Bogoliubov excitations in the quasiperiodic kicked rotor: Stability of a kicked condensate and the quasi–insulator-to-metal transition

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We study the dynamics of a Bose-Einstein condensate in the quasiperiodic kicked rotor described by a Gross-Pitaevskii equation confined in a toroidal trap. As the interactions are increased, Bogoliubov excitations appear and deplete the condensate; we characterize this instability by considering the population of the first Bogoliubov mode, and we show that it does not prevent, for small enough interaction strengths, the observation of the quasi–insulator-to-metal transition that replaces the Anderson transition of the noninteracting case. However, the predicted subdiffusion in momentum space is not observed in the stable region within experimentally accessible times. For higher interaction strengths, the condensate may be strongly depleted before this dynamical regimes set in.

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I. INTRODUCTION

Ultracold atoms are clean, controllable, and flexible systems whose dynamics can be modeled from first principles. Interacting ultracold bosons are often well described by a mean-field approximation leading to the Gross-Pitaevskii equation (GPE) [1,2], which is useful in many situations of experimental interest: superfluidity and vortex formation [3], chaotic behavior [4–7], soliton propagation [8], etc. Ultracold-atom systems are thus increasingly used to realize simple models that are inaccessible experimentally in other areas of physics [9].

Ultracold gases in a disordered optical potential have been used as an emulator for the Anderson model [10], allowing the direct observation of the Anderson localization in one [11,12] and in three dimensions [13–15]. The quantum kicked rotor (QKR), obtained by placing cold atoms in a pulsed standing wave, is also a (less obvious) quantum simulator for Anderson physics [16,17]: It displays *dynamical localization*, a suppression of chaotic diffusion in momentum space, recognized as being equivalent to Anderson localization [16]. Recent studies suggest that interactions (treated in the frame of the GPE) lead to a progressive destruction of the dynamical localization, which is replaced by a subdiffusive regime [18–21], in analogy with what is numerically observed for the one-dimensional (1D) Anderson model with bosonic mean-field interactions itself [22–26], experimentally observed in the ultracold-atom implementation of the interacting Anderson model [27].

Applying standing-wave pulses (kicks) to a Bose-Einstein condensate (BEC) may lead to a dynamical instability that transfers atoms from the condensed to the noncondensed fraction, a phenomenon that is not described by the GPE. The most common correction to the GPE in this context is the Bogoliubov–de Gennes (BdG) approach [28,29]. The BdG theory considers "excitations"–described as independent bosonic quasiparticles–of the Bose gas, and thus it indicates how (and how much) it differs from a perfectly condensed gas. It has been applied both to the description of the dynamical instability in the periodic kicked rotor [30–32] and to the study of a one-dimensional weakly interacting BEC [33,34] in a disordered potential. In the latter case, it was found that the quasiparticles may also display Anderson localization. Interestingly, a modified version of the QKR, the *quasiperiodic* kicked rotor (QPKR), emulates, in the absence of interactions, the dynamics of a 3D Anderson-like model, and displays the Anderson metal-insulator transition [35,36]. With this system, a rather complete theoretical and experimental study of this transition has been performed [37–40]. In the present work, we use the Bogoliubov approach to the QPKR to study the stability of the condensate and to assess the possibility of the observation of the "quasi–insulator-metal transition" that replaces Anderson localization in the presence of interactions, the localized state being replaced by a subdiffusive one [21]. We show that for weak enough interactions, the condensate remains metastable for experimentally relevant times, and that the Bogoliubov quasiparticles also display a phase transition. This shows that the transition can be observed for a low enough nonlinearity, so that the instability timescale is much larger than the duration of the experiment, paving the way for its experimental observation with the quasiperiodic quantum kicked rotor in the presence of interactions.

II. DYNAMICS OF BOGOLIUBOV EXCITATIONS

A kicked rotor is realized by submitting ultracold atoms to short kicks of a standing wave at times separated by a constant interval T_1. If such kicks have a constant amplitude, one obtains the standard (periodic) kicked rotor, which exhibits dynamical localization [17,41], i.e., localization in momentum space. If the amplitude of the kicks is modulated with a quasiperiodic function

$F(t) = 1 + \varepsilon \cos(\omega_2 t + \varphi_2)\cos(\omega_3 t + \varphi_3)$, where $\omega_2 T_1$, $\omega_3 T_1$, and $\hbar \equiv 4\hbar k_L^2 T_1/M$ (the reduced Planck constant) are incommensurable (k_L is the wave vector of the standing wave and M is the mass of the atoms), the QPKR is obtained [35,36]. In the absence of particle-particle interactions, the QPKR Hamiltonian, in conventional normalized units [41,42], is

$$H(t) = \frac{p^2}{2} + KF(t)\cos x \sum_{n\in\mathbb{N}} \delta(t-n), \tag{1}$$

where K is proportional to the average standing-wave intensity. In such units, the time interval between kicks is $T_1 = 1$, and lengths are measured in units of $(2k_L)^{-1}$. Throughout this work, we take $\omega_2 = 2\pi\sqrt{5}$, $\omega_3 = 2\pi\sqrt{13}$, and $\hbar = 2.89$ corresponding to typical experimental values [37–39]. In the absence of interactions, the QPKR displays, for low values of K and ϵ, dynamical localization in momentum space at long times (i.e., $\langle p^2\rangle \sim$ const); for $K \gg 1$, $\epsilon \approx 1$ one observes a diffusive regime $\langle p^2\rangle \sim t$, and in between there is a critical region that displays a subdiffusive behavior $\langle p^2\rangle \sim t^{2/3}$ [42]. In the presence of weak interactions modeled as a mean-field nonlinear potential, the critical and the diffusive regimes are not affected, whereas the localized regime is replaced by a subdiffusive one $\langle p^2\rangle \sim t^{\alpha}$, with $\alpha \sim 0.4$ [21,43]. In the following, we consider low enough interaction strengths and short enough times so that this change of behavior is not significant; we shall thus use the term "quasilocalized" (or "quasi–insulator") to characterize this phase.

We use in the present work a model that is slightly different from the experimentally realized QPKR: The wave function $\phi(x,t)$ obeys periodic boundary conditions over one spatial period of the optical potential: $\phi(L,t) = \phi(0,t)$, where $L = 2\pi$ is the system size, and normalization $\int_0^L |\phi(x,t)|^2 dx = 1$. Because of this boundary condition, the spectrum of the momentum operator $p = i\hbar\partial/\partial x$ is discrete: $n\hbar$, with n an integer, so that the momentum space wave function $\tilde{\phi}(p)$ is given by the Fourier series:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{inx}\tilde{\phi}(n\hbar). \tag{2}$$

In this model, there is no spatial dilution of the boson gas, and the average nonlinear potential, which is proportional to the atom density, does not vary with time. This is not the case in the usual experimental realization of the QPKR, where the atom cloud diffuses with time in *momentum space* (so that, even in the presence of dynamical localization, it is still undergoing spatial dilution), causing a significant diminution of the spatial density; once the system is diluted, the nonlinearity does not play any important role. Our model is thus expected to catch more clearly the physics in the presence of the nonlinearity. Such a model can be realized experimentally by using a tightly confined toroidal trap [44] formed by Laguerre-Gauss (LG) modes in which atoms are confined in the radial direction but are free to rotate. The azimuthal dependence of such modes can be used to create a sinusoidal intensity modulation *along* the torus, analogous to a standing wave [45], in the present case, formed by a superposition of LG_{01} and LG_{0-1} modes. Note that in such a geometry, collective effects can manifest themselves when interactions become strong [46], but here we will be mainly interested in the weak interactions regime.

In this quasi-1D geometry, we take interactions into account via the particle-number-conserving Bogoliubov formalism [47] at zero temperature. The gas of interacting bosons is separated into two parts: (i) The condensed fraction (the "condensate") and (ii) the noncondensed fraction ("excitations" or "quasiparticles"). The condensate is governed by the Gross-Pitaevskii equation

$$i\hbar\frac{\partial\phi(x,t)}{\partial t} = H(t)\phi(x,t) + g|\phi(x,t)|^2\phi(x,t), \tag{3}$$

where the condensate wave function ϕ is normalized to unity: $\int_0^{2\pi} |\phi(x,t)|^2 dx = 1$ ($L = 2\pi$ is the system length) and the rescaled 1D interaction strength [48] $g = 4\hbar k_L a\omega_\perp T_1 N$ is proportional to the S-wave scattering length a, to the number of atoms N in the trap, to the transverse confinement frequency $\omega_\perp$ and to the kick period T_1.

The noncondensed part is described in the Bogoliubov formalism as a set of *independent* bosonic quasiparticles, with a two-component state vector (u_k, v_k), where k is an integer labeling the mode of momentum $k\hbar$ satisfying the normalization condition

$$\int_0^{2\pi} \left(|u_k(x,t)|^2 - |v_k(x,t)|^2\right)dx = 1, \tag{4}$$

and evolving according to the equation

$$i\hbar\partial_t \begin{bmatrix} u_k(x,t) \\ v_k(x,t) \end{bmatrix} = \mathcal{L}\begin{bmatrix} u_k(x,t) \\ v_k(x,t) \end{bmatrix}, \tag{5}$$

where the operator $\mathcal{L}$ is a 2×2 matrix:

$$\mathcal{L} = \begin{bmatrix} Q(t) & 0 \\ 0 & Q(t)^\dagger \end{bmatrix} \mathcal{L}_{\rm GP} \begin{bmatrix} Q(t) & 0 \\ 0 & Q(t)^\dagger \end{bmatrix},$$

$$\mathcal{L}_{\rm GP} = \begin{bmatrix} H + 2g|\phi|^2 - \mu(t) & g\phi^2 \\ -g\phi^{*2} & -H - 2g|\phi|^2 + \mu(t) \end{bmatrix},$$

with $\mu(t) = \int_0^L (\phi^* H\phi + g|\phi|^4)dx$ the time-dependent chemical potential. The presence of the projection operator $Q(t) = 1 - |\phi\rangle\langle\phi|$ ensures the conservation of the total number of particles [47].

The goal of this work is to study the (i) stability and (ii) dynamical localization properties of a quasiperiodically kicked condensate. The stability of the condensate can be assessed by monitoring the number of noncondensed atoms (the quantum depletion) at zero temperature, which is given by $\delta N = \sum_k N_k$, where

$$N_k = \int_0^{2\pi} |v_k(x,t)|^2 dx$$

is the number of excitations in the mode k. To describe the localization properties of the system, we will expand the condensate wave function and the Bogoliubov mode in Fourier series:

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{l\in\mathbb{Z}} e^{ilx}\tilde{f}(l,t), \tag{6}$$

$$\tilde{f}(l,t) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ilx} f(x,t)dx, \tag{7}$$

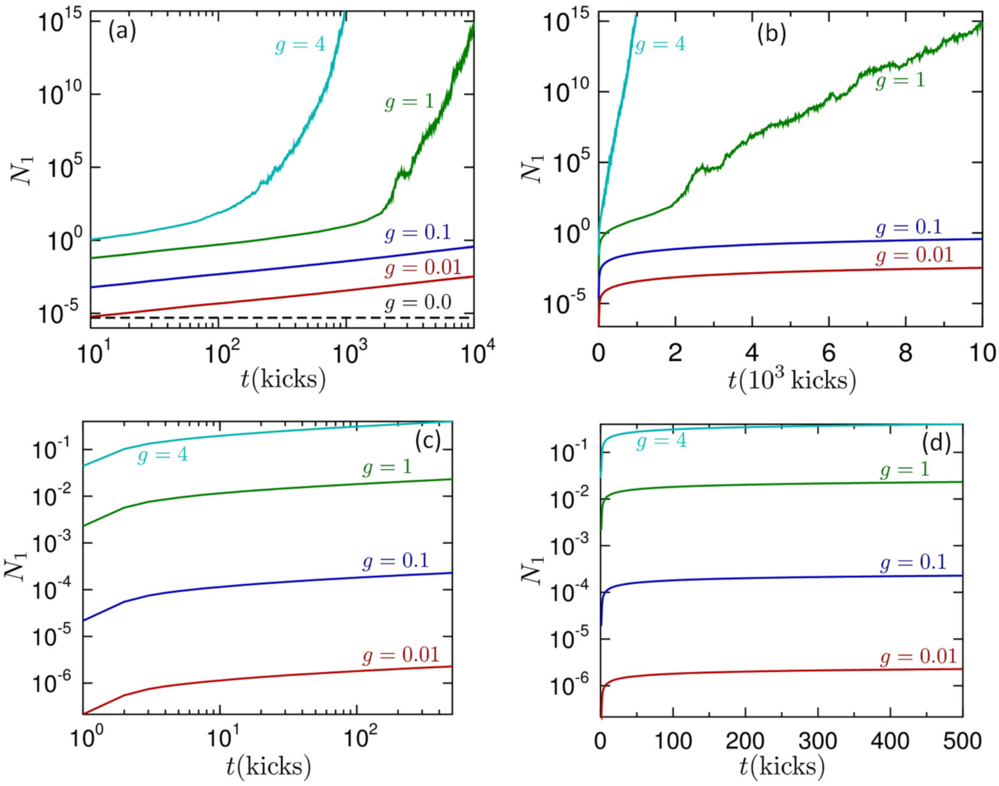

FIG. 1. Evolution of the excitation number N_1 of the $k = 1$ Bogoliubov mode, in different regimes: quasi–insulator [(a) log-log and (b) semilog scale], $K = 4$, $\varepsilon = 0.1$ and metal [(c) log-log and (d) semilog scale], $K = 9$, $\varepsilon = 0.8$ for four values of the interaction strength g increasing from bottom to top: 0.01 (red, bottom curve), 0.1 (blue), 1 (green), and 4 (cyan, top curve). Plot (a) shows a drastic growth of the excitation number for $g \geqslant 1$, indicating the onset of an instability; the dashed line in plot (a) indicates the behavior in the absence of interactions (constant N_1). Plot (b) (semilog scale) shows that for $g = 4$ this growth is exponential. For $g = 1$ it is also roughly exponential, with fluctuations, but for lower values of g it is algebraic, approximately $\propto t$ on the considered time scale of 10^4 kicks. In plots (a) and (b), large values of N_1 (larger than the initial number of atoms in the condensate) are unphysical and signal the failure of the linearized Bogoliubov approach. In the metallic regime, no instability is apparent up to 500 kicks. $\hbar = 2.89$ throughout this work.

where $f = \phi, u_k, v_k$. The momentum distributions of the condensate and of Bogoliubov excitations (in a mode k) then read $n_c(p = \hbar l) = |\tilde{\phi}(l)|^2$ and $n_b(p = \hbar l) = |\tilde{v}_k(l)|^2/N_k$, respectively.

Finally, for our numerical study we will take as initial conditions for the (u_k, v_k) amplitudes the eigenstates of the operator $\mathcal{L}(t = 0)$[49], which are plane waves of momentum $\hbar k$ [47],

$$\begin{bmatrix} \tilde{u}_k(l, t = 0) \\ \tilde{v}_k(l, t = 0) \end{bmatrix} = \sqrt{\frac{\pi}{2}} \begin{bmatrix} \zeta + 1/\zeta \\ \zeta - 1/\zeta \end{bmatrix} \delta_{k,l} \tag{8}$$

with $k \in \mathbb{Z}^*$, and ζ given by

$$\zeta = \left[\frac{k^2}{k^2 + 2g/\pi\hbar^2} \right]^{1/4}. \tag{9}$$

In the example below, we will focus on the evolution of the $k = 1$ Bogoliubov mode, which is initially the most populated; see Eqs. (8) and (9) [50].

III. STABILITY OF THE CONDENSATE

For the periodic kicked rotor, several studies showed the emergence of an instability at large positive values of g (repulsive interactions) [30–32], which manifests itself by an exponential increase of the number of excitations. We shall now study this instability in the *quasiperiodic* kicked rotor for $g > 0$. Equations (3) and (5) can be integrated simultaneously by a split-step method. Numerical data are averaged over 500 random realizations of the phases $\varphi_2, \varphi_3 \in [0, 2\pi)$. As the total number of particles is fixed, the number of condensed particles is $N - \delta N$ and the noncondensed fraction $\delta N/N$. As long as δN is much smaller that the typical number of atoms $N \approx 10^5$ used in an experiment, the kicked condensate will be considered to be stable. The study of the regime where the number of excitations becomes comparable to the atom number is still an open problem, for which interesting approaches can be found in the literature; see, e.g., Ref. [31].

Figures 1(a) and 1(b) display the onset of the instability at a relatively large time (compared to experimentally accessible time scales) of 10^4 kicks in the quasi–insulator region $K = 4, \varepsilon = 0.1$. For low-g values, the number of excitations N_1 increases moderately with time, approximately $\propto t$. For higher values of g, an explosive growth of N_1 indicates that the condensate has lost its stability (and that the Bogoliubov approximation has lost its validity). However, for $g \lesssim 1$, even at the limit of an experiment duration $t = 10^3$, the instability has not set in, which is confirmed by the small number of excitations $N_1(g = 1, t = 10^3) \approx 9.5$. As shown in Appendix,

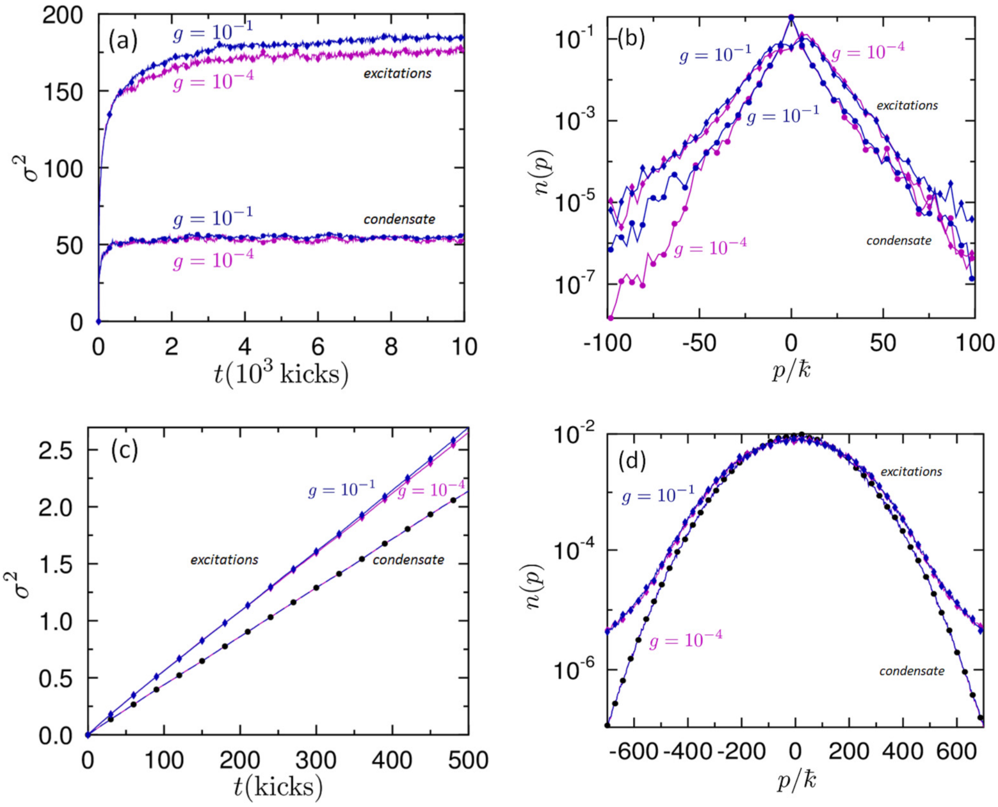

FIG. 2. Dynamics of the condensate and of the Bogoliubov excitations. The momentum variance σ^2 of the condensate (circles) and of the excitations (diamonds) for $g = 10^{-4}$ (magenta) and $g = 10^{-1}$ (blue) are shown in plot (a) for the quasilocalized regime ($K = 4$, $\varepsilon = 0.1$, $t \leqslant 10^4$) and in plot (c) for the diffusive regime ($K = 9$, $\varepsilon = 0.8$, $t \leqslant 500$). The momentum distributions $n(p)$ of the condensate (circles) and of the excitations (diamonds) in logarithmic scale are shown in (b) for the quasilocalized regime (at $t = 10^4$) and in (d) for the diffusive regime (at $t = 500$).

the origin of the dynamical instability can be understood by studying a simplified evolution operator illustrating the competition between the kicks and the interactions that can destroy the condensate. Plots (c) and (d) show the evolution of the number of excitations in the metal (diffusive) regime $K = 9, \varepsilon = 0.8$, up to $t = 500$. In this case, the condensate is less affected by the presence of interactions, as the kinetic energy grows linearly with time and eventually dominates the constant interaction energy $\simeq g/(2\pi)$ (see below).

IV. THE QUASI–INSULATOR-TO-METAL TRANSITION IN THE STABLE REGION

The previous analysis has shown that, for small interaction strengths, the system remains essentially condensed, with populations in the Bogoliubov modes $N_k \ll N$ up to relatively long times. In other words, in this regime, the Bogoliubov formalism is valid and the condensate metastable. We will thus focus on the case $g \leqslant 0.1$ and study the localization properties of the system to find whether the system follows the noninteracting regimes (localized, diffusive) or if the localized phase is replaced by a subdiffusive phase [18,21]. We will also discuss how the critical properties of the transition are changed by interactions.

We prepare the system in its ground state [$\phi(x, t=0) = 1/\sqrt{L}$], at zero temperature. The Bogoliubov modes are then initially populated only from quantum fluctuations. For $g \leqslant 0.1$ and for short enough times, one expects the condensate to display (quasi)localization if $K < K_0$ and diffusion if $K > K_0$, K_0 being the critical point. Our numerical simulations show that this is also the case for the excitations. Figure 2(a) shows the second moment of the distribution $\sigma_i^2 = \langle p^2\rangle_i - \langle p\rangle_i^2$, with $\langle p^2\rangle_i = \hbar^2 \sum_l l^2 n_i(p)$ and $\langle p\rangle_i = \hbar \sum_l l n_i(p)$, for both the condensate ($i = c$, circles) and the excitations ($i = b$, diamonds) in the quasilocalized regime. For the two values of the interaction strength, $g = 10^{-4}$ (magenta curve) and $g = 10^{-1}$ (blue curve), the second moment of the condensate saturates to a constant value $\sigma_c^2 \approx 50$, showing that the wave packet is quasilocalized. More interestingly, the curves with diamond markers in Fig. 2(a) show that variance of the momentum of Bogoliubov excitations *also tend to saturate*, with a larger value $\sigma_b^2 \approx 180$: quasiparticles also (quasi)localize.

The momentum distributions n_c and n_b in the quasilocalized regime at $t = 10^4$ for $g = 10^{-4}$ are shown in Fig. 2(b) [same graphical conventions as in Fig. 2(a)]. Both distributions remain essentially centered around the origin so that their first moment $\langle p\rangle_i$ ($i = c, b$) remains small, while they show an exponential behavior in the wings. For the condensate, assuming an exponential profile $n_c(p) \propto \exp(-|p|/\xi)$ [see Fig. 2(b)], the width ξ of the momentum distribution at $t = 10^4$ is given by $\xi = \sigma_c/\sqrt{2} \approx 5$, which evolves very slowly up to $t = 10^4$. Thus, for very weak interactions and in the time range accessible to experiments, the condensate behaves as a single particle and displays similar behaviors in the vicinity of the Anderson transition. The Bogoliubov distribution presents a double peak near the center. This peculiar shape is probably due to the fact that the initial momentum distribution of the mode $k = 1$ [Eq. (8)] is centered at $p = \hbar$, thus breaking the symmetry between positive and negative momenta. The wings of the Bogoliubov momentum distribution and of the condensate have approximately the same slope, confirming that excitations have the same localization length in this case. The fact that the former has a flatter top than the latter explains why σ_b is significantly larger than σ_c.

Figure 2(c) is the equivalent of Fig. 2(a) in the diffusive regime $K = 9$, $\varepsilon = 0.8$, showing that σ_c^2 and σ_b^2 increase linearly with time, with a diffusion coefficient that is similar for $g = 10^{-4}$ and 10^{-1}, $D_c = \sigma_c^2/2t \approx 20$, and $D_b = \sigma_b^2/2t \approx 25$. Figure 2(d) [equivalent to Fig. 2(b)] represents the corresponding momentum distributions at $t = 500$. Both have the typical Gaussian shape associated with a diffusion process.

These results show that when the condensate is stable and for experimentally relevant times, the system is not affected by the presence of (weak) interactions and that Bogoliubov excitations display the same dynamics as particles. For larger interaction strengths, one expects the presence of a subdiffusive phase instead of the localized regime [18–21]. However, for the range of parameters investigated in this work ($10^{-4} < g < 4$, $K = 4$, $\varepsilon = 0.1$) we found that the condensate is *never stable and subdiffusive at the same time*. For $g \gtrsim 0.1$, interactions appear to be more likely to destroy the condensate than to induce subdiffusion.

The fact that Bogoliubov excitations behave like (noninteracting) particles in the quasi–insulator and metal regimes in the stable region suggests that they display a quantum phase transition of the same nature as the Anderson transition, which can be verified by determining its (universal) critical exponent ν, governing the algebraic divergence of the localization length $\xi(K)$ near the transition point:

$$\xi(K) \propto \frac{1}{(K_c - K)^\nu}. \tag{10}$$

The universality of this second-order phase transition has been demonstrated experimentally in the absence of interactions [39], giving an experimental value for the critical exponent $\nu = 1.63 \pm 0.05$, independent of microscopic parameters and consistent with the numerically predicted value 1.58 ± 0.02 [52,53]. We used a finite-time scaling method [42,53] to extract a critical exponent ν from the dynamics of both the condensate and the excitations. We chose the path in the parameter plane (K, ε) used in [37], $\varepsilon(K) = 0.1 + 0.14(K - 4)$. Figure 3 shows that, for small nonlinearities, the critical exponent is the same for both components and compares very well with the (noninteracting) experimental measurement, but their values tend to become different for higher values of g. For $g \geqslant 0.1$, the value of the critical exponent starts to deviate from the universal value, as the system enters a new regime where the subdiffusive character of the quasilocalized regime becomes important even for short times [21]. The critical point is also the same for both the condensate and the excitations; at $g = 0$ its value is $K_0 \approx 6.38 \pm 0.05$ and changes only slightly up to $g = 0.1$, in accordance with the self-consistent theory prediction of Ref. [21]. Hence, we can conclude that for low values of g, Bogoliubov excitations undergo a second-order phase transition of the same nature as for noninteracting particles, with the same critical exponent.

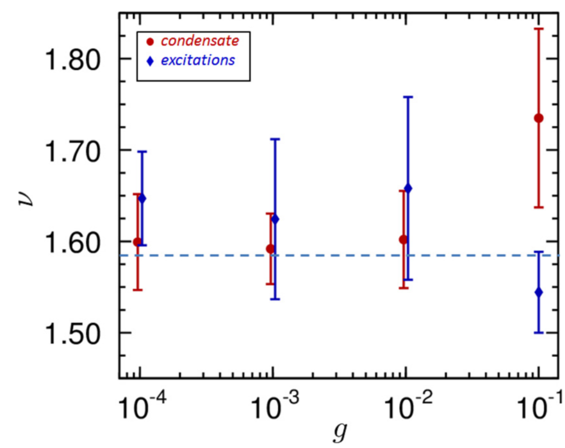

FIG. 3. Critical exponent ν vs interaction strength g for both the condensed fraction (red circles) and the Bogoliubov excitations (blue diamonds). Error bars are calculated via a standard bootstrap method [51]. The points were slightly shifted horizontally so that error bars do not superpose. The blue dashed line indicates the critical exponent value $\nu \approx 1.58$ in the absence of interactions.

The very same equations of evolution (5)—with different initial conditions—describe the dynamics if the Bogoliubov modes are populated by some other process. Experimentally, a specific Bogoliubov mode can be selectively excited using two laser waves whose directions are chosen so that their wave-vector difference is equal to the wave vector k of the desired mode [54]. Then the linear or exponential growth of the mode can be easily monitored experimentally. The above study was restricted to the Bogoliubov mode $k = 1$. Considering another mode $k \neq 1$ is equivalent to a change of the initial condition in the Bogoliubov equations. We checked numerically that other modes display the same behavior, but they are much more affected by finite-time effects, as their initial momentum distribution is more asymmetric [see Eq. (8)].

V. CONCLUSION

In conclusion, in a quasiperiodic kicked rotor the condensate is stable in the weakly interacting regime. For times longer than the experimental timescale (presently 1000 kicks), both the excitations and the condensate display a behavior very close to the Anderson transition of noninteracting particles, with the same critical exponent; the universality of the phase transition is thus valid irrespective of the nature of particles. However, our results also show that it might be difficult to observe a subdiffusive phase at larger interacting strengths due to the emergence of a dynamical instability. Thus, the fate of the transition in the presence of strong interactions remains an open problem. For low positive values of g, the transition can be experimentally observed, and, by increasing interactions via a Feshbach resonance, one can observe the onset of nonlinear effects. This shows that the nonlinear regime might be observed with relatively low values of g and that the transition can be observed within the stability

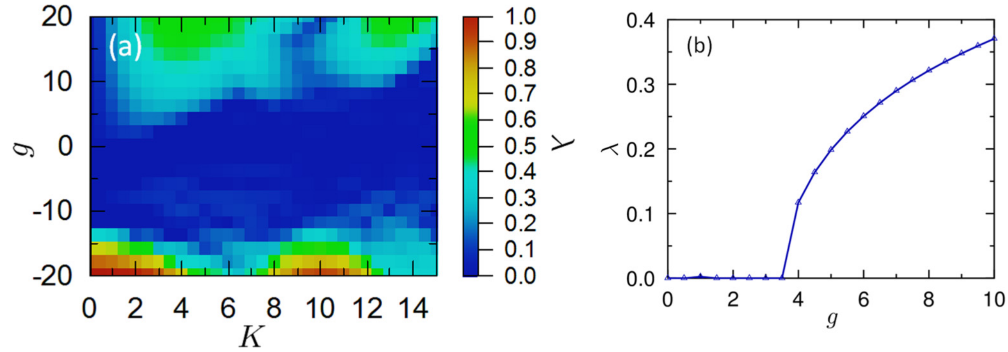

FIG. 4. (a) Largest growth rate λ per kick as a function of the kick amplitude K and of the interaction strength g for $\hbar = 2.89$. For $K = 0$, the system is unstable for $g < -13$ as expected from Bogoliubov theory. For $K > 0$, a dynamical instability can also occur for $g > 0$. For large negative g the condensate is intrinsically instable, which explains the nonzero value of λ even for $K = 0$. Panel (b) shows the emergence of the instability for $K = 4$ and $g > 0$.

regime of the condensate. The present work paves the way for such an experiment, which would represent an important step in our understanding of interacting disordered systems presenting phase transitions.

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APPENDIX: DYNAMICAL INSTABILITY OF THE KR BOGOLIUBOV MODES

The goal of this Appendix is to explain, in a simpler case, the features of Fig. 1 by analyzing the properties of the operator $\mathcal{L}$ in Eq. (5). We use two important simplifications: (i) we set the modulation $\varepsilon = 0$ and consider the standard kicked rotor single-particle Hamiltonian (1) with $K(t) = K =$ const and (ii) we make the assumption that the condensate wave function is homogeneous, $\phi(x) = 1/\sqrt{2\pi}$ so that Eqs. (5) form a closed set of equations. The projector Q then becomes time-independent and the dynamical properties of the system are governed by $\mathcal{L}_{\rm GP}$ only. To study the influence of g on the stability of the system, we consider the evolution operator over one period:

$$U = \exp\left(-\frac{i}{\hbar}\begin{bmatrix}\frac{p^2}{2}+\frac{g}{2\pi} & \frac{g}{2\pi}\\ -\frac{g}{2\pi} & -\frac{p^2}{2}-\frac{g}{2\pi}\end{bmatrix}\right) \times\begin{bmatrix}U_K & 0\\ 0 & U_{-K}\end{bmatrix}. \tag{A1}$$

In momentum space, the kick operator is

$$\langle p=\hbar l|U_K|p=\hbar m\rangle = (-i)^{(m-n)}J_{m-n}(K/\hbar). \tag{A2}$$

The eigenstates of U contain all the dynamical properties of the system: if one of the eigenvalues ϵ is greater than one, the system will develop a dynamical instability with a growth rate $\log|\epsilon|$. In Fig. 4, we represent the growth rate $\lambda = \log_{10}|\epsilon|$ associated to the eigenstate with the largest eigenvalue ϵ as a function of K and g, for $\hbar = 2.89$.

The limiting case $K = 0$ provides a good test for our method as we know from the standard Bogoliubov theory [55] that the system is unstable for $g < -\hbar^2\pi/2 \sim -13$. In the presence of the kicks $K > 0$, the system can now develop a dynamical instability for repulsive interactions $g > 0$. For $K = 4$, we find that the system is indeed unstable at large g which is in qualitative agreement with Fig. 1(b). The critical value $g_c \approx 3.5$ for the instability, however, is overestimated, due to the strong assumptions (i) and (ii). We also find that the system tends to be more stable at large kick amplitudes, which is compatible with Fig. 1(b). Finally in the noninteracting limit obtained for $g = 0$, the system is stable as the eigenvalues of the evolution operator, Eq. (A1) coincides with those of the standard noninteracting kicked rotor. In particular, the results obtained above are compatible with those of Ref. [56].

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