Quantum ϕ synchronization in a coupled optomechanical system with periodic modulation

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Based on the concepts of quantum synchronization and quantum phase synchronization proposed by A. Mari *et al.* [Phys. Rev. Lett. **111**, 103605 (2013)], we introduce and characterize the measure of a more generalized quantum synchronization called quantum ϕ synchronization under which pairs of variables have the same amplitude and possess the same ϕ phase shift. Naturally, quantum synchronization and quantum antisynchronization become special cases of quantum ϕ synchronization. Their relations and differences are also discussed. To illustrate these theories, we investigate the quantum ϕ synchronization and quantum phase synchronization phenomena of two coupled optomechanical systems with periodic modulation and show that quantum ϕ synchronization is more general as a measure of synchronization. We also show the phenomenon of quantum antisynchronization when $\phi = \pi$.

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I. INTRODUCTION

As a collective dynamic behavior in complex systems, synchronization was first proposed by Huygens in the 17th century [1,2]. He noticed that the oscillations of two pendulum clocks with a common support tend to synchronize with each other [3]. Since then, synchronization has been widely studied and applied in classical physics. Furthermore, with the development of quantum mechanics, the concept of quantum synchronization was proposed and widely applied in fields such as cavity quantum electrodynamics [4,5], atomic ensembles [6–8], van der Pol oscillators [5,9–11], Bose-Einstein condensation [12], and superconducting circuit systems [13,14].

In recent years, there has been growing interest in exploiting synchronization [15] for significant applications in microscale and nanoscale systems [16]. For example, synchronization of two anharmonic nanomechanical oscillators is implemented in [17]. And, experimentally, the synchronization measure of the system has been realized through optomechanical devices, including the synchronization of two nanomechanical beam oscillators coupled by a mechanical element [18], two dissimilar silicon nitride micromechanical oscillators coupled by an optical cavity radiation field [19], and two nanomechanical oscillators via a photonic resonator [20]. These ingenious experiments fully test the theoretical prediction of the synchronization of optomechanical systems. In addition, the relationship between quantum synchronization and the collective behavior of classical systems is also widely concerned, such as quantum synchronization of van der Pol oscillators with trapped ions [21] and quantum-classical transition of correlations of two coupled cavities [22]. In addition, the role of the environment and the correlation between subsystems in a system with quantum synchronization, such as entanglement and mutual information, have been discussed as the main influencing factors [23–25].

Another aspect of synchronization drawing much more attention recently is the generalization of its classical concepts into continuous-variable quantum systems, such as complete synchronization [26], phase synchronization [27,28], lag synchronization [29], and generalized synchronization [30]. After Mari *et al.* introduced the concepts of quantum complete synchronization and quantum phase synchronization [31], some interesting efforts have been devoted to enhancing quantum synchronization and quantum phase synchronization by manipulating modulation [32,33], changing the ways of coupling between two subsystems [31,34–36], and introducing nonlinearity [37,38]. Furthermore, the concepts of quantum generalized synchronization, time-delay synchronization, and in-phase and antiphase synchronization have also been mentioned in [39] and [40]. However, other than quantum complete synchronization under which pairs of variables have the same amplitude and phase, the concept of quantum antisynchronization corresponding to classical antisynchronization has not been proposed yet. Moreover, a more generalized quantum synchronization can be defined as follows: "The pairs of variables have the same amplitude and possess the same ϕ phase shift" (hereafter referred to as quantum ϕ synchronization), i.e., for $\phi = \pi$, the pairs of variables, such as positions and momenta, will always have a π phase difference with each other [40]. This type of quantum ϕ synchronization is called quantum antisynchronization. Hence, one will naturally ask how to define and measure quantum ϕ synchronization.

To shed light on this question, in this work we give the definition of quantum ϕ synchronization for continuous-variable quantum systems by combining the concept of quantum synchronization and the phenomenon of transition from in-phase to antiphase synchronization [40]. The paper is organized

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as follows. In Sec. II, we first reexamine the definitions of quantum complete synchronization and phase synchronization. Based on these concepts, the definition of quantum ϕ synchronization is given, under which quantum synchronization and quantum antisynchronization can be treated as special cases of quantum ϕ synchronization. The ϕ synchronization of a coupled optomechanical system with periodic modulation is studied to illustrate our theory in Sec. III. In Sec. IV, a brief discussion and summary are given.

II. MEASURE OF QUANTUM SYNCHRONIZATION AND QUANTUM ϕ SYNCHRONIZATION

Unlike the synchronization in classical systems, the complete synchronization in quantum systems cannot be defined straightforwardly, since fluctuations of the variables in the two subsystems must adhere strictly to the limits brought by the Heisenberg principle. To address this issue, Mari *et al.* proposed the measurement criterion of quantum complete synchronization for continuous-variable systems [31],

$$S_c = \frac{1}{\langle q_-(t)^2 + p_-(t)^2\rangle}, \tag{1}$$

where $q_-(t) = \frac{1}{\sqrt{2}}[q_1(t) - q_2(t)]$ and $p_-(t) = \frac{1}{\sqrt{2}}[p_1(t) - p_2(t)]$ are error operators. In order to study purely quantum mechanical effects, the changes of variables are generally taken as

$$\begin{aligned} q_-(t) &\to \delta q_-(t) = q_-(t) - \langle q_-(t)\rangle, \\ p_-(t) &\to \delta p_-(t) = p_-(t) - \langle p_-(t)\rangle. \end{aligned} \tag{2}$$

Then the contribution of the classical systematic error brought by the mean values $\langle q_-(t)\rangle$ and $\langle p_-(t)\rangle$ in S_c can be dropped, and S_c is replaced by the pure quantum synchronization measure

$$S_q = \frac{1}{\langle \delta q_-(t)^2 + \delta p_-(t)^2\rangle}. \tag{3}$$

This definition requires that the mean values of $q_-(t)$ and $p_-(t)$ are exactly 0, i.e., $\langle q_-(t)\rangle = 0$ and $\langle p_-(t)\rangle = 0$.

Mari *et al.* have explained that if the averaged phase-space trajectories (limit cycles) of the two systems are constant but slightly different from each other, a classical systematic error can be easily excluded from the measure of synchronization [31]. The mean-value synchronization is also regarded as a necessary condition of pure quantum synchronization [38]. It is more reasonable and rigorous to study pure quantum synchronization based on mean-value synchronization. Therefore, we can generalize the definition of quantum complete synchronization into quantum ϕ synchronization

$$S_\phi = \frac{1}{\langle q_-^\phi(t)^2 + p_-^\phi(t)^2\rangle}, \tag{4}$$

which does not require mean-value synchronization. The ϕ error operators are defined as $q_-^\phi(t) = \frac{1}{\sqrt{2}}[q_1^\phi(t) - q_2^\phi(t)]$ and $p_-^\phi(t) = \frac{1}{\sqrt{2}}[p_1^\phi(t) - p_2^\phi(t)]$ with

$$\begin{aligned} q_j^\phi(t) &= q_j(t)\cos(\phi_j) + p_j(t)\sin(\phi_j), \\ p_j^\phi(t) &= p_j(t)\cos(\phi_j) - q_j(t)\sin(\phi_j), \end{aligned} \tag{5}$$

where the phase $\phi_j = \arctan[\langle p_j(t)\rangle/\langle q_j(t)\rangle]$, $\phi_j \in [0, 2\pi]$. The upper limit of S_ϕ is also given by the Heisenberg principle,

$$\begin{aligned} S_\phi &= \frac{1}{\langle q_-^\phi(t)^2 + p_-^\phi(t)^2\rangle} \\ &\leqslant \frac{1}{2\sqrt{\langle q_-^\phi(t)^2\rangle\langle p_-^\phi(t)^2\rangle}} \\ &\leqslant \frac{1}{2\sqrt{[\langle q_-^\phi(t)^2\rangle - \langle q_-^\phi(t)\rangle^2][\langle p_-^\phi(t)^2\rangle - \langle p_-^\phi(t)\rangle^2]}} \\ &\leqslant \frac{1}{\sqrt{\left|\frac{1}{2}[q_1^\phi(t), p_1^\phi(t)] + \frac{1}{2}[q_2^\phi(t), p_2^\phi(t)]\right|^2}} \\ &= 1. \end{aligned} \tag{6}$$

This means that the closer S_ϕ is to 1, the better the quantum ϕ synchronization. Again, let us take the changes of variables

$$\begin{aligned} q_-^\phi(t) &\to \delta q_-^\phi(t) = q_-^\phi(t) - \langle q_-^\phi(t)\rangle, \\ p_-^\phi(t) &\to \delta p_-^\phi(t) = p_-^\phi(t) - \langle p_-^\phi(t)\rangle. \end{aligned} \tag{7}$$

The mean values of $q_-^\phi(t)$ and $p_-^\phi(t)$ are 0 when the average amplitude and period of the mean value of the two variables are the same. In this case, S_ϕ equals the pure quantum ϕ-synchronization measure S_q^ϕ mathematically,

$$\begin{aligned} S_q^\phi &= \frac{1}{\langle \delta q_-^\phi(t)^2 + \delta p_-^\phi(t)^2\rangle} \\ &= \Big\langle \frac{1}{2}[(\delta p_1)^2 + (\delta q_1)^2 + (\delta p_2)^2 + (\delta q_2)^2 \\ &\quad + 2(\delta p_1\delta q_2 - \delta q_1\delta p_2)\sin\phi - 2(\delta p_1\delta p_2 + \delta q_1\delta q_2) \\ &\quad \times \cos\phi]\Big\rangle^{-1}, \end{aligned} \tag{8}$$

where $\phi = \phi_2 - \phi_1$ can be determined by the final steady state. Now let us explain the relationship between quantum ϕ synchronization and quantum complete synchronization, quantum phase synchronization. We can see from Eq. (8) that the definition of S_q^ϕ can be reduced to (a) quantum synchronization—if $\phi = 0$, then $S_q = S_q^\phi$; or (b) quantum phase synchronization—if $\langle \delta q_-^\phi(t)^2\rangle = \langle \delta p_-^\phi(t)^2\rangle$, then $S_p = S_q^\phi$, or (c) if $\phi = \pi$, then $\widetilde{S}_q = S_q^\phi$, where $\widetilde{S}_q$ can be defined as quantum antisynchronization. Therefore, quantum synchronization and quantum antisynchronization are special cases of quantum ϕ synchronization. But the definition of quantum phase synchronization is slightly different [31]:

$$S_p = \frac{1}{2}\langle \delta p_-^\phi(t)^2\rangle^{-1} = \frac{1}{\langle \delta p_-^\phi(t)^2 + \delta p_-^\phi(t)^2\rangle}. \tag{9}$$

Unlike S_ϕ in Eq. (6), the measure of quantum phase synchronization S_p can exceed 1. To illustrate these definitions, we next compare quantum ϕ synchronization with quantum synchronization and quantum phase synchronization in coupled optomechanical systems with periodic modulation.

III. QUANTUM SYNCHRONIZATION, QUANTUM PHASE SYNCHRONIZATION, AND QUANTUM ϕ SYNCHRONIZATION IN COUPLED OPTOMECHANICAL SYSTEMS WITH PERIODIC MODULATION

To examine the relations and differences between quantum synchronization, quantum phase synchronization, and quantum ϕ synchronization, we consider a coupled optomechanical system with periodic modulation [36,37]. Two subsystems are coupled by optical fibers [41]. Each of them consists of a mechanical oscillator coupled with a Fabry-Perot cavity driven by a time-periodic modulated feld (see Fig. 1) [42]. It is noteworthy that the optomechanical device is experimentally possible. On the one hand, the synchronization of two mechanically isolated nanomechanical resonators via a photonic resonator has been implemented [20]. On the other hand, a self-oscillating mechanical resonator in an on-fiber optomechanical cavity excited by a tunable laser with periodically modulated power also has been studied [32]. Then the Hamiltonian of the whole coupled system can be written as

$$H = \sum_{j=1}^{2}\Big\{-\triangle_j[1+A_c\cos(\omega_c t)]a_j^\dagger a_j + \frac{\omega_j}{2}\left(p_j^2+q_j^2\right) - g a_j^\dagger a_j q_j + iE(a_j^\dagger - a_j)\Big\} + \lambda(a_1^\dagger a_2 + a_2^\dagger a_1), \tag{10}$$

where a and $a^\dagger$ are the creation and annihilation operators, and q_j and p_j are the position and momentum operators of the mechanical oscillator with frequency ω_j in the jth subsystem, respectively [43,44]. λ is the optical coupling strength and E is the intensity of the driving field. $\triangle_j$ is the optical detuning, which is modulated with a common frequency ω_c and amplitude A_c. g is the optomechanical coupling constant. To solve the time evolution of the dynamical operators $O = q_j, p_j, a_j$ of the system, we consider the dissipation effects in the Heisenberg picture and utilize the quantum Langevin equation [45]. From Eq. (10), the evolution equations of the operators can be written as

$$\begin{aligned}
\dot{q_j} =& \omega_j p_j,\\
\dot{p_j} =& -\omega_j q_j - \gamma p_j + g a_j^\dagger a_j + \xi_j,\\
\dot{a_j} =& -\{\kappa - i\triangle_j[1+A_c\cos(\omega_c t)]\}a_j + i g a_j q_j + E\\
& - i\lambda a_{3-j} + \sqrt{2\kappa}a_j^{\text{in}},
\end{aligned} \tag{11}$$

where κ is the radiation loss coefficient [46,47] and γ is the mechanical damping rate. a_j^{in} and ξ_j are input bath operators and satisfy the standard correlation

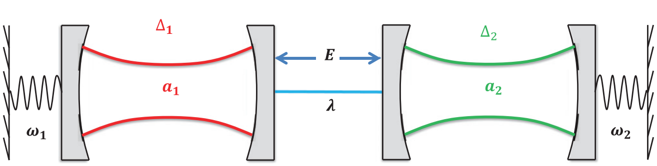

FIG. 1. Schematic of the coupled optomechanical system with periodic modulation.

$\langle a^{\text{in}^\dagger}(t)a^{\text{in}}(t') + a^{\text{in}}(t')a^{\text{in}^\dagger}(t)\rangle = \delta(t-t')$ and $\frac{1}{2}\langle \xi_j(t)\xi_{j'}(t') + \xi_{j'}(t')\xi_j(t)\rangle = \gamma(2\bar{n}_{\text{bath}}+1)\delta_{jj'}\delta(t-t')$ under the Markovian approximation [43,44], where $\bar{n}_{\text{bath}} = 1/[\exp(\hbar\omega_j/k_B T) - 1]$ is the mean occupation number of the mechanical baths, which gauges the temperature T of the system [48–50]. To solve the set of nonlinear differential operator equations, we need to linearize Eq. (11). There are several ways to do that, such as using the stochastic Hamiltonian [51,52] and the mean-field approximation [34,39,53,54]. Here we use the mean-field approximation, since it can uncover the effects of the mean error and quantum fluctuation on quantum synchronization. Namely, the operators are decomposed into a mean value and a small fluctuation, i.e.,

$$O(t) = \langle O(t)\rangle + \delta O(t). \tag{12}$$

And as long as $|\langle O(t)\rangle| \gg 1$, the usual linearization approximation to Eq. (11) can be implemented [35]. Then Eq. (11) can be divided into two different sets of equations, one for the mean value,

$$\begin{aligned}
&\partial_t\langle q_j\rangle = \omega_j\langle p_j\rangle,\\
&\partial_t\langle p_j\rangle = -\omega_j\langle q_j\rangle - \gamma\langle p_j\rangle + g|\langle a_j\rangle|^2,\\
&\partial_t\langle a_j\rangle = -\{\kappa - i\triangle_j[1+A_c\cos(\omega_c t)]\}\langle a_j\rangle + ig\langle a_j\rangle\langle q_j\rangle + E\\
&\qquad - i\lambda\langle a_{3-j}\rangle,
\end{aligned} \tag{13}$$

and the other for the fluctuation,

$$\begin{aligned}
&\partial_t\delta q_j = \omega_j\delta p_j,\\
&\partial_t\delta p_j = -\omega_j\delta q_j - \gamma\delta p_j + g(\langle a_j\rangle\delta a_j^\dagger + \langle a_j\rangle^*\delta a_j) + \xi_j,\\
&\partial_t\delta a_j = -\{\kappa - i\triangle_j[1+A_c\cos(\omega_c t)]\}\delta a_j + ig(\langle a_j\rangle\delta q_j\\
&\qquad + \langle q_j\rangle\delta a_j) - i\lambda\delta a_{3-j} + \sqrt{2\kappa}a_j^{\text{in}}.
\end{aligned} \tag{14}$$

In Eq. (14), the second-order smaller terms of the fluctuation have been ignored. Then, by defining $u = (\delta q_1, \delta p_1, \delta x_1, \delta y_1, \delta q_2, \delta p_2, \delta x_2, \delta y_2)^\top$ with $\delta x_j = \frac{1}{\sqrt{2}}(\delta a_j^\dagger + \delta a_j)$ and $\delta y_j = \frac{i}{\sqrt{2}}(\delta a_j^\dagger - \delta a_j)$, Eq. (14) can be simplified to

$$\partial_t u = Mu + n, \tag{15}$$

where $n = (0, \xi_1, \sqrt{2\kappa}x_1^{\text{in}}, \sqrt{2\kappa}y_1^{\text{in}}, 0, \xi_2, \sqrt{2\kappa}x_2^{\text{in}}, \sqrt{2\kappa}y_2^{\text{in}})^\top$ is the noise vector with $x_1^{\text{in}} = \frac{1}{\sqrt{2}}(a^{\text{in}^\dagger} + a^{\text{in}})$ and $y_1^{\text{in}} = \frac{i}{\sqrt{2}}(a^{\text{in}^\dagger} - a^{\text{in}})$. M is a time-dependent coefficient matrix,

$$M = \begin{pmatrix} M_1 & M_0\\ M_0 & M_2\end{pmatrix}, \tag{16}$$

with

$$M_j = \begin{pmatrix} 0 & \omega_j & 0 & 0 \\ -\omega_j & -\gamma & \sqrt{2}g\mathrm{Re}(\langle a_j\rangle) & \sqrt{2}g\mathrm{Im}(\langle a_j\rangle) \\ -\sqrt{2}g\mathrm{Im}(\langle a_j\rangle) & 0 & -\kappa & -F_j \\ \sqrt{2}g\mathrm{Re}(\langle a_j\rangle) & 0 & F_j & -\kappa \end{pmatrix},$$

and

$$M_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & -\lambda & 0 \end{pmatrix},$$

where $F_j = \Delta_j[1 + A_c\cos(\omega_c t)] + g\langle q_j\rangle$ and the evolution process of matrix element $M(t)$ at any time can be obtained by solving Eq. (13) numerically when the initial conditions are 0. In order to study the contribution of quantum fluctuation to quantum synchronization, we consider the following covariance matrix:

$$V_{ij} \equiv \tfrac{1}{2}\langle u_i u_j + u_j u_i\rangle. \tag{17}$$

The evolution of V over time is governed by [35,53,55,56]

$$\partial_t V = MV + VM^T + N. \tag{18}$$

The noise matrix $N = \mathrm{diag}(0, \gamma(2\bar{n}_{\mathrm{bath}} + 1), \kappa, \kappa, 0, \gamma(2\bar{n}_{\mathrm{bath}} + 1), \kappa, \kappa)$ satisfying $N_{ij}\delta(t - t') = \frac{1}{2}\langle\xi_i(t)\xi_j(t') + \xi_j(t')\xi_i(t)\rangle$. Hence, Eq. (3), Eq. (8), and Eq. (9) can be rewritten in terms of V_{ij},

$$\begin{aligned} S_q &= \left[\tfrac{1}{2}(V_{11} + V_{22} + V_{55} + V_{66} - V_{15} - V_{51} - V_{62} - V_{26})\right]^{-1}, \\ S_q^\phi &= \left[\tfrac{1}{2}(V_{11} + V_{22} + V_{55} + V_{66} + 2V_{25}\sin\phi - 2V_{16}\sin\phi \right. \\ &\quad \left. - 2V_{26}\cos\phi - 2V_{15}\cos\phi)\right]^{-1}, \\ S_p &= [V_{11}(\sin\phi_1)^2 + V_{22}(\cos\phi_1)^2 + V_{55}(\sin\phi_2)^2 \\ &\quad + V_{66}(\cos\phi_2)^2 - 2V_{12}\sin\phi_1\cos\phi_1 - 2V_{15}\sin\phi_1\sin\phi_2 \\ &\quad + 2V_{16}\sin\phi_1\cos\phi_2 + 2V_{25}\cos\phi_1\sin\phi_2 \\ &\quad - 2V_{26}\cos\phi_1\cos\phi_2 - 2V_{56}\cos\phi_2\sin\phi_2]^{-1}, \end{aligned} \tag{19}$$

and their evolutions can be derived by solving Eq. (13), Eq. (15), and Eq. (18). In addition, under different parameters, the calculated time-averaged synchronization

$$\overline{S_o}(t) = \lim_{T\to\infty}\frac{1}{T}\int_0^T S_o(t)dt \tag{20}$$

is used as the synchronization measure in the asymptotic steady state of the system, where $o = \phi, p$. According to the R-H criterion [57], all the eigenvalues of coefficient matrix M will be negative after a temporary evolutionary process. Hence, a stable limit-cycle solution representing a periodic oscillation will exist [58].

As discussed in the last section, if $\phi = 0$, which requires the condition of mean-value complete synchronization, i.e., $\langle q_-(t)\rangle = \langle q_1(t)\rangle - \langle q_2(t)\rangle = 0$, $\langle p_-(t)\rangle = \langle p_1(t)\rangle - \langle p_2(t)\rangle = 0$, the measures of quantum ϕ synchronization and quantum synchronization are equivalent, i.e., $S_q^\phi = S_q$. As shown in Fig. 2(a), $\langle q_1(t)\rangle$ and $\langle q_2(t)\rangle$ are found to oscillate exactly in phase when entering the stable state. The same conclusion holds for $\langle p_1(t)\rangle$ and $\langle p_2(t)\rangle$ in Fig. 2(b). In Fig. 2(c), the evolutions of $\langle q_1(t)\rangle \leftrightharpoons \langle p_1(t)\rangle$ and $\langle q_2(t)\rangle \leftrightharpoons \langle p_2(t)\rangle$ of the two oscillators trend to an asymptotic periodic orbit (i.e., the two limit cycles tend to be consistent), which indicates that the system is stable. And Figs. 2(d) and 2(e) show that the changes in $\overline{S_q}$ and $\overline{S_q^\phi}$ over time are exactly the same. When the mean-value synchronization is not complete, as shown in Figs. 3(a) and 3(b), there exists a phase advance between ϕ_2 and ϕ_1, i.e., $\phi = \phi_2 - \phi_1 = \arctan[\langle p_2(t)\rangle/\langle q_2(t)\rangle] - \arctan[\langle p_1(t)\rangle/\langle q_1(t)\rangle] \approx 0.2\pi$. Similarly, the two consistent limit cycles are shown in Fig. 3(c), indicating that the evolution of the system can still reach a steady state when the mean value is not in complete synchronization. However, quantum synchronization $\overline{S_q}$ and quantum ϕ synchronization are different as shown in Fig. 3(d). This is because the definition of quantum ϕ synchronization takes the effect of mean-value incomplete synchronization into account. Quantum ϕ synchronization is then more general and rigorous than quantum synchronization. As mean-value incomplete synchronization will break the condition for Eq. (2), the contribution of $\langle q_-(t)\rangle$ and $\langle p_-(t)\rangle$ to the quantum complete synchronization S_c is much greater than that of the quantum fluctuation. Besides, mean-value incomplete synchronization will always occur with a change of parameters. As shown in Fig. 4, different phase differences ϕ will be generated by a different modulation frequency ω_c. And similar phenomena have been shown in [40]. Therefore, it is necessary to give the quantum

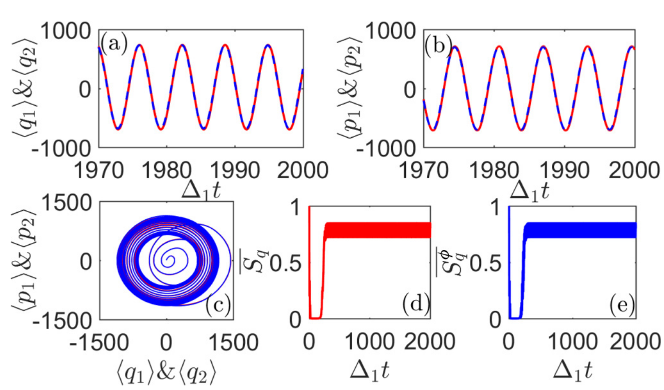

FIG. 2. Time evolution of (a) the mean values $\langle q_1\rangle$ (solid red line) and $\langle q_2\rangle$ (dashed blue line), (b) the mean values $\langle p_1\rangle$ (solid red line) and $\langle p_2\rangle$ (dashed blue line), (c) the limit-cycle trajectories in the $\langle q_1\rangle \leftrightharpoons \langle p_1\rangle$ (red) and $\langle q_2\rangle \leftrightharpoons \langle p_2\rangle$ (blue) spaces, (d) the measure of quantum synchronization $\overline{S_q}$, and (e) the measure of quantum ϕ synchronization $\overline{S_q^\phi}$. Parameters are chosen to refer to [31], [36], [38] and [41]: $\lambda = 0.03\Delta_1$, $A_c = 2$, $\omega_c = 3\Delta_1$. Other parameters are normalized by $\Delta_1 = 1$, $\Delta_2 = 1.005\Delta_1$, $\omega_1 = \Delta_1$, $\omega_2 = \Delta_2$, $g = 0.005\Delta_1$, $\gamma = 0.005\Delta_1$, $\kappa = 0.15\Delta_1$, $E = 100\Delta_1$.

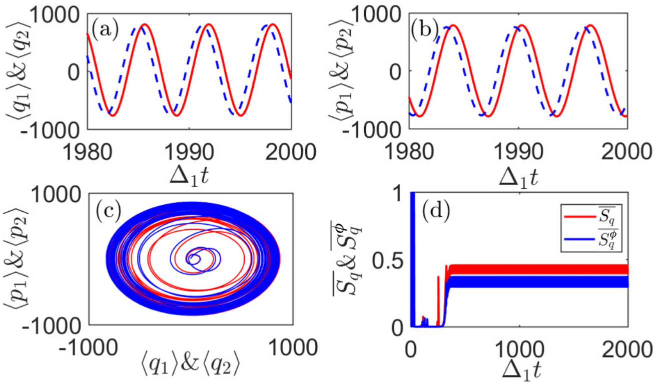

FIG. 3. Time evolution of (a) the mean values $\langle q_1\rangle$ (solid red line) and $\langle q_2\rangle$ (dashed blue line), (b) the mean values $\langle p_1\rangle$ (solid red line) and $\langle p_2\rangle$ (dashed blue line), (c) the limit-cycle trajectories in the $\langle q_1\rangle \leftrightharpoons \langle p_1\rangle$ (red) and $\langle q_2\rangle \leftrightharpoons \langle p_2\rangle$ (blue) spaces, and (d) the time evolution of the measure of quantum synchronization $\overline{S_q}$ [(upper) red line] and quantum ϕ synchronization $\overline{S_q^\phi}$ [(lower) blue line] with $\lambda = 0.14\Delta_1$ and $A_c = 1$, $\omega_c = 2\Delta_1$. Other parameters are the same as in Fig. 2.

synchronization when the mean-value synchronization is not complete, namely, quantum ϕ synchronization. Moreover, quantum ϕ synchronization also can be related to quantum phase synchronization. As shown in Fig. 5(a), both quantum ϕ synchronization $\overline{S_q^\phi}$ and quantum phase synchronization $\overline{S_p}$ first decrease and then increase as the optical coupling strength λ increases, and the changing trend of $\overline{S_p}$ and $\overline{S_q^\phi}$ with λ is accordant. When $\lambda = 0.016\Delta_1$, both $\overline{S_\phi} = 0.58$ and $\overline{S_p} = 0.36$ are minimized. This means that $\langle \delta q_-^\phi(t)^2\rangle$ is approximately proportional to $\langle \delta p_-^\phi(t)^2\rangle$ [$\langle \delta q_-^\phi(t)^2\rangle > \langle \delta p_-^\phi(t)^2\rangle$]. In this case, the measure of ϕ synchronization is in accordance with that of phase synchronization. When $\langle \delta q_-^\phi(t)^2\rangle = \langle \delta p_-^\phi(t)^2\rangle$, the two definitions are the same. However, if $\langle \delta q_-^\phi(t)^2\rangle$ has no linear relation with $\langle \delta p_-^\phi(t)^2\rangle$, the definitions of ϕ synchronization and phase synchronization are quite different as shown in Fig. 5(b). In Fig. 5(b), the quantum ϕ synchronization $\overline{S_q^\phi}$ becomes worse when the modulation amplitude A_c increases, while the quantum phase synchro-

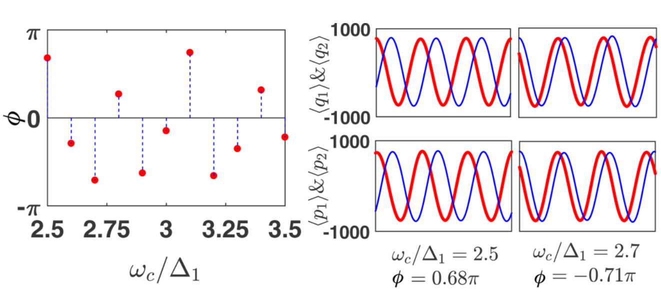

FIG. 4. Discrete point diagram (left) of phase ϕ versus modulation frequency ω_c when the system reaches steady state. Time evolution (right) of the mean values $\langle q_1\rangle$ and $\langle p_1\rangle$ (thick red line) and $\langle q_2\rangle$ and $\langle p_2\rangle$ (thin blue line) with two values of ω_c from the left panel. The parameter $\lambda = 0.14\Delta_1$, $A_c = 1$, and other parameters are the same as in Fig. 2.

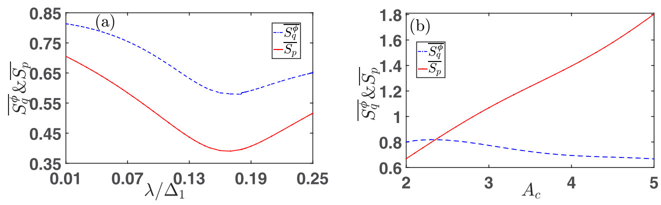

FIG. 5. Mean values of the quantum phase synchronization measure $\overline{S_p}$ (solid red line) and quantum ϕ-synchronization measure $\overline{S_q^\phi}$ (dotted blue line) as a function of (a) the optical coupling coefficient λ with $A_c = 2$, $\omega_c = 3\Delta_1$ and (b) the modulation frequency A_c with $\lambda = 0.03\Delta_1$, $\omega_c = 3\Delta_1$. Other parameters are the same as in Fig. 2.

nization $\overline{S_p}$ is significantly enhanced. This difference is due to the fact that the quantum ϕ synchronization takes both $\langle \delta q_-^\phi(t)^2\rangle$ and $\langle \delta p_-^\phi(t)^2\rangle$ into consideration, while quantum phase synchronization only considers $\langle \delta p_-^\phi(t)^2\rangle$. This also results in quantum phase synchronization $\overline{S_p}$ exceeding 1 as shown in Fig. 5(b). However quantum ϕ synchronization is still less than 1 due to the Heisenberg principle, which has also been demonstrated in Eq. (6).

When $\phi = \pi$, the ϕ error operators become $q_-^\pi(t) = \frac{1}{\sqrt{2}}[q_1(t) + q_2(t)]$ and $p_-^\pi(t) = \frac{1}{\sqrt{2}}[p_1(t) + p_2(t)]$.

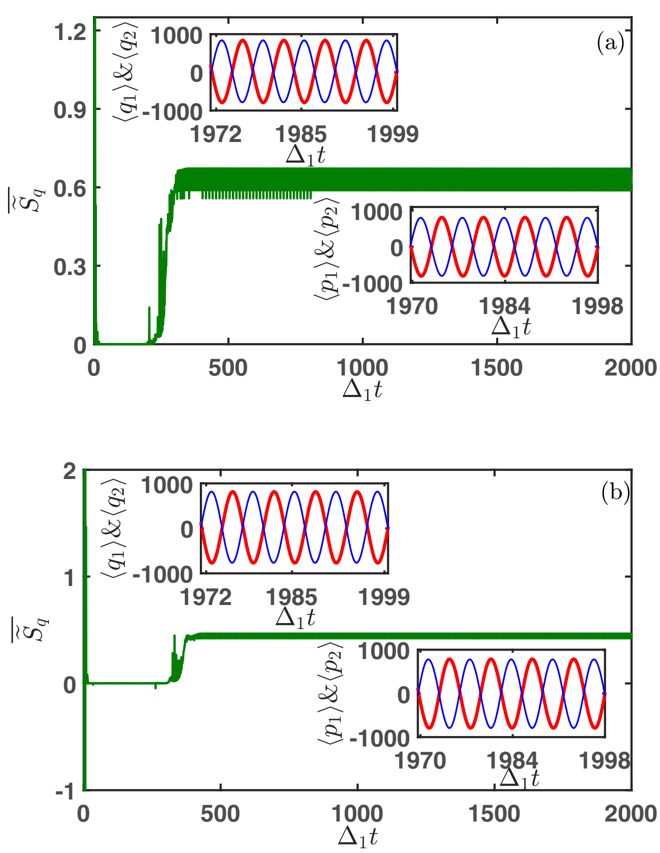

FIG. 6. Evolution of the quantum synchronization $\widetilde{S_q}$ (solid green line) and the mean values $\langle q_1\rangle$ and $\langle p_1\rangle$ (thick red line) and $\langle q_2\rangle$ and $\langle p_2\rangle$ (thin blue line) when the system is stable with (a) $\lambda = 0.3\Delta_1$ and $A_c = 1.5$, $\omega_c = 2\Delta_1$ and (b) $\lambda = 0.2\Delta_1$ and $A_c = 1$, $\omega_c = 2\Delta_1$. Other parameters are the same as in Fig. 2.

The quantum ϕ synchronization becomes quantum antisynchronization, i.e.,

$$\widetilde{S}_q \equiv S_q^{\pi} = \frac{1}{\langle \delta q_-^{\pi}(t)^2 + \delta p_-^{\pi}(t)^2 \rangle} = \left\langle \frac{1}{2}[(\delta p_1 + \delta p_2)^2 + (\delta q_1 + \delta q_2)^2] \right\rangle^{-1}. \tag{21}$$

We can also find this phenomenon of quantum antisynchronization in coupled optomechanical systems under certain parameters. As shown in Fig. 6, in quantum antisynchronization the mean value is antisynchronization and quantum ϕ synchronization is not 0.

IV. CONCLUSIONS

In summary, we have introduced and characterized a more generalized concept called quantum ϕ synchronization. It can be defined as pairs of variables which have the same amplitude and possess the same ϕ phase shift. The measure of quantum ϕ synchronization has also been defined with the phase difference ϕ. Therefore, quantum synchronization and quantum antisynchronization can be treated as special cases of quantum ϕ synchronization. In addition, quantum phase synchronization can also be related with quantum ϕ synchronization. As an example, we have investigated the quantum ϕ synchronization and quantum phase synchronization phenomena of two coupled optomechanical systems with periodic modulation. It has been shown that quantum ϕ synchronization is more general as a measure of synchronization than quantum synchronization. We have also shown the different effects of the optical coupling coefficient and the modulation amplitude on quantum phase synchronization and quantum ϕ synchronization. These two definitions of synchronization are only concordant with each other in the case where $\langle \delta q_-^{\phi}(t)^2 \rangle$ is approximately proportional to $\langle \delta p_-^{\phi}(t)^2 \rangle$. Based on quantum ϕ synchronization, the quantum antisynchronization phenomenon has also been defined and observed for $\phi = \pi$ under some parameters. Therefore, the definition of quantum ϕ synchronization provides a new way to study the quantum synchronization of continuous-variable systems. In addition, it will be interesting in the future to study the linearization method by using the stochastic Hamiltonian [51,52] and its influence on quantum ϕ synchronization.

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