Geometry of generalized Pauli channels

Katarzyna Siudzińska ⬤

Institute of Physics, Faculty of Physics, Astronomy and Informatics,
Nicolaus Copernicus University, ul. Grudziądzka 5/7, 87–100 Toruń, Poland

We analyze the geometry of the generalized Pauli channels constructed from the mutually unbiased bases. The Choi-Jamiołkowski isomorphism allows us to express the Hilbert-Schmidt line and volume elements in terms of the eigenvalues of the generalized Pauli maps. After determining appropriate regions of integration, we analytically compute the volume of generalized Pauli channels and their important subclasses. In particular, we obtain the volumes of the generalized Pauli channels that can be generated by a legitimate generator and are entanglement breaking. We also provide the upper bound for the volume of positive, trace-preserving generalized Pauli maps. This helps us to better understand the structure of high-dimensional sets of quantum channels.

I. INTRODUCTION

Due to their unique properties, both the Pauli channels and the generalized Pauli channels have found many important applications in quantum information theory. Their construction is mainly based on the sets of mutually unbiased bases [1]. The generalization of the Pauli channels, also known as *Pauli diagonal channels constant on axes*, was defined and analyzed by Nathanson and Ruskai [2]. They were interpreted as the depolarizing channels whose degree of depolarization depends on the choice of the axis. Most effort was put into finding the properties of the channels constructed from the maximal number of mutually unbiased bases (MUBs). However, the authors also showed how to define the generalized Pauli channels for nonmaximal numbers of MUBs, and these are the channels that are our main consideration. In recent literature, the generalized Pauli channels have been considered as an example of commutating dynamical maps [3]. It has been shown that their behavior under uniform time dilation of the corresponding memory kernel is linked to their P-divisibility property [4]. This property has also been studied in Refs. [5,6]. It turns out that, under certain conditions, the generalized Pauli channels belong to the class of absolutely separating channels, which means that their output cannot be unitarily transformed into any entangled state [7]. In Ref. [8], the authors have presented estimators for the channel parameters using a quantum process tomography method.

This paper is a continuation of Ref. [9], where we introduced the geometry on the space of positive, trace-preserving Pauli maps. There, we used the Choi-Jamiołkowski isomorphism [10,11] to obtain the the Hilbert-Schmidt metric on the space of Pauli maps. Using this metric, we calculated the volumes of the entanglement-breaking channels, P- and CP-divisible channels, as well as the channels obtainable through time-local generators. The paper was motivated by the question about the extent to which the formalisms of quantum channels and master equations coincide. The relative amount of channels that can be produced using the master equations with time-local generators was established. The

volume formulas allowed us to learn more about the shapes and structures of several subsets of quantum channels.

Other works on the geometry of quantum channels include Ref. [12] where Narang and Arvind obtained the volume of the Pauli channels that can be simulated by a one-qubit mixed state environment. Similar calculations were repeated by Jung *et al.* for the generalized amplitude-damping channels [13]. The Choi-Jamiołkowski isomorphism was also used to compute the volume of unital qubit channels with the Lebesque measure [14]. For the Hilbert-Schmidt measure, Szarek *et al.* [15] listed the bounds for the volume radii of positive, trace-preserving qudit maps and their nested subsets. Recently, the geometry of the sets of quantum channels has been analyzed for the Pauli channels [16] and Weyl channels [17] accessible via the Lindblad semigroup. There also exist first results for continuous variable systems. The geometry of quantum Gaussian channels has been considered with respect to the Hilbert-Schmidt [18] and the Bures-Fisher [19] metric.

In the following sections, we start with the definition of the generalized Pauli channels in dimension d that are constructed from $N \leqslant d + 1$ mutually unbiased bases. We derive the positivity and complete positivity conditions in terms of the channel eigenvalues. Next, we introduce the associated Choi-Jamiołkowski states to find the Hilbert-Schmidt line and volume elements of the channels. Having obtained the necessary positivity conditions, we give the upper bound on the volume of positive, trace-preserving generalized Pauli maps. It is very hard to establish analytical regions of integration for arbitrary N and d, and for this reason we consider three special examples: $N = d + 1$, $N = d$, and $N = 3$, two of which yield the same results. In each case, we present the integration regions in such a way that it is possible to analytically calculate the volume integrals. This way, we get the relative volumes as functions of dimension d for the generalized Pauli channels, as well as the subset of channels that are provided with legitimate generator and entanglement-breaking channels. Due to the generalized Pauli channels being higher dimensional, and therefore harder to analyze, than the Pauli channels, some of our results provide bounds on the volumes rather than exact

values. In Conclusions, we list the open problems that have arisen during our research.

II. GENERALIZED PAULI CHANNELS

Consider mixed unitary qubit evolution given by

$$\Lambda[\rho] = \sum_{\alpha=0}^{3} p_\alpha \sigma_\alpha \rho \sigma_\alpha, \qquad (1)$$

where p_α is a probability distribution and σ_α's denote the Pauli matrices. Observe that Λ is unital; i.e., it preserves the maximally mixed state $\rho_* = \mathbb{I}_2/2$. An important feature of the Pauli channel is that the eigenbases of the Pauli matrices are mutually unbiased. Recall that the two orthonormal bases $\{\psi_\alpha\}$ and $\{\varphi_\beta\}$ in $\mathcal{H} \simeq \mathbb{C}^d$ are mutually unbiased if and only if their vectors satisfy

$$|\langle \psi_\alpha | \varphi_\beta \rangle|^2 = \frac{1}{d}. \qquad (2)$$

One can always find between three and $d + 1$ mutually unbiased bases in a given dimension d [20]. The maximal number $N = d + 1$ of MUBs is reached for d being a prime number or a prime power [1,21]. If $d = d_1 d_2$, then there are at least $N = 1 + \min\{d_1, d_2\}$ mutually unbiased bases [22].

Now, let $P_k^{(\alpha)} := |\psi_k^{(\alpha)}\rangle \langle \psi_k^{(\alpha)}|$ be a rank-1 projector onto the kth vector $|\psi_k^{(\alpha)}\rangle$ of the αth mutually unbiased bases. Then, construct the quantum-classical channels $\Phi_\alpha[\rho] = \sum_{k=0}^{d-1} P_k^{(\alpha)} \rho P_k^{(\alpha)}$. The generalized Pauli map in any dimension d is constructed from $N \leqslant d + 1$ mutually unbiased bases in the following way [2],

$$\Lambda = p_{N+1} \mathbb{1} + \sum_{\alpha=0}^{N} p_\alpha \Phi_\alpha, \qquad (3)$$

where $\Phi_0[\rho] := \frac{1}{d} \mathbb{I}_d \mathrm{Tr}[\rho]$ is the completely depolarizing channel.

For $N < d + 1$, the generalized Pauli map Λ is a quantum channel if and only if

$$
\begin{aligned}
& p_{N+1} + \sum_{\alpha=1}^{N} p_\alpha + p_0 = 1, \\
& d^2 p_{N+1} + d \sum_{\alpha=1}^{N} p_\alpha + p_0 \geqslant 0, \\
& d p_\alpha + p_0 \geqslant 0, \quad \alpha = 1, \dots, N, \\
& p_0 \geqslant 0.
\end{aligned}
\qquad (4)
$$

Indeed, Φ_0 and Φ_α can be alternatively rewritten as

$$\Phi_\alpha[\rho] = \frac{1}{d} \sum_{k=0}^{d-1} U_\alpha^k \rho U_\alpha^k \qquad (5)$$

and

$$\Phi_0[\rho] = \frac{1}{d^2} \left(\rho + \sum_{\alpha=1}^{N} \sum_{k=1}^{d-1} U_\alpha^k \rho U_\alpha^{k\dagger} + \sum_{\beta=1}^{d+1-N} \sum_{k=1}^{d-1} A_{\beta,k} \rho A_{\beta,k}^\dagger \right), \qquad (6)$$

respectively. The set $\{\mathbb{I}_d, U_\alpha^k, A_{\beta,l}\}$ forms an orthogonal operator basis with respect to the Hilbert-Schmidt inner product, whereas $U_\alpha^k := \sum_{j=0}^{d-1} \omega^{jk} P_j^{(\alpha)}$ are the unitary operators constructed from $P_k^{(\alpha)}$ and $\omega := e^{2\pi i/d}$. Therefore, the complete positivity conditions are equivalent to the positivity of all the coefficients in the operator sum representation, which is exactly Eq. (4). The map Λ is mixed unitary if U_α^k are the Weyl operators $W_{kl} := \sum_{m=0}^{d-1} \omega^{km} |m + l\rangle \langle m|$ [23] or their tensor products, as then one can always find unitary operators $A_{\beta,k}$ that complete the orthonormal basis. In this case, the generalized Pauli channel is mixed unitary regardless of the dimension d. Using the Weyl operators, one can always construct at least $N = 3$ mutually unbiased bases [20]. If $N = d + 1$, then $\mathbb{1} = \sum_{\alpha=1}^{N} \Phi_\alpha - d\Phi_0$ is linearly dependent on Φ_α and Φ_0. Hence, p_{N+1} is set to zero as an excessive degree of freedom and the condition $p_0 \geqslant 0$ is dropped [2].

Note that any generalized Pauli channel is unital, as $\Lambda[\mathbb{I}_d] = \mathbb{I}_d$. Moreover,

$$\Phi_\alpha[U_\beta^k] = \delta_{\alpha\beta} U_\beta^k, \quad \Phi_\alpha[A_{\beta,l}] = 0, \qquad (7)$$

and hence the remaining eigenvalue equations read

$$\Lambda[U_\alpha^k] = (p_\alpha + p_{N+1}) U_\alpha^k, \quad \Lambda[A_{\beta,l}] = p_{N+1} A_{\beta,l}. \qquad (8)$$

The complete positivity conditions in terms of the eigenvalues

$$\lambda_\alpha = p_{N+1} + p_\alpha, \quad \lambda_{N+1} = p_{N+1} \qquad (9)$$

are as follows:

$$-\frac{1}{d-1} \leqslant \sum_{\beta=1}^{N} \lambda_\beta + (d + 1 - N) \lambda_{N+1}$$

$$\leqslant 1 + d \min_{\alpha=1,\dots,N+1} \lambda_\alpha. \qquad (10)$$

Note that for $N = d + 1$, one has $\lambda_{N+1} = 0$, which recovers the generalized Fujiwara-Algoet conditions [2,24]. Moreover, conditions for $N = d$ and $N = d + 1$ coincide.

III. LINE AND VOLUME ELEMENTS

Let us analyze the geometry of the generalized Pauli channels Λ by considering the properties of the associated Choi-Jamiołkowski quantum states [10,11]

$$\rho_\Lambda = \frac{1}{d} \sum_{k,l=0}^{d-1} |k\rangle \langle l| \otimes \Lambda[|k\rangle \langle l|]. \qquad (11)$$

In general, it is very hard to explicitly write the state ρ_Λ corresponding to a given Λ, as the choice of the mutually unbiased bases is arbitrary. However, we can still introduce the metric via the Hilbert-Schmidt line element $ds^2 = \mathrm{Tr}(d\rho_\Lambda^2)$ [25]. In terms of the channel eigenvalues, it reads

$$ds^2 = \frac{d-1}{d^2} \left(\sum_{\alpha=1}^{N} d\lambda_\alpha^2 + (d + 1 - N) d\lambda_{N+1}^2 \right). \qquad (12)$$

Now, the metric tensor has the diagonal form $g = \frac{d-1}{d^2} \mathrm{diag}(1, \dots, 1, d + 1 - N)$ with N ones. Hence, the associated volume element $dV = \sqrt{\det g} \prod_{\alpha=1}^{N+1} d\lambda_\alpha$ is given

by

$$dV = \sqrt{d+1-N}\left(\frac{\sqrt{d-1}}{d}\right)^{N+1}\prod_{\alpha=1}^{N+1} d\lambda_\alpha. \quad (13)$$

Notably, for $N = d + 1$, $d\lambda_{N+1}^2$ vanishes, and therefore one has

$$ds^2 = \frac{d-1}{d^2}\sum_{\alpha=1}^{d+1} d\lambda_\alpha^2,$$

$$dV = \left(\frac{\sqrt{d-1}}{d}\right)^{d+1}\prod_{\alpha=1}^{d+1} d\lambda_\alpha. \quad (14)$$

For $d = 2$, the above formulas reproduce the line and volume elements for the Pauli channels that have been recently obtained in Ref. [9].

IV. RELATIVE VOLUMES OF QUANTUM CHANNELS

In this section, we analytically derive the volumes of the generalized Pauli channels and their important subclasses. However, it is a nontrivial task to find the regions of integration for arbitrary N and d. Therefore, for the most part, we limit our consideration to two extreme cases: $N = d + 1$ and $N = 3$.

A. Generalized Pauli positive maps

Before we calculate the volumes of generalized Pauli channels, let us find the normalization factor that makes our results more clear to read. Consider a trace-preserving generalized Pauli map Λ that, in general, is not completely positive. From the definition, Λ is positive if and only if

$$\forall_{P,Q} \quad \text{Tr}(\Lambda[Q]P) \geqslant 0, \quad (15)$$

where P and Q are rank-1 projectors. Unfortunately, this inequality is very hard to check. Therefore, we analyze the necessary conditions

$$\min_{P,Q\in\{P_k^{(\alpha)}, Q_l^{(\beta)}\}} \text{Tr}(\Lambda[Q]P) \geqslant 0, \quad (16)$$

with the minimum taken over

$$P_k^{(\alpha)} = \frac{1}{d}\left[\mathbb{I}_d + \sum_{m=1}^{d-1}\omega^{-mk}U_\alpha^m\right] \quad (17)$$

and

$$Q_l^{(\beta)} = \frac{1}{d}\left[\mathbb{I}_d + \sum_{n=1}^{d-1}q_n^{(\beta,l)}A_{\beta,n}\right]. \quad (18)$$

Using the method presented in Ref. [26], it is straightforward to show that

$$\min_{P,Q\in\{P_k^{(\alpha)}, Q_l^{(\beta)}\}} \text{Tr}(\Lambda[Q]P)$$

$$= \frac{1}{d}[1 + \min\{-\lambda_{\max}, (d-1)\lambda_{\min}\}], \quad (19)$$

where $\lambda_{\max} = \max_\alpha \lambda_\alpha$ and $\lambda_{\min} = \min_\alpha \lambda_\alpha$ for $\alpha = 1, \ldots, N$. Note that for $N = d + 1$, the above formula is the maximal

output ∞-norm [26]. Now, the necessary condition for positivity of trace-preserving generalized Pauli maps is

$$-\frac{1}{d-1} \leqslant \lambda_\alpha \leqslant 1. \quad (20)$$

Unfortunately, numerical calculations indicate that such conditions may lead to maps that are not positive. Hence, condition (16) is not sufficient. However, we can at least calculate the upper estimate on the volume of positive, trace-preserving generalized Pauli maps. Explicitly, it reads

$$V_P(d,N) = \sqrt{d+1-N}\left(\frac{\sqrt{d-1}}{d}\right)^{N+1}\prod_{\alpha=1}^{N+1}\int_{-\frac{1}{d-1}}^{1} d\lambda_\alpha$$

$$= \sqrt{\frac{d+1-N}{(d-1)^{N+1}}}, \quad (21)$$

for $N = 1, \ldots, d$, and

$$V_P(d) = \left(\frac{\sqrt{d-1}}{d}\right)^{d+1}\prod_{\alpha=1}^{d+1}\int_{-\frac{1}{d-1}}^{1} d\lambda_\alpha$$

$$= \frac{1}{\sqrt{(d-1)^{d+1}}}, \quad (22)$$

for $N = d + 1$. Note that for $N = d$, one has $V_P(d,d) = V_P(d)$. In the upcoming subsection, $V_P(d)$ is used for normalization purposes.

B. $N = d + 1$ and $N = d$ mutually unbiased bases

From now on, let us assume, without a loss of generality, that $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{d+1}$. Note that there are $(d+1)!$ possible index permutations for a given dimension d, so the total volume for arbitrarily ordered λ_α is $(d+1)!$ times the volume for nondecreasingly ordered eigenvalues (for details, see Appendix B). This simple trick significantly simplifies the regions of integration. The results of this subsection are calculated for $N = d + 1$, but they also apply to the case with $N = d$.

The volume $V_{\text{CP}}(d)$ of the generalized Pauli channels is calculated by integrating the volume element dV over the complete positivity region (see Appendix A). Due to the nature of lower and upper bounds of λ_α, we are unable to derive the final formula for $V_{\text{CP}}(d)$ directly. Instead, we find the analytical volume ratios $V_{\text{CP}}(d)/V_P(d)$ for small values of d (for more details, see Appendix B),

$$\frac{V_{\text{CP}}(2)}{V_P(2)} = \frac{1}{3}, \quad \frac{V_{\text{CP}}(4)}{V_P(4)} = \frac{1}{30},$$

$$\frac{V_{\text{CP}}(3)}{V_P(3)} = \frac{1}{8}, \quad \frac{V_{\text{CP}}(5)}{V_P(5)} = \frac{1}{144},$$

and conjecture that the general formula reads

$$\frac{V_{\text{CP}}(d)}{V_P(d)} = \frac{d}{(d+1)!}. \quad (23)$$

Quantum channels are used to provide the evolution of open quantum systems. Instead of a single channel Λ, one takes a time-parametrized family $\{\Lambda(t)|t \geqslant 0, \Lambda(0) = \mathbb{1}\}$ that is called the *dynamical map*. The simplest example of a

dynamical map is the Markovian semigroup $\Lambda(t) = e^{t\mathcal{L}}$. It is the solution of the master equation

$$\dot{\Lambda}(t) = \mathcal{L}\Lambda(t), \tag{24}$$

where $\mathcal{L}$ is the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) generator [27,28]

$$\mathcal{L} = \sum_{\alpha=0}^{N} \gamma_\alpha (\Phi_\alpha - \mathbb{1}), \tag{25}$$

with positive decoherence rates $\gamma_\alpha \geqslant 0$. If $N = d + 1$, then $\gamma_0 = 0$ [24]. One way of including non-Markovian effects is to introduce the time-local generator $\mathcal{L}(t)$, which is given via formula (25) but with time-dependent $\gamma_\alpha(t) \ngeqslant 0$. Observe that the generalized Pauli channel Λ belongs to a family of generalized Pauli dynamical maps $\Lambda(t)$ generated by $\mathcal{L}(t)$ if and only if $\Lambda = \Lambda(t_*)$ for a fixed $t_* \geqslant 0$. Equivalently, one has $\lambda_\alpha \geqslant 0$, which provides the integration region

$$\lambda_{k-1} \leqslant \lambda_k \leqslant \frac{1}{d+2-k}\left(1 + d\lambda_1 - \sum_{\beta=1}^{k-1} \lambda_\beta\right),$$
$$k = 2, \ldots, d+1, \tag{26}$$
$$0 \leqslant \lambda_1 \leqslant 1,$$

when paired with the complete positivity condition from Appendix A. Let us call the channels with $\lambda_\alpha \geqslant 0$ *achievable with the time-local generators* and denote their volume by $V_{\mathrm{G}}(d)$. Again, we calculate the analytical volumes $V_{\mathrm{G}}(d)$ for small values of d,

$$\frac{V_{\mathrm{G}}(2)}{V_{\mathrm{CP}}(2)} = \frac{3}{16}, \quad \frac{V_{\mathrm{G}}(4)}{V_{\mathrm{CP}}(4)} = \frac{1215}{4096},$$
$$\frac{V_{\mathrm{G}}(3)}{V_{\mathrm{CP}}(3)} = \frac{64}{243}, \quad \frac{V_{\mathrm{G}}(5)}{V_{\mathrm{CP}}(5)} = \frac{24576}{78125},$$

and conjecture that the general formula reads

$$\frac{V_{\mathrm{G}}(d)}{V_{\mathrm{CP}}(d)} = \frac{d^2-1}{d^2}\left(\frac{d-1}{d}\right)^d. \tag{27}$$

Surprisingly, this is the only relative volume we consider that does not go to zero with the increase of d. Indeed, if we take the limit $d \to \infty$, then Eq. (27) monotonically approaches e from below.

An interesting subclass of the generalized Pauli channels, with many applications in quantum communication and information processing, consists in the channels that are entanglement breaking (EB). A quantum channel Λ is entanglement breaking if and only if its Choi-Jamiołkowski state ρ_Λ is separable. If $\lambda_\alpha \geqslant 0$, then $\sum_{\alpha=1}^{d+1} \lambda_\alpha \leqslant 1$ is the necessary and sufficient condition for breaking the entanglement [2]. Therefore, the generalized Pauli channel achievable with the time-local generators that are also EB corresponds to the
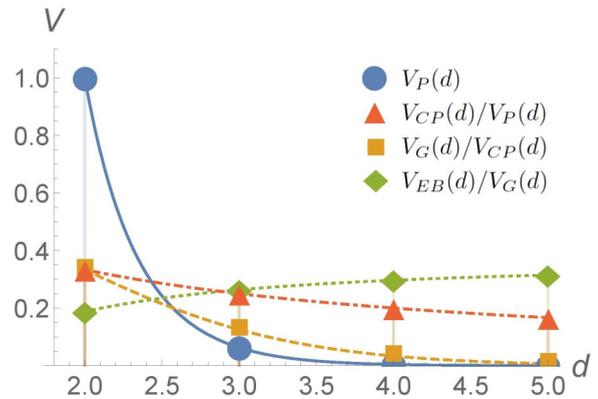

FIG. 1. A graphical representation of the volumes for selected classes of the generalized Pauli trace-preserving maps constructed from d or $d+1$ mutually unbiased bases.

region

$$\lambda_{k-1} \leqslant \lambda_k \leqslant \frac{1}{d+2-k}\left(1 - \sum_{\beta=1}^{k-1} \lambda_\beta\right),$$
$$k = 2, \ldots, d+1, \tag{28}$$
$$0 \leqslant \lambda_1 \leqslant \frac{1}{d+1}.$$

For small dimensions d, we find relatively simple analytical formulas for the volumes $V_{\mathrm{EB}}(d)$ of the entanglement-breaking channels achievable with the time-local generators,

$$\frac{V_{\mathrm{EB}}(2)}{V_{\mathrm{G}}(2)} = \frac{1}{3}, \quad \frac{V_{\mathrm{EB}}(4)}{V_{\mathrm{G}}(4)} = \frac{1}{5},$$
$$\frac{V_{\mathrm{EB}}(3)}{V_{\mathrm{G}}(3)} = \frac{1}{4}, \quad \frac{V_{\mathrm{EB}}(5)}{V_{\mathrm{G}}(5)} = \frac{1}{6},$$

from which we conjecture that

$$\frac{V_{\mathrm{EB}}(d)}{V_{\mathrm{G}}(d)} = \frac{1}{d+1}. \tag{29}$$

The results of this subsection are presented in Fig. 1. The curves have no physical meaning and are plotted only to show the behavior of the discretely valued relative volumes. Note that the only function that does not monotonically decrease with the increase of the dimension d is $V_{\mathrm{G}}(d)/V_{\mathrm{CP}}(d)$, which approaches $1/e$ for $d \to \infty$. This means that the bigger the Hilbert space, the more there are generalized Pauli channels with positive eigenvalues. For $d = 2$, the results coincide with Ref. [9]. Interestingly, for $d > 2$, the volume of positive, trace-preserving maps is no longer equal to unity.

C. $N = 3$ mutually unbiased bases

Now, let us analyze the volumes of the generalized Pauli channels constructed from three mutually unbiased bases. This is another interesting choice, as one can construct three MUBs in any dimension. Once again, in order to simplify the integrals, we assume that $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$. There are $3! = 6$ possible index permutations for $\lambda_\alpha = 1, 2, 3$. However, due to $N < d + 1$, we have one more eigenvalue, λ_4, and there
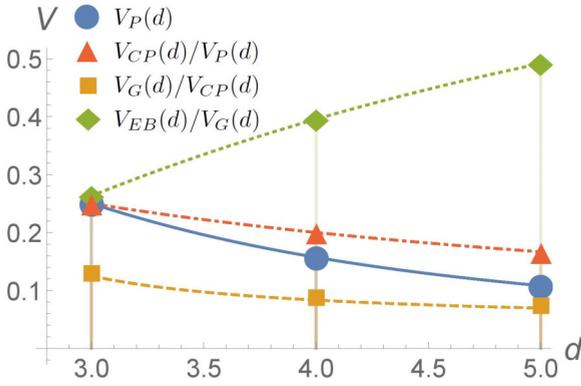
FIG. 2. A graphical representation of the volumes for selected classes of the generalized Pauli trace-preserving maps constructed from three mutually unbiased bases.

are four possible ways that λ_4 can be placed in the above sequence of inequalities. For each possible placement of λ_4, we derive the integration regions separately (see Appendix C). Therefore, the final result is the sum of four integrals multiplied by 6.

As the number of eigenvalues is set, we are able to directly derive the formulas for the relative volumes of the channels considered in the previous subsection. These are

$$
\begin{aligned}
V_{\mathrm{P}}(d) &= \frac{\sqrt{d-2}}{(d-1)^2}, \\
\frac{V_{\mathrm{CP}}(d)}{V_{\mathrm{P}}(d)} &= \frac{d}{24(d-2)}, \\
\frac{V_{\mathrm{G}}(d)}{V_{\mathrm{CP}}(d)} &= \frac{(d^2-1)(d-1)^3}{d^5}, \\
\frac{V_{\mathrm{EB}}(d)}{V_{\mathrm{G}}(d)} &= \frac{1}{d+1},
\end{aligned}
\tag{30}
$$

where we used the notation $V(d,3) \equiv V(d)$ for the sake of simplicity. The only difference is that $V_{\mathrm{EB}}(d)$ is the volume of channels achievable with the time-local generator that satisfy the condition [2]

$$
\sum_{\alpha=1}^{N} \lambda_\alpha + (d+1-N)\lambda_{N+1} \leqslant 1
\tag{31}
$$

for breaking the entanglement, which is necessary but may not be sufficient. Hence, for $d=3$, $V_{\mathrm{EB}}(d)$ gives just the upper bound for the volume of entanglement-breaking channels with positive eigenvalues. Despite this fact, $V_{\mathrm{EB}}(d)/V_{\mathrm{G}}(d)$ for $N = 3$ is given by the same formula as in the case of $N = d, d+1$, which could indicate that condition (31) is sufficient after all.

In Fig. 2, we plot the relative volumes for small dimensions d. One can notice several similarities with Fig. 1. Again, all the functions decrease monotonically with the increase of the dimension d, except for $V_{\mathrm{G}}(d)/V_{\mathrm{CP}}(d)$, which this time approaches identity for $d \to \infty$. For $d = 3$, the results agree with the calculations for the maximal number of MUBs from the previous subsection.

V. CONCLUSIONS

In this paper, we establish the geometry of the generalized Pauli channels constructed from N mutually unbiased bases, where N ranges up to $d+1$. Using the Choi-Jamiołkowski isomorphism between quantum channels and quantum states, we find the Hilbert-Schmidt line and volume elements in the state space. In more detail, we consider two special examples: $N = d+1$ and $N = 3$. In both cases, we express the integration regions for the channels, channels achievable with the time-local generators and their entanglement-breaking subclass in a way that allows us to calculate the volume integrals analytically. Finally, we obtain the relative channel volumes as functions of the dimension d. Some interesting behavior can be observed, like the percentage of channels with positive eigenvalues approaching a constant, nonzero value with the increase of d. Our results can be interpreted as the probability that a randomly selected generalized Pauli channel has the desired properties. Moreover, they show how little channels are accessible with time-local generators and therefore how important it is to develop the memory kernel approach.

One of the open problems is the characterization of integration regions for the generalized Pauli channels constructed from any number of mutually unbiased bases. It would be interesting to obtain the channel volumes with both d and N dependence, even for $\lambda_{N+1} = 0$. Tighter conditions for the positive maps and entanglement-breaking channels would make it possible to find the exact volumes instead of their upper bounds. Also, the problem that requires further study is establishing the system of conditions under which a given channel is a semigroup [29,30]. Another task is to compare our results with the volumes calculated using other metrics, like the Fisher-Rao [31], Bures [32,33], or Hastings [34,35] measure.

Recall that the Hilbert-Schmidt volume is a volume on the set of bipartite quantum states. Therefore, one could compare the volume of all Choi-Jamiołkowski states to the volume of all quantum states. Sommers and Życzkowski [25] obtained the general formula for the volume of states with respect to the Hilbert-Schmidt metric:

$$
V(n) = \sqrt{n}(2\pi)^{n(n-1)/2} \frac{\prod_{k=2}^{n-1} k!}{(n^2-1)!}.
\tag{32}
$$

For $n = d^2$, one recovers the volume of bipartite $d \times d$ states. It is easy to see that this formula does not resemble any of the channel volumes calculated in this paper. One could also analyze the relation between the volumes of all separable states and all separable Choi-Jamiołkowski states, which correspond to entanglement-breaking channels. Due to the complexity of calculations, the most well-studied case is a two-qubit state, for which the Hilbert-Schmidt [36,37], Fisher [38], and Dirichlet [39] volumes are known. These volumes do not coincide, which could indicate that relative channel volumes might also depend on the metric. Interestingly, the Hilbert-Schmidt relative volume of separable two-qubit states (8/33) is almost halfway through in value between the volume of EB Pauli channels (1/6) [9] and the volume of EB Pauli channels achievable with the time-local generators (1/3).

APPENDIX A: REGIONS OF INTEGRATION FOR $N = d + 1$

The complete positivity region for trace-preserving generalized Pauli maps is obtained from the generalized Fujiwara-Algoet condition

$$-\frac{1}{d-1} \leqslant \sum_{\alpha=1}^{d+1} \lambda_\alpha \leqslant 1 + d \min_\alpha \lambda_\alpha. \tag{A1}$$

For nondecreasingly ordered channel eigenvalues, this region consists in λ_α that satisfy one of the following systems of inequalities:

$$\begin{cases} -\frac{1}{d-1} - \sum_{\beta=1}^{d} \lambda_\beta \leqslant \lambda_{d+1} \leqslant 1 + d\lambda_1 - \sum_{\beta=1}^{d} \lambda_\beta, \\ \lambda_{k-1} \leqslant \lambda_k \leqslant -\frac{1}{d+2-k}\left(\frac{1}{d-1} + \sum_{\beta=1}^{k-1} \lambda_\beta\right) \quad \forall_{2 \leqslant k \leqslant d}, \\ -\frac{1}{d-1} \leqslant \lambda_1 \leqslant -\frac{1}{d^2-1}, \end{cases} \tag{A2}$$

or

$$\begin{cases} \lambda_d \leqslant \lambda_{d+1} \leqslant 1 + d\lambda_1 - \sum_{\beta=1}^{d} \lambda_\beta, \\ \lambda_{k-1} \leqslant \lambda_k \leqslant \frac{1}{d+2-k}\left(1 + d\lambda_1 - \sum_{\beta=1}^{k-1} \lambda_\beta\right) \quad \forall_{k>M}, \\ -\frac{1}{d+2-M}\left(\frac{1}{d-1} + \sum_{\beta=1}^{M-1} \lambda_\beta\right) \leqslant \lambda_M \leqslant \frac{1}{d+2-M}\left(1 + d\lambda_1 - \sum_{\beta=1}^{M-1} \lambda_\beta\right), \\ \lambda_{k-1} \leqslant \lambda_k \leqslant -\frac{1}{d+2-k}\left(\frac{1}{d-1} + \sum_{\beta=1}^{k-1} \lambda_\beta\right) \quad \forall_{k<M}, \\ -\frac{1}{d-1} \leqslant \lambda_1 \leqslant -\frac{1}{d^2-1}, \end{cases} \tag{A3}$$

for a fixed M such that $2 \leqslant M \leqslant d$, or

$$\begin{cases} \lambda_{k-1} \leqslant \lambda_k \leqslant \frac{1}{d+2-k}\left(1 + d\lambda_1 - \sum_{\beta=1}^{k-1} \lambda_\beta\right) \quad \forall_{2 \leqslant k \leqslant d+1}, \\ -\frac{1}{d^2-1} \leqslant \lambda_1 \leqslant 1. \end{cases} \tag{A4}$$

APPENDIX B: ANALYTICAL INTEGRATION OF THE VOLUME

Let us observe how the analytical volume integrals are calculated by analyzing the example of the Pauli channels, which correspond to the choices $d = 2$ and $N = d + 1$. Their volume is obtained by integrating the volume element dV from Eq. (14) over the complete positivity region $\mathcal{CP}(2)$,

$$V_{\mathrm{CP}}(2) = \int_{\mathcal{CP}(2)} dV = \frac{1}{8} \int_{\mathcal{CP}(2)} d\lambda_1 d\lambda_2 d\lambda_3. \tag{B1}$$

Now, assume that the eigenvalues are nondecreasingly ordered, so that $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$. According to Appendix A, the region of integration $\mathcal{CP}'(2)$ for ordered λ_α's is provided by the following systems of inequalities:

$$\begin{cases} -1 - \lambda_1 - \lambda_2 \leqslant \lambda_3 \leqslant 1 + \lambda_1 - \lambda_2, \\ \lambda_1 \leqslant \lambda_2 \leqslant -\frac{1}{2}(1 + \lambda_1), \\ -1 \leqslant \lambda_1 \leqslant -\frac{1}{3}, \end{cases} \quad \begin{cases} \lambda_2 \leqslant \lambda_3 \leqslant 1 + \lambda_1 - \lambda_2, \\ -\frac{1}{2}(1 + \lambda_1) \leqslant \lambda_2 \leqslant \frac{1}{2}(1 + \lambda_1), \\ -1 \leqslant \lambda_1 \leqslant -\frac{1}{3}, \end{cases} \quad \begin{cases} \lambda_2 \leqslant \lambda_3 \leqslant 1 + \lambda_1 - \lambda_2, \\ \lambda_1 \leqslant \lambda_2 \leqslant \frac{1}{2}(1 + \lambda_1), \\ -\frac{1}{3} \leqslant \lambda_1 \leqslant 1. \end{cases} \tag{B2}$$

Therefore, the corresponding volume reads as follows:

$$\begin{aligned} V'_{\mathrm{CP}}(2) = \int_{\mathcal{CP}'(2)} dV = \frac{1}{8}\bigg(&\int_{-1}^{-\frac{1}{3}} d\lambda_1 \int_{\lambda_1}^{-\frac{1}{2}(1+\lambda_1)} d\lambda_2 \int_{-1-\lambda_1-\lambda_2}^{1+\lambda_1-\lambda_2} d\lambda_3 + \int_{-1}^{-\frac{1}{3}} d\lambda_1 \int_{-\frac{1}{2}(1+\lambda_1)}^{\frac{1}{2}(1+\lambda_1)} d\lambda_2 \int_{\lambda_2}^{1+\lambda_1-\lambda_2} d\lambda_3 \\ &+ \int_{-\frac{1}{3}}^{1} d\lambda_1 \int_{\lambda_1}^{\frac{1}{2}(1+\lambda_1)} d\lambda_2 \int_{\lambda_2}^{1+\lambda_1-\lambda_2} d\lambda_3 \bigg) = \frac{1}{8}\frac{4}{9} = \frac{1}{18}, \end{aligned} \tag{B3}$$

where the more detailed calculations are omitted due to the integrals being elementary. To obtain the total volume of the Pauli channels $V_{\mathrm{CP}}(2)$, we make another observation. Both the volume element dV and the Fujiwara-Algoet conditions [40]

$$-1 \leqslant \lambda_1 + \lambda_2 + \lambda_3 \leqslant 1 + 2 \min_\alpha \lambda_\alpha \tag{B4}$$

are symmetric with respect to index permutations. Therefore, by permuting the indices in the ordering $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$, we can cover all the possible orders that constitute to the general case of nonordered λ_α's. There are a total of six possible ways to order the channel eigenvalues, and hence we finally get

$$V_{\mathrm{CP}}(2) = 6V'_{\mathrm{CP}}(2) = \frac{1}{3}. \tag{B5}$$

Analogical calculations are performed to derive all the other volumes presented in this paper. For higher dimensions d, the volume $V'_{\mathrm{CP}}(d) = \int_{\mathcal{CP}'(d)} dV$ for nondecreasingly ordered eigenvalues is a sum of $d+1$ polynomial integrals. The region of integration $\mathcal{CP}'(d)$ is again given in Appendix A. Now, there are $(d+1)!$ possible index permutations for ordered λ_α, so the total volume $V_{\mathrm{CP}}(d) = (d+1)! V'_{\mathrm{CP}}(d)$. The difficulty in obtaining the general formula for $V_{\mathrm{CP}}(d)$ lies not in the complexity of integrals but in the fact that this formula is a sum of $d+1$-tuple integrals whose limits depend on d. However, for fixed d, the derivations, even if lengthy, are straightforward.

APPENDIX C: REGIONS OF INTEGRATION FOR $N = 3$

For trace-preserving generalized Pauli maps constructed from three mutually unbiased bases, the complete positivity region is determined by the generalized Fujiwara-Algoet condition

$$-\frac{1}{d-1} \leqslant \lambda_1 + \lambda_2 + \lambda_3 + (d-2)\lambda_4 \leqslant 1 + d \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}. \tag{C1}$$

We assume that $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ and

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{\lambda_{\min}, \lambda_{\mathrm{mid}1}, \lambda_{\mathrm{mid}2}, \lambda_{\max}\}, \tag{C2}$$

where $\lambda_{\min} \leqslant \lambda_{\mathrm{mid}1} \leqslant \lambda_{\mathrm{mid}2} \leqslant \lambda_{\max}$. Now, inequality (C1) holds if and only if the eigenvalues λ_α satisfy one of the following systems of inequalities:

$$\begin{cases} \lambda_{\mathrm{mid}2} \leqslant \lambda_{\max} \leqslant \dfrac{1}{1+(d-3)\delta_{\max,4}}[1+(d-1)\lambda_{\min} - \lambda_{\mathrm{mid}1} - \lambda_{\mathrm{mid}2} - (d-3)(\delta_{\mathrm{mid}1,4} + \delta_{\mathrm{mid}2,4})\lambda_4], \\[2mm] \lambda_{\mathrm{mid}1} \leqslant \lambda_{\mathrm{mid}2} \leqslant \dfrac{1+(d-1)\lambda_{\min} - \lambda_{\mathrm{mid}1} - (d-3)(\delta_{\mathrm{mid}1,4} + \delta_{\min,4})\lambda_4}{d-1-(d-3)(\delta_{\mathrm{mid}1,4} + \delta_{\min,4})}, \\[2mm] \lambda_{\min} \leqslant \lambda_{\mathrm{mid}1} \leqslant \dfrac{1+(d-1)\lambda_{\min} - (d-3)\delta_{\min,4}\lambda_4}{d-(d-3)\delta_{\min,4}}, \\[2mm] -\dfrac{1}{d^2-1} \leqslant \lambda_{\min} \leqslant 1 \end{cases} \tag{C3}$$

or

$$\begin{cases} -\dfrac{1}{1+(d-3)\delta_{\max,4}}\left[\dfrac{1}{d-1} + \lambda_{\min} + \lambda_{\mathrm{mid}1} + \lambda_{\mathrm{mid}2} + (d-3)(1-\delta_{\max,4})\lambda_4\right] \leqslant \lambda_{\max} \\[2mm] \qquad \leqslant \dfrac{1}{1+(d-3)\delta_{\max,4}}[1+(d-1)\lambda_{\min} - \lambda_{\mathrm{mid}1} - \lambda_{\mathrm{mid}2} - (d-3)(\delta_{\mathrm{mid}1,4} + \delta_{\mathrm{mid}2,4})\lambda_4], \\[2mm] \lambda_{\mathrm{mid}1} \leqslant \lambda_{\mathrm{mid}2} \leqslant -\dfrac{\frac{1}{d-1} + \lambda_{\min} + \lambda_{\mathrm{mid}1} + (d-3)(\delta_{\mathrm{mid}1,4} + \delta_{\min,4})\lambda_4}{d-1-(d-3)(\delta_{\mathrm{mid}1,4} + \delta_{\min,4})}, \\[2mm] \lambda_{\min} \leqslant \lambda_{\mathrm{mid}1} \leqslant -\dfrac{\frac{1}{d-1} + \lambda_{\min} + (d-3)\delta_{\min,4}\lambda_4}{d-(d-3)\delta_{\min,4}}, \\[2mm] -\dfrac{1}{d-1} \leqslant \lambda_{\min} \leqslant -\dfrac{1}{d^2-1}, \end{cases} \tag{C4}$$

or

$$\begin{cases} \lambda_{\mathrm{mid}2} \leqslant \lambda_{\max} \leqslant \dfrac{1}{1+(d-3)\delta_{\max,4}}[1+(d-1)\lambda_{\min} - \lambda_{\mathrm{mid}1} - \lambda_{\mathrm{mid}2} - (d-3)(\delta_{\mathrm{mid}1,4} + \delta_{\mathrm{mid}2,4})\lambda_4], \\[2mm] -\dfrac{\frac{1}{d-1} + \lambda_{\min} + \lambda_{\mathrm{mid}1} + (d-3)(\delta_{\mathrm{mid}1,4} + \delta_{\min,4})\lambda_4}{d-1-(d-3)(\delta_{\mathrm{mid}1,4} + \delta_{\min,4})} \leqslant \lambda_{\mathrm{mid}2} \leqslant \dfrac{1+(d-1)\lambda_{\min} - \lambda_{\mathrm{mid}1} - (d-3)(\delta_{\mathrm{mid}1,4} + \delta_{\min,4})\lambda_4}{d-1-(d-3)(\delta_{\mathrm{mid}1,4} + \delta_{\min,4})}, \\[2mm] \lambda_{\min} \leqslant \lambda_{\mathrm{mid}1} \leqslant -\dfrac{\frac{1}{d-1} + \lambda_{\min} + (d-3)\delta_{\min,4}\lambda_4}{d-(d-3)\delta_{\min,4}}, \\[2mm] -\dfrac{1}{d-1} \leqslant \lambda_{\min} \leqslant -\dfrac{1}{d^2-1}, \end{cases} \tag{C5}$$

or

$$
\begin{cases}
\lambda_{\mathrm{mid2}} \leqslant \lambda_{\max} \leqslant \dfrac{1}{1+(d-3)\delta_{\max,4}}[1+(d-1)\lambda_{\min}-\lambda_{\mathrm{mid1}}-\lambda_{\mathrm{mid2}}-(d-3)(\delta_{\mathrm{mid1},4}+\delta_{\mathrm{mid2},4})\lambda_4], \\[3mm]
\lambda_{\mathrm{mid1}} \leqslant \lambda_{\mathrm{mid2}} \leqslant \dfrac{1+(d-1)\lambda_{\min}-\lambda_{\mathrm{mid1}}-(d-3)(\delta_{\mathrm{mid1},4}+\delta_{\min,4})\lambda_4}{d-1-(d-3)(\delta_{\mathrm{mid1},4}+\delta_{\min,4})}, \\[3mm]
-\dfrac{\frac{1}{d-1}+\lambda_{\min}+(d-3)\delta_{\min,4}\lambda_4}{d-(d-3)\delta_{\min,4}} \leqslant \lambda_{\mathrm{mid1}} \leqslant \dfrac{1+(d-1)\lambda_{\min}-(d-3)\delta_{\min,4}\lambda_4}{d-(d-3)\delta_{\min,4}}, \\[3mm]
-\dfrac{1}{d-1} \leqslant \lambda_{\min} \leqslant -\dfrac{1}{d^2-1}.
\end{cases}
\tag{C6}
$$

By $\delta_{\alpha,k}$, we understand the Kronecker delta, where $\alpha = 1, 2, 3, 4$ and $k = \min, \mathrm{mid1}, \mathrm{mid2}, \max$. For example, $\delta_{\min,1}$ does not vanish if and only if $\lambda_1 = \lambda_{\min}$. Now, if one has $\lambda_\alpha \geqslant 0$, then the region of integration for the channels achievable with the time-local generators is determined by

$$
\begin{cases}
\lambda_{\mathrm{mid2}} \leqslant \lambda_{\max} \leqslant \dfrac{1}{1+(d-3)\delta_{\max,4}}[1+(d-1)\lambda_{\min}-\lambda_{\mathrm{mid1}}-\lambda_{\mathrm{mid2}}-(d-3)(\delta_{\mathrm{mid1},4}+\delta_{\mathrm{mid2},4})\lambda_4], \\[3mm]
\lambda_{\mathrm{mid1}} \leqslant \lambda_{\mathrm{mid2}} \leqslant \dfrac{1+(d-1)\lambda_{\min}-\lambda_{\mathrm{mid1}}-(d-3)(\delta_{\mathrm{mid1},4}+\delta_{\min,4})\lambda_4}{d-1-(d-3)(\delta_{\mathrm{mid1},4}+\delta_{\min,4})}, \\[3mm]
\lambda_{\min} \leqslant \lambda_{\mathrm{mid1}} \leqslant \dfrac{1+(d-1)\lambda_{\min}-(d-3)\delta_{\min,4}\lambda_4}{d-(d-3)\delta_{\min,4}}, \\[3mm]
0 \leqslant \lambda_{\min} \leqslant 1.
\end{cases}
\tag{C7}
$$

Finally, if the nondecreasingly ordered eigenvalues of the generalized Pauli channel are non-negative and satisfy condition (31), then

$$
\begin{cases}
\lambda_{\mathrm{mid2}} \leqslant \lambda_{\max} \leqslant \dfrac{1}{1-(d-3)\delta_{\max,4}\lambda_4}[1-\lambda_{\min}-\lambda_{\mathrm{mid1}}-\lambda_{\mathrm{mid2}}-(d-3)(1-\delta_{\max,4})], \\[3mm]
\lambda_{\mathrm{mid1}} \leqslant \lambda_{\mathrm{mid2}} \leqslant \dfrac{1-\lambda_{\min}-\lambda_{\mathrm{mid1}}-(d-3)(\delta_{\min,4}+\delta_{\mathrm{mid1},4})\lambda_4}{d-1-(d-3)(\delta_{\min,4}+\delta_{\mathrm{mid1},4})}, \\[3mm]
\lambda_{\min} \leqslant \lambda_{\mathrm{mid1}} \leqslant \dfrac{1-\lambda_{\min}-(d-3)\delta_{\min,4}\lambda_4}{d-(d-3)\delta_{\min,4}}, \\[3mm]
0 \leqslant \lambda_{\min} \leqslant \dfrac{1}{d+1}.
\end{cases}
\tag{C8}
$$

Note that there are some similarities between these conditions and the conditions in Appendix A— for example, if $d = 3$, the integration regions for $N = 3$ and $N = d, d+1$ coincide.

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