

Effective field theory of bosons with finite-range interaction in a disordered environmentAlberto Cappellaro^{1,*} and Luca Salasnich^{1,2}¹*Dipartimento di Fisica e Astronomia “Galileo Galilei” and CNISM, Università di Padova, via Marzolo 8, 35131 Padova, Italy*²*CNR-INO, via Nello Carrara, 1 - 50019 Sesto Fiorentino, Italy*

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We investigate the low-temperature properties of an ultracold gas made of bosonic alkali-metal atoms with finite-range interaction under the effect of a disordered environment. The statistical characterization of the disorder is investigated within an effective-field-theory formalism for a generic spatial dimension d . Moving to $d = 3$, where all the arising divergences are properly regularized, we focus on the depletion of both the condensate and superfluid densities. At zero temperature we obtain meaningful analytical formulas for the condensate fraction and the superfluid fraction which take into account the interplay among scattering length, effective range, and disorder strength.

DOI: [10.1103/PhysRevA.101.053628](https://doi.org/10.1103/PhysRevA.101.053628)**I. INTRODUCTION**

During the last decades, great interest has been aroused by the interplay between interactions and disorder. The investigation of quantum effects in many-particle systems moving in disordered environments goes back in time to the prediction of a localized phase for free electrons in random-lattice structures [1].

Similarly to other condensed matter issues, significant efforts have been devoted to explore the role of a disordered environment in the framework of ultracold atomic gases [2,3]. The main reason lies in the remarkable experimental control over the relevant physical parameters such as densities and interaction strengths. In order to construct a random potential mimicking a porous medium, a viable strategy consists in superimposing two different optical lattices [4,5]. Experimentally, another possibility is provided by laser-speckle fields, arising from the interference pattern of waves with the same frequency but different phases and amplitudes [6–8].

In this paper we focus on disordered bosons within their superfluid phase but in the presence of a finite-range interaction between the atoms. By taking into account a nonlocal two-body potential, nonuniversal corrections to the thermodynamic potential can be obtained beyond the usual mean-field picture [9–11]. As a consequence, detachments from universality may represent a viable approach to reveal the elusive contribution of quantum fluctuations. These corrections can also be included in a modified Gross-Pitaevskii equation (GPE), leading to nonuniversal dynamical features [12–16].

Our analysis is carried on within the framework of the effective field theory, where one can relate the coupling constants of the Euclidean action functional to the measurable s -wave scattering parameters via the T-matrix technique [10,17]. From a technical point of view, in order to implement a quantum field theory in presence of a disorder (external

potential, different strategies are at disposal. Since we are interested in how disorder and finite-range interactions both contribute to modify the condensate and superfluid depletion, we adopt a perturbative approach: the disorder is assumed to be weak, in a similar way as quantum and thermal fluctuations. Their role is then taken into account up to the one-loop (i.e., Gaussian) level. For a zero-range interaction, it has been shown that also a very weak disorder affects the thermodynamic picture of a bosonic superfluid [18–20].

On the other hand, the case of strong disorder has to be treated nonperturbatively. The most standard approaches all rely upon the so-called replica trick, first proposed in the context of spin glasses [21–24]. In Ref. [25], a replicated Hartree-Fock theory managed to reproduce the phase diagram for interacting bosons where also the localized phase is included. However, in this paper we are going to focus only on the weak disorder regime where the replica trick is not crucial, since it basically reproduces the perturbative results, as shown for dirty superconductors [26] and cold atoms [27].

The paper is organized as follows: first we review how (weak) disorder can be included within a Euclidean functional formulation for a system of interacting bosons in a generic dimension d . Then we move to compute the thermodynamic potential up to the Gaussian level in quantum and thermal fluctuations by taking into account both the presence of an uncorrelated quenched disorder and a finite-range interaction between the atoms. In order to simplify the computational burden, we focus on pointlike defects. The case $d = 3$ is analyzed in more detail, and explicit results are given also for the thermal depletion of the superfluid density.

II. MODELING THE DISORDER

In this paper, we study the interplay between disorder, nonlocal interaction, and fluctuations in a quantum gas made of bosonic particles. To fulfill this task, our analysis will be carried on within the functional integration framework.

In order to properly construct the path integral for an ensemble of identical atomic bosons with mass m , we start

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from the second-quantized Hamiltonian,

$$\hat{K} = \int_{L^d} d^d \mathbf{r} \hat{\psi}^\dagger(\mathbf{r}) \left[-\frac{\hbar^2 \nabla^2}{2m} - \mu - U_D(\mathbf{r}) \right] \hat{\psi}(\mathbf{r}) + \frac{1}{2} \int_{L^d} d^d \mathbf{r} d^d \mathbf{r}' \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') V(|\mathbf{r} - \mathbf{r}'|) \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}), \quad (1)$$

where $V(r)$ is the central two-atom interaction potential (with $r = |\mathbf{r}|$), while $U_D(\mathbf{r})$ is the random external potential, which takes into account the effect of disordered environment. The field operators in Eq. (1) respect the usual bosonic commutation relation $[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = \delta^{(d)}(\mathbf{r} - \mathbf{r}')$. The spatial integration is taken over a large d -dimensional hyper cubic volume L^d ; at the end of the calculation, if interested in the thermodynamic limit, we have to take the limit $L \rightarrow \infty$.

Given Eq. (1) and adopting the functional integration formalism, we can represent the partition function of the system as [28]

$$\mathcal{Z} = e^{-\beta\Omega} = \int \mathcal{D}[\psi, \psi^*] \exp \left\{ -\frac{1}{\hbar} S_E[\psi, \psi^*] \right\}, \quad (2)$$

where Ω is the thermodynamic potential, $\psi(\mathbf{r}, \tau)$ is the coarse-grained complex field modeling the bosons, $\mathcal{D}[\psi, \psi^*]$ is the measure of the functional integration over $\psi(\mathbf{r}, \tau)$ and $\psi^*(\mathbf{r}, \tau)$, and $S_E[\psi, \psi^*]$ denotes the Euclidean (i.e., imaginary time) action, defined as

$$S_E[\psi, \psi^*] = \int_0^{\beta\hbar} d\tau \int_{L^d} d^d \mathbf{r} [\psi^* \hbar \partial_\tau \psi + K(\psi, \psi^*)]. \quad (3)$$

In the equation above, $\beta = 1/(k_B T)$, k_B being the Boltzmann constant and T the absolute temperature.

It is good to characterize the external random potential $U_D(\mathbf{r})$ from a statistical point of view. This implies that its features can be extracted from a probability density function (PDF), according to which single realizations of a disordered environment are distributed. A reasonable choice for $U_D(\mathbf{r})$ can be a centered (i.e., zero average) Gaussian such as the following one:

$$\mathcal{P}[U_D] \propto \exp \left\{ -\frac{1}{2} \int d^d \mathbf{r} d^d \mathbf{r}' U_D(\mathbf{r}) \Delta^{-1}(\mathbf{r}, \mathbf{r}') U_D(\mathbf{r}') \right\}, \quad (4)$$

with a proper normalization factor which is, however, not important in the proceeding [28]. Indeed, we will mostly need the correlator

$$\langle U_D(\mathbf{r}) U_D(\mathbf{r}') \rangle_{\text{dis}} = \Delta(\mathbf{r}, \mathbf{r}'), \quad (5)$$

where $\langle \dots \rangle_{\text{dis}}$ has to be intended as

$$\langle \dots \rangle_{\text{dis}} = \int \mathcal{D}[U_D] \mathcal{P}[U_D] (\dots). \quad (6)$$

From Eq. (4) it is clear that we are assuming that the characteristic timescale of $U_D(\mathbf{r})$ is infinitely long compared to the other ones; we are then restricting ourselves to the (important) case of a *quenched* disordered environment. Obviously, this assumption can be relaxed and the random potential can be taken as time- or temperature-dependent [29].

Among Gaussian-distributed random configurations, the most simple situation is provided by potentials with null correlation length. They all have a δ -like correlator, such as

$$\Delta(\mathbf{r}, \mathbf{r}') = \Delta(\mathbf{r}) \delta^{(d)}(\mathbf{r} - \mathbf{r}'). \quad (7)$$

Despite being an extremely simplifying assumption, the equation above suits well with disorder generated by static and pointlike bosonic impurities [29]. More complicated disordered configurations are discussed, for instance, in Refs. [29,30].

III. DISORDER AND FINITE-RANGE INTERACTIONS

A. The finite-range effective potential

The crucial issue now consists in a proper choice of the two-body interaction. Since the first seminal experiments with the alkali atoms in the degenerate regime, it appeared that reliable analysis and predictions can be carried on by considering a contact interaction (i.e., zero range) $V(r) = g_0 \delta^{(3)}(r)$, where $r = |\mathbf{r}|$ and $g_0 = \int d^d \mathbf{r} V(r)$ is the interaction strength. This approximation leads to a universal thermodynamics, where all the relevant equilibrium features of the system depend only on the coupling parameter [31,32].

However, thanks to Feshbach resonances, it is possible to explore regimes where deviations from universalities become relevant [10–12,15,33]. Moreover, in Ref. [9] it was already pointed out that some divergences in the perturbative expansion of the thermodynamic potential of Bose gases can be healed by involving additional features of the interaction potential such as its characteristic range.

In order to analyze the first correction to the thermodynamic properties of the Bose gas due to a finite-range interaction, we begin by considering the following low-momentum expansion:

$$\tilde{V}(q) = g_0 + g_2 q^2 + \mathcal{O}(q^4), \quad (8)$$

of the Fourier transform $\tilde{V}(q)$ of two-body interaction potential $V(r)$, where $g_0 = \tilde{V}(0)$ and $g_2 = (1/2)\tilde{V}''(0)$.

By making use of the effective-field-theory techniques [9] and of the T-matrix formulation of two-atoms collisions [17], it is possible to define the coupling constants g_0 and g_2 in the equation above in terms of measurable scattering parameters. As discussed in detail in Ref. [11], one obtains

$$g_0 = \frac{4\pi\hbar^2}{m} a_s, \quad g_2 = \frac{2\pi\hbar^2}{m} a_s^2 r_s. \quad (9)$$

It is worth recalling that this recipe can be likewise applied in lower dimensions, namely, $d = 1$ and (with some technical complications) in $d = 2$ [33].

As mentioned above, thanks to Feshbach resonances, one can explore a wide range of values for the s -wave scattering length by tuning an external magnetic fields. The effective range expansion responsible for Eqs. (8) and (9) has to be treated carefully close to narrow resonances or zero crossings of a_s as a function of the magnetic field B . Rigorously speaking, close to a narrow resonance, a multichannel effective theory should be required, as detailed in Ref. [34], naturally reading to finite-range effects. Other multichannel approaches

rely upon the formation of molecular bound states in the regime of resonant interaction [35–37].

However, it has been shown that it is possible to reduce the multichannel scattering problem to a single-channel one [38], at least for a broad resonance. The analysis reported in Ref. [38] is further extended for zero-point crossings and narrow resonances in Ref. [39], where it is also compared with multichannel numerical outcomes. Remarkably, for the isolated resonances in ^{39}K and ^{133}Cs , a very good agreement is found, providing a very good single-channel approximation.

B. δ -correlated Gaussian disorder: perturbative analysis

At the mean-field level, the space-time translationally invariant ground state of the system can be described by a constant v , namely, $\psi(\mathbf{r}, \tau) = v$. For the sake of simplicity the constant v is taken real. By inserting it in Eq. (3), the mean-field thermodynamic potential reads

$$\frac{\Omega_{\text{mf}}(\mu, v)}{L^d} = -[\mu + \langle U_D \rangle_{\text{dis}}]v^2 + \frac{1}{2}\tilde{V}(0)v^4, \quad (10)$$

where we have already considered the disorder average defined in Eq. (6). However, in the following we are going to consider the case of δ -correlated centered Gaussian disorder, so $\langle U_D \rangle_{\text{dis}} = 0$. Equation (10) is made stationary only by $v = 0$ for $\mu < 0$, but v acquires a nonzero value for $\mu > 0$:

$$v^2 = \frac{\mu}{\tilde{V}(0)}. \quad (11)$$

This fact obviously signals the occurring of the superfluid transition, where a $U(1)$ symmetry is spontaneously broken.

It is important to remark that a δ correlation does not affect the mean-field picture of the homogeneous and

stationary ground state. Moreover, if we consider the interaction potential as given by Eq. (8), we immediately realize that only $g_0 = \tilde{V}(0)$ appears up to this level of approximation. As a consequence, in order to explore the role of disorder and nonlocal interactions, one has to consider fluctuations, both quantum and thermal, above the ground state given by Eq. (11).

Let us then consider the following shift of the field:

$$\psi(\mathbf{r}, \tau) = v + \eta(\mathbf{r}, \tau), \quad (12)$$

with $\eta(\mathbf{r}, \tau)$ being the complex fluctuating field.

By replacing Eq. (12) in the Euclidean action $S_E[\psi, \psi^*]$ and retaining terms up to the quadratic (Gaussian) level in η and η^* , the partition function defined in Eq. (2) can be factorized in

$$\mathcal{Z}[U_D] \simeq e^{-\beta\Omega_{\text{mf}}} \int \mathcal{D}[\eta, \eta^*] e^{-\frac{1}{\hbar}(S_g^{(\text{pure})}[v, \eta] + S_g^{(\text{dis})}[v, \eta, U_D])}, \quad (13)$$

where Ω_{mf} is given by Eq. (10), $S_g^{(\text{pure})}$ is the Euclidean action describing the periodic imaginary-time trajectories of the fluctuating fields η and η^* , while the disorder contribution is encoded in $S_g^{(\text{dis})}$.

Within our perturbative scheme, $S_g^{(\text{pure})}$ reads the same expression for a system not subject to an external disordered environment. In the Fourier space, this implies that

$$S_g^{(\text{pure})} = \frac{\hbar}{2} \sum_{\mathbf{q}, \omega_n} \Psi^\dagger(\mathbf{q}, \omega_n) \mathbb{M}(\mathbf{q}, \omega_n, v) \Psi(\mathbf{q}, \omega_n), \quad (14)$$

with the spinor $\Psi^\dagger(\mathbf{q}, \omega_n) = [\tilde{\eta}^*(\mathbf{q}, \omega_n), \tilde{\eta}(-\mathbf{q}, -\omega_n)]$, and

$$\mathbb{M}(\mathbf{q}, \omega_n, v) = \beta \begin{pmatrix} -i\hbar\omega_n + \frac{\hbar^2 q^2}{2m} - \mu + g_0 v^2 + v^2(g_0 + g_2 q^2) & v^2(g_0 + g_2 q^2) \\ v^2(g_0 + g_2 q^2) & i\hbar\omega_n + \frac{\hbar^2 q^2}{2m} - \mu + g_0 v^2 + v^2(g_0 + g_2 q^2) \end{pmatrix} \quad (15)$$

is the inverse of the Gaussian propagator. In this framework, the discrete frequencies labeled by $n \in \mathbb{Z}$ are the usual Matsubara ones, defined as $\omega_n = 2\pi n / (\beta\hbar)$ [28].

On the other hand, the disorder contribution $S_g^{(\text{dis})}$ can be written as [40]

$$S_g^{(\text{dis})} = -v \int_0^{\beta\hbar} d\tau \int d^d \mathbf{r} U_D(\mathbf{r})(1, 1) \cdot \begin{pmatrix} \eta(\mathbf{r}, t) \\ \eta^*(\mathbf{r}, t) \end{pmatrix}. \quad (16)$$

Since Eq. (13) has, by construction, a Gaussian structure, the functional integration over η and η^* can be performed exactly, reading, in Fourier space [27],

$$\begin{aligned} \mathcal{Z}[U_D] &\simeq e^{-\beta\Omega_{\text{mf}}} \exp \left\{ -\frac{1}{2} \sum_{\mathbf{q}, \omega_n} \ln[\det \mathbb{M}(\mathbf{q}, \omega_n)] \right\} \\ &\times \exp \left\{ \frac{1}{2} \sum_{\mathbf{q}} |\tilde{U}_D(\mathbf{q})|^2 (v, v) \mathbb{M}^{-1}(\mathbf{q}, 0) \begin{pmatrix} v \\ v \end{pmatrix} \right\}, \end{aligned} \quad (17)$$

where $\tilde{U}_D(\mathbf{q})$ is the Fourier transform of the random potential $U_D(\mathbf{r})$. Let us remark that, in the equation above, the propagator \mathbb{M} is computed at $\omega_n = 0$, so there is no dependency on the temperature in the disorder contribution. As stated in Sec. II, this is due to the assumption of a quenched disorder, whose characteristic features are frozen compared to other (quantum and thermally) fluctuating quantities.

The pure contribution of Gaussian fluctuations leads us to

$$\Omega_g^{(\text{pure})}(\mu, v) = \frac{1}{2\beta} \sum_{\mathbf{q}, \omega_n} \ln [\beta^2 (\hbar^2 \omega_n^2 + E_q^2)], \quad (18)$$

with

$$E_q^2 = \left[\frac{\hbar^2 q^2}{2m} - \mu + g_0 v^2 + v^2 \tilde{V}(q) \right]^2 - v^4 \tilde{V}^2(q) \quad (19)$$

being the elementary excitation spectrum over the uniform ground state. By performing the Matsubara summation in Eq. (18) we can identify the quantum and thermal contribution

to $\Omega_g^{(\text{pure})}$:

$$\Omega_g^{(\text{pure})} = \frac{1}{2} \sum_{\mathbf{q}} E_{\mathbf{q}} + \frac{1}{\beta} \sum_{\mathbf{q}} \ln(1 - e^{-\beta E_{\mathbf{q}}}). \quad (20)$$

Clearly, $\Omega_g^{(\text{pure})}$ is independent from $U_D(\mathbf{r})$, so Eq. (20) is left untouched by the (functional) disorder average introduced in Eq. (6), i.e., $\langle \Omega_g^{(\text{pure})} \rangle_{\text{dis}} = \Omega_g^{(\text{pure})}$. Obviously, this is not true for the disorder term. Indeed, if we take the logarithm of the second line of Eq. (13) we get the corresponding

$$\Omega_g^{(\text{dis})} = -\frac{v^2}{2} \sum_{\mathbf{q}} |\tilde{U}_D(\mathbf{q})|^2 (1, 1) \cdot \mathbb{M}^{-1}(\mathbf{q}, 0) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (21)$$

Under the assumption of a Gaussian distribution disorder, $\langle \Omega_g^{(\text{dis})} \rangle_{\text{dis}}$ can be easily computed by using Eq. (5), which gives

$$\int \mathcal{D}[U_D] \mathcal{P}[U_D] |U_D(\mathbf{q})|^2 = \langle |U_D(\mathbf{q})|^2 \rangle_{\text{dis}} = \tilde{\Delta}(\mathbf{q}), \quad (22)$$

where $\tilde{\Delta}(\mathbf{q})$ is the Fourier transform of the δ -correlated disorder field $\Delta(\mathbf{r})$. In this way we obtain

$$\langle \Omega_g^{(\text{dis})} \rangle_{\text{dis}} = -\frac{v^2}{2} \sum_{\mathbf{q}} \tilde{\Delta}(\mathbf{q}) (1, 1) \cdot \mathbb{M}^{-1}(\mathbf{q}, 0) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (23)$$

The inverse of Eq. (15) is immediate to compute, such that, at the continuum limit $\sum_{\mathbf{q}} \rightarrow L^d \int d^d \mathbf{q} / (2\pi)^d$,

$$\frac{\langle \Omega_g^{(\text{dis})} \rangle_{\text{dis}}}{L^d} = -v^2 \int \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{\tilde{\Delta}(\mathbf{q})}{\frac{\hbar^2 q^2}{2m} + 2v^2 g_2 q^2 - \mu + 3g_0 v^2}. \quad (24)$$

In the following section we derive analytical and numerical results for the disorder contribution to the thermodynamic potential provided Eq. (24) as a starting point.

IV. POINTLIKE DEFECTS

A. Disorder contributions in d dimensions

In order to proceed with our calculation, it is necessary to specify the spatial behavior (or, equivalently, the dependence on \mathbf{q}) of the disorder correlator $\Delta(\mathbf{r})$. For pointlike defects, Eq. (7) is further simplified, assuming

$$\tilde{\Delta}(\mathbf{q}) = \Delta, \quad (25)$$

where Δ is a real constant. As mentioned earlier, despite this simple assumption, the system, in its superfluid phase, is nevertheless crucially affected by the presence of a disordered external environment.

In order to highlight this feature, given Eq. (25), let us recast Eq. (24) as

$$\frac{\langle \Omega_g^{(\text{dis})} \rangle_{\text{dis}}}{L^d} = -\frac{\Delta v^2}{\lambda(v^2, g_2)} \int \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{1}{\frac{\hbar^2 q^2}{2m} + \frac{3g_0 v^2 - \mu}{\lambda(v^2, g_2)}}, \quad (26)$$

where, in analogy with the notation used in Ref. [11], we have defined

$$\lambda(v^2, g_2) = 1 + \frac{4mg_2}{\hbar^2} v^2. \quad (27)$$

At this point, it is immediate to realize that the disorder contribution $\langle \Omega_g^{(\text{dis})} \rangle_{\text{dis}} / L^d$ diverges for $d \geq 2$. However, it has been extensively shown (see [10,32,41,42] for technical details) that (finite) meaningful information may be extracted by means of the so-called dimensional regularization.

The key point of this method consists in performing the integration in Eq. (26) in a generic complex dimension $d_\epsilon = d - \epsilon$, with values of $\epsilon \in \mathbb{C}$ for which the result is convergent. The last step requires an analytical continuation back to the physical dimension we are interested in, i.e., one to consider a proper limit procedure such that $\epsilon \rightarrow 0$. Within this regularization framework, the following result holds:

$$\mathcal{I}(d) = \int \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{1}{q^2 + M^2} = \frac{\Gamma(1 - d/2)}{2^d \pi^{d/2}} M^{d-2}, \quad (28)$$

$\Gamma(z)$ being the Euler's Γ function. As a consequence, Eq. (26) reads

$$\frac{\langle \Omega_g^{(\text{dis})} \rangle_{\text{dis}}}{L^d} = -\frac{\Gamma(1 - d/2)}{(2\pi)^{d/2}} \left(\frac{m}{\hbar^2} \right)^{d/2} \frac{\Delta v^2 (3g_0 v^2 - \mu)^{d/2-1}}{\lambda^{d/2}(v^2, g_2)}. \quad (29)$$

Moving from Eq. (29), a simple derivative leads us to the disorder contribution to the total density as a function of the condensed one. More technically,

$$\begin{aligned} n_g^{(\text{dis})}(n_0) &= -\frac{1}{L^d} \frac{\partial}{\partial \mu} \langle \Omega_{\text{dis}}^{(0)}(v, \mu) \rangle_{\text{dis}} \Big|_{\substack{\mu = g_0 n_0 \\ v^2 = n_0}} \\ &= \frac{\Gamma(2 - d/2)}{4\pi^{d/2}} \left(\frac{m}{\hbar^2} \right)^{d/2} \frac{\Delta g_0^{d/2-2} n_0^{d/2-1}}{\lambda^{d/2}(n_0, g_2)}. \end{aligned} \quad (30)$$

It is fundamental to note that following a perturbative scheme, we have identified v^2 as given in Eq. (11) with the density n_0 of condensed atoms. Thus, our approach provides an implicit expression for the condensate density of the system. Since the pure and the disorder contributions to the thermodynamic potential are additive, as evident from Eqs. (13) and (20), the same occurs for the contributions to the total number density, i.e.,

$$n = n_0 + n_g^{(0)}(n) + n_g^{(T)}(n) + n_g^{(\text{dis})}(n). \quad (31)$$

Let us also point out that within a perturbative approach for the weakly interacting system, it is possible to approximate $n_0 \approx n$ in the fluctuation corrections $n_g^{(0)}(n)$, $n_g^{(T)}(n)$, and $n_g^{(\text{dis})}(n)$. Thus, in the following we are going to approximate n_0 with n in the perturbatively computed fluctuation contributions. In Eq. (31), $n_g^{(0)}$ and $n_g^{(T)}$ are computed similarly to Eq. (30): first, one has to extract the regularized Gaussian corrections from Eq. (20), then derive them with respect to the chemical potential.

In Ref. [33], an extensive analysis addressed this topic for the pure system. Being interested in the role played by the

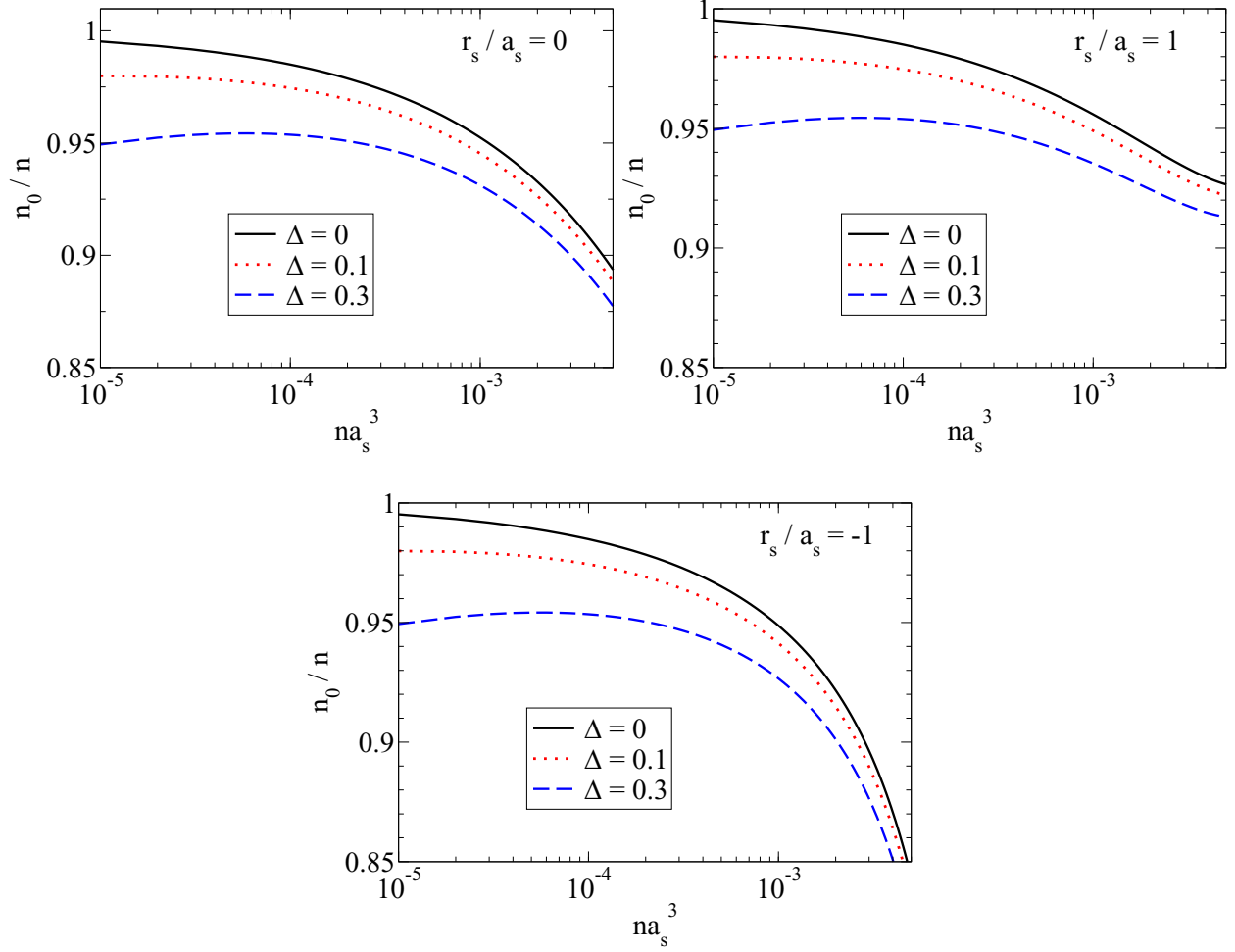


FIG. 1. Condensate fraction n_0/n at zero temperature as a function of the dimensionless gas parameter na_s^3 for increasing disorder strength Δ . The three panels correspond to different values of the ratio r_s/a_s between the effective range r_s and the scattering length a_s . In the legend, Δ is expressed in units of E_B^2/n , with $E_B = \hbar^2 n^{2/3}/m$. The curves are obtained from Eqs. (32) and (33), which provide a reliable thermodynamic picture of the system within the range of diluteness (i.e., the values of na_s^3) chosen for the horizontal axis.

disorder, we report here only the final result, i.e.,

$$n_g^{(0)}(n) = \frac{1}{4\pi^{d/2}\Gamma(d/2)} \left(\frac{m}{\hbar^2}\right)^{d/2} \frac{(g_0 n)^{d/2}}{\lambda^{(d-1)/2}(n, g_2)} \left[2 \frac{1 + \frac{2m}{\hbar^2} g_2 n}{\lambda(n, g_2)} \mathcal{B}\left(\frac{d+1}{2}, -\frac{d}{2}\right) + \mathcal{B}\left(\frac{d-1}{2}, \frac{2-d}{2}\right) \right],$$

$$n_g^{(T)}(n) = \int \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{\frac{\hbar^2 q^2}{2m} (1 + \frac{2m}{\hbar^2} g_2 n) + g_0 n}{\sqrt{\frac{\hbar^2 q^2}{2m} [\lambda(n, g_2) \frac{\hbar^2 q^2}{2m} + 2g_0 n]}} \left(\frac{1}{e^{\beta E_q(n)} - 1} \right), \quad (32)$$

with $\mathcal{B}(x, y)$ the Euler β function, which can be rewritten, after analytic continuation, in terms of the Euler gamma function $\Gamma(x)$ as follows: $\mathcal{B}(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. $E_q(n)$ is the spectrum of collective excitations given by Eq. (19), with $n_0 \approx n$.

V. ANALYTICAL AND NUMERICAL RESULTS IN THREE DIMENSIONS

We now consider the common situation of bosons moving in three spatial dimensions, i.e., $d = 3$. In this case, during the calculation of $\Omega(v, \mu)$, only power divergences arise. Within the dimensional regularization scheme, they are set to zero

and no explicit renormalization is required [32]; this holds for both the pure and disorder terms.

A. Condensate density

By taking the limit $d \rightarrow 3$ in Eqs. (30) and (32), at zero temperature one gets

$$\frac{n_0}{n} = 1 - \frac{32}{3\sqrt{\pi}} \frac{\sqrt{na_s^3}}{1 + 8\pi (na_s^3) \left(\frac{r_s}{a_s}\right)} \left[\frac{1 + 4\pi (na_s^3) \left(\frac{r_s}{a_s}\right)}{1 + 8\pi (na_s^3) \left(\frac{r_s}{a_s}\right)} - \frac{3}{4} \right] - \frac{1}{(4\pi)^{3/2}} \frac{\Delta \left(\frac{m^2}{\hbar^4 n^{1/3}}\right)}{\left[1 + 8\pi (na_s^3) \left(\frac{r_s}{a_s}\right)\right]^{3/2} (na_s^3)^{1/6}}. \quad (33)$$

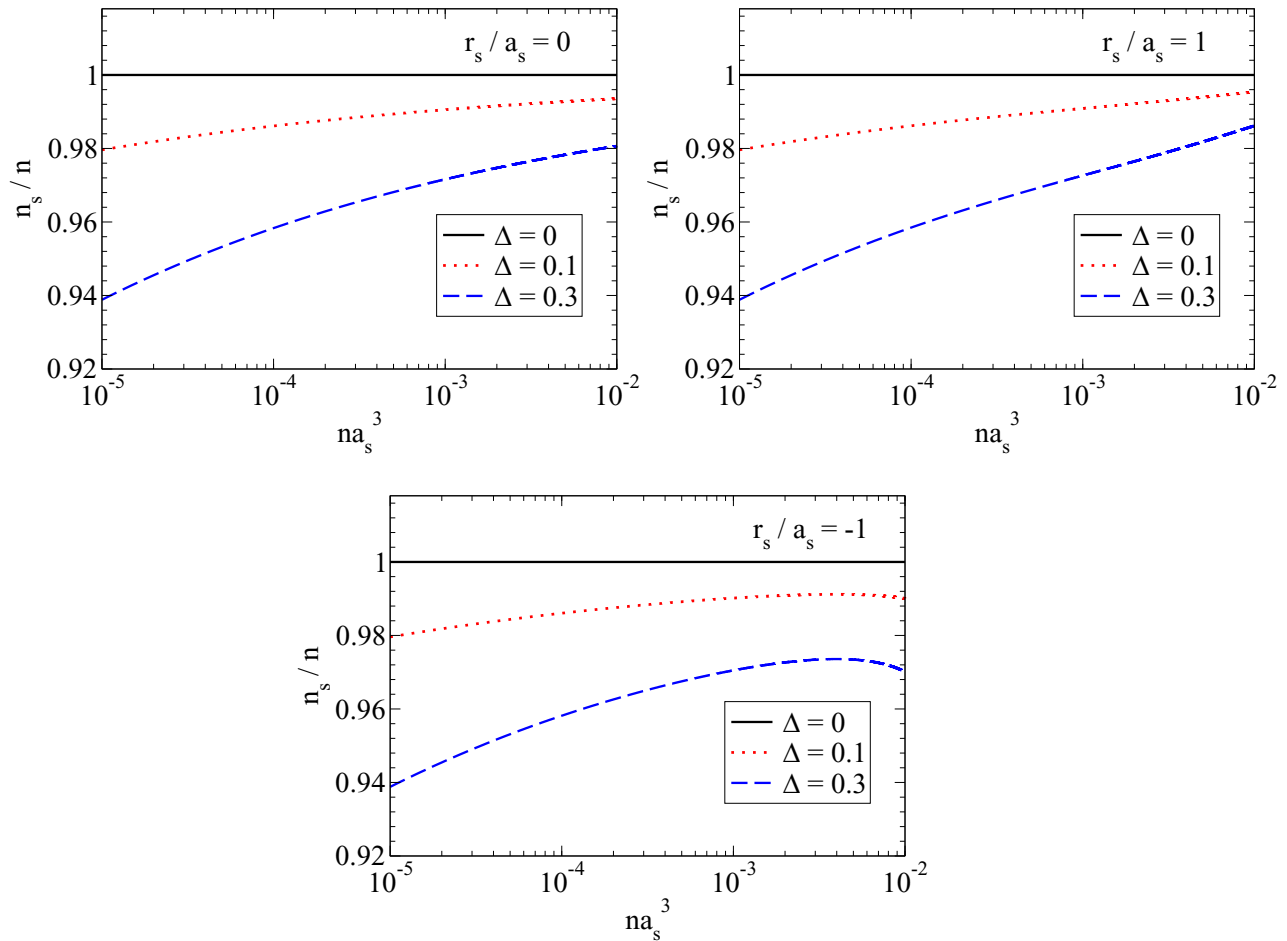


FIG. 2. The superfluid fraction n_s/n as a function of the dimensionless gas parameter na_s^3 for different combinations of Δ and the ratio r_s/a_s . As in Fig. 1, the disorder strength Δ is expressed in units of E_B^2/n with $E_B = \hbar^2 n^{2/3}/m$, and r_s/a_s is the ratio between the effective range r_s and the scattering length a_s . The curves have been obtained from Eq. (37).

In Fig. 1 we report the behavior of the condensate fraction n_0/n at $T = 0$ as a function of the gas parameter na_s^3 for different values of the effective range and increasing the disorder correlator Δ . We immediately realize that the weight of the finite-range corrections is enhanced at higher densities (for a given s -wave scattering length). Indeed, from the middle and the right panel of Fig. 1 one can observe that depending on the sign of the ratio r_s/a_s , the condensate is depleted slower (for $r_s/a_s > 0$) or faster (when the ratio is negative). Moreover, the disorder contribution, depending on r_s/a_s , is relevant also at very low values of na_s^3 , where the particles seem to be much more sensitive to the presence of a disordered external potential. As clearly shown by the dotted red curve and the dashed-dotted blue curve in the panels of Fig. 1, the joint presence of disorder and finite-range corrections modifies the condensate depletion of the system.

Despite that the corrections reported in Fig. 1 are tiny in magnitude, calculations in the pure case [10–12] show that the finite-range (Gaussian) contribution to $\Omega(\mu, \nu)$ removes an artificial thermodynamic instability and consequently expands the applicability range of a Gaussian theory. More precisely, by using Eq. (11) with $\tilde{V}(0) \rightarrow g_0$ (i.e., $r_s \rightarrow 0$), one can express the thermodynamic potential as a function of the chemical potential μ . By recalling that $P(\mu) = -\Omega(\mu)/L^3$

and that the uniform configuration is stable for $\partial_\mu^2 P(\mu) > 0$, one realizes that the uniform configuration is unstable above a critical value of the chemical potential corresponding to $(na_s)^3 \simeq 0.004$. In Ref. [11], analytical results for a Bose gas made of hard spheres (where $r_s/a_s = 2/3$) are compared to the corresponding path-integral ground-state Monte Carlo (MC) simulation [43], finding very good agreement between theoretical predictions and numerical outcomes.

Moreover, already in Ref. [12], a modified version of the GPE, including corrections due to Gaussian fluctuations and finite-range interactions, was found to reproduce numerical simulations based on diffusion MC methods reasonably well compared to its zero-range counterpart. Obviously, refined MC simulations are able to explore the behavior of (bosonic) quantum gases well beyond the range of validity of the Gaussian (i.e., Bogoliubov) approach, at a more dense regime and with higher values of the disorder strengths [44].

B. Superfluid density

The interplay between disorder and nonlocal interactions can be effectively understood by analyzing the depletion of the superfluid density. According to the Landau phenomenological description [45], in the presence of a superflow, the

total density of the system can be split into

$$n = n_s + n_n^{(\text{pure})}(n) + n_n^{(\text{dis})}(n), \quad (34)$$

with n_s the superfluid density and $n_n = n_n^{(\text{pure})} + n_n^{(\text{dis})}$ the normal one. The depletion of the superfluid is usually driven by the thermal activation of the collective excitations. For instance, in $d = 3$ dissipation can occur via density waves (i.e., phonons). As detailed in Refs. [29,46], the original intuition of Landau and Khalatnikov can be adapted to a field-theory analysis, reading the well-known formula

$$n_n^{(\text{pure})}(n, T) = \frac{\beta}{4d} \int \frac{d^d \mathbf{q}}{(2\pi)^d} \left(\frac{\hbar^2 q^2}{m} \right) \frac{e^{\beta E_q}}{(e^{\beta E_q} - 1)^2}, \quad (35)$$

with E_q given, again, by Eq. (19) with $n_0 \approx n$.

However, it is known that the presence of disorder adds another source of depletion, also at $T = 0$. For static pointlike defects, this disorder contribution has a peculiar simple form [27,46], i.e.,

$$n_n^{(\text{dis})}(n) = \frac{4}{d} n_g^{(\text{dis})}(n), \quad (36)$$

where $n_{\text{dis}}(n)$ has been computed in Eq. (30). Thus, if n is known, Eqs. (34)–(36) provide an easy way to extract the superfluid fraction of the system. In the case $d = 3$ one finds

$$\frac{n_s}{n} = 1 - \frac{1}{6\pi^{3/2}} \frac{\Delta \left(\frac{m^2}{\hbar^4 n^{1/3}} \right)}{\left[1 + 8\pi (na_s^3) \left(\frac{r_s}{a_s} \right) \right]^{3/2} (na_s^3)^{1/6}}. \quad (37)$$

In Fig. 2 we report the behavior of n_s/n , according to the equation above, as a function of the gas parameter na_s^3 . In the left panel of Fig. 2 we include the results in absence of a finite-range interaction in order to better understand the case of $r_s \neq 0$. Indeed, for a positive value of the ratio r_s/a_s (middle panel), the superfluid fraction does not seem to be significantly affected. On the contrary, for a negative value of the effective range r_s , the left panel of Fig. 2 shows that the behavior of n_s/n is no longer monotonous within the range of values we have considered for na_s^3 . This is an example of the interplay between disorder and finite-range interactions, both affecting a relevant transport quantity such as the superfluid fraction of the system.

For the sake of completeness, we notice that from Eq. (37) one can also deduce the critical disorder strength

$$\Delta_c = 6\pi^{3/2} \frac{\hbar^4 n^{1/3}}{m^2} \left[1 + 8\pi (na_s^3) \left(\frac{r_s}{a_s} \right) \right]^{3/2} (na_s^3)^{1/6}, \quad (38)$$

above which the superfluidity is destroyed despite the absence of thermal excitations. The formula shows that the effective range r_s induces a nonlinear shift on Δ_c .

VI. CONCLUSIONS

We have considered a bosonic system with a finite-range two-body interaction placed in a disordered environment. We have investigated the superfluid phase of this system and, according to our perturbative field-theoretical analysis valid in any spatial dimension d , we have computed explicitly, Eqs. (33) and (37), the modified depletion of the condensate and superfluid density in the three-dimensional case. Our results show that disorder and nonlocal interactions simultaneously modify the contribution of quantum and thermal fluctuations, leading to nonuniversal corrections. These theoretical predictions become very important when the s -wave effective range r_s of the interatomic potential is of the same order (or larger) with respect to the s -wave scattering length a_s . This regime can be achieved by approaching a zero-point crossing of the scattering where the effective range (otherwise constant and $\sim 10a_0$, with a_0 the Bohr's radius) may vary and change its sign. It has been pointed out [39] that in this regime, a single-channel approximation [38] provides quite reliable results.

In lower spatial dimensions quantum and thermal fluctuations are strongly enhanced and a proper characterization of their contribution is mandatory, also in terms of interaction parameters. Moreover, the resulting thermodynamic picture may serve as a starting point to build an effective Gross-Pitaevskii-like equation in the spirit of the local-density approximation. This has been done, for instance, with strongly magnetic atoms [47,48], binary mixtures [49], or spin-orbit coupling [50]. However, all these papers do not consider a disordered environment, which crucially alters the superfluid dynamics of condensed systems. The physical picture becomes more rich if we allow the possibility of a localized phase, but first we need to understand the interplay between disorder and nonlocal interaction in the fluctuations contribution, which may help] (or not) in driving the system towards the superfluid-to-localized transition.

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