Collective excitations of a BCS superfluid in the presence of two sublattices

M. Iskin
Department of Physics, Koç University, Rumelifeneri Yolu, 34450 Sarıyer, Istanbul, Turkey

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We consider a generic Hamiltonian that is suitable for describing a uniform BCS superfluid on a lattice with a two-point basis and study its collective excitations at zero temperature. For this purpose, we first derive a Gaussian effective action for the pairing fluctuations and then extract the low-energy dispersion relations for the in-phase Goldstone and out-of-phase Leggett modes along with the corresponding amplitude (i.e., the so-called Higgs) ones. We find that while the Goldstone mode is gapless at zero momentum and propagating in general, the Leggett mode becomes undamped only with sufficiently strong interactions. Furthermore, we show that, in addition to the conventional contribution that is controlled by the energy of the Bloch bands, the velocity of the Goldstone mode has a geometric contribution that is governed by the quantum metric tensor of the Bloch states. Our results suggest that the latter contribution dominates the velocity when the former becomes negligible for a narrowband or a flatband.

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I. INTRODUCTION

A deep connection between the quantum geometry of the underlying Bloch states and the superfluid (SF) phase stiffness tensor (often referred to as the SF weight) of some multiband Fermi SFs has recently been revealed in the literature [1–3]. It turns out that the SF stiffness tensor of a multiband SF has two physically distinct mechanisms: While the conventional contribution is due to the intraband processes and has a direct counterpart in the one-band models, the geometric contribution is due to the interband processes and therefore is exclusive to the multiband models. In the particular case of a uniform BCS superfluid with two underlying sublattices [2,4,5], this is such that the geometric contribution is controlled by the so-called quantum metric tensor of the underlying Bloch states [6–8].

Furthermore, in the context of spin-orbit-coupled Fermi SFs in a continuum, we recently showed that the quantum metric tensor of the underlying helicity states has also partial control over the low-energy collective excitations of the system at zero temperature [9]. Motivated by this result and earlier works [2,4,5], here we perform a similar collective-mode analysis of the case of a generic Hamiltonian that is suitable for describing a uniform BCS SF on a lattice with a two-point basis. Allowing that the SF order parameter may fluctuate (around its uniform value) independently on the two sublattices, there are two phase and two amplitude modes which are associated with the total and relative fluctuations of the phase and amplitude degrees of freedoms. For instance, the in-phase fluctuations are phononlike and correspond to the Goldstone mode and the out-of-phase fluctuations are excitonlike and correspond to the Leggett mode. Thus, in comparison to the Goldstone mode that is considered in Ref. [9], the presence of a Leggett mode makes the current analysis somewhat more cumbersome.

We find that while the Goldstone mode is gapless at zero momentum and propagating in general, the low-energy Leggett mode becomes undamped only with sufficiently strong interactions. More importantly, by identifying the quantum-metric contribution to the Goldstone mode, we show that this geometric effect is complementary to the recent works on the geometric contribution to the SF stiffness tensor [2,4,5], i.e., they are both controlled by the effective-mass tensor of the SF carriers. This suggests that an analogous contribution to the collective excitations must be present in many other multiband systems including the twisted bilayer graphene [10–12].

The rest of the paper is organized as follows. In Sec. II we first derive a Gaussian effective action for the pairing fluctuations, then extract the low-energy dispersion relations for the collective modes, and then compare our generic results with those of the honeycomb literature [13–15]. In Sec. III we show that the velocity of the Goldstone mode has a geometric contribution that can be traced back to the same origin as the recent works on the SF stiffness tensor [2,4,5]. The paper ends with a summary of our conclusions in Sec. IV.

II. EFFECTIVE-ACTION APPROACH

In this section we first introduce a generic lattice Hamiltonian that is suitable for describing a uniform BCS SF with two underlying sublattices and then extract its collective excitations from an effective action that is derived up to the Gaussian order in the fluctuations of the SF order parameter.

A. Hamiltonian

Having a general single-particle Hamiltonian on a lattice with a two-point basis in mind, we consider

$$H = \sum_{\sigma\mathbf{k}} (c^\dagger_{\sigma A\mathbf{k}} c^\dagger_{\sigma B\mathbf{k}})[\xi_\mathbf{k}\tau_0 + \mathbf{d}_\mathbf{k}\cdot\boldsymbol{\tau}]\begin{pmatrix} c_{\sigma A\mathbf{k}} \\ c_{\sigma B\mathbf{k}}\end{pmatrix} - U\sum_{S\mathbf{k}\mathbf{k}'\mathbf{q}} c^\dagger_{\uparrow S,\mathbf{k}} c^\dagger_{\downarrow S,-\mathbf{k}+\mathbf{q}} c_{\downarrow S,-\mathbf{k}'+\mathbf{q}} c_{\uparrow S,\mathbf{k}'}, \quad (1)$$

where $c^\dagger_{\sigma S\mathbf{k}}$ ($c_{\sigma S\mathbf{k}}$) creates (annihilates) a spin-σ fermion on sublattice $S \in \{A, B\}$ with quasimomentum $\mathbf{k}$, i.e., in units of $\hbar \to 1$, the Planck constant. In the first line where $\xi_\mathbf{k} = \epsilon_\mathbf{k} - \mu$ and τ_0 is a 2×2 identity matrix, $\epsilon_\mathbf{k}$ is due to the intrasublattice hoppings and μ is the chemical potential. The intersublattice hoppings are taken into account by the second term, where $\boldsymbol{\tau} = \sum_i \tau_i \hat{\boldsymbol{i}}$ is a vector of Pauli matrices for the sublattice sector and the sublattice-coupling field $\mathbf{d}_\mathbf{k} = \sum_i d^i_\mathbf{k} \hat{\boldsymbol{i}}$ is a generic one with $\hat{\boldsymbol{i}}$ denoting a unit vector along the $i = (x, y, z)$ direction. Thus, the single-particle problem is described by the Hamiltonian density $h^0_\mathbf{k} = \epsilon_\mathbf{k}\tau_0 + \mathbf{d}_\mathbf{k} \cdot \boldsymbol{\tau}$, leading to a two-band energy spectrum $\epsilon_{s\mathbf{k}} = \epsilon_\mathbf{k} + s d_\mathbf{k}$ with $d_\mathbf{k} = |\mathbf{d}_\mathbf{k}|$, where $s = \pm$ labels the upper and lower bands, respectively. For instance, in the case of a honeycomb lattice [4], one finds $\epsilon_\mathbf{k} = -2t' \cos(\sqrt{3}k_x a) - 4t' \cos(\sqrt{3}k_x a/2) \cos(3k_y a/2)$, $d^x_\mathbf{k} = -t\cos(k_y a) - 2t\cos(k_y a/2)\cos(\sqrt{3}k_x a/2)$, $d^y_\mathbf{k} = t\sin(k_y a) - 2t \sin(k_y a/2)\cos(\sqrt{3}k_x a/2)$, and $d^z_\mathbf{k} = 0$. Similarly, in the case of a Mielke lattice [5], one finds $\epsilon_\mathbf{k} = -2(t' + t'')\cos(k_x a)\cos(k_y a)$, $d^x_\mathbf{k} = -2t\cos(k_x a) - 2t\cos(k_y a)$, $d^z_\mathbf{k} = 2(t' - t'')\sin(k_x a)\sin(k_y a)$, and $d^y_\mathbf{k} = 0$. Note that while $\epsilon_\mathbf{k} = \epsilon_{-k_x,k_y} = \epsilon_{k_x,-k_y}$ and the τ_x field $d^x_\mathbf{k} = d^x_{-k_x,k_y} = d^x_{k_x,-k_y}$ are parity even functions of both k_x and k_y and the τ_z field $d^z_\mathbf{k} = -d^z_{-k_x,k_y} = -d^z_{k_x,-k_y}$ is an odd function of both k_x and k_y, the τ_y field $d^y_\mathbf{k} = d^y_{-k_x,k_y} = -d^y_{k_x,-k_y}$ is an even (odd) function of k_x (k_y).

In the second line of Eq. (1), $U \geqslant 0$ corresponds to the strength of the on-site attraction between $\uparrow$ and $\downarrow$ particles and we decouple this quartic term (in the fermionic degrees of freedom) using the Grassmann functional-integral formalism [13,16]. For this purpose, we first express the partition function $\mathcal{Z} = \int \mathcal{D}[c^\dagger, c] e^{-\mathcal{S}}$ with the associated action $\mathcal{S} = \int_0^{1/T} d\tau[\sum_{\sigma S\mathbf{k}} c^\dagger_{\sigma S\mathbf{k}}(\tau)\partial_\tau c_{\sigma S\mathbf{k}}(\tau) + H(\tau)]$, where T is the temperature in units of $k_B \to 1$, the Boltzmann constant. Then we introduce a Hubbard-Stratonovich transformation at the expense of introducing a complex bosonic field Δ_{Sq} and integrate out the remaining terms that are quadratic in the fermionic degrees of freedom. This leads to $\mathcal{Z} = \int \mathcal{D}[\Delta^*, \Delta] e^{-\mathcal{S}_\mathrm{eff}}$, where Δ_{Sq} plays the role of a fluctuating order parameter for pairing and $\mathcal{S}_\mathrm{eff}$ is the effective bosonic action for the resultant pairs of fermions. Here the collective index $q = (\mathbf{q}, i\nu_n)$ denotes both the pair momentum $\mathbf{q}$ and the bosonic Matsubara frequency $\nu_n = 2\pi n T$.

Finally, by decomposing $\Delta_{Sq} = \Delta_0 + \Lambda_{Sq}$ in terms of a q-independent stationary field Δ_0 and q-dependent fluctuations around it, one may in principle obtain $\mathcal{S}_\mathrm{eff}$ at the desired order in Λ_{Sq}. This decomposition implies that our discussion is restricted to inversion-symmetric systems, and we numerically checked that the order parameter is indeed uniform for the entire lattice. Due to the simplicity of the uniform SF phase, we can make analytical progress and reveal a direct connection with the recent literature on the SF stiffness tensor of the uniform SFs [2,4,5,13,14]. In contrast, when the inversion symmetry is broken, for instance, by the presence of an energy offset between sublattices, the analogous calculations are not as tractable [15].

In this paper we include only the first nontrivial term and obtain the Gaussian effective action $\mathcal{S}_\mathrm{Gauss} = \mathcal{S}_0 + \mathcal{S}_2$, as the first-order term $\mathcal{S}_1$ trivially vanishes due to the saddle-point condition discussed next.

B. Saddle-point approximation

The effective-action approach is a standard tool in many-body physics and it leads to $\mathcal{S}_0 = \Delta_0^2/TU + (1/N_l T)\sum_{s\mathbf{k}} \xi_{s\mathbf{k}} - (1/N_l)\sum_k \ln[\det(\mathbf{G}_k^{-1}/T)]$, where the collective index $k = (\mathbf{k}, i\omega_\ell)$ denotes both the particle momentum $\mathbf{k}$ and the fermionic Matsubara frequency $\omega_\ell = (2\ell + 1)\pi T$. Here N_l is the number of lattice sites and $\mathbf{G}_k^{-1} = i\omega_\ell \mathbf{1} - H^0_\mathbf{k}$ is the inverse Green's function for the mean-field Hamiltonian density $H^0_\mathbf{k}$, i.e.,

$$\mathbf{G}_k^{-1} = \begin{bmatrix} (i\omega_\ell - \xi_\mathbf{k})\tau_0 - \mathbf{d}_\mathbf{k}\cdot\boldsymbol{\tau} & -\Delta_0\tau_0 \\ -\Delta_0\tau_0 & (i\omega_\ell + \xi_\mathbf{k})\tau_0 + \mathbf{d}_\mathbf{k}\cdot\boldsymbol{\tau} \end{bmatrix}.$$

Here we use $\epsilon_{-\mathbf{k}} = \epsilon_\mathbf{k}$ and $\mathbf{d}_{-\mathbf{k}} \cdot \boldsymbol{\tau}^* = \mathbf{d}_\mathbf{k} \cdot \boldsymbol{\tau}$ and the Hamiltonian is given by $H_0 = \sum_\mathbf{k} \Psi^\dagger_\mathbf{k} H^0_\mathbf{k} \Psi_\mathbf{k}$, where $\Psi^\dagger_\mathbf{k} = (c^\dagger_{\uparrow A\mathbf{k}} c^\dagger_{\uparrow B\mathbf{k}} c_{\downarrow A,-\mathbf{k}} c_{\downarrow B,-\mathbf{k}})$. After the summation over ω_ℓ, we obtain

$$\mathcal{S}_0 = \frac{\Delta_0^2}{TU} + \frac{1}{N_l}\sum_{s\mathbf{k}}\left\{\frac{\xi_{s\mathbf{k}} - E_{s\mathbf{k}}}{T} + 2\ln[f(-E_{s\mathbf{k}})]\right\}, \tag{2}$$

where $\xi_{s\mathbf{k}} = \epsilon_{s\mathbf{k}} - \mu$, $E_{s\mathbf{k}} = \sqrt{\xi^2_{s\mathbf{k}} + \Delta_0^2}$ is the quasiparticle energy spectrum, and $f(x) = 1/(e^{x/T} + 1)$ is the Fermi-Dirac distribution.

The saddle-point order parameter Δ_0 can also be expressed as $\Delta_0 = U\langle c_{\uparrow S\mathbf{k}} c_{\downarrow S,-\mathbf{k}}\rangle$, with $\langle \cdots \rangle$ denoting a thermal average, and we take it to be a real parameter throughout the paper without loss of generality. Using the saddle-point condition $\partial\mathcal{S}_0/\partial\Delta_0 = 0$ for the action and the thermodynamic relation $N_0 = -T\partial\mathcal{S}_0/\partial\mu$ for the number of particles, we find [4,5,13]

$$\frac{1}{U} = \frac{1}{N_l}\sum_{s\mathbf{k}} \frac{1 - 2f(E_{s\mathbf{k}})}{2E_{s\mathbf{k}}}, \tag{3}$$

$$N_0 = \sum_{s\mathbf{k}}\left\{\frac{1}{2} - \frac{\xi_{s\mathbf{k}}}{2E_{s\mathbf{k}}}[1 - 2f(E_{s\mathbf{k}})]\right\}. \tag{4}$$

In order to evaluate the collective excitations, we need self-consistent solutions for Δ_0 and μ as a function of U and hopping parameters. In addition, for the $T = 0$ of interest in this paper, these mean-field solutions turns out to be sufficient for a qualitative description of the many-body problem.

C. Gaussian fluctuations

Going beyond the saddle-point action $\mathcal{S}_0$, we calculate the first nontrivial term in the expansion and find $\mathcal{S}_2 = \sum_{Sq} |\Lambda_{Sq}|^2/2TU + (1/2N_l)\mathrm{Tr}\sum_{kq} \mathbf{G}_k \boldsymbol{\Sigma}_q \mathbf{G}_{k+q} \boldsymbol{\Sigma}_{-q}$, where Tr denotes a trace over the sublattice and spin sectors. The matrix elements of $\mathbf{G}_k$ can be written as

$$G^{11}_k = \frac{1}{2}\sum_s \frac{i\omega_\ell + \xi_{s\mathbf{k}}}{(i\omega_\ell)^2 - E^2_{s\mathbf{k}}}(\tau_0 + s\hat{\mathbf{d}}_\mathbf{k}\cdot\boldsymbol{\tau}), \tag{5}$$

$$G^{22}_k = \frac{1}{2}\sum_s \frac{i\omega_\ell - \xi_{s\mathbf{k}}}{(i\omega_\ell)^2 - E^2_{s\mathbf{k}}}(\tau_0 + s\hat{\mathbf{d}}_\mathbf{k}\cdot\boldsymbol{\tau}), \tag{6}$$

$$G^{12}_k = \frac{1}{2}\sum_s \frac{\Delta_0}{(i\omega_\ell)^2 - E^2_{s\mathbf{k}}}(\tau_0 + s\hat{\mathbf{d}}_\mathbf{k}\cdot\boldsymbol{\tau}), \tag{7}$$

where $\hat{\mathbf{d}}_\mathbf{k} = \mathbf{d}_\mathbf{k}/d_\mathbf{k}$ and $G_k^{21} = G_k^{12}$. In addition, the matrix elements of the fluctuation field $\mathbf{\Sigma}_q$ are $\Sigma_q^{11} = \Sigma_q^{22} = 0$, $\Sigma_q^{12} = -\Lambda_{Tq}\tau_0 - \Lambda_{Rq}\tau_z$, and $\Sigma_q^{21} = -\Lambda_{T,-q}^*\tau_0 - \Lambda_{R,-q}^*\tau_z$. Motivated by the earlier works on two-band SFs, we define $\Lambda_{Tq} = (\Lambda_{Aq} + \Lambda_{Bq})/2$ for the total and $\Lambda_{Rq} = (\Lambda_{Aq} - \Lambda_{Bq})/2$ for the relative fluctuations.

After the summation over ω_ℓ, we obtain $\mathcal{S}_2 = (1/2N_l T)\sum_q \bar{\mathbf{\Lambda}}_q^\dagger \mathbf{M}_q \bar{\mathbf{\Lambda}}_q$, where $\bar{\mathbf{\Lambda}}_q^\dagger = (\Lambda_{Tq}^*\Lambda_{T,-q}\Lambda_{Rq}^*\Lambda_{R,-q})$ is a vector of fluctuation fields and

$$\mathbf{M}_q = \begin{pmatrix} \mathbf{T}_q & \mathbf{C}_q \\ \mathbf{C}_q^* & \mathbf{R}_q \end{pmatrix}$$

stands for the inverse fluctuation propagator. Here, while the submatrices $\mathbf{T}_q$ and $\mathbf{R}_q$ describe the purely total and purely relative fluctuations, respectively, the submatrix $\mathbf{C}_q$ is responsible for their coupling. The submatrix $\mathbf{C}_q^*$ is related to $\mathbf{C}_q$ via a complex conjugate acting only on the multiplying factors, i.e., its matrix elements are determined by Eq. (12) but with $[d_z + d_z' + i(d_x d_y' - d_y d_x')]$.

In order to simplify their expressions, we denote $\xi_{s\mathbf{k}}$ by ξ, $\xi_{s',\mathbf{k+q}}$ by ξ', $E_{s\mathbf{k}}$ by E, $E_{s',\mathbf{k+q}}$ by E', $sd_\mathbf{k}^i/d_\mathbf{k}$ by d_i, and $s'd_{\mathbf{k+q}}^i/d_{\mathbf{k+q}}$ by d_i' and define the functions $u^2 = (1 + \xi/E)/2$, $u'^2 = (1+\xi'/E')/2$, $v^2 = (1-\xi/E)/2$, $v'^2 = (1-\xi'/E')/2$, $f = 1/(e^{E/T}+1)$, and $f' = 1/(e^{E'/T}+1)$. In addition, we also define

$$r_1 = (1-f-f')\left(\frac{u^2u'^2}{i\nu_n - E - E'} - \frac{v^2v'^2}{i\nu_n + E + E'}\right) + (f-f')\left(\frac{v^2u'^2}{i\nu_n + E - E'} - \frac{u^2v'^2}{i\nu_n - E + E'}\right), \quad (8)$$

$$r_2 = (1-f-f')\left(\frac{uvu'v'}{i\nu_n + E + E'} - \frac{uvu'v'}{i\nu_n - E - E'}\right) + (f-f')\left(\frac{uvu'v'}{i\nu_n + E - E'} - \frac{uvu'v'}{i\nu_n - E + E'}\right) \quad (9)$$

for a compact presentation of the matrix elements of $\mathbf{M}_q$ as well. Using the simpler notation and definitions, we find

$$T_q^{1j} = \frac{\delta_{1j}}{U} + \frac{1}{2N_l}\sum_{ss'\mathbf{k}} r_j(1 + d_xd_x' + d_yd_y' + d_zd_z'), \quad (10)$$

$$R_q^{1j} = \frac{\delta_{1j}}{U} + \frac{1}{2N_l}\sum_{ss'\mathbf{k}} r_j(1 - d_xd_x' - d_yd_y' + d_zd_z'), \quad (11)$$

$$C_q^{1j} = \frac{1}{2N_l}\sum_{ss'\mathbf{k}} r_j[d_z + d_z' - i(d_xd_y' - d_yd_x')], \quad (12)$$

where δ_{ij} is the Kronecker delta [17]. The remaining elements of $\mathbf{T}_q$, $\mathbf{R}_q$, and $\mathbf{C}_q$ are all related to the given ones as follows: $T_q^{22} = T_{-q}^{11}$, $T_q^{21} = T_q^{12}$, $R_q^{22} = R_{-q}^{11}$, $R_q^{21} = R_q^{12}$, $C_q^{22} = C_{-q}^{11*}$, and $C_q^{21} = C_q^{12}$. Here the complex conjugate again acts only on the multiplying factor of Eq. (12). We note that while T_q^{12} and R_q^{12} are even under both $\mathbf{q} \to -\mathbf{q}$ and $i\nu_n \to -i\nu_n$, C_q^{12} is even under only $i\nu_n \to -i\nu_n$ and T_q^{11} and R_q^{11} are even under only $\mathbf{q} \to -\mathbf{q}$. In addition, we also note that a familiar factor $d_xd_x' + d_yd_y' + d_zd_z' = ss'\hat{\mathbf{d}}_\mathbf{k}\cdot\hat{\mathbf{d}}_{\mathbf{k+q}}$ is appearing in the elements of $\mathbf{T}_q$ [9].

Next we reexpress the fluctuation fields $\Lambda_{T\,(R)q} = \alpha_{T\,(R)q}e^{i\gamma_{T\,(R)q}}$ in terms of real functions $\alpha_{T\,(R)q}$ and $\gamma_{T\,(R)q}$ and associate $\lambda_{T\,(R)q} = \sqrt{2}\alpha_{T\,(R)q}\cos(\gamma_{T\,(R)q})$ with the amplitude degrees of freedom and $\theta_{T\,(R)q} = \sqrt{2}\alpha_{T\,(R)q}\sin(\gamma_{T\,(R)q})$ with the phase ones in the small-$\gamma_{T\,(R)q}$ limit. Such a unitary transformation can be achieved by [13,16]

$$\bar{\mathbf{\Lambda}}_q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix}\begin{pmatrix} \lambda_{Tq} \\ \theta_{Tq} \\ \lambda_{Rq} \\ \theta_{Rq} \end{pmatrix}, \quad (13)$$

where $\lambda_{T\,(R)q}$ and $\theta_{T\,(R)q}$ are real functions. Furthermore, assuming $\lambda_{T\,(R),-q} = \lambda_{T\,(R)q}^*$ and $\theta_{T\,(R),-q} = \theta_{T\,(R)q}^*$, we finally obtain the desired action

$$\mathcal{S}_2 = \frac{1}{2T}\sum_q (\lambda_{Tq}^*\theta_{Tq}^*\lambda_{Rq}^*\theta_{Rq}^*)\begin{pmatrix} T_{q,E}^{11} + T_q^{12} & iT_{q,O}^{11} & C_{q,E}^{11} + C_q^{12} & iC_{q,O}^{11} \\ -iT_{q,O}^{11} & T_{q,E}^{11} - T_q^{12} & -iC_{q,O}^{11} & C_{q,E}^{11} - C_q^{12} \\ C_{q,E}^{11*} + C_q^{12*} & iC_{q,O}^{11*} & R_{q,E}^{11} + R_q^{12} & iR_{q,O}^{11} \\ -iC_{q,O}^{11*} & C_{q,E}^{11*} - C_q^{12*} & -iR_{q,O}^{11} & R_{q,E}^{11} - R_q^{12} \end{pmatrix}\begin{pmatrix} \lambda_{Tq} \\ \theta_{Tq} \\ \lambda_{Rq} \\ \theta_{Rq} \end{pmatrix}. \quad (14)$$

Here we split the matrix elements T_q^{11}, R_q^{11}, and C_q^{11} into two in terms of an even and an odd function in $i\nu_n$, e.g., such that $T_q^{11} = T_{q,E}^{11} + T_{q,O}^{11}$, where $T_{q,E}^{11} = (T_q^{11} + T_q^{22})/2$ is the even and $T_{q,O}^{11} = (T_q^{11} - T_q^{22})/2$ is the odd part.

Having derived the Gaussian effective action, next we are ready to analyze it in detail and extract the collective modes of the system.

D. Collective excitations at $T = 0$

The dispersions $\omega_\mathbf{q}$ for the collective modes are determined by the poles of the propagator matrix $\mathbf{M}_q^{-1}$ for the pair fluctuation fields by setting $\det \mathbf{M}_q = 0$ after an analytic continuation $i\nu_n \to \omega + i0^+$ to the real axis. Since the quasiparticle-quasihole terms with the prefactor $(f - f')$ have the usual Landau singularity for $q \to (\mathbf{0}, 0)$ causing the collective modes to decay into the two-quasiparticle continuum, a small-q expansion is possible only in two cases: (i) just below the critical SF transition temperature provided that $\Delta_0 \to 0 \ll |\omega|$ and (ii) at $T = 0$ provided that $|\omega| \ll \min(E + E')$. In this work we are interested in the latter case, and by setting $T = 0$ in Eqs. (9) and (8) we find

$$T_{q,E}^{11} = \frac{1}{U} + \frac{1}{2N_l}\sum_{ss'\mathbf{k}} \frac{(\xi\xi' + EE')(E + E')}{2EE'[(i\nu_n)^2 - (E+E')^2]} \times (1 + d_xd_x' + d_yd_y' + d_zd_z'), \quad (15)$$

$$T_{q,O}^{11} = \frac{1}{2N_l}\sum_{ss'\mathbf{k}} \frac{(\xi E' + E\xi')i\nu_n}{2EE'[(i\nu_n)^2 - (E+E')^2]} \times (1 + d_x d_x' + d_y d_y' + d_z d_z'), \quad (16)$$

$$T_q^{12} = -\frac{1}{2N_l}\sum_{ss'\mathbf{k}} \frac{\Delta_0^2(E+E')}{2EE'[(i\nu_n)^2 - (E+E')^2]} \times (1 + d_x d_x' + d_y d_y' + d_z d_z'). \quad (17)$$

The matrix elements of the $\mathbf{R}_q$ and $\mathbf{C}_q$ sectors have similar forms except for the multiplying factors in the second lines [17].

We note that, in the limit when $\mathbf{q} \to \mathbf{0}$, while the multiplying factor for $\mathbf{C}_q$ directly vanishes, i.e., $d_z + d_z' - i(d_x d_y' - d_y d_x') \to 0$, for the honeycomb lattice, these terms sum to 0 for the Mielke lattice as $d_\mathbf{k}^z$ is an odd function of both k_x and k_y, suggesting that the total and relative fluctuations are always uncoupled. In addition, in the limit when $\mathbf{q} \to \mathbf{0}$ and $i\nu_n \to 0$, we find $T_{q,E}^{11} - T_q^{12} \to 1/U - \sum_\mathbf{k} 1/2N_l E_{s\mathbf{k}} = 0$ due to the saddle-point condition, suggesting that the total phase mode is gapless at $\mathbf{q} = \mathbf{0}$ and we identify it as a Goldstone mode. Similarly, in the limit when $\mathbf{q} \to \mathbf{0}$ and $|i\nu_n| \to 2\Delta_0$, we find $T_{q,E}^{11} + T_q^{12} \to 0$ due again to the saddle-point condition, suggesting that the total amplitude mode is gapped with $2\Delta_0$ (i.e., this holds only in the weakly interacting BCS limit for which the amplitude and phase fields are weakly coupled due to the negligible contribution from $T_{q,O}^{11}$) at $\mathbf{q} = \mathbf{0}$, and we identify it as the so-called Higgs mode. Therefore, we conclude that the $\mathbf{q} \to \mathbf{0}$ limit is consistent with our physical intuition.

We are also aware of several numerical works where the collective excitations of a BCS SF are analyzed on the two-dimensional honeycomb lattice [13–15], and next we check the consistency of our generic results with those of the honeycomb literature.

1. Zhao and Paramekanti's work

As a first benchmark, we consider the static limit when $i\nu_n \to 0$, for which all of the matrix elements that couple amplitude and phase fields go to zero, i.e., $\{T_{q,O}^{11}, T_{q,O}^{11}, C_{q,O}^{11}\} \to 0$, making the analysis of $\mathcal{S}_2$ a much simpler task. For instance, the phase fluctuations are described purely by the action

$$(\theta_{Tq}^* \theta_{Rq}^*)\begin{pmatrix} T_{q,E}^{11} - T_q^{12} & C_{q,E}^{11} - C_q^{12} \\ C_{q,E}^{11*} - C_q^{12*} & R_{q,E}^{11} - R_q^{12}\end{pmatrix}\begin{pmatrix}\theta_{Tq} \\ \theta_{Rq}\end{pmatrix}. \quad (18)$$

In order to confirm that Eq. (18) reproduces the results of Ref. [13], one first needs to reexpress their Eq. (4) in terms of the total and relative phases and then match their matrix elements $u_\mathbf{q}$ and $v_\mathbf{q}$ in such a way that $u_\mathbf{q} + \mathrm{Re}[v_\mathbf{q}] = \Delta_0^2(T_{\mathbf{q},E}^{11} - T_\mathbf{q}^{12})$, $u_\mathbf{q} - \mathrm{Re}[v_\mathbf{q}] = \Delta_0^2(R_{\mathbf{q},E}^{11} - R_\mathbf{q}^{12})$, and $\mathrm{Im}[v_\mathbf{q}] = -\Delta_0^2(C_{\mathbf{q},E}^{11} - C_\mathbf{q}^{12})$. The origin of the prefactor Δ_0^2 is due to the difference in the definitions of the fluctuations fields, i.e., they substitute $\Lambda_{Sq} = \Delta_0(\lambda_{Sq} + i\theta_{Sq})$. We note that since $d_\mathbf{k}^z = 0$ for the honeycomb model, their $\gamma_\mathbf{k} = d_\mathbf{k}^x - i d_\mathbf{k}^y$ leads to $\gamma_\mathbf{k}^* \gamma_{\mathbf{k}+\mathbf{q}} = d_\mathbf{k}^x d_{\mathbf{k}+\mathbf{q}}^x + d_\mathbf{k}^y d_{\mathbf{k}+\mathbf{q}}^y + i(d_\mathbf{k}^x d_{\mathbf{k}+\mathbf{q}}^y - d_\mathbf{k}^y d_{\mathbf{k}+\mathbf{q}}^x)$, and this expression corresponds to $[d_x d_x' + d_y d_y' + i(d_x d_y' - d_y d_x')]/ss'$ in our notation. However, we note that there must be a typo in Eq. (5) of [13] and the first term must read as Δ_0^2/U instead of $2\Delta_0^2/U$. This is also evident from the discussion given below Eq. (17), i.e., noting that $\mathrm{Im}[v_\mathbf{q}] \to 0$ in the $\mathbf{q} \to \mathbf{0}$ limit, $u_\mathbf{q} + \mathrm{Re}[v_\mathbf{q}]$ must also vanish in order to recover the total phase mode as a gapless Goldstone one. For completeness, the amplitude fluctuations are described purely by the action

$$(\lambda_{Tq}^* \lambda_{Rq}^*)\begin{pmatrix} T_{q,E}^{11} + T_q^{12} & C_{q,E}^{11} + C_q^{12} \\ C_{q,E}^{11*} + C_q^{12*} & R_{q,E}^{11} + R_q^{12}\end{pmatrix}\begin{pmatrix}\lambda_{Tq} \\ \lambda_{Rq}\end{pmatrix} \quad (19)$$

in the static limit.

2. Tsuchiya, Ganesh, and Nikuni's work

As a second benchmark, we consider a two-band lattice whose energy bands are completely symmetric around the zero energy, i.e., $\xi_{s\mathbf{k}} = -\xi_{-s,\mathbf{k}}$, which requires that $\mu = 0$ and $\epsilon_\mathbf{k} = 0$. For instance, this particular discussion is relevant in the context of a pair of Dirac cones at half filling. When this is the case, by setting $\xi_{s\mathbf{k}} = s d_\mathbf{k}$ and $E_{s\mathbf{k}} = \sqrt{d_\mathbf{k}^2 + \Delta_0^2} = E_\mathbf{k}$ in Eqs. (15)–(17), we find

$$T_{q,E}^{11} = \frac{1}{U} + \frac{1}{N_l}\sum_\mathbf{k} \frac{E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}}}{E_\mathbf{k}E_{\mathbf{k}+\mathbf{q}}[(i\nu_n)^2 - (E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})^2]} \times \left(E_\mathbf{k}E_{\mathbf{k}+\mathbf{q}} + d_\mathbf{k}^x d_{\mathbf{k}+\mathbf{q}}^x + d_\mathbf{k}^y d_{\mathbf{k}+\mathbf{q}}^y + d_\mathbf{k}^z d_{\mathbf{k}+\mathbf{q}}^z\right), \quad (20)$$

$$R_{q,E}^{11} = \frac{1}{U} + \frac{1}{N_l}\sum_\mathbf{k} \frac{E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}}}{E_\mathbf{k}E_{\mathbf{k}+\mathbf{q}}[(i\nu_n)^2 - (E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})^2]} \times \left(E_\mathbf{k}E_{\mathbf{k}+\mathbf{q}} - d_\mathbf{k}^x d_{\mathbf{k}+\mathbf{q}}^x - d_\mathbf{k}^y d_{\mathbf{k}+\mathbf{q}}^y + d_\mathbf{k}^z d_{\mathbf{k}+\mathbf{q}}^z\right), \quad (21)$$

$$T_q^{12} = -\frac{1}{N_l}\sum_\mathbf{k} \frac{\Delta_0^2(E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})}{E_\mathbf{k}E_{\mathbf{k}+\mathbf{q}}[(i\nu_n)^2 - (E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})^2]}, \quad (22)$$

$$C_{q,E}^{11} = -\frac{i}{N_l}\sum_\mathbf{k} \frac{(E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})\left(d_\mathbf{k}^x d_{\mathbf{k}+\mathbf{q}}^y - d_\mathbf{k}^y d_{\mathbf{k}+\mathbf{q}}^x\right)}{E_\mathbf{k}E_{\mathbf{k}+\mathbf{q}}[(i\nu_n)^2 - (E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})^2]}, \quad (23)$$

$$C_{q,O}^{11} = \frac{1}{N_l}\sum_\mathbf{k} \frac{\left(d_\mathbf{k}^z E_{\mathbf{k}+\mathbf{q}} + E_\mathbf{k} d_{\mathbf{k}+\mathbf{q}}^z\right)i\nu_n}{E_\mathbf{k}E_{\mathbf{k}+\mathbf{q}}[(i\nu_n)^2 - (E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})^2]}. \quad (24)$$

The remaining terms are such that $R_q^{12} = T_q^{12}$ and $T_{q,O}^{11} = R_{q,O}^{11} = C_q^{12} = 0$. We also note that the saddle-point condition (3) becomes $1/U = \sum_\mathbf{k} 1/N_l E_\mathbf{k}$. Since $C_{q,O}^{11}$ vanishes for the honeycomb lattice and it sums to 0 for $d_\mathbf{k}^z$ that is odd in k_x or k_y, we find for these cases that the amplitude and phase fields are completely decoupled, i.e., they are purely described by Eqs. (18) and (19). Setting the corresponding determinants to 0, we find

$$\left\{\frac{1}{U} + \sum_\mathbf{k} \frac{(E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})\left(E_\mathbf{k}E_{\mathbf{k}+\mathbf{q}} \pm \Delta_0^2 + d_\mathbf{k}^z d_{\mathbf{k}+\mathbf{q}}^z\right)}{N_l E_\mathbf{k}E_{\mathbf{k}+\mathbf{q}}[(i\nu_n)^2 - (E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})^2]}\right\}^2 = \left\{\frac{1}{N_l}\sum_\mathbf{k} \frac{(E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})\left(d_\mathbf{k}^x d_{\mathbf{k}+\mathbf{q}}^x + d_\mathbf{k}^y d_{\mathbf{k}+\mathbf{q}}^y\right)}{E_\mathbf{k}E_{\mathbf{k}+\mathbf{q}}[(i\nu_n)^2 - (E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})^2]}\right\}^2 + \left\{\frac{1}{N_l}\sum_\mathbf{k} \frac{(E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})\left(d_\mathbf{k}^x d_{\mathbf{k}+\mathbf{q}}^y - d_\mathbf{k}^y d_{\mathbf{k}+\mathbf{q}}^x\right)}{E_\mathbf{k}E_{\mathbf{k}+\mathbf{q}}[(i\nu_n)^2 - (E_\mathbf{k} + E_{\mathbf{k}+\mathbf{q}})^2]}\right\}^2 \quad (25)$$

for the poles of the propagator matrices given in Eqs. (18) and (19), where $\pm$ is for the phase and amplitude modes, respectively. Note that since $C^{11}_{q,E} \to 0$ in the limit when $\mathbf{q} \to \mathbf{0}$, the total and relative fields are not coupled, leading to a gapless Goldstone mode and a gapped Leggett mode as discussed below and in Sec. III B.

In the half-filled honeycomb case, Eq. (25) is in partial agreement with the work of Tsuchiya *et al.* [14], i.e., our $\pm$ results are similar to their expressions (12) and (11), respectively, when their $F = 0$ [18]. This discrepancy is amusing given that the collective modes for the usual one-band models that are found from the Gaussian fluctuations and random-phase approximation are known to be consistent with each other and with that of the kinetic theory [19–21]. Furthermore, they conclude that the Goldstone and Leggett modes are degenerate and therefore they both become gapless at $\mathbf{q} = \mathbf{0}$.

When we set $\mathbf{q} = \mathbf{0}$ in Eq. (25), we find two solutions for the phase modes and two solutions for the amplitude ones, which can be written, respectively, as

$$0 = \frac{1}{N_l} \sum_{\mathbf{k}} \frac{(i\nu_n)^2}{E_{\mathbf{k}}\left[(i\nu_n)^2 - 4E_{\mathbf{k}}^2\right]}, \tag{26}$$

$$0 = \frac{1}{N_l} \sum_{\mathbf{k}} \frac{(i\nu_n)^2 - 4d_{\mathbf{k}}^2 + 4\left(d_{\mathbf{k}}^z\right)^2}{E_{\mathbf{k}}\left[(i\nu_n)^2 - 4E_{\mathbf{k}}^2\right]}, \tag{27}$$

$$0 = \frac{1}{N_l} \sum_{\mathbf{k}} \frac{(i\nu_n)^2 - 4\Delta_0^2}{E_{\mathbf{k}}\left[(i\nu_n)^2 - 4E_{\mathbf{k}}^2\right]}, \tag{28}$$

$$0 = \frac{1}{N_l} \sum_{\mathbf{k}} \frac{(i\nu_n)^2 - 4d_{\mathbf{k}}^2 + 4\left(d_{\mathbf{k}}^z\right)^2 - 4\Delta_0^2}{E_{\mathbf{k}}\left[(i\nu_n)^2 - 4E_{\mathbf{k}}^2\right]}. \tag{29}$$

Here Eq. (26) suggests that the total phase (Goldstone) mode is gapless when $i\nu_n \to 0$ and Eq. (28) suggests that the total amplitude mode is gapped with $|i\nu_n| \to 2\Delta_0$. In addition, the relative phase (Leggett) mode is gapped as Eq. (27) is not satisfied for $i\nu_n \to 0$, and assuming $|i\nu_n| \ll \min(2E_{\mathbf{k}}) = 2\Delta_0$, its finite frequency is determined by $(i\nu_n)^2 = \{\sum_{\mathbf{k}}[d_{\mathbf{k}}^2 - (d_{\mathbf{k}}^z)^2]/E_{\mathbf{k}}^3\}/\{\sum_{\mathbf{k}}[\Delta_0^2 + (d_{\mathbf{k}}^z)^2]/4E_{\mathbf{k}}^5\}$. This is in agreement with the low-frequency expansion that is presented in Sec. III B, where $\omega_L^2 \to \tilde{P}/\tilde{R}$ at $\mathbf{q} = \mathbf{0}$. Applying a similar analysis to Eq. (29), we find that the finite frequency of the relative amplitude mode is determined by $(i\nu_n)^2 = \{\sum_{\mathbf{k}}[E_{\mathbf{k}}^2 - (d_{\mathbf{k}}^z)^2]/E_{\mathbf{k}}^3\}/[\sum_{\mathbf{k}}(d_{\mathbf{k}}^z)^2/4E_{\mathbf{k}}^5]$ and it is much larger than $2\Delta_0$. This clearly suggests that this mode is always damped and it decays into the two-quasiparticle continua. For instance, in the strong-coupling BEC limit when $\Delta_0 \gg \max d_{\mathbf{k}}$, these frequencies can be approximated by $(i\nu_n)^2 = (8/N_l)\sum_{\mathbf{k}}[d_{\mathbf{k}}^2 - (d_{\mathbf{k}}^z)^2]$ for the undamped Leggett mode and by $(i\nu_n)^2 = 2N_l\Delta_0^4/\sum_{\mathbf{k}}(d_{\mathbf{k}}^z)^2$ for the damped relative amplitude one. The former result is consistent with the recent literature, where undamped Leggett modes are found for sufficiently strong interactions away from the weak-coupling BCS limit [13,15]. As a final remark, setting $d_{\mathbf{k}}^z = 0$ in Eq. (29) for the honeycomb case, we simply find $0 = 1/U$, suggesting that the relative amplitude branch disappears at $\mathbf{q} = \mathbf{0}$.

III. GEOMETRIC INTERPRETATION

As discussed in Sec. II D, the total and relative fluctuations turn out to be uncoupled from each other in the limit when $\mathbf{q} \to \mathbf{0}$. Next we consider this limit and discuss purely total and purely relative fluctuations in detail due to their analytical simplicity.

A. Purely total fluctuations

For this purpose, it is sufficient to take into account the following terms in the small-$\mathbf{q}$ and -ω expansions: $T^{11}_{q,E} + T^{12}_q = A + \sum_{ij} C_{ij}q_iq_j - D\omega^2 + \cdots$, $T^{11}_{q,E} - T^{12}_q = \sum_{ij} Q_{ij}q_iq_j - R\omega^2 + \cdots$, and $T^{11}_{q,O} = -B\omega + \cdots$. Since $B \neq 0$ in general, it couples the total phase and total amplitude fields and therefore we derive a total phase-only (amplitude-only) action by integrating out the total amplitude (phase) fields. This leads to a phononlike gapless in-phase (Goldstone) mode and an excitonlike gapped amplitude (Higgs) mode [22]

$$\omega_{G\mathbf{q}}^2 = \sum_{ij} \frac{Q_{ij}}{R + B^2/A} q_iq_j, \tag{30}$$

$$\omega_{H\mathbf{q}}^2 = \frac{A + B^2/R}{D} + \sum_{ij}\left(\frac{C_{ij}}{D} + \frac{B^2Q_{ij}/R}{B^2 + AR}\right)q_iq_j. \tag{31}$$

Here the nonkinetic coefficients are given by $A = \sum_{s\mathbf{k}} \Delta_0^2/2N_lE_{s\mathbf{k}}^3$, $B = \sum_{s\mathbf{k}} \xi_{s\mathbf{k}}/4N_lE_{s\mathbf{k}}^3$, $D = \sum_{s\mathbf{k}} \xi_{s\mathbf{k}}^2/8N_lE_{s\mathbf{k}}^5$, and $R = \sum_{s\mathbf{k}} 1/8N_lE_{s\mathbf{k}}^3$. We note that these expressions are simply summations over their conventional counterparts for the usual one-band problem, i.e., they are due entirely to intraband mechanisms.

On the other hand, the kinetic coefficients have a tensor structure and they consist of both an intraband and an interband contribution in such a way that $C_{ij} = C_{ij}^{\text{intra}} + C_{ij}^{\text{inter}}$ and $Q_{ij} = Q_{ij}^{\text{intra}} + Q_{ij}^{\text{inter}}$. A compact way to express these coefficients are

$$C_{ij}^{\text{intra}} = \frac{1}{N_l}\sum_{s\mathbf{k}} \frac{1}{8E_{s\mathbf{k}}^3}\left(1 - \frac{5\Delta_0^2\xi_{s\mathbf{k}}^2}{E_{s\mathbf{k}}^4}\right)\frac{\partial \xi_{s\mathbf{k}}}{\partial k_i}\frac{\partial \xi_{s\mathbf{k}}}{\partial k_j}, \tag{32}$$

$$Q_{ij}^{\text{intra}} = \frac{1}{N_l}\sum_{s\mathbf{k}} \frac{1}{8E_{s\mathbf{k}}^3}\frac{\partial \xi_{s\mathbf{k}}}{\partial k_i}\frac{\partial \xi_{s\mathbf{k}}}{\partial k_j}, \tag{33}$$

$$C_{ij}^{\text{inter}} = -\frac{1}{N_l}\sum_{s\mathbf{k}} \frac{d_{\mathbf{k}}}{4s\xi_{\mathbf{k}}E_{s\mathbf{k}}}\left(1 + \frac{2\Delta_0^2}{d_{\mathbf{k}}^2}\right)g_{\mathbf{k}}^{ij}, \tag{34}$$

$$Q_{ij}^{\text{inter}} = -\frac{1}{N_l}\sum_{s\mathbf{k}} \frac{d_{\mathbf{k}}}{4s\xi_{\mathbf{k}}E_{s\mathbf{k}}}g_{\mathbf{k}}^{ij}. \tag{35}$$

We again note that while Eqs. (32) and (33) can be expressed as a sum over their conventional counterparts, Eqs. (34) and (35) do not have counterparts in the usual one-band problem. It turns out that the interband contributions are controlled by the quantum metric tensor $g_{\mathbf{k}}^{ij}$ of the underlying quantum states in $\mathbf{k}$ space [6–8]. For our generic two-band lattice model, the quantum metric tensor of the Bloch states can be written as $2g_{\mathbf{k}}^{ij} = -\hat{\mathbf{d}}_{\mathbf{k}} \cdot (\partial^2\hat{\mathbf{d}}_{\mathbf{k}}/\partial k_i\partial k_j)$ or equivalently $2g_{\mathbf{k}}^{ij} = (\partial\hat{\mathbf{d}}_{\mathbf{k}}/\partial k_i)\cdot(\partial\hat{\mathbf{d}}_{\mathbf{k}}/\partial k_j)$. Alternatively, it can be expressed as

$$g_{\mathbf{k}}^{ij} = \frac{1}{2d_{\mathbf{k}}^2}\sum_{\ell=(x,y,z)}\frac{\partial d_{\mathbf{k}}^\ell}{\partial k_i}\frac{\partial d_{\mathbf{k}}^\ell}{\partial k_j} - \frac{1}{2d_{\mathbf{k}}^2}\frac{\partial d_{\mathbf{k}}}{\partial k_i}\frac{\partial d_{\mathbf{k}}}{\partial k_j}, \tag{36}$$

without loss of generality.

B. Purely relative fluctuations

Similar to Sec. III A, it may again be sufficient to take into account the following terms in the small-$\mathbf{q}$ and -ω expansions: $R_{q,E}^{11} + R_q^{12} = \tilde{A} + \sum_{ij} \tilde{C}_{ij} q_i q_j - \tilde{D}\omega^2 + \cdots$, $R_{q,E}^{11} - R_q^{12} = \tilde{P} + \sum_{ij} \tilde{Q}_{ij} q_i q_j - \tilde{R}\omega^2 + \cdots$, and $R_{q,O}^{11} = -\tilde{B}\omega + \cdots$. None of these expansion coefficients have a conventional counterpart in the usual one-band model. For instance, the nonkinetic coefficients are given by

$$\tilde{A}(\tilde{P}) = \frac{1}{U} - \frac{1}{N_l} \sum_{ss'\mathbf{k}} \frac{\xi_{s\mathbf{k}}\xi_{s'\mathbf{k}} + E_{s\mathbf{k}}E_{s'\mathbf{k}} \mp \Delta_0^2}{4E_{s\mathbf{k}}E_{s'\mathbf{k}}(E_{s\mathbf{k}} + E_{s'\mathbf{k}})} x_{ss'}^{\mathbf{k}}, \quad (37)$$

$$\tilde{D}(\tilde{R}) = \frac{1}{N_l} \sum_{ss'\mathbf{k}} \frac{\xi_{s\mathbf{k}}\xi_{s'\mathbf{k}} + E_{s\mathbf{k}}E_{s'\mathbf{k}} \mp \Delta_0^2}{4E_{s\mathbf{k}}E_{s'\mathbf{k}}(E_{s\mathbf{k}} + E_{s'\mathbf{k}})^3} x_{ss'}^{\mathbf{k}}, \quad (38)$$

$$\tilde{B} = \frac{1}{N_l} \sum_{ss'\mathbf{k}} \frac{\xi_{s\mathbf{k}}E_{s'\mathbf{k}} + E_{s\mathbf{k}}\xi_{s'\mathbf{k}}}{4E_{s\mathbf{k}}E_{s'\mathbf{k}}(E_{s\mathbf{k}} + E_{s'\mathbf{k}})^2} x_{ss'}^{\mathbf{k}}, \quad (39)$$

where we define $x_{ss'}^{\mathbf{k}} = 1 - ss'[(d_{\mathbf{k}}^x)^2 + (d_{\mathbf{k}}^y)^2 - (d_{\mathbf{k}}^z)^2]/d_{\mathbf{k}}^2$. The kinetic coefficients $\tilde{C}_{ij}$ and $\tilde{Q}_{ij}$ are more involved and are not presented here.

This expansion suggests that the Leggett mode is gapped as long as $\tilde{P} \neq 0$ and its finite frequency is determined by $\omega_L^2 = \tilde{P}/\tilde{R}$, when the coupling between the relative phase and relative amplitude fields is negligible. Here we note an intuitive result that $\tilde{P} = 0$ when the two bands are identical, i.e., when the sublattice-coupling field $\mathbf{d}_{\mathbf{k}} = 0$ vanishes so that $\xi_{s\mathbf{k}} = \xi_{-s,\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$. In addition, in the strong-coupling BEC limit when $\Delta_0 \gg \max|\epsilon_{s\mathbf{k}}|$, we note that $\tilde{P} \to 0$ as well. This is because since $\mu \lessgtr 0$ and $|\mu| \gg \max|\epsilon_{s\mathbf{k}}|$ in the dilute limit of particles or holes when $0.5 - N_0/N_l \approx \pm 0.5$, and $|\mu| \approx 0$ around half filling when $N_0/N_l \approx 0.5$, one can substitute $\xi_{s\mathbf{k}} \to -\mu$ and $E_{s\mathbf{k}} \to \sqrt{\mu^2 + \Delta_0^2}$. Thus, we conclude that the Leggett mode becomes undamped for sufficiently strong interactions with a negligibly smaller gap in the strong-coupling limit. This result is also intuitive given that the sublattice structure of the noninteracting particles should not play a primary role in the regime of tightly bound molecules.

Since $\tilde{B} \neq 0$ in most cases, we derive a relative phase-only (amplitude-only) action by integrating out the relative amplitude (phase) fields. This leads to an excitonlike out-of-phase (Leggett) mode and an excitonlike higher-energy amplitude (Higgs) mode [23]

$$\omega_{L(H)\mathbf{q}}^2 = \frac{\tilde{B}^2 + \tilde{A}\tilde{R} + \tilde{P}\tilde{D} \mp \tilde{W}}{2\tilde{D}\tilde{R}} + \sum_{ij} \left[\frac{\tilde{C}_{ij}}{2\tilde{D}} \left(1 \mp \frac{\tilde{B}^2 + \tilde{A}\tilde{R} - \tilde{P}\tilde{D}}{\tilde{W}} \right) + \frac{\tilde{Q}_{ij}}{2\tilde{R}} \left(1 \mp \frac{\tilde{B}^2 - \tilde{A}\tilde{R} + \tilde{P}\tilde{D}}{\tilde{W}} \right) \right] q_i q_j, \quad (40)$$

where we define $\tilde{W} = [(\tilde{B}^2 + \tilde{A}\tilde{R} + \tilde{P}\tilde{D})^2 - 4\tilde{A}\tilde{P}\tilde{D}\tilde{R}]^{1/2}$. Here the leading nonzero contribution to β_5 is approximated by $\tilde{D}\tilde{R}$ [23] and it must be replaced with the proper factor coming from the higher-order expansion coefficients in those exceptional cases when $\tilde{D} = 0$. One such example is the honeycomb lattice that is considered in Sec. II D 2, for which case we find $\tilde{B} = 0$ and set $\tilde{W} = \tilde{A}\tilde{R} - \tilde{P}\tilde{D}$, where $\tilde{A} = 1/U = \sum_{\mathbf{k}} 1/N_l E_{\mathbf{k}}$, $\tilde{P} = \sum_{\mathbf{k}} d_{\mathbf{k}}^2/N_l E_{\mathbf{k}}^3$, $\tilde{R} = \sum_{\mathbf{k}} \Delta_0^2/4N_l E_{\mathbf{k}}^5$, and $\tilde{D} = 0$.

C. SF phase stiffness tensor

At $T = 0$, we verify that the SF phase stiffness tensor $\mathcal{D}_{ij}$ is directly proportional to the kinetic coefficient Q_{ij} of the total phase fluctuations, i.e., $\mathcal{D}_{ij} = 8N_l(\Delta_0^2/\mathcal{A})Q_{ij}$, with $\mathcal{A}$ the area of the lattice, in such a way that [2,4,5]

$$\mathcal{D}_{ij}^{\text{conv}} = \frac{\Delta_0^2}{\mathcal{A}} \sum_{s\mathbf{k}} \frac{1}{E_{s\mathbf{k}}^3} \frac{\partial \xi_{s\mathbf{k}}}{\partial k_i} \frac{\partial \xi_{s\mathbf{k}}}{\partial k_j}, \quad (41)$$

$$\mathcal{D}_{ij}^{\text{geom}} = -\frac{2\Delta_0^2}{\mathcal{A}} \sum_{s\mathbf{k}} \frac{d_{\mathbf{k}}}{s\xi_{\mathbf{k}} E_{s\mathbf{k}}} g_{\mathbf{k}}^{ij}. \quad (42)$$

This is in accordance with the Landau criterion for superfluidity, i.e., the lowest-lying collective excitations of the SF ground state have a linear dispersion whose finite velocity is characterized by the SF stiffness. In addition to those given in Secs. II D 1 and II D 2, such an association between the Landau criterion for the Goldstone modes of the SF phase and the nonvanishing of the SF phase stiffness, which is compatible with the recent literature, may be considered as a third benchmark for the consistency of our results. The direct link between the quantum metric tensor and the SF stiffness tensor is relatively new in the literature [1,2], revealing the geometric origin of superconductivity in the presence of other bands. This result is particularly illuminating for a narrow-band or flatband superconductivity for which the geometric contribution clearly dominates the SF stiffness tensor when the conventional one is negligible. Motivated by these works, there have been many studies on the subject exploring a variety of multiband Hamiltonians, including most recently that of the twisted bilayer graphene [10–12].

Furthermore, it has been proposed that the quantum metric tensor has partial control over all those SF properties that depend explicitly on the effective-mass tensor of the SF carriers, i.e., of the corresponding (two- or many-body) bound state [4,5]. In the context of two-band SFs, our finding (30) for the velocity of the Goldstone mode is in complete agreement with our earlier work [9], suggesting that an analogous contribution to the collective excitations must be present in many other multiband systems as well.

IV. CONCLUSION

In summary, we have considered a generic lattice Hamiltonian that is suitable for describing a uniform BCS SF with two underlying sublattices and then extracted its collective excitations from an effective action that is derived up to the Gaussian order in the fluctuations of the SF order parameter. Allowing for independent fluctuations on the two sublattices, there are phononlike in-phase (Goldstone) and excitonlike out-of-phase (Leggett) modes in this system. While the Goldstone mode is gapless at zero momentum and propagating in general, the Leggett mode becomes undamped only with sufficiently strong interactions.

Furthermore, we showed that, in addition to the conventional contribution, the velocity of the Goldstone mode has a geometric contribution that is governed by the quantum metric tensor of the Bloch states. This suggests that the latter contribution dominates the velocity when the former becomes negligible for a narrowband or a flatband model. We traced the origin of the geometric contribution to the Goldstone mode back to the recent works on the geometric contribution to the SF stiffness tensor and argued that these geometric effects are complementary to each other, i.e., they are both controlled by the effective-mass tensor of the SF carriers. This suggests that an analogous contribution to the collective excitations must be present in many other multiband systems including the twisted bilayer graphene [10–12].

As an outlook, our approach can be generalized to a uniform BCS SF on a lattice with an n-point basis and verified that the kinetic coefficient Q_{ij} of the total phase fluctuations is directly proportional to the SF phase stiffness tensor $\mathcal{D}_{ij}$ that is found in the literature [2,3]. It is expected that the exact proportionality that we found in Sec. III C must hold in general. Furthermore, one may also like to study the dampings of the Goldstone and Leggett modes at finite temperatures as well [24].

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[1] S. Peotta and P. Törmä, Superfluidity in topologically nontrivial flat bands, Nat. Commun. **6**, 8944 (2015).

[2] L. Liang, T. I. Vanhala, S. Peotta, T. Siro, A. Harju, and P. Törmä, Band geometry, Berry curvature, and superfluid weight, Phys. Rev. B **95**, 024515 (2017).

[3] M. Iskin, Berezinskii-Kosterlitz-Thouless transition in the time-reversal-symmetric Hofstadter-Hubbard model, Phys. Rev. A **97**, 013618 (2018).

[4] M. Iskin, Superfluid stiffness for the attractive Hubbard model on a honeycomb optical lattice, Phys. Rev. A **99**, 023608 (2019).

[5] M. Iskin, Origin of flat-band superfluidity on the Mielke checkerboard lattice, Phys. Rev. A **99**, 053608 (2019).

[6] J. P. Provost and G. Vallee, Riemannian structure on manifolds of quantum states, Commun. Math. Phys. **76**, 289 (1980).

[7] M. V. Berry, in *Geometric Phases in Physics*, edited by A. Shapere and F. Wilczek, Advanced Series in Mathematical Physics Vol. 5 (World Scientific, Singapore, 1989).

[8] R. Resta, The insulating state of matter: A geometrical theory, Eur. Phys. J. B **79**, 121 (2011).

[9] M. Iskin, Geometric contribution to the Goldstone mode in spin-orbit-coupled Fermi superfluids, Phys. B: Condensed Matter **592**, 412260 (2020).

[10] X. Hu, T. Hyart, D. I. Pikulin, and E. Rossi, Geometric and Conventional Contribution to Superfluid Weight in Twisted Bilayer Graphene, Phys. Rev. Lett. **123**, 237002 (2019).

[11] A. Julku, T. J. Peltonen, L. Liang, T. T. Heikkilä, and P. Törmä, Superfluid weight and Berezinskii-Kosterlitz-Thouless transition temperature of twisted bilayer graphene, Phys. Rev. B **101**, 060505(R) (2020).

[12] F. Xie, Z. Song, B. Lian, and B. A. Bernevig, Topology-Bounded Superfluid Weight in Twisted Bilayer Graphene, Phys. Rev. Lett. **124**, 167002 (2020).

[13] E. Zhao and A. Paramekanti, BCS-BEC Crossover on the Two-Dimensional Honeycomb Lattice, Phys. Rev. Lett. **97**, 230404 (2006).

[14] S. Tsuchiya, R. Ganesh, and T. Nikuni, Higgs mode in a superfluid of Dirac fermions, Phys. Rev. B **88**, 014527 (2013).

[15] Y.-C. Zhang, Z. Xu, and S. Zhang, Topological superfluids and the BEC-BCS crossover in the attractive Haldane-Hubbard model, Phys. Rev. A **95**, 043640 (2017).

[16] M. Iskin and C. A. R. Sá de Melo, BCS-BEC crossover of collective excitations in two-band superfluids, Phys. Rev. B **72**, 024512 (2005).

[17] Here the multiplying factors follow from the trace over the sublattice sector, where $\mathrm{Tr}[(\tau_0 + s\hat{\mathbf{d}}_\mathbf{k}\cdot\boldsymbol{\tau})\tau_0(\tau_0 + s'\hat{\mathbf{d}}_{\mathbf{k+q}}\cdot\boldsymbol{\tau})\tau_0] = 2(1 + d_x d'_x + d_y d'_y + d_z d'_z)$ for the elements of the submatrix $\mathbf{T}_q$, $\mathrm{Tr}[(\tau_0 + s\hat{\mathbf{d}}_\mathbf{k}\cdot\boldsymbol{\tau})\tau_z(\tau_0 + s'\hat{\mathbf{d}}_{\mathbf{k+q}}\cdot\boldsymbol{\tau})\tau_z] = 2(1 - d_x d'_x - d_y d'_y + d_z d'_z)$ for the elements of the submatrix $\mathbf{R}_q$, $\mathrm{Tr}[(\tau_0 + s\hat{\mathbf{d}}_\mathbf{k}\cdot\boldsymbol{\tau})\tau_z(\tau_0 + s'\hat{\mathbf{d}}_{\mathbf{k+q}}\cdot\boldsymbol{\tau})\tau_0] = 2[d_z + d'_z - i(d_x d'_y - d_y d'_x)]$ for the elements of the submatrix $\mathbf{C}_q$, and $\mathrm{Tr}[(\tau_0 + s\hat{\mathbf{d}}_\mathbf{k}\cdot\boldsymbol{\tau})\tau_0(\tau_0 + s'\hat{\mathbf{d}}_{\mathbf{k+q}}\cdot\boldsymbol{\tau})\tau_z] = 2[d_z + d'_z + i(d_x d'_y - d_y d'_x)]$ for the elements of the submatrix $\mathbf{C}_q^*$.

[18] Using the notation of Ref. [14], in contrast to their expressions $1/U = -(C + D) + |R|$ for the degenerate amplitude modes and $1/U = -(C - D) + \sqrt{4|F|^2 + |R|^2}$ for the degenerate phase modes, our Eq. (25) gives two distinct solutions for the amplitude modes that are determined by $1/U = -(C + D) \mp |R|$ and two distinct solutions for the phase modes that are determined by $1/U = -(C - D) \mp |R|$. Thus, while our total amplitude mode coincides with their Higgs solution, we find a nondegenerate solution for the relative amplitude mode. Similarly, while our solution for the total phase mode coincides with their AB Leggett solution when their $F = 0$, e.g., in the $\mathbf{q} = \mathbf{0}$ limit, we again find a nondegenerate solution for the relative phase mode. In the text we set $\mathbf{q} = \mathbf{0}$ for its analytical simplicity and discuss the resultant solutions, i.e., Eqs. (28) and (29) correspond to the amplitude modes and Eqs. (26) and (27) correspond to the phase ones.

[19] I. O. Kulik, O. Entin-Wohlman, and R. Orbach, Pair susceptibility and mode propagation in superconductors: A microscopic approach, J. Low Temp. Phys. **43**, 591 (1981).

[20] L. Belkhir and M. Randeria, Crossover from Cooper pairs to composite bosons: A generalized RPA analysis of collective excitations, Phys. Rev. B **49**, 6829 (1994).

[21] J. R. Engelbrecht, M. Randeria, and C. A. R. Sá de Melo, BCS to Bose crossover: Broken-symmetry state, Phys. Rev. B **55**, 15153 (1997).

[22] For the purely total fluctuations, one needs to derive the characteristic equation up to fourth order in the expansion, e.g., $\beta_5\omega^4 + \sum_{ijkl}\beta_4^{ijkl} q_i q_j q_k q_l + \sum_{ij}\beta_3^{ij}\omega^2 q_i q_j + \beta_2\omega^2 + \sum_{ij}\beta_1^{ij} q_i q_j = 0$ such that the Goldstone mode $\omega_{G\mathbf{q}}^2 = \sum_{ij} x_G^{ij} q_i q_j$ is determined by $x_G^{ij} = -\beta_1^{ij}/\beta_2$ and the Higgs mode $\omega_{H\mathbf{q}}^2 = \omega_H^2 + \sum_{ij} x_H^{ij} q_i q_j$ is determined by $\omega_H^2 = -\beta_2/\beta_5$ and $x_H^{ij} = -\beta_3^{ij}/\beta_5 + \beta_1^{ij}/\beta_2$. While our quadratic expansion in the text fully determines x_G^{ij}, i.e., our Eq. (30) is exact, we neglect

the additional corrections to ω_H^2 and x_H^{ij} that are coming from the higher-order terms to β_3^{ij} and β_5. To be more precise, we substitute $\beta_1^{ij} = AQ_{ij}$, $\beta_2 = -AR - B^2$, $\beta_3^{ij} \approx -RC_{ij} - DQ_{ij}$, and $\beta_5 \approx DR$.

[23] For the purely relative fluctuations, one needs to derive the characteristic equation up to fourth order in the expansion, e.g., $\beta_5\omega^4 + \sum_{ijkl} \beta_4^{ijkl} q_i q_j q_k q_l + \sum_{ij} \beta_3^{ij} \omega^2 q_i q_j + \beta_2\omega^2 + \sum_{ij} \beta_1^{ij} q_i q_j + \beta_0 = 0$ such that the Leggett and Higgs modes $\omega^2_{L\,(H)\mathbf{q}} = \omega^2_{L\,(H)} + \sum_{ij} x^{ij}_{L\,(H)} q_i q_j$ are determined by $\omega_{L\,(H)} = (-\beta_2 \mp \sqrt{\beta_2^2 - 4\beta_0\beta_5})/2\beta_5$ and $x^{ij}_{L\,(H)} = [-\beta_3^{ij} \pm (\beta_2\beta_3^{ij} - 2\beta_5\beta_1^{ij})/\sqrt{\beta_2^2 - 4\beta_0\beta_5}]/2\beta_5$. Note that the equation in Ref. [22] is recovered in the $\beta_0 \to 0$ limit. In our quadratic expansion presented in the text, we neglect the additional corrections to $\omega^2_{L\,(H)}$ and $x^{ij}_{L\,(H)}$ that are coming from the higher-order terms to β_3^{ij} and β_5. To be more precise, we substitute $\beta_0 = \tilde{A}\tilde{P}$, $\beta_1^{ij} = \tilde{A}\tilde{Q}_{ij} + \tilde{P}\tilde{C}_{ij}$, $\beta_2 = -\tilde{A}\tilde{R} - \tilde{P}\tilde{D} - \tilde{B}^2$, $\beta_3^{ij} \approx -\tilde{R}\tilde{C}_{ij} - \tilde{D}\tilde{Q}_{ij}$, and $\beta_5 \approx \tilde{D}\tilde{R}$.

[24] S. N. Klimin, H. Kurkjian, and J. Tempere, Leggett collective excitations in a two-band Fermi superfluid at finite temperatures, New J. Phys. **21**, 113043 (2019).