Gravitational vortex mass in a superfluid

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We consider superfluid hydrodynamics of two-dimensional Bose-Einstein condensates. Interpreting the curvature of the macroscopic condensate wave function as an effective gravity in such a superfluid universe, we argue for a superfluid equivalence principle: that the gravitational mass of a quantized vortex should be equal to the inertial vortex mass. In this model, gravity and electromagnetism have the same origin and are emergent properties of the superfluid universe, which itself emerges from the underlying collective structure of more elementary particles, such as atoms. The Bose-Einstein condensate is identified as the elusive dark matter of the superfluid universe with vortices and phonons, respectively, corresponding to massive charged particles and massless photons. Implications of this cosmological picture of superfluids to the physics of dense vortex matter are considered.

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I. INTRODUCTION

The apparent equivalence of the inertial mass m_{i} and the gravitational mass m_{g} has puzzled scientists at least since the times of Newton [1]. Einstein's equivalence principle is a cornerstone of general relativity that necessitates this equality, albeit provides no explanation for its origin [2]. More recently, the repeated attempts to unify quantum mechanics and general relativity, together with the growing mystery of the dark matter paradigm [3–5], are calling for revision to our understanding of the nature of gravity and the fabric of spacetime.

Here we bypass such grand challenges and translate the question of the equivalence between inertial and gravitational masses to a superfluid toy universe, whose properties are based on firm theoretical and experimental foundations. The full superfluid spacetime of the atoms is $2+1$ dimensional whereas the spacetime of a particle dual of vortices of this theory is $1+1$ dimensional. A remarkable property of such low dimensional spacetimes is that they may allow for a variety of well founded formulations of quantum gravity [6–9]. Further to this, it has been suggested that Einstein's field equations, and thereby gravity, could emerge as a consequence of quantum fluctuations of a regular quantum field theory over a background spacetime metric [10,11]. Meanwhile, a correspondence between two-dimensional superfluid hydrodynamics and relativistic electrodynamics is established [12–17] and underpins the concept of the inertial mass of a vortex [12,18–20]. In this work, we draw inspiration from these theoretical considerations with a specific focus on aiming to investigate the equivalence principle and the gravitational mass of a quantized vortex within the context of such an emergent superfluid universe.

It is prudent to digress here to emphasize that this work is strictly concerned with analogs and does not address the aforementioned issues of real quantum gravity or dark matter. As such, all fundamental physics concepts and terminology deployed in this work refer to analogs, unless explicitly stated otherwise or made clear by the context.

In classical hydrodynamic theory of fluids the vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$, the curl of the velocity potential, plays a pivotal role. It is already at this elementary level that the connection between hydrodynamics of fluids and classical electromagnetic theory, as quantified by Maxwell's equations, seems to appear, since the magnetic field is equal to the vorticity of the magnetic vector potential. This connection presumably prompted Maxwell to contemplate the hypothesis of molecular vortices and to state that "*under the action of magnetic forces something belonging to the same mathematical class as angular* velocity... *forms a part of the phenomenon*" [21]. Presently, it is thought that the unified electromagnetic field describes all of the classical electromagnetic phenomena, including the propagation of light. Quantum electrodynamics further explains how charged particles may be spawned as excitations of the electromagnetic field [22]. However, neither Maxwell's electrodynamics or quantum electrodynamics are able to shed any light on the nature of the "substrate" (infamously known as the aether) of the electromagnetic field.

Two-dimensional (plus one time dimension) superfluids provide a mathematically appealing analogy to relativistic electrodynamics [12,13]. In such a superfluid universe, the electric field is associated with the superflow as determined by the gradient of the spatial phase of the condensate order parameter of the superfluid. The magnetic field may be associated with the rate of change of the dynamic phase of the condensate, and the quantized vortices (more precisely the kelvons which are quasiparticles associated with the vortex) correspond to the massive charged particles, analogous to the electrons. In contrast to electromagnetic fields of our Universe, the "substrate" of the electromagnetic fields of this superfluid universe correspond to a well defined entity—a Bose-Einstein condensate (BEC) composed of the underlying "trans-Planckian" constituent particles such as rubidium atoms [16].

Once the Bose-Einstein condensate forms, the quasiparticles of the condensate are elevated to the status of the elementary particles of the superfluid universe, and its vacuum, the substrate for all fields, is the condensate itself. In this sense, the whole superfluid universe together with all of the fundamental forces are emergent. The trans-Planckian atoms respect Galilean invariance but the quasiparticles of the superfluid with the linear phonon quasiparticle dispersion relation allow for an interpretation in terms of an acoustic metric with an effective Lorentz invariance [23]. In this picture, the quantum field theory of the normal state atoms realize the grand unified theory (GUT) of the superfluid universe and the Bose-Einstein condensate corresponds to a low energy state that emerges via a spontaneous symmetry breaking mechanism as the system cools. The superfluid universe features a peculiar anti-GUT property whereby effective symmetries emerge in the "low energy corner" of the quasiparticles [16], as quantified by the Bogoliubov dispersion relation

$$E(p)=\sqrt{(pc_s)^2+\left(\frac{p^2}{2m}\right)^2}, \tag{1}$$

where c_s is the speed of sound and p is the quasiparticle momentum. These are the "relativistic" Bogoliubov phonons with an acoustic metric associated with the linear dispersion relation at low momenta $p\to 0$ that results in the emergent Lorentz invariance. In contrast, the effective Lorentz invariance violating term, $\propto p^4$ in the square root, results in a quadratic dispersion relation for high momenta and at high temperatures.

The true zero mode of this system, the Nambu-Goldstone boson, is the vacuum (the condensate) of this theory. The topological excitations (quantized vortices) with angular momentum quantum number $\ell=-1$ kelvons are the charged particles of this superfluid universe. Their dispersion relation $\omega=\omega_k+k^2\ln(1/k)$ is approximately linear at low momenta and may be viewed as the effective relativistic particles of the theory [12]. The kelvon based inertial mass of a vortex, which is the electron of the superfluid universe, is [19]

$$m_{\rm i}^v=\frac{2\pi\hbar n}{\omega_k}, \tag{2}$$

where ω_k is the zero-point frequency of the kelvon and n is the two-dimensional (2D) background condensate particle density. The sound waves in the superfluid are the photons of the superfluid universe and the fluctuating condensate density gives rise to an emergent gravitational field via the resulting nonvanishing quantum pressure field, as discussed in detail later.

In Sec. II, we associate each term in the generalised Gross-Pitaevskii energy functional with fields of the superfluid universe. Sections III and IV discuss the emergence of electromagnetism and gravity, respectively. In Sec. V we argue for the vortex correspondence principle proposing the equality of the gravitational and inertial vortex mass. In a $(2+1)$-dimensional superfluid the motion of quantized vortices may be modeled in terms of Hamilton's phase-space equations for a one-dimensional massive particle. The resulting vortex-particle duality is considered in Sec. VI. In Sec. VII we consider quantum Hall physics in the superfluid universe. Concluding remarks are provided in Sec. VIII.

II. A (2 + 1)-DIMENSIONAL SUPERFLUID UNIVERSE

We consider a two-dimensional (plus one time dimension) superfluid universe governed by the complex valued order parameter $\Phi(\mathbf{r},t)$ normalized to the atom number $N_a=\int|\Phi|^2d\mathbf{r}^2$ and the usual Gross-Pitaevskii energy functional [24,25]

$$\mathcal{E}=\int\left(\frac{\hbar^2}{2m}|\nabla\Phi|^2+\frac{c_0}{2}|\Phi|^4+2c_0\tilde{n}|\Phi|^2-\mu_{\rm DE}|\Phi|^2\right)d\mathbf{r}^2, \tag{3}$$

where m is the mass of the "trans-Planckian" particles (e.g., atoms) and c_0 is the coupling constant that relates the condensate density to the energy per particle (chemical potential) $\mu_{\rm DE}$ of the superfluid vacuum. The evolution of this superfluid universe is determined by the generalized Gross-Pitaevskii equation

$$i\hbar\partial_t\Phi(\mathbf{r},t)=\mathcal{H}\Phi(\mathbf{r},t), \tag{4}$$

where the Hamiltonian $\mathcal{H}=(1/\Phi)\delta\mathcal{E}/\delta\Phi^*$ is

$$\mathcal{H}=\left(-\frac{\hbar^2}{2m}\nabla^2+c_0n(\mathbf{r},t)+2c_0\tilde{n}(\mathbf{r},t)-\mu_{\rm DE}\right), \tag{5}$$

$n(\mathbf{r},t)=|\Phi(\mathbf{r},t)|^2$, and $\tilde{n}(\mathbf{r},t)$ is the particle density of the fluid not included in the condensate. Using the Madelung transformation [26] $\Phi(\mathbf{r},t)=|\Phi(\mathbf{r},t)|e^{iS(\mathbf{r},t)}$, in the case of static thermal cloud, Eq. (4) may be expressed equivalently in its hydrodynamic form in terms of the continuity equation

$$\frac{\partial n}{\partial t}=-\nabla\cdot(n\boldsymbol{v}_s) \tag{6}$$

and an Euler-like equation

$$-\hbar\frac{\partial S}{\partial t}=-\frac{\hbar^2}{2m|\Phi|}\nabla^2|\Phi|+\frac{m}{2}v_s^2+c_0|\Phi|^2+2c_0\tilde{n}-\mu_{\rm DE}, \tag{7}$$

where $\boldsymbol{v}_s=\hbar\nabla S/m$ is the superfluid velocity and S is the real-valued phase function of the condensate [24,25].

To draw a distinction between the standard terminology used in cold atom physics and that of the superfluid universe, we change the notation in Eq. (3), expressing it as

$$\mathcal{E}={\rm GEM}+\int\left[\left(\frac{c_0}{2}\Psi_{\rm DM}^2+2c_0\Psi_{\rm NM}^2-\mu_{\rm DE}\right)\Psi_{\rm DM}^2\right]d\mathbf{r}^2,$$

where GEM, defined later, stands for gravity and electromagnetism, the dark matter density $\Psi_{\rm DM}^2=|\Phi(\mathbf{r},t)|^2$, and the normal matter density $\Psi_{\rm NM}^2=\tilde{n}(\mathbf{r},t)$. In equilibrium, the normal matter

$$\tilde{n}(\mathbf{r})=\sum_q\{f(T,E_q)[|u_q(\mathbf{r})|^2+|v_q(\mathbf{r})|^2]+|v_q(\mathbf{r})|^2\} \tag{8}$$

exists in the form of Bogoliubov quasiparticles with energies E_q and quasiparticle amplitudes u_q and v_q. The Bose-Einstein distribution $f(T,E_q)$ determines the dependence of the normal matter density, and thereby also the dark matter density via conservation of total atom number, on temperature T of the atoms. As such, the atomic temperature T may also be interpreted as the temperature of the cosmic microwave background of the superfluid universe, which controls the ratio of ordinary matter to dark matter. The two main types of quasiparticle matter in this theory are phonons (massless particles) and kelvons (charged, massive particles), the latter being confined to and carried along by quantized vortices. The quasiparticles are oblivious to the existence of the trans-Planckian world of atoms out of which the condensate and thereby the whole superfluid universe emerged. Although the whole superfluid universe including gravity and electrodynamics is emergent, the rules of quantum mechanics are nevertheless inherited by the quasiparticles of the superfluid from the laws governing the trans-Planckian world of true atoms.

Next we reemploy the Madelung transformation to split the GEM into a part involving superflow and another one accounting for the effects of quantum pressure. This yields

$$\mathrm{GEM} = \int \frac{\hbar^2}{2m}|\nabla S(\mathbf{r})|^2|\Phi(\mathbf{r})|^2 + \frac{\hbar^2}{2m}\{\nabla|\Phi(\mathbf{r})|\}^2 d\mathbf{r}^2, \quad (9)$$

which we may also express as

$$\mathrm{GEM} = \int \Psi_{\mathrm{EM}}^2\Psi_{\mathrm{DM}}^2 d\mathbf{r}^2 + \int \Psi_{\mathrm{G}}^2\Psi_{\mathrm{DM}}^2 d\mathbf{r}^2. \quad (10)$$

The electromagnetic interaction Ψ_{EM}^2 emerges due to the motion (flow of the true atoms) of the vacuum and the gravitational interaction Ψ_{G}^2 emerges due to the curvature (variation in the density of the true atoms) of the superfluid spacetime. The effective interaction between gravitational and electromagnetic fields is mediated by the dark matter Ψ_{DM}^2. We will first consider the Ψ_{EM}^2 facet of the GEM.

III. EMERGENT ELECTROMAGNETISM

Let us first consider the superfluid density $n = n_0 + \delta n$ to be uniform (yet, oxymoronically allowing infinitesimal density fluctuations δn) and the trans-Planckian atoms to be confined to a quasi-2D regime such that the embedding space is three dimensional but the vortex dynamics is planar (two dimensional). This is a typical setting for instance in experimental studies on two-dimensional quantum turbulence in Bose-Einstein condensates [27,28]. The term

$$\mathrm{GEM}_{\mathrm{EM}} = \int \Psi_{\mathrm{EM}}^2|\Phi(\mathbf{r})|^2 d\mathbf{r}^2, \quad (11)$$

corresponds to the electromagnetic energy of the superfluid universe and the classical electrodynamics are obtained by considering the superfluid hydrodynamics within the aforementioned uniform condensate density approximation such that the continuity equation (6) becomes

$$\frac{\partial n_0}{\partial t} = -n_0(\nabla_\perp \cdot \boldsymbol{v}_s), \quad (12)$$

where we have introduced the subscript $\perp$ to remind us of the fact that spatial gradients only exist in the two-dimensional plane, and the equation for the phase evolution (7) may be approximated by

$$-\hbar\frac{\partial S}{\partial t} = \frac{1}{2}mv_s^2 + c_0 n_0 + 2c_0\tilde{n} - \mu_{\mathrm{DE}}. \quad (13)$$

The superfluid electric and magnetic fields

$$\mathbf{E}_{\mathrm{sf}} = mn_0\boldsymbol{v}_{\mathrm{s}} \times \mathbf{e}_z \quad \text{and} \quad \mathbf{B}_{\mathrm{sf}} = \frac{\hbar m}{c_0}\frac{\partial S}{\partial t}\mathbf{e}_z, \quad (14)$$

where $\mathbf{e}_z$ is the unit vector normal to the condensate plane, correspond to, respectively, spatial and temporal gradients of the condensate phase. Defining the magnetic field to be proportional to the phase change $\partial_t S$, rather than the condensate density n_0, ensures that the mean value of the magnetic field of the vacuum (ground state condensate) vanishes since then $-\hbar\partial_t S = c_0 n_0 - \mu_{\mathrm{DE}} = 0$. The vortex current

$$\mathbf{j}_v = \rho_v \boldsymbol{v}_v,$$

where $\boldsymbol{v}_v$ is the velocity field of the vortex phase singularities, is expressed in terms of the vortex density

$$\rho_v = (\nabla_\perp \times \boldsymbol{v}_{\mathrm{s}}) \cdot \mathbf{e}_z. \quad (15)$$

The superfluid vacuum constants are

$$\epsilon_v = \frac{1}{mn_0} \quad \text{and} \quad \mu_v = \frac{m^2}{c_0} \quad (16)$$

such that the speed of sound is

$$c_s = \sqrt{\frac{c_0 n_0}{m}} = \frac{1}{\sqrt{\mu_v\epsilon_v}}. \quad (17)$$

With these definitions, all of the classical electrodynamic theory for the superfluid universe can be derived starting from the generalized Gross-Pitaevskii energy functional.

A. Gauss-like E law

The Gauss-like law

$$\nabla_\perp \cdot \mathbf{E}_{\mathrm{sf}} = \frac{\rho_v}{\epsilon_v} \quad (18)$$

states that the vortex "charges" are the sources of the electric field, and is merely a restatement of the quantization of circulation $\oint \boldsymbol{v}_s \cdot d\mathbf{l} = \kappa w$, where w is an integer winding number and $\kappa = 2\pi\hbar/m$ is the quantum of circulation. Using Eq. (14), the divergence of the electric field is

$$\begin{aligned}\nabla_\perp \cdot \mathbf{E}_{\mathrm{sf}} &= mn_0\nabla_\perp \cdot (\boldsymbol{v}_{\mathrm{s}} \times \mathbf{e}_z)\\ &= mn_0\nabla_\perp \cdot (v_y\mathbf{e}_x - v_x\mathbf{e}_y)\\ &= mn_0(\partial_x v_y - \partial_y v_x). \quad (19)\end{aligned}$$

The vortex density

$$\begin{aligned}\rho_v &= \sum_{i=1}^{N_v} w_i\kappa\delta(r - r_i)\\ &= \boldsymbol{\omega} \cdot \mathbf{e}_z = (\nabla_\perp \times \boldsymbol{v}_{\mathrm{s}}) \cdot \mathbf{e}_z\\ &= (\partial_x v_y - \partial_y v_x), \quad (20)\end{aligned}$$

where N_v is the total number of vortices in the system and w_i is the integer winding number of the ith vortex, is equal to the divergence of the electric field divided by the permittivity of the vacuum, as stipulated by Gauss's law. In the continuum limit the Feynman criterion for the areal vortex density yields

$$\rho_v = \kappa n_v = \kappa\frac{\Omega_{\mathrm{rot}} m}{\hbar\pi} = 2\Omega_{\mathrm{rot}}, \quad (21)$$

which states that the magnitude of the vorticity of a rigidly rotating body equals twice its angular rotation frequency $\Omega_{\rm rot}$.

B. Gauss-like B law

The Gauss-like law for the magnetic field

$$\nabla_\perp \cdot \mathbf{B}_{\rm sf} = \nabla_\perp \cdot [B(x,y)\mathbf{e}_z] = 0 \tag{22}$$

is trivially satisfied because $\mathbf{B}$ has only one component and it is orthogonal to the x-y plane. In words, this superfluid universe has no monopoles of magnetic kind.

C. Faraday-like law

The law of induced electric fields due to changing magnetic field

$$\nabla_\perp \times \mathbf{E}_{\rm sf} = -\frac{\partial \mathbf{B}_{\rm sf}}{\partial t} \tag{23}$$

may be derived using the continuity equation for the superflow of atoms. The curl of the electric field is

$$\nabla_\perp \times \mathbf{E}_{\rm sf} = mn_0 \nabla_\perp \times (\boldsymbol{v}_s \times \mathbf{e}_z) = -mn_0(\nabla_\perp \cdot \boldsymbol{v}_s)\mathbf{e}_z, \tag{24}$$

where the second equality follows from a vector identity. The negative of the time derivative of the magnetic field

$$-\frac{\partial \mathbf{B}_{\rm sf}}{\partial t} = -\frac{\hbar m}{c_0}\frac{\partial}{\partial t}\left(\frac{\partial S}{\partial t}\right)\mathbf{e}_z = -mn_0(\nabla_\perp \cdot \boldsymbol{v}_s)\mathbf{e}_z \tag{25}$$

is thus equal to the curl of the electric field. This can be shown by differentiating the Euler-like equation (13) to yield

$$-\hbar\frac{\partial}{\partial t}\left(\frac{\partial S}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}mv_s^2 + c_0 n_0 + 2c_0\tilde{n} - \mu_{\rm DE}\right). \tag{26}$$

The assumption of a uniform condensate density implies that $mv_s^2 \ll c_0 n_0$ and since at ultralow temperatures $\tilde{n} \ll n_0$, it follows that for a constant $\mu_{\rm DE}$

$$-\frac{\hbar m}{c_0}\frac{\partial}{\partial t}\left(\frac{\partial S}{\partial t}\right) = m\frac{\partial n_0}{\partial t} = -mn_0\nabla_\perp \cdot \boldsymbol{v}_s, \tag{27}$$

where the second equality is just the continuity equation.

D. Ampere-Maxwell-like law

The law of induced magnetic fields due to electric current or changing electric field is

$$\nabla_\perp \times \mathbf{B}_{\rm sf} = \mu_v \mathbf{j}_v + \mu_v \epsilon_v \frac{\partial \mathbf{E}_{\rm sf}}{\partial t}, \tag{28}$$

which is consistent with the vortex current continuity equation $\nabla \cdot \mathbf{j}_v + \partial_t \rho_v = 0$. As in classical electrodynamics, the charges and currents must be explicitly introduced while in the full theory they emerge naturally as excitations of the superfluid. Once the charges have been introduced, Eq. (28) may be derived by considering a transformation to a reference frame moving at a local vortex velocity

$$i\hbar\partial_t \Phi_v(\mathbf{r},t) = [i\hbar\partial_t - \mathbf{J}_v \cdot \mathbf{A}]\Phi(\mathbf{r},t). \tag{29}$$

For the case of a uniformly rotating vortex lattice with $\mathbf{p} = m\boldsymbol{v}_s = m\Omega_{\rm rot} r\mathbf{e}_z \times \mathbf{e}_r$, Eq. (29) reduces to the usual transformation to a rigidly rotating frame,

$$\mathbf{J}_v \cdot \mathbf{A} = \mathbf{\Omega}_{\rm rot} \cdot (\mathbf{r} \times \mathbf{p}). \tag{30}$$

Since for a generic scalar function $A(r)$

$$\nabla_\perp \times [A(r)\mathbf{e}_z] = [\nabla_\perp A(r)] \times \mathbf{e}_z, \tag{31}$$

the curl of the superfluid magnetic field is

$$\begin{aligned}\nabla_\perp \times \mathbf{B}_{\rm sf} &= \nabla_\perp \times \left\{\frac{\hbar m}{c_0}\left(\frac{\partial S}{\partial t} - \mathbf{J}_v \cdot \mathbf{A}/\hbar\right)\mathbf{e}_z\right\} \\ &= \frac{\hbar m}{c_0}\nabla_\perp\left(\frac{\partial S}{\partial t}\right) \times \mathbf{e}_z - \frac{\Omega_{\rm rot}^2 m^2}{c_0}\nabla_\perp\left(r^2\right) \times \mathbf{e}_z. \end{aligned} \tag{32}$$

The first term on the previous row is equal to

$$\mu_v \epsilon_v \frac{\partial \mathbf{E}_{\rm sf}}{\partial t} = \mu_v \frac{\partial \boldsymbol{v}_s}{\partial t} \times \mathbf{e}_z = \frac{\hbar m}{c_0}\nabla_\perp\left(\frac{\partial S}{\partial t}\right) \times \mathbf{e}_z \tag{33}$$

and the second term yields the vortex current

$$\mu_v \mathbf{j}_v = \mu_v 2\Omega_{\rm rot}\boldsymbol{v}_s = -\frac{\Omega_{\rm rot}^2 m^2}{c_0}\nabla_\perp\left(r^2\right) \times \mathbf{e}_z. \tag{34}$$

In Eq. (34) we have used the Feynman criterion $\rho_v = 2\Omega_{\rm rot}$ and the approximation that the vortices would move at the local superfluid velocity such that $\boldsymbol{v}_v = \boldsymbol{v}_s$, which is not true in general when "gravitational" effects due to the spatial condensate density variations become important such as in the case of nonuniform condensates caused by external trapping [29] or rapid rotation [30].

E. Lorentz-like force law

The exact equation of motion for a vortex is [29]

$$\boldsymbol{v}_v = \boldsymbol{v}_S + \boldsymbol{v}_n, \tag{35}$$

where $\boldsymbol{v}_v$ is the vortex velocity and the background superfluid velocity

$$\boldsymbol{v}_S = \hbar\nabla_\perp S(r)/m|_{r_v} \tag{36}$$

equals the background condensate phase gradient (electric field) at the location of the vortex after the vortex self-induction velocity field is removed. The velocity component

$$\boldsymbol{v}_n = -\left.\frac{\hbar\mathbf{e}_z \times \nabla e(\mathbf{r})}{2me(\mathbf{r})}\right|_{r_v} \tag{37}$$

due to the background condensate density gradient (gravity) is expressed in terms of the function $e(r)$ defined, to lowest order in the multipole expansion of the condensate density in the vicinity of the vortex core, by $n(r) = e(r)(r - r_v)^2$. Equation (35) can be cast as a force equation by multiplying from the right by the factor $mn_0 \times \boldsymbol{\kappa}$ to yield the superfluid counterpart of the Lorentz force,

$$\mathbf{F}_{\rm B} = \mathbf{F}_{\rm E} + \mathbf{F}_{\rm G}, \tag{38}$$

where the three forces are

$$\mathbf{F}_{\rm B} = \kappa\boldsymbol{v}_v \times \mathbf{B}_{\rm sf}, \quad \mathbf{F}_{\rm E} = \kappa\mathbf{E}_{\rm sf}, \quad \text{and } \mathbf{F}_{\rm G} = m_v\mathbf{G}_{\rm sf}. \tag{39}$$

The first two are obtained by direct substitution of Eq. (14) with $\mathbf{B}_{\rm sf} \approx mn_0\mathbf{e}_z$ since the dynamical phase evolution at the location of the vortex phase singularity where $n|_{r_v} = 0$ is $\partial S/\partial t|_{r_v} \approx c_0 n_0/\hbar$. The gravitational force is discussed in detail in Sec. V.

Equation (38) may be regarded as the geodesic equation for the vortex and may be expressed as Newton's second law that provides a definition for the vortex mass m_v via

$$m_v \mathbf{a} = \mathbf{F}_\mathrm{E} + \mathbf{F}_\mathrm{G}, \tag{40}$$

where $\mathbf{a}$ is the acceleration of the vortex and is perpendicular to the velocity of the vortex. There are two real forces $\mathbf{F}_\mathrm{E}$ and $\mathbf{F}_\mathrm{G}$ acting on the vortex and they determine how the velocity of the vortex changes when it is subjected to the external forces. The equation of motion, Eq. (38), may also be expressed in terms of the the Magnus force $\mathbf{F}_\mathrm{Mag} = \mathbf{F}_\mathrm{E} - \mathbf{F}_\mathrm{B}$, as $\mathbf{F}_\mathrm{G} + \mathbf{F}_\mathrm{Mag} = 0$. In strictly uniform systems $\mathbf{F}_\mathrm{G} = 0$ and therefore also $\mathbf{F}_\mathrm{Mag} = 0$, such that $\mathbf{F}_\mathrm{E} = \mathbf{F}_\mathrm{B} = m_v \mathbf{a}$, which results in the statement that vortices are frozen in the superfluid since $\boldsymbol{v}_v = \boldsymbol{v}_s$.

F. Electromagnetic waves

The wave equations

$$\nabla_\perp^2 \mathbf{E}_\mathrm{sf} = \mu_v \epsilon_v \frac{\partial^2 \mathbf{E}_\mathrm{sf}}{\partial t^2} \tag{41}$$

and

$$\nabla_\perp^2 \mathbf{B}_\mathrm{sf} = \mu_v \epsilon_v \frac{\partial^2 \mathbf{B}_\mathrm{sf}}{\partial t^2} \tag{42}$$

may be derived directly using the hydrodynamic Maxwell equations, Eqs. (18), (22), (23), and (28) in the usual way. In the long-wavelength limit the waves described by these equations are associated with the linearised (infinitesimal) density perturbations of the Gross-Pitaevskii equation [24,25]

$$\nabla^2 \delta n(\mathbf{r}) = \frac{1}{c_s^2} \frac{\partial^2 \delta n(\mathbf{r})}{\partial t^2}. \tag{43}$$

These Bogoliubov phonons have the well known dispersion relation

$$\omega_\mathrm{phonon} = \sqrt{\left(\frac{\hbar k^2}{2m}\right)^2 + (c_s k)^2}, \tag{44}$$

which, in the long-wavelength limit, is a linear function $E = pc_s$ of momentum $p = \hbar k$ with the constant of proportionality $c_s = \sqrt{c_0 n/m}$ equal to the speed of sound. These sound waves have the remarkable property that they also correspond to a propagating density perturbation of the condensate and thereby a spatiotemporal variation in the quantum pressure, which will later be associated with gravity. As the density varies sinusoidally, the magnetic field oscillates in the axial direction while the electric field oscillates in the plane of the superfluid such that the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_v$ is parallel with the superflow and is directed in the propagation direction of the phonon.

G. Quantum electrodynamics

The preceding assumptions meant that vortex charges were frozen in the superfluid and this led to the Maxwellian approximation of classical electrodynamics. However, accounting for the fact that the superfluid is compressible intrinsically enables vortex-antivortex pair creation and annihilation events that are accompanied by emission and absorption of phonon radiation [31,32] and enable relativistic effects such as Zitterbewegung and particle production via Kibble-Zurek mechanism. Indeed, the full theory, Eqs. (3) and (4), self-consistently describe the processes of relativistic electrodynamics [12], as can readily be observed in numerical simulations, shown, e.g., in the supplementary movie of Ref. [33].

In order to generate more interesting kinds of stable matter of composite particles, one needs to go beyond the superfluid QED picture and introduce a diversity of particle types. This can be achieved via vectorial extensions to the scalar BEC universe, which are a natural way to incorporate "new physics". For instance, Rabi coupled two-component condensates host vortices that are predicted to result in a phenomenon similar to quark confinement [34,35]. It is also possible to go beyond quantum electrodynamics by introducing a true spin degree of freedom to the superfluid, which enables the creation of especial kinds of vortex particles, some of which may also behave as non-Abelian anyons [36]. The resulting superfluid quantum chromodynamics has interesting connections to the field of topological quantum computation [37].

We close the discussion on the electrodynamics of the superfluid universe by mentioning that the above considerations make it clear that the *atomtronics* applications [38–43], where the flow of superfluid atoms is traditionally considered to be analogous to the flow of electrons, could naturally be described from the perspective of *vortextronics*, where the vortices, instead of atoms, take the place of the charged current carrying electrons and the flow of the superfluid atoms simply corresponds to radiation and electromagnetic fields in a cavity that is used for trapping the superfluid.

We will next move on to consider the Ψ_G^2 term and the emergence of gravity.

IV. EMERGENT GRAVITY

The Einstein field equations of general relativity can be derived using a Lagrangian variational principle with the matter-free part of the four-dimensional spacetime generated by the Einstein-Hilbert action [11],

$$S_\mathrm{EH} = \int \mathcal{L}\, d\mathbf{r}^4 = \frac{c^4}{16\pi G} \int R\sqrt{-g}\, d\mathbf{r}^4, \tag{45}$$

where R is the Ricci scalar, c is the speed of light, G is Newton's gravitational constant, and the integration is over four-dimensional spacetime coordinates. Sakharov took the viewpoint that the Lagrangian $\mathcal{L}$ would be generated by an underlying quantum field theory and expressed the vacuum quantum fluctuations as a series expansion

$$\mathcal{L} = \lambda + \alpha R + \beta R^2 + \cdots \tag{46}$$

where the first term in the right corresponds to the cosmological constant, the second term gives rise to the Einstein-Hilbert action that yields the Einstein's field equations and the remaining terms result in higher order corrections to general relativity [10,11]. In this picture, gravity and general relativity are emergent phenomena generated by vacuum fluctuations of the underlying quantum field theory. Gravity in the superfluid universe and the gravitational mass of quantized vortices have similar origin.

The linear Bogoliubov phonon dispersion relation may be described in terms of an acoustic metric [23,44] of the superfluid universe,

$$g_{\mu\nu} = \Omega^2 \left(\begin{array}{c|c} -(c_s^2 - v_s^2) & -v_j \\ \hline -v_i & \delta_{ij} \end{array} \right), \tag{47}$$

where the conformal factor Ω is constant for flat spacetime. For the curved spacetime with two space dimensions, relevant to our discussion, $\Omega = mn/c_s$ and

$$\sqrt{-g} = \Omega^2 c_s = \frac{m^2 n^2}{c_s}, \tag{48}$$

where $g = \det(g_{\mu\nu})$ denotes the determinant of the metric tensor. The spacetime interval [23,44]

$$ds^2 = \Omega^2[-c_s^2 dt^2 + (d\mathbf{x} - \boldsymbol{v}_s dt)^2] \tag{49}$$

accounts for the linear part of the Bogoliubov phonon dispersion relation.

Gravity in the emergent matter-free superfluid universe arises due to the quantum fluctuations that result in the condensate density fluctuations even at zero temperature due to the fluctuating quantum depletion, caused by the trans-Planckian (atom-atom) particle interactions [24,25,45]. We begin by expressing the gravitational energy

$$\mathrm{GEM_G} = \int \frac{\hbar^2}{2m} (\nabla|\Phi(\mathbf{r})|)^2 d\mathbf{r}^2, \tag{50}$$

in terms of the quantum pressure

$$P_q = -\frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} = -\frac{\hbar^2}{4m} \nabla^2 \ln\left(\frac{n}{n_0}\right) - \frac{\hbar^2}{2m} \left(\frac{|\nabla\sqrt{n}|}{\sqrt{n}}\right)^2. \tag{51}$$

This yields

$$\mathrm{GEM_G} = -\int n\left[\frac{\hbar^2}{4m}\nabla^2 \ln\left(\frac{n}{n_0}\right) + P_q\right] d\mathbf{r}^2. \tag{52}$$

In two-dimensional space the Riemann tensor reduces to the Ricci scalar, which is related by $K = R/2$ to the geometric Gaussian curvature K. Combining Liouville's equation of differential geometry,

$$\nabla^2 \ln(\tilde{\Omega}) = -K\tilde{\Omega}^2, \tag{53}$$

where $\tilde{\Omega} = \Omega/\Omega_0$, with Eq. (52) we obtain

$$\mathrm{GEM_G} = \int n\left[\frac{\hbar^2}{8m} R\tilde{\Omega}^2 - P_q\right] d\mathbf{r}^2, \tag{54}$$

which may also be expressed as

$$\mathrm{GEM_G} = \frac{\hbar^2 c_s}{8m^3 n_0^2} \int \sqrt{-g}\left[nR + \frac{4\sqrt{n}}{\tilde{\Omega}^2}\nabla^2\sqrt{n}\right] d\mathbf{r}^2. \tag{55}$$

Associating the Lagrangian of the two-dimensional superfluid universe with the quantum kinetic energy $\mathrm{GEM_G} = \int \mathcal{L}_{\mathrm{G}}^{(2+1)} d\mathbf{r}^2$ brings about the connection to the superfluid Einstein-Hilbert action

$$S_{\mathrm{SEH}} = \int \mathcal{L}_{\mathrm{G}}^{(2+1)} d\mathbf{r}^2 dt = \frac{c_s^4}{16\pi} \int \sqrt{-g}\left[\phi R + \omega \frac{\nabla^2\sqrt{\phi}}{\sqrt{\phi}}\right] d\mathbf{r}^2 dt, \tag{56}$$

where

$$\phi = \frac{2\pi\hbar^2 n}{m^3 c_s^3 n_0^2} \quad \text{and} \quad \omega = \frac{16\pi\hbar^2}{nm^3 c_s^3}. \tag{57}$$

The structure of this action where curvature is coupled to a (dark matter) scalar field bears similarity to the Liouville quantum gravity [6,9] and Jordan-Brans-Dicke dilaton theories [46,47], which are a broader class of scalar-tensor theories of gravity considered already by Nordström [48,49]. Here the scalar field ϕ couples to all matter and energy fields and in fact is the "source of everything" in the superfluid universe. Indeed, it is straightforward to add all of the matter and energy fields $\mathcal{L}_{\mathrm{EM,DM,NM,DE}}$ in Eq. (3) to the "vacuum" action (56), including the effects of the dark energy and/or cosmological constant via μ_{DE}. The superfluid gravitational field is nonzero in any region of space where the condensate particle density is spatially varying. As such, the density fluctuations inherent in two-dimensional quantum turbulence may be interpreted as a form of quantum gravity in the superfluid universe.

In a matter-free universe quantum fluctuations yield spacetime curvature (via modulation of n) locally. Gravitation in the large-scale structure of the universe can be "added" by introducing an "external" potential. For example, a harmonic trapping potential would yield a non-uniform Thomas-Fermi condensate density $n = \mu_{\mathrm{DE}}[1 - (r/R_{\mathrm{TF}})]^2/c_0$, where as a self-gravitating universe could be induced by long-ranged dipole-dipole interactions that have been observed to generate self-bound droplets [50]. Antitrapping external potentials (cosmologies) are also frequently used in cold-atom experiments. Generically, it is possible to imprint any density landscapes in the laboratory condensates, such as a "Bose-Einstein cosmology" [51].

A. Gravitational waves

In the context of general gravity in 2 + 1 dimensions, the "folklore" states that, due to the lack of degrees of freedom, the theory should be trivial and that there should be no gravitational waves, and that gravity would then be manifest only via topological effects [7,52]. However, in Eq. (56) compressibility of the dark matter field provides the local degrees of freedom absent in the Einsteinian (2 + 1)-dimensional gravity. It is therefore reasonable to anticipate the possibility of wave motion akin to gravitational waves to exits in this superfluid universe similar to the higher dimensional Jordan-Brans-Dicke theories. The question is then what should such waves physically correspond to in laboratory experiments?

The "desirable" properties of such waves might be that their speed of propagation be close to the speed of sound c_s and that the generalized angular momentum they carry would be 2. For a ground state scalar BEC there exists only one gapless excitation branch linear in momentum: the usual Bogoliubov phonon. However, those quasiparticles, although producing spatial modulations of the condensate density, are plane waves carrying an angular momentum of zero and therefore the otherwise plausible idea of associating the longest wavelength phonons as gravitational waves in this system seems tenuous. This is also the case for other sounds in superfluids such as the second sound.

Another idea would be to associate other quasiparticle excitations, such as the scissors modes, pertinent to the quadrupole operator with the gravitons because these have angular momentum quantum number 2 but this would trade off the gapless linear spectrum. The next possible direction could be to consider gravitational waves as a genuinely nonlinear effect and to associate them with two-dimensional Jones-Roberts solitons, or other vortexonium-like rarefaction pulses, that have a dispersion relation at low momenta whose slope does coincide with that of the phonons [32,53].

In the presence of matter (e.g., vortex lattices) yet another possibility arises. The Kelvin-Tkachenko (KTK) vortex shear waves are excitation modes below the phonon line, also linear in momentum in the "stiff" limit [54–56]. These are transverse shear waves that correspond to the collective motion of the vortex particles and may be viewed as the mean-field precursors for the collective degrees of freedom that yield a geometric description of the fractional quantum-Hall effect [57,58]. However, we shall deem more rigorous contemplations of the nature of gravitons in the superfluid universe model to be outside the scope of this study.

B. Gravity, topology, and enstrophy

In the superfluid universe, topology of the spacetime is inherently linked to the spacetime curvature. At zero temperature the condensate ground state is smooth, to the extent that vacuum fluctuations may be neglected, and the universe is composed of dark matter only. If an instability, such as a parameter quench or tunneling to a lower energy state occurs, particles (vortices) and an electromagnetic field (condensate phase gradients) emerge. The vortices then puncture the condensate, changing its topology. The topology of such a multiply connected condensate and the resulting gravity are linked by the Gauss-Bonnet theorem

$$\int_{\mathcal{M}} K\,dS + \int_{\partial\mathcal{M}} k_g dl = 2\pi\zeta(\mathcal{M}), \tag{58}$$

which is a statement that the sum of the total curvature of a compact 2D Riemannian manifold $\mathcal{M}$ and the rotation of its smooth surface $\partial\mathcal{M}$ is proportional to the Euler characteristic ζ of $\mathcal{M}$. The surface of a planar BEC with vortices is homeomorphic to an n-torus without a boundary for which $\zeta = 2 - 2g_t$ and $\int_{\partial\mathcal{M}} k_g dl = 0$, and therefore Eq. (58) reduces to

$$\int_{\mathcal{M}} K\,dS = 4\pi(1 - g_t), \tag{59}$$

where g_t is the genus of the surface. For a ground state condensate $g_t = 0$, and generically $g_t = N_v$ for a BEC with N_v vortices. Within the point-vortex approximation the relevant surface would be homeomorphic to a closed unit disk with g_t holes cut out, for which $\int_{\mathcal{M}} K\,dS = 0$ and $\zeta = 1 - 1g_t$. Hence, gravity is absent in the point-vortex dual picture of the BEC due to the absence of condensate density modulations.

In general, due to the quantization of circulation, the number of vortices is also related to the enstrophy

$$\mathbb{E} = \int |\nabla \times \boldsymbol{v}_s|^2 dS \tag{60}$$

of the system of quantized vortices. Direct substitution of the vortex density, Eq. (20), yields

$$\mathbb{E}_v = \int |\rho_v|^2 dS = \kappa^2 g_t f, \tag{61}$$

where the generalized function (distribution) $f = \int \delta(\mathbf{r} - \mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')dS$. At first sight f seems ill defined and it remains unclear if a well defined interpretation of f could be obtained, similar to the generalized functions used in the derivation of Tan's contact [59,60]. Nevertheless, to be consistent with the Feynman rule, Eq. (21), that provides a link between the singular microscopic vorticity and the macroscopic smoothed vorticity fields, requires that the coarse grained average $\langle f \rangle = 4n_v$. The function $\kappa^2 f$ may also be associated with the onstrophy of a single quantum vortex [61].

Combining the Gauss-Bonnet theorem with the enstrophy equation (61) shows that the enstrophy

$$\mathbb{E}_v \approx 4\kappa^2\left(1 - \frac{1}{8\pi}\int R\,dS\right)^2, \tag{62}$$

a purely topological entity here, is also a measure of the total curvature, linking an important hydrodynamical quantity to gravity. In the theory of two-dimensional turbulence, the conservation law of enstrophy, $\partial\mathbb{E}/\partial t = 0$, underpins the inverse energy cascade, which ultimately leads to the phase separation of vortices and antivortices into Onsager vortex clusters, and a seeming matter-antimatter asymmetry in the theory [33].

Equation (62) shows that regions of high enstrophy, such as occurs within Onsager vortices [27,28], may also correspond to regions of high curvature. This naturally leads to the interpretation that the Einstein-Bose condensation transition [33,62] at negative absolute temperature would be expected to lead to the formation of a black hole analog with the associated phenomenology such as event horizons, ergo regions, Hawking radiation, and black hole thermodynamics.

V. GRAVITATIONAL VORTEX MASS

Equipped with the preceding considerations we are in a position to discuss the gravitational mass of a vortex. Adding a quantized vortex in an otherwise flat superfluid universe, $n(r) = n_0 = \text{const}$, changes the topology of the spacetime and influences its dynamics. The qualitative features brought about by the nucleation of a vortex include

(i) the topology of the condensate changes from being singly connected to being multiply connected,

(ii) a vortex core bound quasiparticle—a kelvon, that is a component of the normal matter of the superfluid universe—emerges in the elementary excitation spectrum,

(iii) the vortex acquires a mass due to its coupling to the dark matter field,

(vi) the superfluid vacuum begins to flow due to the phase gradient of the condensate and this superflow corresponds to an emergent electric field,

(vii) motion of the vortex (kelvon) induces a magnetic field due to the time variation of the condensate density,

(viii) an accelerating vortex may radiate phonons, and

(ix) a condensate density gradient due to the structure of the vortex core results in quantum pressure that gives rise to a gravitational field.

A vortex centered at the origin may be described by the wave function

$$\psi_v(r) = n_0\chi(r)e^{iS(r,t)}, \tag{63}$$

where χ is the vortex core structure function [61,63],

$$\chi(r) = \tanh\left(\frac{r}{\sqrt{2}r_c}\right) \approx \frac{r}{\sqrt{r^2+r_c^2}}, \tag{64}$$

and the phase function $S = \arctan(x, y)$ has a singularity at the origin. In the vicinity of the vortex core the condensate density $n(r)$ forms a harmonic oscillator potential:

$$n(r)_{r\to 0} = n_0\chi^2_{r\to 0} = n_0 r^2/r_c^2. \tag{65}$$

A test vortex with circulation q_2 placed distance r from the origin is influenced by two forces due to the presence of the source vortex of circulation q_1 at the origin: (i) the electric force $\mathbf{F}_{\rm E}$ due to the phase gradient (superflow) and (ii) the gravitational (Magnus) force $\mathbf{F}_{\rm G}$ due to the local density gradient (quantum pressure); see Sec. III E.

The forces $\mathbf{F}_{\rm E}$ and $\mathbf{F}_{\rm G}$ may be obtained by considering the difference in energy, $\Delta\text{GEM} = (\mathcal{E}_1 - \mathcal{E}_0)$, between universes with and without a vortex [24]:

$$\Delta\text{GEM} = \int \frac{\hbar^2 n_0}{2m}\left[\left(\frac{\chi}{r}\right)^2 + \left(\frac{\partial\chi}{\partial r}\right)^2 + \frac{1}{2r_c^2}(1-\chi^2)^2\right]d\mathbf{r}^2. \tag{66}$$

We associate the first two terms in Eq. (66), respectively, with the electric and gravitational fields produced by the vortex particle, while the last term is due to the change in the dark matter energy density. It would be tempting to elevate these terms into potentials, the negative gradients of which would then yield the conservative forces on the vortex. In the case of the electric force, the integral of the first integrand in Eq. (66) does indeed yield the usual electrostatic potential proportional to $\ln(r/r_c)$ in two dimensions. However, for the second, gravitational term this approach only works for the short-distance limit, as clarified further below.

In contrast to classical electrodynamics where the Lorentz force is determined solely by the electromagnetic fields acting on the charged particle and is independent of all other forces, Eq. (38) completely determines the dynamics of a vortex. In words, gravity, electricity and magnetism are the "three sides of the same coin."

A strictly uniform system, $\boldsymbol{v}_n = 0$, corresponds to a zero gravity $\mathbf{F}_{\rm G} = 0$ because the Magnus force vanishes and the vortex is frozen in the superfluid, traveling at the speed of local superfluid velocity such that $\boldsymbol{v}_v = \boldsymbol{v}_S$. Consequently, $\mathbf{F}_{\rm B} = \mathbf{F}_{\rm E} = m_i^v\mathbf{a}$, where $\mathbf{a}$ is the inertial acceleration of the vortex. For nonuniform systems, such as harmonically trapped condensates, gravitational effects become important and the Magnus force has a nonzero value. For an infinite system with a vortex placed in a region of a parabolic underdensity, $\boldsymbol{v}_S = 0$ due to the image of the vortex being infinitely far from the vortex such that $\boldsymbol{v}_v = \boldsymbol{v}_n$. In this case $\mathbf{F}_{\rm E} = 0$ and $\mathbf{F}_{\rm B} = \mathbf{F}_{\rm G} = m_g^v\mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration of the vortex. The resulting periodic circular vortex motion is then entirely due to the curvature of the condensate density. We may then consider placing the vortex in "Einstein's elevator" such that it is not possible to distinguish between the two aforementioned cases, which may be set up such that $\mathbf{a} = \mathbf{g}$. Hence we arrive at a vortex equivalence principle: the equality of the inertial and gravitational vortex masses,

$$m_{\rm g}^v = m_{\rm i}^v. \tag{67}$$

This may not be surprising since both gravitation and electromagnetism in this theory are generated by the same emergent quantity: the Laplacian of the matterwave of the Bose-Einstein condensate. In a periodic circular motion the vortex experiences the usual centripetal acceleration, irrespective of whether it is caused by electrical or gravitational effects, such that $qv_vB_{\rm sf} = m_v v_v^2/r$, and thus the vortex mass, Eq. (2), may be expressed as

$$m^v = \frac{2\pi\hbar^2}{c_0\omega_k}\frac{\partial S}{\partial t}. \tag{68}$$

Furthermore, the mass m^v and charge q_v of the vortex are not independent quantities but are related by

$$\frac{q_v}{m^v} = \epsilon_v\omega_k = \frac{h}{m_a m_v}, \tag{69}$$

where $m_a = m$ is the mass of the atom. The permittivity and the kelvon frequency may thus be viewed as vacuum "constants" that link the masses of the dark matter particles that form the superfluid and the elementary quasiparticles of the superfluid universe.

The explicit forms of the forces on a test vortex are obtained directly from the respective terms in Eq. (38):

$$\mathbf{F}_{\rm E} = -mn_{bg}\boldsymbol{\kappa}\times\boldsymbol{v}_S = \frac{2\pi\hbar^2 n_0\chi^2}{m}|\nabla_\perp S|\mathbf{e}_{12} = \frac{q_1q_2}{2\pi\epsilon_v}\frac{r}{r^2+r_c^2}\mathbf{e}_{12}, \tag{70}$$

where $\mathbf{e}_{12}$ is a unit vector from the source vortex to the test vortex, $q_i = \pm h/m$, and the gravitational force

$$\mathbf{F}_{\rm G} = -mn_{bg}\boldsymbol{\kappa}\times\boldsymbol{v}_n = -\frac{\pi\hbar^2 n_0\chi^2}{m}\frac{\partial_r(\chi^2)}{\chi^2}\mathbf{e}_{12} = -G_v m_1 m_2\frac{rr_c^2}{\left(r^2+r_c^2\right)^2}\mathbf{e}_{12}. \tag{71}$$

As in Einsteinian gravity, here too we have judiciously defined the gravitational constant, $G_v = \omega_k^2/4\pi mn_0$, in such way that the gravitational vortex mass is, by construction, equal to the inertial vortex mass as stipulated by the superfluid equivalence principle. Equation (71) also serves to define the gravitational field generated by a vortex of mass m_v as $\mathbf{G}_{\rm sf} = \mathbf{F}_{\rm G}/m_v = -4\pi G_v m_v\boldsymbol{e}_z\times\boldsymbol{v}_n/\kappa$.

The electric force is repulsive if the test vortex has the same sign of circulation as the source vortex, and attractive if the test vortex has an opposite sign of circulation with respect to the source vortex. All vortices for which $\omega_k < 0$ have a negative mass [19,64] such that the product m_1m_2 is always positive and the vortex-vortex gravity is always an attractive interaction. This is because all vortices create a parabolic underdensity in the condensate. For short distances, $r < r_c$, the forces reduce to

$$\mathbf{F}_{\rm E} = \frac{q_1q_2}{2\pi\epsilon_v}\frac{r}{r_c^2}\mathbf{e}_{12} \tag{72}$$

and

$$\mathbf{F}_G = -G_v m_1 m_2 \frac{r}{r_c^2}\mathbf{e}_{12}. \tag{73}$$

These approximate "Newtonian" forces, Eqs. (72) and (73), can be obtained, in the $r < r_c$ limit, as the negative gradients of the respective potentials, both being $\propto r^2$ [the first two terms of Eq. (66)]. However, for large distances the $1/r^3$ behavior of gravity is very different from the $1/r$ force law of the electromagnetic field for a single vortex. Moreover, in contrast to the electric field, the total gravitational field is not simply a sum of the fields generated by individual vortices. Instead, the total gravitational field depends crucially on the distribution of the global dark matter energy density, the last term in Eq. (66).

VI. VORTEX-PARTICLE DUALITY

Dualities in physics are a powerful concept that grant multiple viewpoints for the same physical phenomenon. Prominent examples of such dualities include the anti–de Sitter and conformal field theory (AdS-CFT) correspondence and its variants, the holographic principle, gauge-gravity duality, bulk-edge correspondence, and fluid-gravity duality [65,66].

A particularly relevant example for this work is the *particle-vortex duality* that was originally discussed in the context of bosons [67–69], and has recently been extended to fermions and a broader web of dualities [70–74]. In simple terms, the particle-vortex duality provides a link between the action of particles Φ,

$$\mathcal{S}_a = \int |(\partial_\mu - iA_\mu)\Phi|^2 - V(\Phi)\, d^3\boldsymbol{x}, \tag{74}$$

and the action of vortices (kelvon quasiparticles) Υ,

$$\mathcal{S}_k = \int |(\partial_\mu - ia_\mu)\Upsilon|^2 - \tilde{V}(\Upsilon)\, d^3\boldsymbol{x}, \tag{75}$$

where, respectively, A_μ and a_μ are background and dynamical gauge fields and V and $\tilde{V}$ incorporate model specific potentials and duality dependent Chern-Simons terms.

In the context of the low-energy effective theories considered here, the particles of the superfluid described by the Gross-Pitaevskii theory, see Sec. II, realize an atom dual, Eq. (74), with the the background gauge field A_μ generated for instance by rotating the superfluid. The atoms carry a "charge" equal to their mass and these atoms "see" the vortices as flux quanta such that the flux density of the background gauge field A_μ equals the kelvon (vortex) density ρ_v. The vortex dual, Eq. (75), describes the kelvon quasiparticles (vortices) of Sec. III coupled to the dynamical (electromagnetic) gauge field a_μ. The kelvons "see" the atoms as flux quanta such that the flux density of the dynamical gauge field a_μ equals the atom density ρ_a. The atom dual describes the quantum hydrodynamics of the Bose-Einstein condensate. The vortex dual describes the electrodynamics of the kelvons in a curved spacetime.

In addition to this $(2+1)$- to $(2+1)$-dimensional particle-vortex duality, the superfluid universe also features a *vortex-particle duality* which maps the $(2+1)$-dimensional vortices to $(1+1)$-dimensional particles. As such, the system features a particle-vortex-particle duality "thread" where, in the case of a uniform BEC that neglects gravitational effects, the first duality links the XY model to 2D Coulomb gas and the second one further links the 2D Coulomb gas to the sine-Gordon quantum field theory [75,76]. The $(2+1)$- and the $(1+1)$-dimensional aspects of this duality thread are briefly characterized below.

A. Two-dimensional weakly interacting classical field theory with gravity

The dynamics of the superfluid (atoms) moving in two-dimensional space is described by the Gross-Pitaevskii equation, Eq. (4). Each of the atoms has four (q_x, q_y, p_x, p_y) canonical phase space coordinates such that Hamilton's equations of motion are

$$\dot{q}_x = -\frac{\partial H_{\text{atom}}}{\partial p_x}, \quad \dot{q}_y = -\frac{\partial H_{\text{atom}}}{\partial p_y},$$
$$\dot{p}_x = \frac{\partial H_{\text{atom}}}{\partial q_x}, \quad \dot{p}_y = \frac{\partial H_{\text{atom}}}{\partial q_y}, \tag{76}$$

where H_{atom} is a Hamiltonian for the atoms.

When quantized vortices are nucleated in the superfluid, the condensate order parameter becomes topologically multiply connected. The vortices that puncture the condensate are thus not part of the superfluid although their motion is fully correlated with it. The interaction between the superfluid and the fluid of vortices is mediated by the dark matter (atoms). The superfluid "experiences" the vortices as obstructions that constrain its dynamics and the vortices "experience" the fluid as an obstruction that they have to plough through. The vortices acquire mass due to their interaction with the Higgs-like dark matter field (the condensate of atoms).

The description of the two-dimensional fluid with its four-dimensional phase space corresponds to a $(2+1)$-dimensional weakly gravitating classical field theory for which the vortex degrees of freedom also realize a $(1+1)$-dimensional boundary quantum field theory. The gravity that originates from the quantum pressure of the condensate is emergent. We cannot overemphasize the importance of the fact that the fluid atoms and the vortex particles (kelvons) may formally exist in spacetimes of different dimensionality such that in $(1+1)$ case the kelvons are associated with the sine-Gordon instantons [75,76].

B. One-dimensional strongly interacting quantum field theory without gravity

The dynamics of the vortices is described by the vortex equation of motion (35), to which the Onsager point vortex model provides a rather good approximation in a uniform system in the dilute vortex gas limit. Each of the vortices has two (q_x, p_x) canonical phase space coordinates such that Hamilton's equations of motion are

$$\dot{q}_x = -\frac{\partial H_{\text{vortex}}}{\partial p_x}, \quad \dot{p}_x = \frac{\partial H_{\text{vortex}}}{\partial q_x}. \tag{77}$$

The function $H_{\rm vortex}$ is the well known two-dimensional Coulomb gas pseudo-Hamiltonian

$$H_{\rm vortex} = -\sum_{i<j}^{N_v} s_i s_j \ln(|\mathbf{r}_i - \mathbf{r}_j|/r_c), \tag{78}$$

where $\mathbf{r}_i = \sqrt{\mathbf{q}_{x,i}^2 + \alpha^2 \mathbf{p}_{x,i}^2}$ and α has the dimensions of time divided by mass. The particle-vortex duality provides the mapping from the neutral 2D Coulomb gas (point-vortex picture) to the one-dimensional sine-Gordon quantum field theory (particle picture) [75,76]. Quantum mechanically, the vortices correspond to the quantized kelvon quasiparticles entering Eq. (8), which in the $(1+1)$ dual present themselves as instantons. In the presence of a circular boundary and in the vicinity of the Einstein-Bose condensation transition, the point-vortex model can be mapped onto an inverted, strongly interacting one-dimensional harmonic oscillator Hamiltonian [62].

VII. QUANTUM HALL EFFECTS

A Hamiltonian for an electron in a uniform magnetic field $\mathbf{B} = \nabla \times \mathbf{A}_e$ is

$$H_e = \frac{(\mathbf{p} - q_e \mathbf{A}_e)^2}{2m_e}, \tag{79}$$

where m_e is the mass of the electron, q_e its charge, $\mathbf{p}$ its momentum, and $\mathbf{A}_e$ is the vector potential. When the magnetic field strength B is sufficiently increased, a two-dimensional electron gas with fixed number of electrons undergoes successive topological quantum phase transitions to strongly correlated integer and fractional (when Coulomb interactions are accounted for) quantum Hall liquids [77]. Such topological states of matter are anticipated to emerge when the filling fraction

$$\nu_e = \frac{N_e}{\Phi/\Phi_0} = \frac{N_e}{N_\Phi} \lesssim 1, \tag{80}$$

where N_e and N_Φ are the number of electrons and the number of magnetic flux quanta, respectively, and $\Phi = B\mathcal{A}$ is the magnetic flux piercing area $\mathcal{A}$ and quantized in units of $\Phi_0 = h/2e$.

A great effort has been expended in trying to observe bosonic quantum Hall states using rapidly rotating neutral superfluids [24,78–82]. This has been prompted by the observation that the Hamiltonian of such systems can be mapped onto that of the two-dimensional electron problem, Eq. (79). Specifically, a Bose-Einstein condensate in a harmonic oscillator potential, expressed in the rotating frame of reference, has the "single-particle" Hamiltonian

$$\begin{aligned} H_a &= \frac{\mathbf{p}^2}{2m_a} + \frac{1}{2} m\omega_{\rm osc}^2 r^2 + gn - \Omega_{\rm rot} L_z \\ &= \frac{(\mathbf{p} - q_a \mathbf{A}_a)^2}{2m_a} + \frac{1}{2} m[\omega_{\rm osc}^2 - \Omega_{\rm rot}^2] r^2 + gn, \end{aligned} \tag{81}$$

where m_a is the mass of the atom, $\omega_{\rm osc}$ is the harmonic oscillator frequency, $\Omega_{\rm rot}$ is the external rotation frequency, and L_z is the axial component of the orbital angular momentum operator. When $\Omega_{\rm rot}$ is increased and is approaching the value of $\omega_{\rm osc}$ such that $\omega_{\rm osc}^2 - \Omega_{\rm rot}^2 \to 0$, ever larger number of vortices are nucleated in the system while the atom cloud expands radially, becoming ever more dilute such that $n \to 0$. The result is that the last two terms in Eq. (81) become negligible with respect to the first term such that H_a becomes mathematically identical to H_e with

$$q\mathbf{A} = -m_a \Omega_{\rm rot} \mathbf{r} \times \mathbf{e}_z. \tag{82}$$

Here we consider an external rotation as a specific source of an effective gauge field, although the conclusions drawn apply equally well to generic synthetic gauge fields.

Based on this observation, it is often stated that the rotation frequency $\Omega_{\rm rot}$ of a neutral superfluid would correspond to the magnetic field B in the problem of a two-dimensional electron gas. Consequently, to realize quantum Hall states, the goal would be to try to make $\Omega_{\rm rot}$ large in order to reach the limit of strong effective magnetic fields, analogously to the case of degenerate two-dimensional electron gas in a strong external magnetic field. This corresponds to the "view of the atoms" [Eq. (74)], but the "view of the electrons" [Eq. (79)] corresponds equally well to the vortex dual [Eq. (75)], in which the kelvons are the charged particles that "see" the atoms as magnetic flux. Hence, increasing $\Omega_{\rm rot}$ actually diminishes the effective magnetic field and increases the filling factor, which in correspondence with Eq. (80) should be

$$\nu_v = \frac{N_v}{N_a}, \tag{83}$$

in contrast to its inverse in the view of the atom dual [79,80], where N_a is the number of atoms and the flux quantum corresponds to the mass of an atom m_a. Zeroing the effective trapping potential and particle interactions in Eq. (81) can clearly be achieved by setting $\Omega_{\rm rot} = \omega_{\rm osc}$, but this is not the same as increasing $\mathbf{A}_e$ alone in Eq. (79). The analog of electric charge in the vortex dual should be the quantum of circulation [83], such that for a rotating BEC the effective total charge

$$q_v = \kappa N_v. \tag{84}$$

Substituting this and the Feynman rule, Eq. (21), into Eq. (82), we obtain the vector potential per condensate particle,

$$\mathbf{A} = \mathbf{A}_v / N_a = -\frac{m_a}{2\mathcal{A}} \mathbf{r} \times \mathbf{e}_z, \tag{85}$$

such that the magnetic field is

$$\mathbf{B}_v = \nabla \times \mathbf{A}_v = \frac{m_a N_a}{\mathcal{A}} \mathbf{e}_z \tag{86}$$

and has no explicit dependence on $\Omega_{\rm rot}$. Similarly, the charge q is only implicitly dependent on $\Omega_{\rm rot}$ and it is only in the product of $q\mathbf{A}$ that the rotation frequency makes an explicit appearance. We also note that $\mathbf{B}_v = m_a n \mathbf{e}_z$ as expressed in Eq. (86) follows directly from the definition of $\mathbf{B}_{\rm sf}$ in Eq. (14); see also below Eq. (39). Had we taken the atom dual point of view, $q_v \to q_a, B_v \to B_a$, the roles of q and B would simply have been reversed such that κN_v would be replaced by $m_a N_a$ and vice versa with the product qA unaffected. Consequently, $\nu_v \nu_a = 1$, which explains the inversion of the filling factor criterion under the duality transformation.

When $\Omega_{\rm rot}$ increases, qA increases because q_v grows, even though both A_v and B_v *decrease* due to the increasing area $\mathcal{A}$

occupied by the superfluid. Nevertheless, N_v grows faster than $\mathcal{A}$ because also the vortex density increases. Hence, the filling factor criterion, Eq. (80), to achieve the quantum Hall limit can be expressed as

$$\nu_v = \frac{N_v}{B_v \mathcal{A}/m_a} = \frac{N_v}{N_a} \lesssim 1, \tag{87}$$

which in practice means that this criterion is immediately satisfied when the first vortex is nucleated in the system. However, for low rotation frequencies the lowest Landau level states are not degenerate and thus do not form a flat band because of the non-negligible influence of the last two terms in Eq. (81) that lift the degeneracy, and it is for this reason that the system needs to be rotated rapidly to make the kinetic energy overwhelm the scalar potentials, instead of creating a large number of vortices per se. In fact, as previously mentioned, when the system is rotated faster, the vortex number goes up and this causes the value of ν_v to *increase*.

This also means that (in the electron/vortex picture) the rotation frequency $\Omega_{\rm rot}$ should not be associated with a magnetic field; rather, the rotating drive creates a strong electric field, Eq. (14) (superflow), which destabilizes the vacuum and nucleates an increasing number of charges (vortices) in the system in accordance with Eq. (21). In contrast, the number of electrons does not change when a two-dimensional electron gas is placed in a magnetic field in typical quantum Hall effect experiments.

Associating vortices with charges and magnetic flux with atoms (rather than vice versa) is further supported by the prediction that a vortex transported around a loop $\mathcal{C}$ in a superfluid accumulates a Berry phase [84,85]

$$\gamma_{\mathcal{C}} = \frac{2\pi}{m_a} \int_{\mathcal{A}_{\mathcal{C}}} \mathbf{B}_v \cdot d\mathcal{A} = 2\pi N_a(\mathcal{C}), \tag{88}$$

where N_a is the number of atoms enclosed by the vortex path. As such, the atoms are the Aharonov-Bohm flux quanta for the vortex. The force on a vortex executing circular motion in a perpendicular magnetic field is $F = \kappa v_v B_{\rm sf} = m_v v_v^2/r$ such that the vortex mass can be expressed in terms of the geometric phase as

$$m_v = \frac{\gamma_{\mathcal{C}}}{\mathcal{A}_{\mathcal{C}}} \frac{\hbar}{\omega_v}. \tag{89}$$

Although the possibility of fractional statistics for a single vortex was excluded in Ref. [84], it may be possible that many vortex KTK states would allow it as the vortices then form collective composite quasiparticles in analogy with the flux attachment to many-electron states in the fractional quantum Hall effect (FQHE).

The above reasoning leads to a proposition for an interpretation of the Bogoliubov quasiparticle excitation spectrum of rotating condensates. Since in the quantum-Hall limit the number of lowest Landau level (LLL) eigenstates should correspond to the number of flux quanta, we are motivated to redefine the vortex filling factor by associating the number of populated LLL states with the effective magnetic flux,

$$\nu_{\rm LLL} = \frac{N_v}{N_{\rm LLL}}, \tag{90}$$

so that the FQHE states with $\nu_{\rm LLL}$ should be associated with condensates of elementary droplets consisting of N_v vortices bound to $N_{\rm LLL}$ one-particle Landau level states [57,86]. Thus we arrive at the following interpretation of the physics of a rotating BEC: in addition to the phonons, the low-lying quasiparticle excitation spectrum comprises two types of modes: (i) the surface modes which approximately correspond to single-particle harmonic oscillator angular momentum eigenstates that ultimately, as $\Omega_{\rm rot} \to \omega_{\rm osc}$, will form the LLL; and (ii) N_v transverse vortex shear modes (KTK modes) visualized, e.g., in the supplementary videos of Ref. [56]. Each of these vortex shear modes can be reconstructed as a product of $N_{\rm LLL}$ nondegenerate LLL states, and as such should be viewed as the mean-field precursors to the many-body FQHE states that could be realized experimentally as metastable excited states of rotating BECs.

Treating the BEC (the Nambu-Goldstone mode) as a single composite boson of effective mass $\tilde{m} = m_a N_a$ that has been selected out as the mean-field ground state of the rotating BEC corresponds to a Wigner crystal of N_v vortex charges. In the vortex dual this is a mean-field integer quantum Hall state with a filling factor of $\tilde{\nu}_v = N_v$. In the atom dual it is a mean-field fractional quantum Hall state with filling factor $\tilde{\nu}_a = 1/N_v$, where the boson $\tilde{m}$ has captured or attached N_v flux quanta internally to itself in order to reduce the effective magnetic field it experiences by the amount of $\Omega_{\rm rot} L_z$. The boson with charge $\tilde{m}$ thus sees the KTK modes as its quasiparticle excitations composed of elementary droplets with fractional charge $\tilde{\nu}_a \tilde{m}$ and quantised phase factors $\Delta\varphi = 2\pi \tilde{\nu}_a$ [87]. When $\Omega_{\rm rot} \to \omega_{\rm osc}$ the atoms in the boson $\tilde{m}$ are depleted over a growing number of pseudo-Nambu-Goldstone zero modes and the Wigner crystal state will no longer be singled out as the many-particle ground state. We aim to return to this point in more detail elsewhere, merely emphasising again in this context that rapidly rotating a neutral superfluid should ultimately lead to

(i) the formation of Onsager vortex clusters and the associated absolute negative temperature states,

(ii) the formation of fractional quantum-Hall-like states due to the quasiparticle condensation in the hierarchy of transverse vortex wave modes, and

(iii) the formation of a black hole analog due to the increasing mass density of vortex particles.

This means that the dense vortex matter may be described in terms of at least three complementary pictures: the hydrodynamical, the electromagnetic, and the gravitational. Indeed, in light of identifying the quantised vortices as charged massive particles, the connections between the physics of black holes and FQHE [57,88,89] are perhaps less surprising.

VIII. CONCLUSIONS

We have considered an emergent $(2+1)$-dimensional superfluid universe where both gravity and electromagnetism originate from the quantum kinetic energy of the superfluid. The Bose-Einstein condensate represents the fabric of the superfluid spacetime. This condensate also corresponds to a dark matter field, which fills the vacuum and is the mediator of all force fields including gravitation and electromagnetism. In this superfluid universe, electric field corresponds to the superflow of the Bose-Einstein condensate, magnetic field corresponds to the condensate phase evolution, vortices are massive charged particles, and the sound waves correspond to the massless photons. Gravity is associated with condensate density gradients and the condensate is identified as the elusive dark matter.

The vortices have two possible signs of circulation and therefore the electromagnetic interaction between two vortices may be attractive or repulsive. Both vortices and antivortices have the same condensate density depletion in their cores and, correspondingly, the density gradients produced by them are identical. Therefore, the gravitational interaction between any two vortices is always attractive. The vortex acquires a mass by interacting with a Higgs-like dark matter field whose density together with the fundamental kelvon excitation frequency determine the inertial vortex mass [19]. Here we have further argued in favor of equality between the inertial and gravitational masses of the quantized vortices. The unified descripion of electromagnetism and gravity and the association of quantized vortices with massive charged particles leads to the picture where the quantum Hall physics of a rapidly rotating neutral superfluid, condensation of elementary vortices into high density negative absolute temperature Onsager vortex clusters, and black hole thermodynamics with emergent quantum gravity are complementary ways to describe the states of dense vortex matter.

Recently, experiments on weakly interacting Bose-Einstein condensates have been used for simulating spacetime analogs including inflationary cosmology [90] and Hawking radiation [91,92], and it seems that quantum turbulent Bose-Einstein condensates [27,28] may provide a fruitful platform for further studies of emergent analog phenomena of gravity, dark matter physics, and AdS-CFT correspondence [93]. Stretching the analogy in the opposite direction, it is amusing to contemplate the implications if the Universe were a superfluid hologram, gravity merely a manifestation of its quantum fluctuations, and the sought after dark matter just a terminal point of the photon dispersion relation—a Bose-Einstein condensate of ultraweakly interacting photons—and that the seeming matter-antimatter asymmetry would be caused by evaporative heating induced negative temperature Onsager vortex clustering. Finally, it is interesting to witness how modeling the dark matter in our Universe deploying quantum mechanical scalar fields is steadily making its way into mainstream cosmology [94].

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