Quantifying coherence in terms of the pure-state coherence

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(Received 9 January 2020; revised manuscript received 26 April 2020; accepted 28 May 2020; published 26 June 2020)

Quantifying quantum coherence is a key task in the resource theory of coherence. Here we establish a good coherence monotone in terms of a state conversion process, which automatically endows the coherence monotone with an operational meaning. We show that any state can be produced from some input pure states via the corresponding incoherent channels. It is found that the coherence of a given state can be well characterized by the least coherence of the input pure states, so a coherence monotone is established by only effectively quantifying the input pure states. In particular, we show that our proposed coherence monotone is the supremum of all the coherence monotones that give the same coherence for any given pure state. We also prove that our coherence monotone is continuous. Considering the convexity, we prove that our proposed coherence measure is a subset of the coherence measure based on the convex roof construction. The similarities and differences between our coherence monotone and coherence cost are studied in detail. As applications, we give a concrete expression of our coherence measure by employing the geometric coherence of a pure state. We also give a thorough analysis of the states of the qubit and finally obtain a series of analytic coherence measures. The numerical examples are also given to show the difference between our coherence monotone and that based on the convex roof construction.

DOI: 10.1103/PhysRevA.101.062114

I. INTRODUCTION

Coherence, as the most fundamental nature of quantum mechanics, is necessary for almost all the other quantum features, such as entanglement [1–7], quantum correlation [8–11], nonlocality [12–16], asymmetry [17–19], and so on. It also plays an important role in many fields including quantum thermodynamics [20–25], quantum biology [26–29], quantum metrology [30–34], quantum phase transitions [35–39], etc. Recently, the resource theory of coherence [40,41] has been well developed based on different free operations [42–49]. It not only provides a strict mathematical framework to effectively quantify coherence [42,50], but also establishes a platform to understand quantum mechanical features from a different perspective.

Up to now, many methods have been proposed to quantify quantum coherence. The most intuitive method could be the coherence measure based on the distance [42,51–54] between the state of interest and the closest incoherent state since the corresponding incoherent operations and incoherent states can be unambiguously defined. The remarkable examples are the coherence measure based on the l_1 norm and quantum relative entropy [42]. However, it has been shown that the strong monotonicity in the resource theory requirements has ruled out many convenient norms such as the trace norm and other l_p norms ($p \neq 1$) [51]. In addition, the usual applications of the commutation such as the skew information and the Tsallis relative α entropy serve as good coherence measures [38,55–59]. The distinguished feature of the above coherence measures is that they can be analytically calculated for a general state. In addition, the relative entropy coherence has the obvious operational meaning due to its connection with the optimal rate for distilling a maximally coherent state from given states [48], the coherence based on the skew information can be related to the quantum metrology [38,55], and the robustness of coherence is shown to be able to describe the advantage enabled by a quantum state in a phase discrimination task [60]. The convex roof construction, a traditional and effective method in the quantification of entanglement measure [61–63], can also be used to quantify coherence [43,48,64–67]. It is obvious that different quantifications not only provide different computability, but also imply different operational meanings. How to explore the new understanding of coherence has been still a significant and attractive topic in the resource theory [68–71].

In this paper, we present the coherence monotone and coherence measure from an entirely different perspective. We consider that some pure states undergo incoherent channels [72] and finally become the common objective state. It is shown that the coherence of the objective state can be well described by the least coherence of the pure input states. Given any certain coherence monotone F defined on pure states, the coherence monotone extended to mixed states through our method serves as the supremum of all the coherence monotones equal to F for pure states. We prove our coherence monotone is continuous based on a redefined distance norm. Considering the convexity, we prove that our coherence measure is a particular subset of the coherence measure based on the convex roof construction. As applications, we select the geometric coherence measure [6] to quantify the coherence of pure states and then establish a coherence measure, which is equivalent to the geometric coherence. Hence we endow an operational understanding to the geometric coherence. In

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particular, we thoroughly analyze the states of the qubit. We find the optimal pure state, give the easier method to choosing the coherence measure of pure states, and finally find out the analytic coherence measure for a general quantum state of a qubit. Finally, we numerically illustrate the difference between our coherence monotone and that based on the convex roof construction, which indicates that our approach can induce new coherence monotones. This paper is organized as follows. In Sec. II, we elucidate how to describe the coherence based on our incoherent operation process and establish the corresponding coherence monotone. In Sec. III, we consider the convexity of the coherence monotone and show the connection with the coherence measure based on the convex roof construction. In Sec. IV, we prove the continuity of our coherence monotone. In Sec. V, we study the similarities and differences between our monotone and the coherence cost. In Sec. VI, we first consider the geometric coherence measure as the pure-state coherence to establish our coherence measure. Then we thoroughly deal with the states of the qubit and give a series of analytic coherence measures and, finally, numerically illustrate the difference between our coherence monotone and those in terms of convex roof construction. The discussion and the conclusion are given in Sec. VII.

II. THE COHERENCE MONOTONE VIA PURE-STATE COHERENCE

To begin with, let us give a brief introduction of the framework of the resource theory, especially of coherence. The resource theory is well defined by the free state and the free operation [41,68,73]. For the coherence as a resource, the free state is the incoherent quantum states, which can be given as $\delta = \sum_i \delta_i |i\rangle\langle i|$ with respect to the basis $\{|i\rangle\}$. The set of incoherent states is denoted by $\mathcal{I}$. The free operation (or the incoherent operation) is given by the completely positive trace-preserving (CPTP) map defined in the Kraus representation as $\varepsilon(\cdot) = \sum_n K_n(\cdot)K_n^\dagger$, with $K_n\delta K_n^\dagger \in \mathcal{I}$ for an incoherent state δ. Thus a good coherence measure $C(\rho)$ of a density matrix ρ should satisfy the following conditions [42]: (A_1) *non-negativity*: $C(\rho) \geqslant 0$ is saturated iff $\rho \in \mathcal{I}$; (A_2) *monotonicity*: $C(\varepsilon(\rho)) \leqslant C(\rho)$ for any incoherent operation $\varepsilon(\cdot)$; (A_3) *strong monotonicity*: $\sum_n p_n C(K_n\rho K_n^\dagger/p_n) \leqslant C(\rho)$, with $p_n = \mathrm{Tr}[K_n\rho K_n^\dagger]$ and $\rho_n = K_n\rho K_n^\dagger/p_n$; (A_4) *convexity*: $C(\rho) \leqslant \sum_i p_i C(\rho_i)$ for any $\rho = \sum_i p_i\rho_i$; (A_5) *only maximally coherent states (MCS) reach the maximum*: $C(\rho)$ is maximal only for $\rho = |\Phi_d\rangle\langle\Phi_d|$, where $|\Phi_d\rangle = \frac{1}{\sqrt{d}}\sum_{n=1}^{d} e^{i\theta_n}|n\rangle$ with real θ_n [74].

In general, $C(\cdot)$ is a good coherence measure if it satisfies all of the above conditions. However, $C(\cdot)$ will be called a coherence monotone if it satisfies all the conditions but (A_4), which is similar to the entanglement monotone [75]. Here we would like to emphasize that a monotone is sometimes as important as a measure, since it is shown that $C(\cdot)$ (similar to the entanglement monotone) has its operational meaning [76,77]. In this sense, the convexity is usually understood as a mathematical convenience [1,41].

In fact, the resource theory can always be established as long as free states and free operations are defined. Based on the different considerations for coherence, it has been shown that the free operations include at least five types, such as physically incoherent operations (PIO) defined by the operations implemented only by incoherent unitary, incoherent ancillary system and incoherent projective measurement; maximally incoherent operations (MIO) defined by the operations that can convert one incoherent state to another incoherent state; dephasing-covariant incoherent operations (DIO) defined by the set of all maps that commute with the dephasing map; incoherent operations (IO), which are defined as $\varepsilon(\cdot)$; and strictly incoherent operations (SIO) defined as the subset of IO with the additional condition that $\varepsilon^\dagger(\cdot)$ is also IO. Here we are mainly interested in the IO and the SIO.

It is not difficult to understand that a pure state can always be converted into a mixed state by some SIO(IO). On the contrary, for a mixed state, one can always find a pure state which can be converted into the given mixed state by SIO(IO) (note that SIO is a subset of IO). A typical example is that any mixed state can be considered as the pure state with the same diagonal entries as the given mixed state undergoes a series of purely dephasing channels to reduce the modulus of the off-diagonal entries and undergoes some proper phase operations to adjust the phases. In particular, an incoherent mixed state can also correspond to an incoherent pure state in this sense. Note that the corresponding pure states for a given mixed state are not generally unique. In this sense, one can collect all these pure states as a set $R(\rho)$ corresponding to the certain mixed state ρ. In other words, *the set $R(\rho)$ is not empty for any given state ρ*. Next, we will show that the coherence of the state ρ can be well described by the minimal coherence achieved by the pure state $|\phi\rangle \in R(\rho)$.

To do so, we have to first consider a coherence measure of a pure state. Let $\mu(|\psi\rangle) = (|\langle 1|\psi\rangle|^2, |\langle 2|\psi\rangle|^2, \dots)^T$ denote the coherence vector with respect to some basis $\{|i\rangle\}$. Denote $f(\mu)$ as a symmetric concave function with two additional conditions: (1) $f = 0$ whenever μ is a permutation of $(1, 0, \cdots, 0)$, and (2) f reaches the maximum only when every element of μ equals $1/d$ (d is the dimension of μ). It is shown that any good coherence measure can always be reduced to a symmetric concave function $f(\mu)$ of $\mu(|\psi\rangle)$ if applied on a pure state $|\psi\rangle$ [67,78]. *Throughout the paper, we specify $F(|\psi\rangle)$ as a good pure-state coherence measure, which means $F(|\psi\rangle)$ is defined only for pure states by $f(\mu(|\psi\rangle))$ mentioned above and satisfies (A_1)–(A_3) and (A_5) for pure states*. In this sense, $F(|\psi\rangle)$ does not pertain to mixed states, and therefore the convexity given by (A_4) makes no sense. With the pure-state coherence measure $F(|\psi\rangle)$, we can further propose our coherence monotone for any mixed state in the following rigorous way.

Theorem 1. If $R(\rho)$ is the set of pure states that can be converted into the given state ρ by IO, then $\mathscr{C}(\rho)$ is a coherence monotone with

$$\mathscr{C}(\rho) = \inf_{|\phi\rangle \in R(\rho)} F(|\phi\rangle), \tag{1}$$

where $F(|\phi\rangle)$ is a good pure-state coherence measure mentioned above.

Proof. To prove the theorem, we will have to show that $\mathscr{C}(\cdot)$ satisfies all the conditions (A_1)–(A_3) and (A_5).

(A_1) *Non-negativity*. Suppose $\sigma = \sum_j \sigma_{jj}|j\rangle\langle j|$ is an arbitrary incoherent state, and $|1\rangle\langle 1|$ is an incoherent pure state. Define an SIO(IO) $\varepsilon_W = \{W_j\}$ as

$$W_j = \sum_\gamma b_\gamma^{(j)}|h_j(\gamma)\rangle\langle\gamma|, \tag{2}$$

$$\sum_j |b_\gamma^{(j)}|^2 = 1, \tag{3}$$

where h_j is a permutation function with $h_j(\gamma) = \beta$ for the integers γ, β and $h_j(\gamma_1) \neq h_j(\gamma_2)$ if $\gamma_1 \neq \gamma_2$. Then,

$$\begin{aligned}\varepsilon_W(|1\rangle\langle 1|) &= \sum_j W_j|1\rangle\langle 1|W_j^\dagger \\ &= \sum_j |b_1^{(j)}|^2|h_j(1)\rangle\langle h_j(1)|.\end{aligned} \tag{4}$$

If we let $h_j(1) = j$ and $|b_1^{(j)}|^2 = \sigma_{jj}$, it is obvious that

$$\varepsilon_W(|1\rangle\langle 1|) = \sigma,$$

which shows that for any incoherent state σ, one can always find a corresponding incoherent pure state $|1\rangle$ such that $|1\rangle$ can be converted to σ by SIO(IO). It implies that for any incoherent state, $\mathscr{C} = 0$. On the contrary, IO cannot convert an incoherent state into a coherent state, so for any coherent state, $\mathscr{C} > 0$.

(A_2) *Monotonicity*. Let Λ be an arbitrary IO and ρ denote any state. Suppose $|\psi\rangle \in R(\rho)$ is the optimal pure state subject to $\mathscr{C}(\rho) = F(|\psi\rangle)$; then it is implied that $\varepsilon(|\psi\rangle\langle\psi|) = \rho$. Define $\rho_0 = \Lambda(\rho)$, i.e., $\rho_0 = \Lambda[\varepsilon(|\psi\rangle\langle\psi|)]$. Based on the definition of $\mathscr{C}$ given in Eq. (1), one can easily find $F(|\psi\rangle) \geqslant \mathscr{C}(\rho_0)$, that is, $\mathscr{C}(\rho) \geqslant \mathscr{C}(\Lambda(\rho))$.

(A_3) *Strong monotonicity*. Let $\varepsilon_K(\cdot) = \sum_l K_l(\cdot)K_l^\dagger$ be an IO. For a state ρ, define

$$\begin{aligned}p_l &= \mathrm{Tr}(K_l\rho K_l^\dagger), \\ \rho_l &= K_l\rho K_l^\dagger/p_l.\end{aligned} \tag{5}$$

The strong monotonicity is equivalent to $\mathscr{C}(\rho) \geqslant \sum_l p_l\mathscr{C}(\rho_l)$.

Suppose $|\psi\rangle$ is the optimal state in $R(\rho)$ such that $\mathscr{C}(\rho) = F(|\psi\rangle)$. It is implied that the following relation holds:

$$|\psi\rangle \xrightarrow{\text{IO}} \rho \xrightarrow{\{K_l\}} \{p_l, \rho_l\}. \tag{6}$$

Equation (6) indicates that there exists an IO such that

$$|\psi\rangle \xrightarrow{\text{IO}} \{t_i, |\varphi_i\rangle\} \xrightarrow{\{K_l\}} \{t_iq_{il}, |\phi_{il}\rangle\}, \tag{7}$$

where $\rho = \sum_i t_i|\varphi_i\rangle\langle\varphi_i|$ with $t_i > 0$ and $q_{il} = \mathrm{Tr}[K_l|\varphi_i\rangle\langle\varphi_i|K_l^\dagger]$ and $|\phi_{il}\rangle = K_l|\varphi_i\rangle/\sqrt{q_{il}}$, $q_{il} \neq 0$. In other words, $|\psi\rangle$ can be converted into $\{t_iq_{il}, |\phi_{il}\rangle\}$ by IO, which, based on Ref. [79], is equivalent to

$$\mu^\downarrow(|\psi\rangle) \prec \sum_{i,l} t_iq_{il}\mu^\downarrow(|\phi_{il}\rangle), \tag{8}$$

where $\mu^\downarrow(|\psi\rangle)$ is the coherence vector in decreasing order. Define the pure state $|\psi_l\rangle$ such that

$$\mu^\downarrow(|\psi_l\rangle) = \sum_i \frac{t_iq_{il}}{p_l}\mu^\downarrow(|\phi_{il}\rangle). \tag{9}$$

It is obvious that $p_l = \sum_i t_iq_{il}$, $\rho_l = \sum_i \frac{t_iq_{il}}{p_l}|\phi_{il}\rangle\langle\phi_{il}|$. One can directly arrive at $\mu^\downarrow(|\psi_l\rangle) \prec \sum_i \frac{t_iq_{il}}{p_l}\mu^\downarrow(|\phi_{il}\rangle)$ which, based on Ref. [79], shows that $|\psi_l\rangle$ can be converted into ρ_l by IO. According to the definition of $\mathscr{C}$, we have

$$F(|\psi_l\rangle) \geqslant \mathscr{C}(\rho_l). \tag{10}$$

Substituting Eq. (9) into Eq. (8), one can obtain

$$\begin{aligned}\mu^\downarrow(|\psi\rangle) &\prec \sum_l p_l \sum_i \frac{t_iq_{il}}{p_l}\mu^\downarrow(|\phi_{il}\rangle) \\ &= \sum_l p_l\mu^\downarrow(|\psi_l\rangle).\end{aligned} \tag{11}$$

It states that $|\psi\rangle$ can be converted to $\{p_l, |\psi_l\rangle\}$ by IO, so the strong monotonicity of the selected measure $F(\cdot)$ gives

$$F(|\psi\rangle) \geqslant \sum_l p_lF(|\psi_l\rangle). \tag{12}$$

Substituting Eq. (10) into Eq. (12), one can obtain

$$\begin{aligned}\mathscr{C}(\rho) = F(|\psi\rangle) &\geqslant \sum_l p_lF(|\psi_l\rangle) \\ &\geqslant \sum_l p_l\mathscr{C}(\rho_l),\end{aligned} \tag{13}$$

which is the exact strong monotonicity of $\mathscr{C}$.

(A_5) *Only MCS reach the maximum.* Suppose ρ is not the MCS. Reference [74] shows that there is decomposition $\rho = \sum_i p_i|\varphi_i\rangle\langle\varphi_i|$ with at least one pure state $|\varphi_{i_0}\rangle$ which is not the MCS. Thus, $\mu^\downarrow(|\Phi_d\rangle) \prec \sum_i p_i\mu^\downarrow(|\varphi_i\rangle)$ and $\mu^\downarrow(|\Phi_d\rangle) \neq \sum_i p_i\mu^\downarrow(|\varphi_i\rangle)$, which implies $|\Phi_d\rangle$ can be converted into ρ. Define $|\psi\rangle$ such that $\mu^\downarrow(|\psi\rangle) = \sum_i p_i\mu^\downarrow(|\varphi_i\rangle)$. One will obtain that $|\psi\rangle \neq |\Phi_d\rangle$, $|\psi\rangle$ can be converted to ρ, and $|\Phi_d\rangle$ can be converted into $|\psi\rangle$. From properties (A_2), (A_5) of F and the definition of $\mathscr{C}$, one can see that $F(|\Phi_d\rangle) > F(|\psi\rangle) \geqslant \mathscr{C}(\rho)$. It shows any state ρ which is not the MCS cannot reach the maximum. Conversely, from Eq. (1), $\mathscr{C}$ inherits property (A_5) of F for pure states. ■

With the above theorem, next we will show that our proposed coherence monotone $\mathscr{C}(\rho)$ serves as the supremum of all the coherence monotones which can be reduced to F for pure states.

Corollary 1. For any coherence monotone $C(\cdot)$ with $C(|\psi\rangle) = \mathscr{C}(|\psi\rangle)$ for any pure state $|\psi\rangle$, $\mathscr{C}(\rho) \geqslant C(\rho)$ holds for any state ρ.

Proof. Given a density matrix ρ, based on the definition of $\mathscr{C}(\rho)$, one can always find the corresponding optimal pure state $|\psi\rangle$ such that $\mathscr{C}(\rho) = F(|\psi\rangle)$ with ρ obtained by IO on the optimal pure state $|\psi\rangle$. Note that it is also valid to write $\mathscr{C}(\rho) = \mathscr{C}(|\psi\rangle) = C(|\psi\rangle) = F(|\psi\rangle)$. Since $C(\cdot)$ is also a coherence monotone, we have $C(|\psi\rangle) \geqslant C(\rho)$, which implies $\mathscr{C}(\rho) \geqslant C(\rho)$. The proof is completed. ■

Up to now, a valid coherence monotone $\mathscr{C}(\cdot)$ has been completely established if a pure-state coherence measure $F(\cdot)$ is given. Based on our definition of $\mathscr{C}(\cdot)$, one can see that $\mathscr{C}(\rho)$ of the state ρ is obtained by the minimal pure-state coherence optimized in the set $R(\rho)$. We will show that the set $R(\rho)$ in the above minimization can actually be replaced by its subset denoted by $Q(\rho)$. So our coherence measure $\mathscr{C}(\rho)$ can be rewritten based on $Q(\rho)$. For clarity, we would like to give the explicit forms of both $Q(\rho)$ and $\mathscr{C}(\rho)$ in the following rigorous way.

Theorem 2. The coherence monotone $\mathscr{C}(\rho)$ of a density matrix ρ can be rewritten as

$$\mathscr{C}(\rho) = \inf_{|\psi\rangle\in Q(\rho)} F(|\psi\rangle), \tag{14}$$

where $F(|\psi\rangle)$ is defined the same as Theorem 1, and $Q(\rho) \subset R(\rho)$ is the set of all pure states $|\phi\rangle$ which fulfill

$$\mu^{\downarrow}(|\phi\rangle) = \sum_i p_i \mu^{\downarrow}(|\varphi_i\rangle), \tag{15}$$

where $\{p_i, |\varphi_i\rangle\}$ is a pure-state decomposition of ρ.

Proof. Let $|\psi\rangle \in R(\rho)$; then there exists a decomposition $\{p_i, |\varphi_i\rangle\}$ of ρ such that $\mu^{\downarrow}(|\psi\rangle) \prec \sum_i p_i\mu^{\downarrow}(|\varphi_i\rangle)$ [79]. Define a pure state $|\psi_0\rangle$ such that $\mu^{\downarrow}(|\psi_0\rangle) = \sum_i p_i\mu^{\downarrow}(|\varphi_i\rangle)$, which actually implies $\mu^{\downarrow}(|\psi_0\rangle) \prec \sum_i p_i\mu^{\downarrow}(|\varphi_i\rangle)$ and $\mu^{\downarrow}(|\psi\rangle) \prec \mu^{\downarrow}(|\psi_0\rangle)$; then we have that $|\psi\rangle$ can be converted into $|\psi_0\rangle$, and $|\psi_0\rangle$ can be converted to ρ. Correspondingly, it follows that $F(|\psi_0\rangle) \leqslant F(|\psi\rangle)$. Thus, all the pure states $|\psi_0\rangle$ can form the subset $Q(\rho)$. In particular, one can find that the minimal $F(\cdot)$ can be achieved by those in the subset $Q(\rho)$. ■

III. THE CONVEXITY

In the previous section, we do not address the convexity. Now we will study the requirements of $F(\cdot)$ such that our proposed coherence monotone can become a good coherence measure, that is, $\mathscr{C}(\cdot)$ is convex.

Theorem 3. $\mathscr{C}$ is convex if and only if for any ensemble $\{p_i, |\psi_i\rangle\}$ ($\sum_i p_i = 1$, and let $\varrho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$), there always exists a pure state $|\varphi_0\rangle \in R(\varrho)$ such that

$$F(|\varphi_0\rangle) \leqslant \sum_i p_i F(|\psi_i\rangle). \tag{16}$$

Proof. Suppose $\{p_i, |\psi_i\rangle\}$ is an arbitrary ensemble and $\varrho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$. If there exists $|\varphi_0\rangle \in R(\varrho)$ satisfying Eq. (16), then

$$\mathscr{C}(\varrho) \leqslant F(|\varphi_0\rangle) \leqslant \sum_i p_i F(|\psi_i\rangle) = \sum_i p_i \mathscr{C}(|\psi_i\rangle).$$

Corollary 1 shows that $\mathscr{C}$ is the upper bound of any coherence monotone which gives the same coherence as $\mathscr{C}$ for pure states; hence, $\mathscr{C}$ is not less than C_f, the coherence measure based on the convex roof construction, i.e.,

$$\mathscr{C}(\varrho) \geqslant C_f(\varrho) = \inf_{\{t_l,|\chi_l\rangle\}} \sum_l t_l \mathscr{C}(|\chi_l\rangle), \tag{17}$$

with $\varrho = \sum_l t_l|\chi_l\rangle\langle\chi_l|$ and $\sum_l t_l = 1$, $t_l > 0$. If $\{p_i, |\psi_i\rangle\}$ happens to be the optimal decomposition that achieves $C_f(\varrho)$ in Eq. (17), one can easily obtain that $\mathscr{C}(\varrho) = C_f(\varrho)$. It implies that $\mathscr{C}(\varrho)$ inherits the convexity of $C_f(\varrho)$.

Conversely, let $\varrho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ and $|\varphi_0\rangle \in R(\varrho)$ be the optimal state such that $F(|\varphi_0\rangle) = \mathscr{C}(\varrho)$. If $\mathscr{C}$ is convex, then

$$F(|\varphi_0\rangle) = \mathscr{C}(\varrho) \leqslant \sum_i p_i F(|\psi_i\rangle). \tag{18}$$

The proof is completed. ■

Theorem 3 shows that if the conditions given by Eq. (16) are satisfied, the proposed coherence measure $\mathscr{C}$ is a good coherence measure. In fact, if $\mathscr{C}$ is convex, $\mathscr{C}$ can own more general important properties.

Theorem 4. For a state ρ, $\mathscr{C}(\rho) = C_f(\rho)$ is equivalent to that $\mathscr{C}$ is convex, where $C_f(\rho)$ is the coherence measure in terms of the convex roof construction.

Proof. The proof actually is given in the proof of Theorem 3, so it is not repeated here. ■

As mentioned at the beginning of the last section, the main results are only restricted to the case of IO(SIO), so the coherence strong monotone can be established first, and then in the current section, we mainly consider the convexity. However, if $\mathscr{C}$ satisfying the convexity is a prerequisite, one will find from the following theorem that our approach is also suitable for the establishment of the coherence measure in the sense of MIO, DIO, and PIO.

Theorem 5. Considering the different free operations, i.e., IO, MIO, DIO, SIO, or PIO, the set $R_\Lambda(\cdot)$ similar to Eq. (1) and the corresponding $\mathscr{C}_{F_\Lambda}(\cdot)$ can be defined based on a pure-state measure $F_\Lambda(\cdot)$, where Λ denoting IO, MIO, DIO, SIO, or PIO indicates the corresponding free operation. If $\mathscr{C}_{F_\Lambda}(\cdot)$ is convex and $F_\Lambda(\cdot)$ for pure states satisfies the strong monotonicity under the corresponding free operation, then $\mathscr{C}_{F_\Lambda}(\cdot)$ will also satisfy the strong monotonicity under these corresponding free operations, respectively.

Proof. Let us focus on a given free operation Λ. For a certain ρ, let $|\psi\rangle$ be the optimal pure state such that $F_\Lambda(|\psi\rangle) = \mathscr{C}_{F_\Lambda}(\rho)$, with $\rho = \varepsilon_1(|\psi\rangle\langle\psi|)$ and ε_1 denoting the considered Λ operation. Suppose that ε_2 is another Λ operation; it is clear that $\varepsilon_T(\cdot) = \varepsilon_2 \circ \varepsilon_1(\cdot)$ must be a Λ operation. Let the Kraus operators of ε_2, ε_1, and ε_T be denoted, respectively, by $\{K_i\}$, $\{M_l\}$, and $\{T_{il}\}$, with $q_{il} = \langle\psi|T_{il}^\dagger T_{il}|\psi\rangle$ and $p_i = \langle\psi|K_i^\dagger K_i|\psi\rangle$. Then,

$$\begin{aligned}
\mathscr{C}_{F_\Lambda}(\rho) &= F_\Lambda(|\psi\rangle) \\
&\geqslant \sum_{il} q_{il} F_\Lambda(T_{il}|\psi\rangle\langle\psi|T_{il}^\dagger/q_{il}) \\
&= \sum_i p_i \sum_l \frac{q_{il}}{p_i} F_\Lambda(T_{il}|\psi\rangle\langle\psi|T_{il}^\dagger/q_{il}) \\
&\geqslant \sum_i p_i \mathscr{C}_{F_\Lambda}\left(\sum_l \frac{q_{il}}{p_i} K_i M_l|\psi\rangle\langle\psi|M_l^\dagger K_i^\dagger/q_{il}\right) \\
&= \sum_i p_i \mathscr{C}_{F_\Lambda}(K_i\rho K_i^\dagger/p_i),
\end{aligned} \tag{19}$$

where the first inequality is due to the strong monotonicity of $F_\Lambda(\cdot)$ under Λ operations, and the second inequality is due to the convexity of $\mathscr{C}_{F_\Lambda}(\cdot)$. Equation (19) shows the strong monotonicity of $\mathscr{C}_{F_\Lambda}(\cdot)$. ■

Before ending this section, we would like to emphasize that $R(\cdot)$ defined under SIO is the same as IO [79]. But $R(\cdot)$ defined under MIO, DIO, or PIO is not the same as IO(SIO), which can be seen from the counterexamples given in Appendix B. These examples show that the definition of $R(\rho)$ depends on the free operations. Hence the monotonicity of $\mathscr{C}$ in Eq. (1) naturally depends on free operations. For example, if one selects $F(\cdot)$ in Eq. (1) as the l_1 norm coherence C_{l_1}, then from Theorem 1, $\mathscr{C}_{l_1}(\rho) = \inf_{|\phi\rangle\in R(\rho)} C_{l_1}(|\phi\rangle)$ is a monotone under IO. But it is not a monotone under DIO (MIO). It can be seen from the following inequality:

$$\mathscr{C}_{l_1}(|\psi\rangle) = C_{l_1}(|\psi\rangle) < C_{l_1}(\Phi(|\psi\rangle\langle\psi|)) \leqslant \mathscr{C}_{l_1}(\Phi(|\psi\rangle\langle\psi|)), \tag{20}$$

where $|\psi\rangle$ and Φ (a DIO) are given in Eq. (B3), the first inequality is a result from Ref. [80], and the second inequality comes from Corollary 1.

IV. THE CONTINUITY

The continuity is a desirable property of a resource measure which indicates that the measure has no sudden transition with a small perturbation on the states. In this section, we will prove that our coherence monotone $\mathscr{C}$ is continuous. To do so, we will first present a measure of the distance of two states.

Given a d-dimensional density matrix ρ, its eigendecomposition can be given as $\rho = \tilde{\Phi}M\tilde{\Phi}^\dagger = \Phi\Phi^\dagger$, where the columns of $\tilde{\Phi}$ correspond to the eigenstates of ρ, the diagonal entries of the diagonal matrix M are the eigenvalues of ρ, and $\Phi = \tilde{\Phi}\sqrt{M}$. Similarly, any decomposition of ρ can be written as $\rho = \Upsilon\Upsilon^\dagger$. Based on the Hughston-Jozsa-Wootters (HJW) theorem [81,82], we have $\Upsilon = \Phi T$, where T is a right-unitary matrix with $TT^\dagger = \mathbf{I}_d$. Let us consider another density matrix σ in the d-dimensional Hilbert space with the eigendecomposition given by $\sigma = \Psi\Psi^\dagger$. Here we would like to emphasize that we keep the zero eigenvalues in Φ and Ψ, so they are both the $(d \times d)$-dimensional matrices. Thus a $(d \times n)$-dimensional right-unitary matrix T can lead them to any decomposition similar to Υ. Throughout this section, we write a density matrix in its eigendecomposition form as $\rho = \Phi\Phi^\dagger$ without further repeated definitions. Now the distance between the states ρ and σ can be defined in the following way.

Theorem 6. For two density matrices $\rho = \Phi\Phi^\dagger$ and $\sigma = \Psi\Psi^\dagger$, the distance between them can be defined by

$$D(\rho, \sigma) = \max_T \sum_k L_2(|\Phi T\rangle_k, |\Psi T\rangle_k),$$

where $|\Phi T\rangle_k$ denotes the kth column of the matrix (ΦT) and $L_2(|\psi\rangle, |\varphi\rangle) = \sqrt{\mathrm{Tr}(|\psi\rangle\langle\psi| - |\varphi\rangle\langle\varphi|)^2}$ is the l_2 norm for two pure states $|\psi\rangle$ and $|\varphi\rangle$ (non-normalized).

Proof. We prove D is a metric. At first, one can easily find that $D(\rho, \sigma) = 0$ if and only if $\rho = \sigma$. Second, it is easy to see that $D(\rho, \sigma) = D(\sigma, \rho)$. Next, we will show that $D(\rho, \sigma)$ satisfies the triangle inequality.

For two density matrices with the eigendecomposition form as $\rho_1 = \Phi_1\Phi_1^\dagger$ and $\rho_2 = \Phi_2\Phi_2^\dagger$, suppose T is the optimal right-unitary matrix such that $D(\rho_1, \rho_2) = \sum_k L_2(|\Phi_1 T\rangle_k, |\Phi_2 T\rangle_k)$. Considering a third density matrix $\sigma = \Psi\Psi^\dagger$, we have $D(\rho_{1\backslash 2}, \sigma) \geqslant \sum_k L_2(|\Phi_{1\backslash 2} T\rangle_k, |\Psi T\rangle_k)$ with the subscript $1\backslash 2$ denoting 1 or 2. Thus the triangle inequality of the l_2 norm $[L_2(|\psi\rangle, |\varphi\rangle)]$ implies that

$$\begin{aligned} &D(\rho_1, \sigma) + D(\rho_2, \sigma) \\ &\quad\geqslant \sum_k L_2(|\Phi_1 T\rangle_k, |\Psi T\rangle_k) + \sum_k L_2(|\Phi_2 T\rangle_k, |\Psi T\rangle_k) \\ &\quad\geqslant \sum_k L_2(|\Phi_1 T\rangle_k, |\Phi_2 T\rangle_k) = D(\rho_1, \rho_2), \end{aligned} \tag{21}$$

which completes the proof. ■

Lemma 1. For two density matrices $\rho_1 = \Phi_1\Phi_1^\dagger$ and $\rho_2 = \Phi_2\Phi_2^\dagger$ with a right-unitary matrix T, define two pure states $|\phi^{(1)}\rangle$ and $|\phi^{(2)}\rangle$ based on the coherence vector such that

$$\mu^\downarrow(|\phi^{(1)}\rangle) = \sum_k \mu^\downarrow(|\Phi_1 T\rangle_k), \tag{22}$$

$$\mu^\downarrow(|\phi^{(2)}\rangle) = \sum_k \mu^\downarrow(|\Phi_2 T\rangle_k). \tag{23}$$

Then,

$$\sqrt{\sum_i |\mu_i^\downarrow(|\phi^{(1)}\rangle) - \mu_i^\downarrow(|\phi^{(2)}\rangle)|^2} \leqslant D(\rho_1, \rho_2), \tag{24}$$

with $\mu_i^\downarrow(\cdot)$ representing the ith element of $\mu^\downarrow(\cdot)$.

Proof. Let $\phi^{(l,k)}$ denote the density matrix of the pure state $|\Phi_l T\rangle_k$ $(l = 1, 2)$; then, $\phi_{ij}^{(l,k)}$ is the (i, j)-th element of $\phi^{(l,k)}$. Since $\mu_i^\downarrow(\cdot)$ denotes the decreasing order, we have

$$\sum_i \mu_i^\downarrow(|\Phi_1 T\rangle_k)\mu_i^\downarrow(|\Phi_2 T\rangle_k) \geqslant \sum_i \phi_{ii}^{(1,k)}\phi_{ii}^{(2,k)},$$

which further implies

$$\begin{aligned} &\sum_i \left|\phi_{ii}^{(1,k)} - \phi_{ii}^{(2,k)}\right|^2 \\ &\quad\geqslant \sum_i |\mu_i^\downarrow(|\Phi_1 T\rangle_k) - \mu_i^\downarrow(|\Phi_2 T\rangle_k|^2. \end{aligned} \tag{25}$$

Therefore,

$$\begin{aligned} &L_2(|\Phi_1 T\rangle_k, |\Phi_2 T\rangle_k) \\ &\quad= \sqrt{\sum_{ij} \left|\phi_{ij}^{(1,k)} - \phi_{ij}^{(2,k)}\right|^2} \\ &\quad\geqslant \sqrt{\sum_i \left|\phi_{ii}^{(1,k)} - \phi_{ii}^{(2,k)}\right|^2} \\ &\quad\geqslant \sqrt{\sum_i |\mu_i^\downarrow(|\Phi_1 T\rangle_k) - \mu_i^\downarrow(|\Phi_2 T\rangle_k)|^2}. \end{aligned} \tag{26}$$

Based on Eqs. (22) and (23), we have

$$\begin{aligned} &\sqrt{\sum_i |\mu_i^\downarrow(|\phi^{(1)}\rangle) - \mu_i^\downarrow(|\phi^{(2)}\rangle)|^2} \\ &\quad= \sqrt{\sum_i \left|\sum_k \mu_i^\downarrow(|\Phi_1 T\rangle_k) - \sum_k \mu_i^\downarrow(|\Phi_2 T\rangle_k)\right|^2} \\ &\quad\leqslant \sum_k \sqrt{\sum_i |\mu_i^\downarrow(|\Phi_1 T\rangle_k) - \mu_i^\downarrow(|\Phi_2 T\rangle_k)|^2} \\ &\quad\leqslant \sum_k L_2(|\Phi_1 T\rangle_k, |\Phi_2 T\rangle_k) \leqslant D(\rho_1, \rho_2), \end{aligned} \tag{27}$$

where the first inequality is due to the triangle inequality of the l_2 norm for vectors, the second inequality comes from Eq. (26), and the last inequality is based on the definition of the distance D. The proof is finished. ■

Based on Theorem 6 and Lemma 1, we can show that our coherence measure is continuous so long as a continuous F is employed.

Theorem 7. If F is continuous for pure states, then $\mathscr{C}$ is continuous.

Proof. To prove the continuity, we need to show that for $\forall\varepsilon > 0$, there exists $\delta > 0$ such that $|\mathscr{C}(\rho_1) - \mathscr{C}(\rho_2)| < \varepsilon$ holds for $D(\rho_1, \rho_2) < \delta$.

Denote $|\alpha\rangle$ and $|\beta\rangle$ are the optimal pure states in $R(\rho_1)$ and $R(\rho_2)$ such that

$$\mathscr{C}(\rho_1) = F(|\alpha\rangle), \tag{28}$$

$$\mathscr{C}(\rho_2) = F(|\beta\rangle). \tag{29}$$

Then, based on the eigendecompositions $\rho_1 = \Phi_1\Phi_1^\dagger$ and $\rho_2 = \Phi_2\Phi_2^\dagger$, there must exist the right-unitary matrices T_1 and T_2 such that

$$\mu^\downarrow(|\alpha\rangle) = \sum_k \mu^\downarrow(|\Phi_1 T_1\rangle_k), \tag{30}$$

$$\mu^\downarrow(|\beta\rangle) = \sum_{k'} \mu^\downarrow(|\Phi_2 T_2\rangle_{k'}). \tag{31}$$

Define another pair of pure states $|\tilde{\alpha}\rangle$ and $|\tilde{\beta}\rangle$ such that

$$\mu(|\tilde{\alpha}\rangle) = \sum_{k'} \mu^\downarrow(|\Phi_1 T_2\rangle_{k'}), \tag{32}$$

$$\mu(|\tilde{\beta}\rangle) = \sum_{k} \mu^\downarrow(|\Phi_2 T_1\rangle_{k}). \tag{33}$$

Lemma 1 shows that

$$\sqrt{\sum_i |\mu_i^\downarrow(|\alpha\rangle) - \mu_i^\downarrow(|\tilde{\beta}\rangle)|^2} \leqslant D(\rho_1, \rho_2), \tag{34}$$

$$\sqrt{\sum_i |\mu_i^\downarrow(|\beta\rangle) - \mu_i^\downarrow(|\tilde{\alpha}\rangle)|^2} \leqslant D(\rho_1, \rho_2). \tag{35}$$

Based on the continuity of F, for $\forall\varepsilon > 0$, we have a δ_0 such that $|F(|\psi^{(1)}\rangle) - F(|\psi^{(2)}\rangle)| < \varepsilon$ holds for $L_2(|\psi^{(1)}\rangle, |\psi^{(2)}\rangle) < \delta_0$. Let $D(\rho_1, \rho_2) < \delta_0^2/4$. Then, based on Lemma 1, for any i, we have

$$\begin{aligned}&\left(\sqrt{\mu_i^\downarrow(|\alpha\rangle)} - \sqrt{\mu_i^\downarrow(|\tilde{\beta}\rangle)}\right)^2\\ &\leqslant |\mu_i^\downarrow(|\alpha\rangle) - \mu_i^\downarrow(|\tilde{\beta}\rangle)|\\ &\leqslant \sqrt{\sum_i |\mu_i^\downarrow(|\alpha\rangle) - \mu_i^\downarrow(|\tilde{\beta}\rangle)|^2} < \frac{\delta_0^2}{4}.\end{aligned} \tag{36}$$

Denote

$$\begin{aligned}|\nu_\alpha\rangle &= \sum_i \sqrt{\mu_i^\downarrow(|\alpha\rangle)}|i\rangle,\\ |\nu_{\tilde{\beta}}\rangle &= \sum_i \sqrt{\mu_i^\downarrow(|\tilde{\beta}\rangle)}|i\rangle.\end{aligned} \tag{37}$$

Then, from Eq. (36), we have

$$\begin{aligned}&[L_2(|\nu_\alpha\rangle, |\nu_{\tilde{\beta}}\rangle)]^2\\ &= \sum_{i,j} \left(\sqrt{\mu_i^\downarrow(|\alpha\rangle)\mu_j^\downarrow(|\alpha\rangle)} - \sqrt{\mu_i^\downarrow(|\tilde{\beta}\rangle)\mu_j^\downarrow(|\tilde{\beta}\rangle)}\right)^2\\ &= \sum_{i,j} \Big[\sqrt{\mu_i^\downarrow(|\alpha\rangle)}\left(\sqrt{\mu_j^\downarrow(|\alpha\rangle)} - \sqrt{\mu_j^\downarrow(|\tilde{\beta}\rangle)}\right)\\ &\quad + \sqrt{\mu_j^\downarrow(|\tilde{\beta}\rangle)}\left(\sqrt{\mu_i^\downarrow(|\alpha\rangle)} - \sqrt{\mu_i^\downarrow(|\tilde{\beta}\rangle)}\right)\Big]^2\\ &< \sum_{i,j} \frac{\left[\sqrt{\mu_i^\downarrow(|\alpha\rangle)} + \sqrt{\mu_j^\downarrow(|\tilde{\beta}\rangle)}\right]^2 \delta_0^2}{4}\\ &\leqslant \sum_{i,j} \frac{[\mu_i^\downarrow(|\alpha\rangle) + \mu_j^\downarrow(|\tilde{\beta}\rangle)]\delta_0^2}{2}\\ &= \frac{\left[\sum_i \mu_i^\downarrow(|\alpha\rangle) + \sum_j \mu_j^\downarrow(|\tilde{\beta}\rangle)\right]\delta_0^2}{2} = \delta_0^2,\end{aligned} \tag{38}$$

i.e.,

$$L_2(|\nu_\alpha\rangle, |\nu_{\tilde{\beta}}\rangle) < \delta_0.$$

Thus we have

$$|F(|\alpha\rangle) - F(|\tilde{\beta}\rangle)| = |F(|\nu_\alpha\rangle) - F(|\nu_{\tilde{\beta}}\rangle)| < \varepsilon, \tag{39}$$

and similarly,

$$|F(|\tilde{\alpha}\rangle) - F(|\beta\rangle)| < \varepsilon,$$

which is equivalent to

$$|\mathscr{C}(\rho_1) - F(|\tilde{\beta}\rangle)| < \varepsilon, |\mathscr{C}(\rho_2) - F(|\tilde{\alpha}\rangle)| < \varepsilon. \tag{40}$$

Equation (40) leads to

$$\mathscr{C}(\rho_2) - \varepsilon < F(|\tilde{\beta}\rangle) - \varepsilon < \mathscr{C}(\rho_1), \tag{41}$$

$$\mathscr{C}(\rho_1) - \varepsilon < F(|\tilde{\alpha}\rangle) - \varepsilon \leqslant \mathscr{C}(\rho_2), \tag{42}$$

which can be rewritten as

$$|\mathscr{C}(\rho_1) - \mathscr{C}(\rho_2)| < \varepsilon. \tag{43}$$

Thus we have shown that for $\forall\varepsilon > 0$, there exists $\delta = \delta_0^2/4$ such that $|\mathscr{C}(\rho_1) - \mathscr{C}(\rho_2)| < \varepsilon$ holds for $D(\rho_1, \rho_2) < \delta$. That is, $\mathscr{C}(\cdot)$ is continuous. The proof is completed. ■

We have shown the continuity of our proposed coherence measure from the general perspective. Here we would like to give an example to demonstrate this property explicitly. Let $F(\cdot) = C_r(\cdot)$ be the coherence measure based on the relative entropy. We know that for a pure state $|\psi\rangle$, $C_r(|\psi\rangle) = S(\Delta[|\psi\rangle\langle\psi|])$, where $\Delta[\cdot]$ represents the diagonal matrix by deleting all off-diagonal entries of the considered matrix, and $S(\rho) = -\mathrm{Tr}\rho\log\rho$ is the von Neuman entropy of a density matrix ρ. Thus we can easily show that $\mathscr{C}_r(\rho) = \inf_{|\phi\rangle\in R(\rho)} C_r(|\phi\rangle)$ is continuous. To do so, we consider two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ and the distance between them, $D(|\psi_1\rangle, |\psi_2\rangle) = L_2(|\psi_1\rangle, |\psi_2\rangle)$. Denote $\epsilon' = tr||\Delta[|\psi_1\rangle\langle\psi_1|] - \Delta[|\psi_2\rangle\langle\psi_2|]||_1$. One can easily find that

$D(|\psi_1\rangle, |\psi_2\rangle) \to 0$ leads to $\epsilon' \to 0$ and utilize Fannes' inequality [83],

$$\begin{aligned}|C_r(|\psi_1\rangle) - C_r(|\psi_2\rangle)| \\ = |S(\Delta[|\psi_1\rangle\langle\psi_1|]) - S(\Delta[|\psi_2\rangle\langle\psi_2|])| \\ \leqslant \epsilon' \log d - \epsilon' \log \epsilon',\end{aligned} \tag{44}$$

and thus $|C_r(|\psi_1\rangle) - C_r(|\psi_2\rangle)| \to 0$, which further implies $\mathscr{C}_r(\rho)$ is continuous.

V. RELATION BETWEEN $\mathscr{C}$ AND COHERENCE COST

In this section, we discuss the relation between $\mathscr{C}$ and the coherence cost. Their differences are apparent. The asymptotic (and one-shot) coherence cost can be understood as the least number of MCS in the asymptotic limit [48] (and the least length of MCS in the one-shot case [70,71]) required to prepare the given state by the incoherent operations. The coherence cost happened to be equal to the coherence of formation, a single-letter formula, which is equivalent to the convex roof construction based on the relative entropy coherence of a pure state [48]. In this sense, the coherence cost can be understood as the specific coherence quantifier based on the particular coherence measure (relative entropy) for pure states. However, $\mathscr{C}$ in the current paper essentially is given by the least coherence of the pure state that can be converted to the given state by incoherent operations. The establishment of $\mathscr{C}$ in Eq. (1) strongly depends on the coherence measure for a pure state, which is similar to (but not the same as) the convex roof construction. Therefore, $\mathscr{C}$ can be built by providing any good coherence measure F for pure states (e.g., l_1 norm, relative entropy, skew information, and so on) and the different selected F leads to different $\mathscr{C}$. It is especially noted that $\mathscr{C}$ is not equal to the coherence cost even though F is chosen as the relative entropy, which will be demonstrated in Sec. VI. Therefore, the coherence cost and $\mathscr{C}$ are generally two different coherence quantifiers. However, in some particular cases (select some specific pure-state coherence quantifier F), $\mathscr{C}$ can be closely related to coherence cost. Next, we will show these relations.

One-shot scenario. Denote $B_\epsilon(\rho)$ as the set of all the states σ satisfying $\mathscr{F}(\rho,\sigma) = (\mathrm{Tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}})^2 \geqslant 1-\epsilon$; then, the one-shot coherence cost can be defined as [70]

$$C_{\mathrm{IO}}^{\epsilon}(\rho) = \min_{\Lambda_{\mathrm{IO}}} \{\log M | \Lambda_{\mathrm{IO}}(|\Psi_M\rangle\langle\Psi_M|) \in B_\epsilon(\rho)\}, \tag{45}$$

with $|\Psi_M\rangle$ being the MCS of M dimension and the subscript representing the IO operation Λ_{IO}, and we use log to indicate $\log_2$. It is obvious that for a pure state $|\psi\rangle$, $C_{\mathrm{IO}}^0(|\psi\rangle) = \log M$, with M equivalent to the number of nonzero elements of $\mu(|\psi\rangle)$. If we choose $F(|\psi\rangle) = C_{\mathrm{IO}}^0(|\psi\rangle)$ to establish coherence monotone as $\mathscr{C}_{\mathrm{IO}}(\rho) = \inf_{|\phi\rangle\in R(\rho)} C_{\mathrm{IO}}^0(|\phi\rangle)$, then the smoothing version can be written as

$$\mathscr{C}_{\mathrm{IO}}^{\epsilon}(\rho) = \min_{\sigma\in B_\epsilon(\rho)} \mathscr{C}_{\mathrm{IO}}(\sigma). \tag{46}$$

Theorem 8. For any density matrix ρ, $\mathscr{C}_{\mathrm{IO}}^{\epsilon}(\rho) = C_{\mathrm{IO}}^{\epsilon}(\rho)$.

Proof. Given a density matrix ρ, let ρ_x be the optimal state such that $\mathscr{C}_{\mathrm{IO}}^{\epsilon}(\rho) = \mathscr{C}_{\mathrm{IO}}(\rho_x)$ with $\mathscr{F}(\rho,\rho_x) \geqslant 1-\epsilon$. Suppose $|\varphi_x\rangle$ to be the optimal state in $R(\rho_x)$ such that

$$\mathscr{C}_{\mathrm{IO}}(\rho_x) = F(|\varphi_x\rangle) = C_{\mathrm{IO}}^0(|\varphi_x\rangle) = \log M,$$

where $M = M[|\varphi_x\rangle]$. The definition of $C_{\mathrm{IO}}^0(|\varphi_x\rangle)$ indicates that $|\Psi_M\rangle$ can be converted into $|\varphi_x\rangle$ by IO operations. The definition of $\mathscr{C}(\rho_x)$ shows that $|\varphi_x\rangle$ can be converted into ρ_x by IO operations. Thus we have that $|\Psi_M\rangle$ can be converted into ρ_x by IO, so one can find

$$\mathscr{C}_{\mathrm{IO}}^{\epsilon}(\rho) = \log M \geqslant C_{\mathrm{IO}}^{\epsilon}(\rho). \tag{47}$$

Conversely, suppose $|\Psi_{M'}\rangle$ is the M'-dimensional MCS such that the state $\rho' = \Lambda_{\mathrm{IO}}(|\Psi_{M'}\rangle\langle\Psi_{M'}|)$ achieves the exact one-shot coherence cost of ρ, so we have $C_{\mathrm{IO}}^{\epsilon}(\rho) = \log M'$. Let $|\varphi'\rangle$ be the optimal state in $R(\rho')$ in the sense of $\mathscr{C}_{\mathrm{IO}}(\rho')$. Note that $|\Psi_{M'}\rangle \in R(\rho')$ for ρ'. One can find that $M[|\varphi'\rangle] > M'$ implies that $|\varphi'\rangle$ will not be the optimal state $R(\rho')$ because at least $|\Psi_{M'}\rangle$ is more suitable than $|\varphi'\rangle$. If $M[|\varphi'\rangle] < M'$, one can always find an MCS $|\Psi_{M[|\varphi'\rangle]}\rangle$ which can be converted into $|\varphi'\rangle$ and, further, be converted into ρ' by incoherent operations. It means that $|\Psi_{M'}\rangle$ is not the optimal MCS for $C_{\mathrm{IO}}^{\epsilon}(\rho)$, which is a contradiction. So we can have $M[|\varphi'\rangle] = M'$, which directly leads to

$$\mathscr{C}_{\mathrm{IO}}(\rho') = F(|\varphi'\rangle) = \log M' = C_{\mathrm{IO}}^{\epsilon}(\rho). \tag{48}$$

In addition, the minimization in $\mathscr{C}_{\mathrm{IO}}^{\epsilon}(\rho)$ implies $\mathscr{C}_{\mathrm{IO}}(\rho') \geqslant \mathscr{C}_{\mathrm{IO}}^{\epsilon}(\rho)$ for the particular ρ' that is optimal for $C_{\mathrm{IO}}^{\epsilon}(\rho)$ instead of $\mathscr{C}_{\mathrm{IO}}^{\epsilon}(\rho)$. So, based on Eq. (48), we have $C_{\mathrm{IO}}^{\epsilon}(\rho) \geqslant \mathscr{C}_{\mathrm{IO}}^{\epsilon}(\rho)$, which, combined with Eq. (47), follows that

$$C_{\mathrm{IO}}^{\epsilon}(\rho) = \mathscr{C}_{\mathrm{IO}}^{\epsilon}(\rho).$$

The proof is completed. ∎

Asymptotic regime. Let $F(|\phi\rangle) = C_r(|\phi\rangle)$ be the coherence measure based on the relative entropy. Then we have $\mathscr{C}_r(\rho) = \inf_{|\phi\rangle\in R(\rho)} C_r(|\phi\rangle)$. The smoothing version is given by $\mathscr{C}_r^{\epsilon}(\rho) = \min_{\sigma\in B_\epsilon(\rho)} \mathscr{C}_r(\sigma)$.

Theorem 9. For a d-dimensional density matrix ρ,

$$\lim_{\epsilon\to 0^+} \lim_{n\to\infty} \frac{\mathscr{C}_r^{\epsilon}(\rho^{\otimes n})}{n} = C_f^r(\rho), \tag{49}$$

where $C_f^r(\rho) = \min_{\{p_i,|\varphi_i\rangle\}} \sum p_i C_r(|\varphi_i\rangle)$ is the coherence of formation.

Proof. From Ref. [48], for $\forall\epsilon$, there is a $|\Psi_2^{\otimes n\mathscr{R}}\rangle$ which can be incoherently converted into $\rho^{(n)} \in B_\epsilon(\rho^{\otimes n})$ when n is large enough with $\mathscr{R} \to C_f^r(\rho)$ for $n\to\infty$, $\epsilon \to 0$. Thus,

$$C_r\left(\left|\Psi_2^{\otimes n\mathscr{R}}\right\rangle\right) \geqslant \mathscr{C}_r(\rho^{(n)}) \geqslant \mathscr{C}_r^{\epsilon}(\rho^{\otimes n}), \tag{50}$$

where the second inequality comes from the minimization required for $\mathscr{C}_r^{\epsilon}(\cdot)$. Denote ρ' as the exact state such that $\mathscr{C}_r^{\epsilon}(\rho^{\otimes n}) = \mathscr{C}_r(\rho')$. Since Corollary 1 shows that $\mathscr{C}_r(\rho')$ is the maximal coherence monotone for the given state ρ', for the particular coherence monotone $C_f^r(\rho')$, we have $\mathscr{C}_r(\rho') \geqslant C_f^r(\rho')$. Considering the asymptotic continuity of $C_f^r(\cdot)$ given by Ref. [48], one can obtain

$$\mathscr{C}_r^{\epsilon}(\rho^{\otimes n}) \geqslant C_f^r(\rho') \geqslant C_f^r(\rho^{\otimes n}) - n\xi(\epsilon),$$

where $\xi(\epsilon) = \epsilon' \log d + \frac{1+\epsilon'}{n} h(\frac{\epsilon'}{1+\epsilon'})$, $\epsilon' = \sqrt{2(1-\sqrt{1-\epsilon})}$, and $h(x) = -x\log x - (1-x)\log(1-x)$. Let us take the limit with respect to $n \to \infty$, $\epsilon \to 0$; then,

$$C_f^r(\rho) = \lim_{\epsilon\to 0^+}\lim_{n\to\infty}\frac{C_r\left(\left|\Psi_2^{\otimes n\mathscr{R}}\right\rangle\right)}{n} \geqslant \lim_{\epsilon\to 0^+}\lim_{n\to\infty}\frac{\mathscr{C}_r^\epsilon(\rho^{\otimes n})}{n} \geqslant \lim_{\epsilon\to 0^+}\lim_{n\to\infty}\left\{\frac{C_f^r(\rho^{\otimes n})}{n} - \xi(\epsilon)\right\} = C_f^r(\rho), \quad (51)$$

where $\lim_{\epsilon\to 0^+}\lim_{n\to\infty}\xi(\epsilon) = 0$ is used. Thus, Eq. (51) implies $\lim_{\epsilon\to 0^+}\lim_{n\to\infty}\frac{\mathscr{C}_r^\epsilon(\rho^{\otimes n})}{n} = C_f^r(\rho)$. The proof is completed. ■

In addition, we would like to mention that since Ref. [70] has shown $\lim_{\epsilon\to 0^+}\lim_{n\to\infty}\frac{C_{\rm IO}^\epsilon(\rho^{\otimes n})}{n} = C_f^r(\rho)$ and our Theorem 8 has shown $\mathscr{C}_{\rm IO}^\epsilon(\rho) = C_{\rm IO}^\epsilon(\rho)$, it is obvious that

$$\lim_{\epsilon\to 0^+}\lim_{n\to\infty}\frac{\mathscr{C}_{\rm IO}^\epsilon(\rho^{\otimes n})}{n} = C_f^r(\rho), \quad (52)$$

which means that the asymptotic case $\mathscr{C}_{\rm IO}^\epsilon(\cdot)$ with $F(|\psi\rangle) = C_{\rm IO}^0(|\psi\rangle)$ is consistent with $C_f^r(\cdot)$.

VI. EXAMPLES

A. The geometric coherence as *F*

In the previous sections, we have given the general form of our coherence measure. Next, we will give a concrete example by selecting an exact coherence measure $F(\cdot)$ for pure states. Here we would like to choose the geometric coherence C_g [6] for pure states as the candidate, which is defined as

$$C_g(|\psi\rangle) = 1 - \sup_{\sigma\in\mathcal{I}}\mathscr{F}(|\psi\rangle, \sigma), \quad (53)$$

where $\mathscr{F}(|\psi\rangle, \sigma) = \langle\psi|\sigma|\psi\rangle$ is the Uhlmann fidelity between the pure state $|\psi\rangle$ and the state σ. Then we have the following theorem.

Theorem 10. The coherence of a state ρ can be well measured by

$$\mathscr{C}_g(\rho) = \inf_{|\phi\rangle\in Q(\rho)} C_g(|\phi\rangle) = \inf_{\{p_i,|\psi_i\rangle\}}\sum p_i C_g(|\psi_i\rangle),$$

with $\rho = \sum p_i|\psi_i\rangle\langle\psi_i|$. It is equivalent to the geometric coherence [6].

Proof. Obviously, the geometric coherence *per se* is a coherence monotone, so the key task is to prove $\mathscr{C}_g$ is convex. However, we do not directly show that the geometric coherence C_g satisfies Theorem 3, but we will prove that $\mathscr{C}_g$ is actually a coherence measure based on the convex roof construction, which will imply that $\mathscr{C}_g$ is a good coherence measure (especially satisfies the convexity).

Without loss of generality, let us consider the coherence in the framework defined by the computational basis $\{|i\rangle\}$. It is obvious that for a pure state $|\varphi\rangle$, we have

$$C_g(|\varphi\rangle) = 1 - \sup_i |\langle\varphi|i\rangle|^2 = 1 - \mu_1^\downarrow(|\varphi\rangle), \quad (54)$$

with $\mu_1^\downarrow(|\varphi\rangle)$ denoting the first element of $\mu^\downarrow(|\varphi\rangle)$. Now we take the geometric coherence $C_g(\rho)$ as the pure-state coherence measure $F(\cdot)$. Then,

$$\mathscr{C}_g(\rho) = \inf_{|\varphi\rangle\in Q(\rho)} C_g(|\varphi\rangle) = 1 - \sup_{|\varphi\rangle\in Q(\rho)}\mu_1^\downarrow(|\varphi\rangle). \quad (55)$$

Based on Theorem 2, one can note that $|\varphi\rangle \in Q(\rho)$ means that there exists a decomposition $\{p_i, |\psi_i\rangle\}$ of ρ such that

$$\mu_1^\downarrow(|\varphi\rangle) = \sum_i p_i\mu_1^\downarrow(|\psi_i\rangle). \quad (56)$$

Thus, $\mathscr{C}_g(\rho)$ can be rewritten as

$$\mathscr{C}_g(\rho) = 1 - \sup_{\{p_i,|\psi_i\rangle\}}\sum_i p_i\mu_1^\downarrow(|\psi_i\rangle) = \inf_{\{p_i,|\psi_i\rangle\}} p_i C_g(|\psi_i\rangle), \quad (57)$$

which shows that $\mathscr{C}_g$ is the coherence measure based on the convex roof construction, i.e., the geometric coherence [6]. So it automatically satisfies the convexity. ■

B. Analytical expressions for qubits

Now we will study the potential analytic expression of our proposed coherence measure. Based on our definition, one can quickly note that the key to calculate our coherence measure is whether one could find out the optimal pure state $|\psi\rangle \in Q(\rho)$ such that $\mathscr{C}(\rho) = F(|\psi\rangle)$. First we would like to give the following theorem.

Theorem 11. If there exists an optimal decomposition $\{\tilde{p}_j, |\tilde{\phi}_j\rangle\}$ for the state ρ such that

$$\sum_i p_i\mu^\downarrow(|\phi_i\rangle) \prec \sum_j \tilde{p}_j\mu^\downarrow(|\tilde{\phi}_j\rangle),$$

with $\{p_i, |\phi_i\rangle\}$ denoting any decomposition of ρ, the optimal pure state $|\psi\rangle$ can be defined by

$$\mu^\downarrow(|\psi\rangle) = \sum_j \tilde{p}_j\mu^\downarrow(|\tilde{\phi}_j\rangle). \quad (58)$$

Proof. To show this, let us suppose there exists the optimal decomposition $\{\tilde{p}_j, |\tilde{\phi}_j\rangle\}$ of σ subject to the above equations. Thus we can denote $\mu^\downarrow(|\psi\rangle) = \sum_j \tilde{p}_j\mu^\downarrow(|\tilde{\phi}_j\rangle)$. Considering any state $|\varphi\rangle \in Q(\rho)$, there always exists a decomposition $\{p_i, |\phi_i\rangle\}$ such that

$$\mu^\downarrow(|\varphi\rangle) = \sum_i p_i\mu^\downarrow(|\phi_i\rangle) \prec \sum_j \tilde{p}_j\mu^\downarrow(|\tilde{\phi}_j\rangle) = \mu^\downarrow(|\psi\rangle).$$

It shows that $|\varphi\rangle$ can be converted into $|\psi\rangle$ by IO, that is, $F(|\varphi\rangle) \geqslant F(|\psi\rangle)$. In other words, $|\psi\rangle$ can achieve the least coherence of the pure states in $Q(\rho)$, namely, Eq. (58) holds. ■

One can find that to obtain the analytic expression, whether there exists a decomposition as the above theorem is the key. However, it is not easy to prove whether there always exists such an optimal decomposition for a general quantum state ρ. But we can show that such an optimal decomposition can always be found in qubit states. That is, one can always establish the analytic coherence measure.

Theorem 12. Given a density matrix σ of a qubit with b denoting its off-diagonal element, the optimal decomposition subject to Theorem 11 can be given by

$$\sigma = \lambda\sigma^{(+)} + (1-\lambda)\sigma^{(-)}, \tag{59}$$

where $\sigma^{(\pm)} = \begin{pmatrix} \frac{1\pm z}{2} & b \\ b^* & \frac{1\mp z}{2} \end{pmatrix}$, with $z = \sqrt{1-4|b|^2}$, and $\lambda \in [0, 1]$ is some weight parameter determined by the state σ. In this sense, the coherence can be given by $\mathscr{C}(\sigma) = F(\sigma^{(\pm)})$.

Proof. Suppose that $\{p_i, |\psi_i\rangle\}$ is any decomposition of σ. Then, similar to the state $\sigma^{(\pm)}$, we can use $\{b_i, z_i\}$ to express the states σ and $|\psi_i\rangle\langle\psi_i|$ with $z = \sqrt{1-4|b|^2}$, $z_i = \sqrt{1-4|b_i|^2}$. Considering $\sigma = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, we have

$$\begin{aligned} z =& \sqrt{1-4|b|^2} = \sqrt{1-4\left|\sum_i p_i b_i\right|^2} \\ \geqslant& \sqrt{1-4\left(\sum_i p_i|b_i|\right)^2} \geqslant \sum_i p_i\sqrt{1-4|b_i|^2} \\ =& \sum_i p_i z_i. \end{aligned} \tag{60}$$

According to the definition of the coherence vector $\mu^{\downarrow}$ for pure states, from Eq. (60) one can obtain

$$\begin{aligned} &\lambda\mu^{\downarrow}(\sigma^{(+)}) + (1-\lambda)\mu^{\downarrow}(\sigma^{(-)}) \\ &\quad = \left(\frac{1+z}{2}, \frac{1-z}{2}\right) \\ &\quad \succ \left(\frac{1+\sum_i p_i z_i}{2}, \frac{1-\sum_i p_i z_i}{2}\right) \\ &\quad = \sum_i p_i \mu^{\downarrow}(|\psi_i\rangle), \end{aligned} \tag{61}$$

which proves the existence of the required optimal decomposition. In addition, based on Theorem 11, one can know that $\sigma^{(\pm)}$ is the exact optimal pure state in $Q(\sigma)$ such that $\mathscr{C}(\sigma) = F(\sigma^{(\pm)})$. The proof is completed. ■

Theorem 13. For a qubit density matrix, the conditions for convexity given in Theorem 3 are equivalent to that $F(|\varphi\rangle) = \tilde{f}(|b|)$ is a convex function on $|b|$ for the pure state $|\varphi\rangle\langle\varphi| = \begin{pmatrix} \frac{1\pm z}{2} & b \\ b^* & \frac{1\mp z}{2} \end{pmatrix}$ with $z = \sqrt{1-4|b|^2}$.

Proof. Any pure state of a qubit can be written as the form of $|\varphi\rangle$, so its coherence vector can be given as $\mu^{\downarrow}(|\varphi\rangle) = (\frac{1+z}{2}, \frac{1-z}{2})$. Similarly, for another pure state $|\psi\rangle$, we can denote its coherence vector as $\mu^{\downarrow}(|\psi\rangle) = (\frac{1+z'}{2}, \frac{1-z'}{2})$. For a good coherence monotone F, one has $F(|\psi\rangle) \leqslant F(|\varphi\rangle)$ if $\mu^{\downarrow}(|\varphi\rangle) \prec \mu^{\downarrow}(|\psi\rangle)$, which implies that $|\varphi\rangle$ can be converted to $|\psi\rangle$ by IO. Thus we can easily find that $z' \geqslant z$, which indicates that $F(|\varphi\rangle)$ is a monotonically decreasing function on z. Since $z = \sqrt{1-4|b|^2}$, we can equivalently say that $F(|\varphi\rangle) = \tilde{f}(|b|)$ is a monotonically increasing function on $|b|$.

To prove $\tilde{f}$ is a convex function, we consider a particular state $\sigma = \sum_i p_i|\psi_i\rangle\langle\psi_i|$, where we denote the off-diagonal entries of $|\psi_i\rangle\langle\psi_i|$ by $|b_i|$. So the off-diagonal entry b of σ can be written as $|b| = b = \sum_i p_i|b_i|$. Let $|\varphi_0\rangle \in Q(\sigma)$ and $\{p_i, |\psi_i\rangle\}$ be the exact pure state and the corresponding decomposition of σ required in Theorem 3. Then, Theorem 3 shows

$$F(|\varphi_0\rangle) \leqslant \sum_i p_i F(|\psi_i\rangle).$$

Let $|\varphi\rangle$ be the optimal state such that $\mathscr{C}(\sigma) = F(|\varphi\rangle)$, with $\mathscr{C}(\sigma) \leqslant F(|\varphi_0\rangle)$ due to $|\varphi_0\rangle \in Q(\sigma)$. Then, one will immediately find

$$F(|\varphi\rangle) \leqslant \sum_i p_i F(|\psi_i\rangle).$$

Theorem 12 implies that the off-diagonal element of $|\varphi\rangle\langle\varphi|$ can be the same as σ, so we can write

$$\tilde{f}(|b|) = F(|\varphi\rangle) \leqslant \sum_i p_i F(|\psi_i\rangle) = \sum_i p_i \tilde{f}(|b_i|),$$

which shows the convex $\tilde{f}$.

Conversely, we first assume $\tilde{f}$ is convex. Suppose $\{p_i, |\psi_i\rangle\}$ is an ensemble with b_i denoting the off-diagonal entries of $|\psi_i\rangle\langle\psi_i|$, and denote $\sigma = \sum_i p_i|\psi_i\rangle\langle\psi_i|$. Let $|\phi\rangle$ be the optimal state in $Q(\sigma)$ such that $F(|\phi\rangle) = \mathscr{C}(\sigma)$. Then, Theorem 12 shows that $|\phi\rangle\langle\phi|$ can have the same off-diagonal entry b as σ, and we have

$$\begin{aligned} F(|\phi\rangle) =& \tilde{f}(|b|) = \tilde{f}\left(\left|\sum_i p_i b_i\right|\right) \\ \leqslant& \tilde{f}\left(\sum_i p_i|b_i|\right) \leqslant \sum_i p_i \tilde{f}(|b_i|) \\ =& \sum_i p_i F(|\psi_i\rangle), \end{aligned} \tag{62}$$

where the first inequality comes from the monotonically increasing function $\tilde{f}$ on $|b|$. Obviously, inequality is attributed to the convexity. Equation (62) is the same as Eq. (16). The proof is completed. ■

Based on the above theorems, we have known the optimal pure state $|\varphi\rangle$ for a mixed state σ of the qubit. So one can easily select the coherence measure F for the pure state, then use F to measure the coherence of $|\varphi\rangle$, and, finally, obtain the coherence $\mathscr{C}(\sigma) = F(|\varphi\rangle)$. Here we would like to emphasize that almost all the known coherence measures based on the l_1 norm, relative entropy, geometric coherence, skew information, and so on are convex on $|b|$ for a pure state of a qubit. Therefore, all these measures can be safely employed for F. The concrete expressions are omitted because they become trivially simple due to our theorems.

C. $\mathscr{C}$ as the new coherence monotone and numerical illustrations

Even though we have shown that if the convexity is satisfied by our $\mathscr{C}(\cdot)$, $\mathscr{C}(\cdot)$ will become equivalent to the coherence C_f in terms of the convex roof construction. However, as mentioned previously, the convexity has been considered as a mathematical requirement, so it could not be a necessity for a good coherence quantifier which can also be found in the above mentioned one-shot coherence cost. Now, we will show that our $\mathscr{C}(\cdot)$ can provide a new coherence monotone. Let us consider an explicit example that shows the difference between $\mathscr{C}(\cdot)$ and $C_f(\cdot)$.

Theorem 14. Given a density matrix $\tau = \lambda|\phi\rangle\langle\phi| + (1-\lambda)|3\rangle\langle 3|$, where $|\phi\rangle = \sqrt{\frac{1+z}{2}}|1\rangle + \sqrt{\frac{1-z}{2}}|2\rangle$ $(0 \leqslant z < 1)$, with $|i\rangle_{i=1,2,3}$ denoting the computational basis of the 3-dimension, it is shown that

$$\mathscr{C}(\tau) = F(|\theta\rangle), \tag{63}$$

where $|\theta\rangle$ is a pure state with $\mu^{\downarrow}(|\theta\rangle) = \lambda\mu^{\downarrow}(|\phi\rangle) + (1-\lambda)\mu^{\downarrow}(|3\rangle)$, and $F(\cdot)$ is any given coherence measure defined for pure states satisfying A_1–A_3 and A_5.

Proof. Suppose $\{p_i, |\varphi_i\rangle\}$ is a pure-state decomposition of τ. One can always formally express $|\varphi_i\rangle$ as $|\varphi_i\rangle = x_i|1\rangle + y_i|2\rangle + z_i|3\rangle$, with $\tau = \sum_i p_i|\varphi_i\rangle\langle\varphi_i|$ and p_i denoting the corresponding weights. In particular, the parameters x_i, y_i, z_i satisfy

$$\sum_i p_i|x_i|^2 = \frac{\lambda(1+z)}{2}, \tag{64}$$

$$\sum_i p_i|y_i|^2 = \frac{\lambda(1-z)}{2}, \tag{65}$$

$$\sum_i p_i|z_i|^2 = 1-\lambda, \tag{66}$$

$$\sum_i p_i x_i y_i^* = \frac{\lambda\sqrt{1-z^2}}{2}. \tag{67}$$

It shows that $|\sum_i p_i x_i y_i^*|^2 = (\sum_i p_i|x_i|^2)(\sum_i p_i|y_i|^2)$. The saturation of the Cauchy-Schwarz inequality [84] requires $x_i = gy_i$ for any i, with g being a nonzero constant. Thus it is not difficult to find

$$|g|^2 = \frac{\sum_i p_i|x_i|^2}{\sum_i p_i|y_i|^2} = \frac{1+z}{1-z} \geqslant 1, \tag{68}$$

which implies $|x_i|^2 \geqslant |y_i|^2$. So the first element of $\mu^{\downarrow}(|\varphi_i\rangle)$ satisfies $\mu_1^{\downarrow}(|\varphi_i\rangle) = |x_i|^2$ or $\mu_1^{\downarrow}(|\varphi_i\rangle) = |z_i|^2$. Considering the following inequality:

$$\sum_i p_i\mu_1^{\downarrow}(|\varphi_i\rangle) \leqslant \sum_i p_i|x_i|^2 + \sum_i p_i|z_i|^2 = \frac{\lambda(1+z)}{2} + 1 - \lambda, \tag{69}$$

one can obtain

$$\sum_i p_i\mu^{\downarrow}(|\varphi_i\rangle) \prec \left[\frac{\lambda(1+z)}{2} + 1 - \lambda, \frac{\lambda(1-z)}{2}, 0\right] = \lambda\mu^{\downarrow}(|\phi\rangle) + (1-\lambda)\mu^{\downarrow}(|3\rangle), \tag{70}$$

which means $\tau = \lambda|\phi\rangle\langle\phi| + (1-\lambda)|3\rangle\langle 3|$ is the optimal decomposition in the sense of Theorem 11. Thus we can define a pure state $|\theta\rangle$ such that $\mu^{\downarrow}(|\theta\rangle) = \lambda\mu^{\downarrow}(|\phi\rangle) + (1-\lambda)$ $\mu^{\downarrow}(|3\rangle)$, and from Theorem 11, we have $\mathscr{C}(\tau) = F(|\theta\rangle)$. ■

Based on the above theorem, we can show the difference between $\mathscr{C}(\tau)$ and convex roof $C_f(\tau)$. Reference [50] gives that the convex roof related to F satisfies $C_f(\tau) = \lambda F(|\phi\rangle)$, which is usually not equal to $F(|\theta\rangle)$, that is, $C_f(\tau) \neq \mathscr{C}(\tau)$. To explicitly demonstrate the difference, let us select $F(\cdot) = C_{l_1}(\cdot)$ the coherence measure based on the l_1 norm. Then our coherence monotone for τ can be given by $\mathscr{C}_{l_1}(\tau) = \inf_{|\phi\rangle\in R(\tau)} C_{l_1}(|\phi\rangle)$. Similarly, the coherence convex roof based on the l_1 norm can be given [66] by $C_f^{l_1}(\tau) = \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_{l_1}(|\varphi_i\rangle)$, with $\tau = \sum_i p_i|\varphi_i\rangle\langle\varphi_i|$. The numerical comparison is plotted in Fig. 1(a), which demonstrates the apparent difference. In addition, we also consider the coherence based on the relative entropy as the coherence measure for pure states. Thus, $\mathscr{C}(\cdot)$ based on the relative entropy is given by comparing it with $\mathscr{C}_r(\tau) = \inf_{|\phi\rangle\in R(\tau)} C_r(|\phi\rangle)$ and the coherence measure in terms of the convex roof construction is exactly given by $C_f^r(\tau)$. Their numerical comparison is shown in Fig. 1(b), which also illustrates their difference.

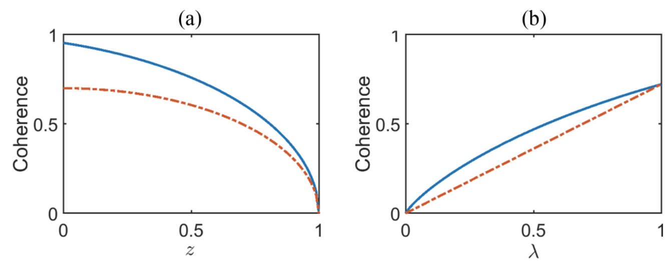

FIG. 1. Comparisons (a) between $\mathscr{C}_{l_1}(\tau)$ (solid line) and $C_f^{l_1}(\tau)$ (dot-dashed line) with $\lambda = 0.7$, $z \in [0, 1]$, and (b) between $\mathscr{C}_r(\tau)$ (solid line) and $C_f^r(\tau)$ (dot-dashed line) with $z = 0.6$, $\lambda \in [0, 1]$.

VII. DISCUSSION AND CONCLUSIONS

In conclusion, we have presented an approach to quantifying quantum coherence. Our coherence measure can be understood as the least coherence of the pure states which can be converted to the state of interest. In particular, we have shown that our coherence monotone is the supremum of all the coherence monotones with the same coherence for any given pure state. Our coherence measure is proven to be a subset of the coherence measure in terms of the convex roof construction, which gives a different understanding of the coherence measure. In addition, we have shown that our coherence monotone is continuous. The comparison with coherence cost shows that our coherence monotone is generally not the same as coherence cost, but with some particular coherence measure for pure states, our coherence monotone can also be equivalent to the coherence cost. As a demonstration, we give concrete examples for our coherence measure. It is especially important that we have thoroughly analyzed the case of qubit states and have given a series of analytic expressions of coherence. We have also given an example to explicitly illustrate the difference between our coherence monotone and that based on the convex roof construction by selecting the l_1 norm and relative entropy as the coherence measure for pure states, which further indicates that our approach can induce new coherence monotones. Finally, we would like to emphasize that the same approach could also be suitable for other resource theories, which will be studied in further work.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China under Grants No. 11775040, No. 12011530014, and No. 11375036.

APPENDIX A: AN ALTERNATIVE PROOF OF STRONG MONOTONICITY SUBJECT TO SIO

Lemma 2. Given an SIO operated on the state ρ as $\varepsilon_K(\rho)=\sum_i K_i\rho K_i^\dagger$, there always exists a corresponding SIO on the pure state $|\psi\rangle\in R(\rho)$ as $\varepsilon_T(|\psi\rangle\langle\psi|)=\sum_i T_i|\psi\rangle\langle\psi|T_i^\dagger$ such that

$$\mathrm{Tr}(T_i|\psi\rangle\langle\psi|T_i^\dagger)=\mathrm{Tr}(K_i\rho K_i^\dagger) \tag{A1}$$

for any given i. In addition, the state $T_i|\psi\rangle\langle\psi|T_i^\dagger/\mathrm{Tr}(T_i|\psi\rangle\langle\psi|T_i^\dagger)$ can be converted into $K_i\rho K_i^\dagger/\mathrm{Tr}(K_i\rho K_i^\dagger)$ by SIO.

Proof. For an n-dimensional pure state $|\psi\rangle$ given, with respect to the computational basis, by

$$|\psi\rangle\langle\psi|=\sum_m |c_m|^2|m\rangle\langle m|+\sum_{m\neq n}c_mc_n^*|m\rangle\langle n|, \tag{A2}$$

let us consider three SIO $\varepsilon_M=\{M_l\}$, $\varepsilon_K=\{K_i\}$ and $\varepsilon_T=\{T_j\}$ defined, respectively, by

$$M_l=\sum_\gamma a_\gamma^{(l)}|\pi_l(\gamma)\rangle\langle\gamma|,\quad l=1,2,\dots, \tag{A3}$$

$$K_i=\sum_\gamma \tau_\gamma^{(i)}|f_i(\gamma)\rangle\langle\gamma|,\quad i=1,2,\dots, \tag{A4}$$

and

$$T_j=\sum_\gamma d_\gamma^{(j)}|\gamma\rangle\langle\gamma|, \tag{A5}$$

where $\{|\gamma\rangle\}$ is the computational basis, π_l and f_i are the permutation function labeled by l and i, respectively, $\sum_l|a_\gamma^{(l)}|^2=\sum_i|\tau_\gamma^{(i)}|^2=1$, and

$$\left|d_\gamma^{(i)}\right|^2=\sum_l\left|a_\gamma^{(l)}\right|^2\left|\tau_{\pi_l(\gamma)}^{(i)}\right|^2. \tag{A6}$$

Thus, one can easily find that $\sum_i|d_\gamma^{(i)}|^2=\sum_l|a_\gamma^{(l)}|^2\sum_i|\tau_{\pi_l(\gamma)}^{(i)}|^2=1$. With the above SIO, we have

$$\begin{aligned}
&K_iM_l|\psi\rangle\langle\psi|M_l^\dagger K_i^\dagger\\
&=\sum_m|c_m|^2\left|a_m^{(l)}\right|^2K_i|\pi_l(m)\rangle\langle\pi_l(m)|K_i^\dagger\\
&\quad+\sum_{m\neq n}c_mc_n^*a_m^{(l)}a_n^{(l)*}K_i|\pi_l(m)\rangle\langle\pi_l(n)|K_i^\dagger\\
&=\sum_m|c_m|^2\left|a_m^{(l)}\right|^2\left|\tau_{\pi_l(m)}^{(i)}\right|^2|\pi_l(m)\rangle\langle\pi_l(m)|\\
&\quad+\sum_{m\neq n}c_mc_n^*a_m^{(l)}a_n^{(l)*}\tau_{\pi_l(m)}^{(i)}\tau_{\pi_l(n)}^{(i)*}|f_i[\pi_l(m)]\rangle\langle f_i[\pi_l(n)]|.
\end{aligned} \tag{A7}$$

Therefore,

$$\begin{aligned}
\mathrm{Tr}(K_i\rho K_i^\dagger)&=\sum_l\mathrm{Tr}(K_iM_l|\psi\rangle\langle\psi|M_l^\dagger K_i^\dagger)\\
&=\sum_l\sum_m|c_m|^2\left|a_m^{(l)}\right|^2\left|\tau_{\pi_l(m)}^{(i)}\right|^2.
\end{aligned} \tag{A8}$$

Similarly,

$$\begin{aligned}
&\mathrm{Tr}[T_i|\psi\rangle\langle\psi|T_i^\dagger]\\
&=\sum_m|c_m|^2\left|d_m^{(i)}\right|^2\langle m|m\rangle+\sum_{m\neq n}c_mc_n^*d_m^{(i)}d_n^{(i)*}\langle n|m\rangle\\
&=\sum_m|c_m|^2\left|d_m^{(i)}\right|^2.
\end{aligned} \tag{A9}$$

Based on Eq. (A6), it is obvious that Eqs. (A8) and (A9) imply $\mathrm{Tr}(K_i\rho K_i^\dagger)=\mathrm{Tr}[T_i|\psi\rangle\langle\psi|T_i^\dagger]$. Thus, Eq. (A1) is proved.

To proceed, let us consider an SIO $\varepsilon_N^{(i)}=\{N_l\}$ defined by

$$N_l=\sum_\gamma\frac{a_\gamma^{(l)}\tau_{\pi_l(\gamma)}^{(i)}}{d_\gamma^{(i)}}|f_i[\pi_l(\gamma)]\rangle\langle\gamma|, \tag{A10}$$

with

$$\begin{aligned}
\sum_l N_l^\dagger N_l&=\sum_l\sum_\gamma\frac{\left|a_\gamma^{(l)}\right|^2\left|\tau_{\pi_l(\gamma)}^{(i)}\right|^2}{\left|d_\gamma^{(i)}\right|^2}|\gamma\rangle\langle\gamma|\\
&=\sum_\gamma|\gamma\rangle\langle\gamma|\frac{\sum_l\left|a_\gamma^{(l)}\right|^2\left|\tau_{\pi_l(\gamma)}^{(i)}\right|^2}{\left|d_\gamma^{(i)}\right|^2}\\
&=\sum_\gamma|\gamma\rangle\langle\gamma|.
\end{aligned} \tag{A11}$$

Then, one can see that

$$\begin{aligned}
&\varepsilon_N(T_i|\psi\rangle\langle\psi|T_i^\dagger)\\
&=\sum_l N_lT_i|\psi\rangle\langle\psi|T_i^\dagger N_l^\dagger\\
&=\sum_l N_l\Bigg(\sum_m|c_m|^2\left|d_m^{(i)}\right|^2|m\rangle\langle m|\\
&\quad+\sum_{m\neq n}c_mc_n^*d_m^{(i)}d_n^{(i)*}|m\rangle\langle n|\Bigg)N_l^\dagger\\
&=\sum_l\sum_m|c_m|^2\left|a_m^{(l)}\right|^2\left|\tau_{\pi_l(m)}^{(i)}\right|^2|f_i[\pi_l(m)]\rangle\langle f_i[\pi_l(m)]|\\
&\quad+\sum_l\sum_{m\neq n}c_mc_n^*a_m^{(l)}a_n^{(l)*}\tau_{\pi_l(m)}^{(i)}\tau_{\pi_l(n)}^{(i)*}|f_i[\pi_l(m)]\rangle\langle f_i[\pi_l(n)]|\\
&=K_i\rho K_i^\dagger,
\end{aligned} \tag{A12}$$

which completes the proof. ■

To show the strong monotonicity of SIO, we need to prove

$$\mathscr{C}(\rho)\geqslant\sum_i p_i\mathscr{C}\left(\frac{K_i\rho K_i^\dagger}{p_i}\right),$$

where $\{K_i\}$ are Kraus operators of arbitrarily given SIO ε_K and $p_i=\mathrm{Tr}(K_i\rho K_i^\dagger)$.

Let $|\psi\rangle\in R(\rho)$, corresponding to the state ρ. According to Lemma 2, there exists an SIO $\varepsilon_T=\{T_i\}$ such that

$$\mathrm{Tr}(T_i|\psi\rangle\langle\psi|T_i^\dagger)=\mathrm{Tr}(K_i\rho K_i^\dagger)=p_i,$$

and $\frac{T_i|\psi\rangle\langle\psi|T_i^\dagger}{p_i}$ can be converted into $\frac{K_i\rho K_i^\dagger}{p_i}$ by SIO. Thus,

$$\begin{aligned}\mathscr{C}(\rho) = F(|\psi\rangle) &\geqslant \sum_i p_i F\left(\frac{T_i|\psi\rangle\langle\psi|T_i^\dagger}{p_i}\right) \\ &= \sum_i p_i \mathscr{C}\left(\frac{T_i|\psi\rangle\langle\psi|T_i^\dagger}{p_i}\right) \geqslant \sum_i p_i \mathscr{C}\left(\frac{K_i\rho K_i^\dagger}{p_i}\right),\end{aligned} \tag{A13}$$

where the first inequality results from the strong monotonicity of F, and the second inequality is due to $\frac{T_i|\psi\rangle\langle\psi|T_i^\dagger}{p_i} \rightarrow \frac{K_i\rho K_i^\dagger}{p_i}$ by SIO and the monotonicity of $\mathscr{C}$.

APPENDIX B: THE EXAMPLES FOR THE DIFFERENCE OF THE SET $R(\cdot)$

1. $R(\cdot)$ based on PIO is different from that based on IO

Consider the state

$$|\phi\rangle = (\sqrt{1+z}|1\rangle + \sqrt{1-z}|2\rangle)/\sqrt{2}, \tag{B1}$$

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z' & p\sqrt{1-z'^2} \\ p\sqrt{1-z'^2} & 1-z' \end{pmatrix}, \tag{B2}$$

with $0 < z < z' < 1$ and $\frac{(1-z')\sqrt{1-z^2}}{(1-z)\sqrt{1-z'^2}} < p < 1$. It is easy to check that $|\phi\rangle$ can be converted to ρ by IO from the theorems in Refs. [49,85], but PIO cannot achieve the transformation from the theorems in Ref. [86]. Thus, $|\phi\rangle$ is not an element in $R(\rho)$ under PIO, but an element in $R(\rho)$ under IO.

2. $R(\cdot)$ based on DIO (MIO) is different from that based on IO

Let

$$|\psi\rangle = (\sqrt{1+z}|1\rangle + \sqrt{1-z}|2\rangle)/\sqrt{2},$$

$$\sigma = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{1-z^2}}{4\sqrt{3}} & 0 & \frac{\sqrt{1-z^2}}{4\sqrt{3}} \\ \frac{\sqrt{1-z^2}}{4\sqrt{3}} & \frac{1}{4} & \frac{\sqrt{1-z^2}}{4\sqrt{3}} & 0 \\ 0 & \frac{\sqrt{1-z^2}}{4\sqrt{3}} & \frac{1}{4} & -\frac{\sqrt{1-z^2}}{4\sqrt{3}} \\ \frac{\sqrt{1-z^2}}{4\sqrt{3}} & 0 & -\frac{\sqrt{1-z^2}}{4\sqrt{3}} & \frac{1}{4} \end{pmatrix}. \tag{B3}$$

From Ref. [80], there exists a DIO (naturally, an MIO) Φ satisfying $\sigma = \Phi(|\psi\rangle\langle\psi|)$, but σ cannot be converted from $|\psi\rangle$ by IO. Thus, $R(\sigma)$ defined under DIO (MIO) including $|\psi\rangle$ does not belong to $R(\sigma)$ under IO. The Kraus operations of Φ are given as [80]

$$K_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \end{pmatrix}, \quad K_4 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 & -\frac{\sqrt{6}}{3} \\ -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

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