for some values sampling off-resonance and resonance parameter sets that may point to an exceptional state similar to our educated guess.

VI. CONCLUSIONS

We propose that the mechanism behind spectral collapse, the transition from a discrete to a continuous spectrum, in the two-photon quantum Rabi model is analogous to a transition from a harmonic to an inverted oscillator, with the critical point being analogous to a free particle at a critical coupling parameter of half the boson field frequency. It is straightforward to show this mechanism in the degenerate qubit regime using an optical phase-space representation. We provide an analytic form for the eigenstates of the two-photon quantum Rabi model in this regime. In addition, we show that it is possible to diagonalize the model in the qubit basis by dividing the Hilbert space in four subspaces with well-defined parity. In this representation, the spectral collapse mechanism seems to remains unchanged for the whole model outside the degenerate qubit regime. Our numerical experiments show no change in the critical coupling for surveys that explore variations in both the photon and qubit frequencies, and they confirm the existence of an exceptional solution at the critical coupling that aligns with the ground state of the boson subspaces in the regions with discrete spectra. We adopt a semiclassical and a variational approach to provide a better understanding of the exceptional state. The semiclassical Born-Oppenheimer approximation shows that at the critical point, the effective potential structure may support the exceptional state. A squeezed coherent cat state ansatz for the variational approach provides an energy that is within 2% of those obtained by numerical diagonalization.

In summary, our results confirm the critical parameter set found in the literature and propose a visualization of the mechanism behind the spectral collapse, assuming a transition from a harmonic to an inverted oscillator might be useful in predicting the behavior in quantum simulators. For example, in neutral atoms within optical lattices, where second-order coupling may be implemented by means of an external field [31,32], exploring the spectral collapse could mean losing the atoms as the trapping potential disappears. Each quantum simulation platform—superconducting systems or ion traps—might require its own analysis to discover the implications of the spectral collapse.

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