Classical dynamics of three-body systems in the Efimov potential

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We investigate the classical nonrelativistic dynamics of three bodies in Efimov's $-1/R^2$ potential in three spatial dimensions, with a view towards semiclassical quantization and insight into the geometry of Efimov states. Without the short-distance cutoff, these dynamics are superintegrable, which allows an exact integration of the equations of motion for arbitrary initial conditions. We show that periodic orbits necessarily lead to exactly vanishing binding energy of the bound states, in disagreement with Efimov's quantum-mechanical results. A scaling anomaly demands that the quantum dynamics of three bodies in this potential be augmented by additional boundary conditions affecting all three particles at the short-distance cutoff point, i.e., near the triple collision. We discuss the inherent difficulties in the definition of appropriate three-body boundary conditions in three spatial dimensions and briefly discuss their consequences for (quasi)periodic orbits. Consequently, the classical orbits corresponding to Efimov states cannot be exactly periodic, but must have a finite timescale (lifetime), associated with the time it takes the system's hyperradius to fall to zero, or to the cutoff value, which is typically much longer than the (quasi)period of the hyperangular motion. The scaling properties of the lifetime are in agreement with the quantum-mechanical predictions of the half-life (width) of Efimov states surrounded by an ultracold gas. Detailed spatiotemporal evolution of the system is generally unpredictable beyond the three-body collision point, even though global conservation laws ensure that the system's hyperradius must be periodic.

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I. INTRODUCTION

The quantum-mechanical three-body problem in weak but singular two-body potentials has well-known three-body negative-energy (bound state or resonance) solutions due to Efimov [1–6]. Over the past 50 years, these predictions have been confirmed first by other theorists [7–9] and then in experiments. The first albeit indirect detection of Efimov states was by Kraemer *et al.* [10]. Then three other groups reported observations of Efimov states in bosonic quantum gases near a Feshbach resonance. The field of experimental Efimov physics has expanded beyond that of Bose gas physics after the discovery of the first helium trimer Efimov states [11].

One particularly remarkable aspect of Efimov (quantum) states is their discrete dilation symmetry, by a constant common factor of 22.7. This is in close relation to the continuous dilation symmetry of the $-1/R^2$ potential, which is the prime example of dilation-symmetric interaction in classical nonrelativistic physics, harking back to the 19th century [12–15].

The Zaccanti *et al.* experiment [16] reported the first measurement of Efimov's scaling factor of 25 ± 4, which is consistent with Efimov's prediction of 22.7, within experimental uncertainties. In the meantime, the first dilated Efimov states have been detected [17] at the predicted energies and extending over length scales of around 1 μm, which is much larger than the largest Rydberg states observed thus far (up to 100 Å; see Sec. 1.3.2 in [18] as well as [19–21]).

The next (third) triatomic resonance has the size of order 22.7 μm [22] and the fourth is of order 22.7^2 $\mu\text{m} \simeq 515$ $\mu\text{m} = 0.5$ mm, i.e., practically of macroscopic size, which might make it almost visible by the naked eye, provided it is visible at optical wavelengths. Thus one may ask, for example, the following questions. What is the semiclassical limit of Efimov states? Can one observe their shape(s) and/or the motion(s) of the three atoms? If yes, then what should one expect?

Another interesting facet of Efimov states dilated n times is that their dynamics slows down as (negative) powers of 22.7: Velocities of particles in the nth state are 22.7^{-n} times smaller than in the $n = 1$ one and the classical period T_n of periodic motion grows as $T_n \sim 22.7^{2n} T_1$. This suggests that the $n = 3$ state should be (quasi)static, which raises many questions. For example, what is the relation of such large-scale slow states to truly static and macroscopic continuum properties of an ultracold dilute Bose gas? What kind of real or gedanken experiments can be devised to test this (predicted) slowing down?

The above questions indicate a need for a semiclassical study of Efimov states. Moreover, in an elaboration [5] of his original work in [1], Efimov emphasized certain qualitative aspects of his three-body resonance states and suggested a hyperradial $-1/R^2$ effective potential in which to study their semiclassical properties. This is supposed to correspond to the unitary limit of the original quantum treatment of Efimov effect. So far, the only published semiclassical study of Efimov states is that by Bhaduri *et al.* [23], which does not even begin to address the above kinds of questions however. Rather, the Bhaduri *et al.* WKB analysis, which is reduced to a

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one-dimensional one-body dynamics in the $-1/r^2$ potential with a short-distance boundary condition, which was introduced to prevent a fall to the center, showed that an Efimov-like energy spectrum appears as a consequence of semiclassical quantization of a periodic orbit that corresponds to a breathing mode of three bodies with a fixed shape. This particular orbit expands at first to its maximal size and then collapses back to the lower cutoff, only to bounce back again, etc.

Such an orbit can only be periodic in one spatial dimension. In two dimensions the particle would not bounce back in the direction from which it arrived, but would rather scatter in an arbitrary direction (see Sec. III B). Three-dimensional periodic orbits in the Efimov potential do not involve any kind of collapse. Indeed, we will show that the opposite must be true: The hyperradius, which effectively describes the size of periodic orbits, must remain constant in time. A proper extension of the WKB method to three bodies moving in three spatial dimensions requires either (i) a precise definition of the boundary conditions for all three particles at the small-hyperradius cutoff, which we show is impossible, or (ii) a relaxation of the requirement of strict periodicity.

The question of whether the three-body system can reemerge (bounce) from the three-body collision intact and then complete the period is subject to the (nonuniversal) physics at the hard-core cutoff. The answer depends crucially on the boundary conditions. Indeed, there are mathematical obstacles to a (simple) definition of such boundary conditions (see Refs. [24,25]). In Sec. VI we briefly discuss some intuitive sets of such boundary conditions, but do not try to implement them numerically, due to complications that they involve.

The second option, which we prefer here, is to relax the condition of strict periodicity: Nonstationary states can also be dealt with semiclassically. Indeed, a classical orbit with negative energy in the $-1/R^2$ potential, corresponding to a (dilated) Efimov state, must necessarily lead to a collapse of the system in a triple collision, subsequent to which the classical evolution of the system becomes to a large degree unpredictable, i.e., highly dependent on the (nonuniversal) boundary conditions at the cutoff. Global conservation laws of energy and hyperangular momentum ensure that the system's hyperradius must rebound after the triple collision, along with a possibly discontinuous change of the triangle's shape at the singularity or at the boundary condition. Thus, the orbit is periodic in the hyperradius and (mostly) smooth in the hyperangles, interrupted at periodic intervals by instantaneous random changes.

The free-fall time τ, associated with the collapse of the classical three-body system to the triple collision or to cutoff, as opposed to the period T of motion in the hyperangular shape space, determines the energy E spectrum of the semiclassical Efimov states, as shown by the Bhaduri *et al.* analysis [23], as well as the (possible) half-life (width) of the quantum-mechanical state, whether it is due to the interaction with the surrounding gas or with the electromagnetic (EM) radiation.[1]

The classical orbits that correspond to semiclassical Efimov states collapse increasingly slowly as the absolute value of the energy of the system is reduced. The time τ it takes for the classical three-body system to fall to the center presents also an upper bound on the half-life $\tau = 2/\Gamma$ of semiclassical Efimov states. Scaling arguments lead to the inverse proportionality of the half-life τ_n and the energy E_n of the nth Efimov state: $\tau_n E_n = \text{const}$. This result is in agreement with the quantum-mechanical predictions (see Refs. [26,27]) of the half-life (i.e., of the quantum-mechanical width) of Efimov states, which are longer lived as n increases and indeed become perpetual in the $\lim_{n\to\infty} E_n = 0$ limit. These predictions form a coherent and at least partially falsifiable semiclassical picture of Efimov states which has not yet been tested in experiment.

The paper is organized as follows. In Sec. II we briefly introduce Efimov's effective potential and discuss its purpose and limitations. In Sec. III we discuss some general properties of the classical mechanics of N bodies moving in a hyperradial $-1/R^2$ potential. In Sec. IV we present the necessary mathematical background, where we show that Efimov's dynamics are superintegrable in the absence of a cutoff. Then in Sec. V we find (all) periodic orbits satisfying Efimov's initial or boundary conditions and show that they cannot lead to Efimov states' energy spectrum; however, we show that a class of quasiperiodic adiabatically shrinking orbits leads to Efimov's spectrum. In Sec. VI we discuss ways to introduce an appropriate boundary condition at the small hyperradius cutoff, which would prevent the collapse (fall to the center) and lead to a bounce of the system, as well as the scaling properties of Efimov state's half-life. Finally, in Sec. VII we summarize our results and draw conclusions.

II. EFIMOV'S EFFECTIVE POTENTIAL: PURPOSE AND LIMITATIONS

Efimov's original work [1] was a study of solutions to the three-body Schrödinger equation in the short-range singular Dirac δ-function potential. He used the so-called hyperspherical harmonic expansion of the wave functions and of the potential to solve the three-body Schrödinger equation. This work led to the remarkable prediction of a three-body bound-state energy spectrum, even though the two-body states are not bound, which was experimentally corroborated some 35 years later. Classical motion, on the other hand, in the Dirac δ-function potential is ill-defined at best, so there is no hope for semiclassical quantization of these equations of motion or for any intuition to be gained on the basis of classical concepts such as shape or trajectory.

In the meantime Efimov [5] had tried to reach precisely such a classically intuitive understanding of his perplexing results. That led him to an effective attractive hyperra-

[1]Dilated Efimov states in realistic situations, i.e., in experiments, are expected to decay towards the ground state (see Refs. [26,27]), even though that feature has not yet been observed in experiment. The dominant decay mechanism of such dilated Efimov states depends on the environment surrounding the three-body system and may be either of the following: mechanical inelastic collisions leading to deep-lying two-body bound states or (multi)photon radiation in the deep infrared region.

dial three-body $-1/R^2$ potential, which is not singular in the hyperangular variables, but is homogeneous, just as the original δ-function one. Indeed, Efimov had noted [2] that his equation of motion (EOM) "has the form of the radial Schrödinger equation of [the] two-dimensional problem with s_i^2/R^2." The transition from a one-body problem in a radial $-1/R^2$ potential to the three-body problem in a hyperradial $-1/R^2$ potential appears immediate, though some differences remain. The overwhelming similarity and the main difficulties underlying these two classical systems remain: Both of their dynamics are strictly speaking undefined beyond the free-fall time. Nevertheless, global conservation laws can be used to extract some information about the temporal evolution, of at least one variable, beyond the free-fall time. This lack of predictability (chaoticity) is easier to illustrate in the one-body system, which we do in Sec. III B, than in the three-body system, to which we devote Secs. IV and V.

Efimov's potential depends only on the hyperradius $R = \sqrt{\frac{1}{3}\sum_{i>j}^{3}(\mathbf{r}_i - \mathbf{r}_j)^2}$ of three bodies but not on the hyperangular degrees of freedom, which implies integrability (Sec. V C) in any number of spatial dimensions. Moreover, the $-1/R^2$ nature of this potential leads to yet another conservation law, which implies superintegrability of its classical three-body dynamics [14,28,29], a fact rarely mentioned and never practically used in the Efimov physics literature.[2] In three spatial dimensions there are $15 + 1$ constants of motion, one each for the 15 components of the hyperangular momentum tensor, which include the three components of the (standard) angular momentum vector plus one virial [12,15], which is the generator of dilations and contractions, besides the usual $3 + 1$ (c.m. plus energy) ones (Sec. IV). This additional constant of motion, also known as the virial, ensures the superintegrability of this dynamical system [14,28,29].

Of course, even mere integrability generally allows for (usually numerical) integration of periodic solutions to the classical equations of motion and also to their semiclassical quantization (which is the way things turned out for the harmonic oscillator and the hydrogen atom). Semiclassical quantization of a three-body system (viz., the He atom) in the Coulomb potential, which is not integrable, was first successfully completed by Wintgen and co-workers [30–34] using Gutzwiller's semiclassical quantization methods based on periodic orbits [35], which suggests that we apply the same program here.

There is one significant difference between three-body dynamics in the Efimov potential and other examples of superintegrable systems, such as the harmonic oscillator: The dilational (conformal) symmetry is anomalous in the sense[3] that it is in fundamental conflict with the quantization conditions [15,36,37]. There are various ways of implementing this anomaly in simple one-body, one-dimensional[4] quantum-mechanical systems, usually by way of a cutoff [36,38,39], which has a counterpart in the semiclassical quantization [23] as well. The periodic one-body orbit of Bhaduri *et al.* [23] involves a fall to the center and a subsequent bounce from the hard-wall boundary condition assumed at short distances, which corresponds to the quantum-mechanical cutoff. This sort of perfectly elastic bounce can be arranged to happen only in one spatial dimension however. For one particle moving in two dimensions, we show in Sec. III B that the particle bounces elastically from the center, albeit in an arbitrary direction.

For three particles moving in three dimensions, there are fundamental theorems [24,25] that preclude the possibility of continuation of dynamics beyond the three-body collision. In Sec. VI we briefly discuss inherent difficulties in the definition of appropriate three-body boundary conditions at the short-distance cutoff.

It should be noted that both the scaling (conformal) anomaly and the Efimov effect depend on the dimensionality of space. There is no Efimov effect in one or two spatial dimensions, as the quantum-mechanical problem of three bodies in the pairwise sum of Dirac δ-function two-body potentials in one spatial dimension is exactly solvable [40], goes by the name of the Lieb-Liniger model, and does not display the characteristic Efimov energy spectrum.[5]

Thus, the Efimov potential approach is ineffective in one spatial dimension. In two spatial dimensions the quantum-mechanical two-body problem in the Dirac δ-function potential has been treated in Refs. [36,38] and references cited therein, where a connection with the conformal anomaly is established as well.[6] At this point however it may be noted that Ref. [23] did not discuss the differences between the one-body system in one dimension and the three-body system in three dimensions, which we do below.

All strictly periodic three-body three-dimensional (3D) orbits in the Efimov potential, with or without a cutoff, must have a constant hyperradius (size) throughout their periods and exactly vanishing energy. Thus, all strictly periodic three-dimensional three-body orbits are in manifest conflict with Efimov's quantum-mechanical eigenenergies, which do not vanish. Therefore, one must look for other classes of orbits in order to reproduce the Efimov spectrum.

We suggest to take such three-body orbits that have the hyperangular dependence of a strictly periodic orbit, but with a (nonconstant) periodic hyperradial dependence, as these two degrees of freedom decouple in the Efimov potential. Hyperradial periodicity of such an orbit is ensured by global conservation laws, but the hyperangular evolution may undergo

[2]Efimov did not emphasize this aspect of his potential either, but rather discussed certain qualitative properties of solutions to the EOM in his potential [5], which we now put to the test.

[3]This quantum anomaly is (presumably) also the reason for the appearance of the remnant discrete dilation symmetry in the quantum-mechanical spectrum.

[4]We are not aware of any attempts at a solution to the quantum-mechanical three-body problem in the Efimov potential however.

[5]This should not surprise us, as the configuration spaces of Lieb-Liniger and Efimov systems have different dimensionalities. The classical motion of three bodies in Efimov's effective potential can also be solved in one dimension; however, the corresponding WKB spectrum is not a good approximation to the Lieb-Liniger spectrum.

[6]The three-body problem in the Dirac δ-function potential in two spatial dimensions has not been discussed.

an abrupt discontinuous change at the moment of vanishing hyperradius, i.e., of three-body collision (collapse). We say "may" here because the actual dynamics (solution to the equations of motion) at the time of collapse is strictly speaking undefined (see the example in Sec. III B). Once the system bounces from the singularity at $R = 0$, the hyperangular dependence resumes its predictability.

Thus, such a solution would not be strictly periodic, as its temporal evolution beyond the free fall or the cutoff time is strictly speaking arbitrary. This suggests that orbits corresponding to Efimov states are not strictly periodic, but change their hyperangular dependences at regular (strictly periodic) time intervals that equal twice the long (as compared with the period of the hyperangular motion) albeit finite free-fall time (lifetime), which increases as the binding energy decreases.

We show in Sec. VI that the free-fall time, i.e., the time it takes the system to reach the triple collision, or the short-distance cutoff, which is much longer than the hyperangular period of the orbit, is also an upper bound on the half-life of semiclassical Efimov states, which is in agreement with the quantum-mechanical predictions of the half-life (inverse of the width) of Efimov states (see Refs. [9,26]).

III. CLASSICAL MECHANICS OF N BODIES IN EFIMOV'S POTENTIAL

The main feature of N-body dynamics in the homogeneous potential $V_{\text{total}}(R) \simeq \frac{1}{R^2}$, also known as the strong or Jacobi-Poincaré potential, is its dilational or conformal symmetry. It determines the nature of its periodic orbits, as having vanishing energy, irrespective of the number of bodies N.

A. Dilation and conformal symmetry of the problem

The Lagrange-Jacobi identity for any homogeneous potential $V_{\text{total}}(R) \simeq \frac{1}{R^\alpha}$,

$$m\frac{d^2}{dt^2}\frac{R^2}{2} = 2T_{\text{total}} + \alpha V_{\text{total}}, \tag{1}$$

gives a relation between the kinetic T_{total} and the potential energy V_{total}, where R is the hyperradius and α is a real number. Here the left-hand side of Eq. (1),

$$m\frac{d^2}{dt^2}\frac{R^2}{2} = \frac{d}{dt}G(R),$$

is the time derivative of the so-called virial

$$G = \sum_{i=1}^{N}\mathbf{q}_i\cdot\mathbf{p}_i = m\frac{d}{dt}\sum_{i=1}^{N}\mathbf{q}_i\cdot\mathbf{q}_i = \frac{m}{2}\frac{dR^2}{dt} = mR\frac{dR}{dt},$$

which is (twice) the generator of dilations and contractions. Note that, for $\alpha = 2$, the equations of motion plus the Lagrange-Jacobi identity lead to the conservation law [12–14]

$$\frac{dD}{dt} = 0,$$

where

$$D = \frac{G}{2} - Et = \text{const}.$$

As, for $\alpha = 2$, the right-hand side of Eq. (1) is identical to twice the total energy E [12–14],

$$\lim_{\alpha=2}\frac{d}{dt}G(R) = m\frac{d^2}{dt^2}\frac{R^2}{2} = 2T_{\text{total}} + 2V_{\text{total}} = 2E,$$

which is a constant of motion itself. This conservation law can be integrated as

$$G(t) - G(t_0) = 2\int_{t_0}^{t} E\,dt = 2E(t - t_0),$$

where $G(t_0) = mR(t_0)\frac{dR}{dt}|_{t=t_0}$ is the initial value of the virial, which can have either sign, depending on the sign of $\dot{R}(t_0) = \frac{dR}{dt}|_{t=t_0}$.[7]

As $G(t) = mR(t)\dot{R}(t)$ and $R(t) \geqslant 0$ for all t, this conservation law is equivalent to

$$\frac{dR}{dt} = \frac{1}{mR}[G(t_0) + 2E(t - t_0)].$$

Thence, for $E < 0$ and for a sufficiently long time $t > t_0 + \frac{G(t_0)}{2E}$, the factor $G(t_0) + 2E(t - t_0) < 0$ must be negative, irrespective of the sign of the initial value $G(t_0)$. Therefore, this equation always leads to $\dot{R}(t) < 0$, which in turn leads to the collapse of the system for a sufficiently long time $t \gg \frac{G(t_0)}{2E}$. Similarly, irrespective of the sign of $G(t_0)$, positive energy $E > 0$ always leads to an infinite expansion of the system.

Thus, in the remaining case $E = 0$, the virial $G(t)$ is an integral of motion $G = G(t_0) = \text{const}$.[8] Accordingly, at zero energy $E = 0$ there are three kinds of motion: (a) unbounded, with $G = mR\frac{dR}{dt} > 0$, which leads to endless expansion of the system $\lim_{t\to\infty} R(t) = \infty$; (b) bounded, with $G = mR\frac{dR}{dt} < 0$, which leads to collapse of the system to a point (zero extension) within a finite time t_c, $\lim_{t\to t_c} R(t) = 0$; and (c) bounded, with $G = mR\frac{dR}{dt} = 0$, which leads to endless, including periodic, motions of the system at a constant extension (size) $R(t) = R(0)$. The additional constant of motion $R = R_0$ makes this system superintegrable, similarly to the Kepler problem (or the hydrogen atom problem) and the harmonic oscillator, but it also makes its quantization procedure more subtle than that in Ref. [23] (see Sec. V F).

B. Periodic orbits of the one-body problem in the strong potential

Periodic orbits of the one-body problem in a central power-law potential are well studied and known and thus are straightforward to picture and understand. The two-body problem is equivalent to the one-body problem, after removal of the c.m. variable, but the three-body problem, even in a hyperradial potential, is more complicated. For this reason we start with the one-body problem.

[7] Of course, the above is nothing but the statement of conservation of the dilation generator D and of the conservation of the conformal boost K from the general theory of conformal symmetry in nonrelativistic systems [28,29]. We avoid this specialist terminology here so as to keep the discussion accessible to a larger number of readers.

[8] This theorem is due to Jacobi [12].

As we are interested only in periodic orbits of one body in the strong potential $-1/R^2$, we may impose $E = 0$ and $\dot{R} = 0$ as an initial condition. This demands one of the following: (i) vanishing angular momentum $L = 0$ and vanishing potential energy $1/R_0^2 = 0$, i.e., $R_0 = \text{const} = \infty$, or (ii) nonvanishing angular momentum $L \neq 0$ and nonvanishing potential energy $1/R_0^2 \neq 0$, i.e., $R_0 = \text{const} < \infty$. The former case is trivial (no motion), whereas the latter allows circular motion, which is the only periodic orbit.

If one insists on a vanishing angular momentum $L = 0$ and finite initial (hyper)radius $R_0 \neq \infty$, one must end up in a collapsed state (see Sec. III A above). The temporal evolution of this system is not defined beyond the free-fall (collapse) time. If one nevertheless wishes to explore possible scenarios for the time evolution beyond the free-fall time, one must modify the potential, either by changing the power (order of homogeneity) in the potential or by introducing a cutoff.

(i) If we view the strong potential $-1/R^2$ as the limiting case of the homogeneous potential $V_{\text{total}}(R) \simeq -\frac{1}{R^\alpha}$ as $\alpha \to 2$, there are periodic solutions with $E < 0$ and $L = 0$. The orbit's precession angle $\Delta\phi$ for a particle moving in the effective potential $V_{L,\alpha}(R) := \frac{L^2}{R^2} - \frac{g}{R^\alpha}$ was determined in Ref. [41] as

$$\Delta\phi(E, L) = 2\int_{R_{\min}}^{\infty} \frac{L}{R^2\sqrt{2[E - V_{L,\alpha}(R)]}} dR. \tag{2}$$

In the limit of vanishing angular momentum $L \to 0$, the precession angle $\lim_{L\to 0} \Delta\phi(E, L, \alpha)$ is energy independent and equals

$$\Delta\phi(E, L = 0, \alpha) = 2\int_{1}^{\infty} \frac{du}{u\sqrt{u^{2-\alpha} - 1}} = \frac{2\pi}{2 - \alpha},$$

which goes to infinity in the limit $\alpha \to 2$,

$$\lim_{L\to 0, \alpha\to 2} \Delta\phi(E, L, \alpha) = \infty.$$

As the angle $\Delta\phi(E, L = 0, \alpha)$ can only be defined modulo 2π, the above result tells us that a free-fall trajectory in the strong potential $-1/R^2$ scatters at an arbitrary angle $\Delta\phi(E, L = 0, \alpha = 2)$ from the center of the potential. Thus, in this sense, a free-fall orbit can be continued beyond the free-fall time, but the resulting trajectory precesses through an arbitrary (incalculable) angle after every free fall to the center. Whether such an orbit can or should be called periodic is left to the reader to decide.

This situation ought to be compared with the corresponding one in the Coulomb potential, $\alpha = 1$, where (a) the deflection angle (2) equals 2π in the $L \to 0$ limit, i.e., the particle backscatters after free fall, and (b) the period of the circular (or any elliptic) orbit $T_{\alpha=1}$ does not depend on the angular momentum and equals twice the free-fall time $T_{\alpha=1} = 2\tau_{\alpha=1}$. This time is only associated with the (binding) energy of the orbit, which is not the case with $\alpha = 2$. Thus, the motion in the $\alpha = 2$ power-law potential has two independent time scales: $T_{\alpha=2}$ and $\tau_{\alpha=2}$. The former is associated (only) with the periodic motion, whereas the latter is independent and not necessarily periodic.

(ii) By introducing a spherically symmetrical cutoff, the problem of arbitrary orbits at vanishing angular momentum $L = 0$ can be solved *per fiat*. This modification of the potential breaks its homogeneity property however and thus leads to the breaking of the Lagrange-Jacobi identity (1) together with all of its consequences in Sec. III A.

An extension of the (hyper)spherical cutoff to three-particles moving in 3D space is problematic; however, the construction of a hyperspherical cutoff (in six dimensions) for three bodies (moving in 3D space) with good or meaningful kinematic and geometric properties remains an open question (see Sec. VI). Whereas one may readily postulate the existence of such a cutoff in a quantum-mechanical calculation, the corresponding classical equations of motion remain unsolved (see Sec. VI). For this reason, the classical EOM cannot be solved beyond the three-body collision time (point) and thus a different timescale is introduced into the problem.

IV. CLASSICAL MECHANICS OF THREE BODIES IN EFIMOV'S POTENTIAL

The first order of business is to find all symmetries, i.e., integrals of motion of three particles moving in Efimov's potential, so as to maximally simplify the integration of equations of motion. Here we show the (super)integrability of dynamics in the hyperspherical $1/R^2$ potential: There are $15 + 1$ (hyperangular momentum tensor components, which include three components of the usual angular momentum vector plus hyperradius R) constants of motion besides the usual $3 + 1$ (c.m. plus energy) ones.

A. Kinematics

The kinetic energy of three-body motion can and must be divided into the c.m. and the internal parts

$$T = \frac{1}{2}\sum_{i=1}^{3} m_i\dot{\mathbf{r}}_i^2 = \frac{1}{2}(m_1 + m_2 + m_3)\mathbf{R}_{\text{c.m.}}^2 + T_{\text{internal}},$$

and the same holds for the angular momentum

$$\mathbf{L} = \sum_{i=1}^{3} m_i\mathbf{r}_i \times \dot{\mathbf{r}}_i$$
$$= (m_1 + m_2 + m_3)\mathbf{R}_{\text{c.m.}} \times \mathbf{V}_{\text{c.m.}} + \mathbf{L}_{\text{internal}}.$$

The internal motion is what interests us here, so we remove the c.m. motion, by introducing two relative-motion three-vectors. There are infinitely many ways of choosing these relative vectors, but we will follow convention and use Jacobi vectors in the equal mass limit. When the masses are unequal one may use Smith's [42–45] version of mass-dependent Jacobi vectors. Coordinates of three (identical) particles (with equal masses) in the c.m. rest frame are given by two Jacobi three-vectors

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3), \tag{3}$$

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2). \tag{4}$$

B. Three-body hyperspherical variables

The kinetic energy in the rest frame is of the form

$$T_{\text{internal}} = \frac{m}{2}(\dot{\boldsymbol{\lambda}}^2 + \dot{\boldsymbol{\rho}}^2). \tag{5}$$

It possesses an O(6) symmetry that is made manifest by introducing the six-dimensional coordinate hypervector $x_\mu = (\boldsymbol{\lambda}, \boldsymbol{\rho})$: The kinetic energy (5) can be written as

$$T_{\text{internal}} = \frac{m}{2}\dot{R}^2 + \frac{K_{\mu\nu}^2}{2mR^2}, \tag{6}$$

where the grand angular momentum tensor $K_{\mu\nu}$, $\mu, \nu = 1, 2, \ldots, 6$, reads

$$\begin{aligned} K_{\mu\nu} &= m(\mathbf{x}_\mu \dot{\mathbf{x}}_\nu - \mathbf{x}_\nu \dot{\mathbf{x}}_\mu) \\ &= (\mathbf{x}_\mu \mathbf{p}_\nu - \mathbf{x}_\nu \mathbf{p}_\mu) \end{aligned} \tag{7}$$

and has 15 linearly independent components that include three components of the ordinary total angular momentum: $\mathbf{L} = \mathbf{l}_\rho + \mathbf{l}_\lambda = m(\boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} + \boldsymbol{\lambda} \times \dot{\boldsymbol{\lambda}})$. Efimov's potential energy $V(R) = -1/R^2$ is hyperradial and this large symmetry remains unbroken, with an additional dilational symmetry.

C. The O(6) symmetry of the problem

The group of ordinary rotations $SO(3)_{\text{rot}}$ is the diagonal SO(3) subgroup of the six-dimensional rotations $SO(3)_{\text{rot}} = SO(3)_{\text{diag}} \subset SO(6)$, which means that the rotations act equally on the first three coordinates ($\boldsymbol{\lambda}$) and the last three coordinates ($\boldsymbol{\rho}$) of the six-dimensional vector x_μ. Then, as demonstrated in Ref. [46], the 15 generators of the SO(6) Lie algebra decompose as $(3)_{\text{rot}} + (3) + (3) + (5) + (1)$ with respect to $SO(3)_{\text{rot}}$, so there is only one extra generator of SO(6) that commutes with the rotations. Upon introduction of complex coordinates,

$$\mathbf{X}_i^{\pm} = \boldsymbol{\lambda}_i \pm i\boldsymbol{\rho}_i, \quad i = 1, 2, 3. \tag{8}$$

The tensor $K_{\mu\nu}$ [Eq. (7)] can be written in terms of the new coordinates as

$$L_{ij} \equiv -i\left(\mathbf{X}_i^+ \frac{\partial}{\partial \mathbf{X}_j^+} + \mathbf{X}_i^- \frac{\partial}{\partial \mathbf{X}_j^-} - \mathbf{X}_j^+ \frac{\partial}{\partial \mathbf{X}_i^+} - \mathbf{X}_j^- \frac{\partial}{\partial \mathbf{X}_i^-}\right), \tag{9}$$

$$\Delta L_{ij} \equiv -i\left(\mathbf{X}_i^+ \frac{\partial}{\partial \mathbf{X}_j^-} + \mathbf{X}_i^- \frac{\partial}{\partial \mathbf{X}_j^+} - \mathbf{X}_j^+ \frac{\partial}{\partial \mathbf{X}_i^-} - \mathbf{X}_j^- \frac{\partial}{\partial \mathbf{X}_i^+}\right), \tag{10}$$

$$W_{ij} \equiv \left(\mathbf{X}_i^+ \frac{\partial}{\partial \mathbf{X}_j^-} - \mathbf{X}_i^- \frac{\partial}{\partial \mathbf{X}_j^+} - \mathbf{X}_j^+ \frac{\partial}{\partial \mathbf{X}_i^-} + \mathbf{X}_j^- \frac{\partial}{\partial \mathbf{X}_i^+}\right), \tag{11}$$

$$Q_{ij} \equiv \frac{1}{2}\left(\mathbf{X}_i^+ \frac{\partial}{\partial \mathbf{X}_j^+} - \mathbf{X}_i^- \frac{\partial}{\partial \mathbf{X}_j^-} + \mathbf{X}_j^+ \frac{\partial}{\partial \mathbf{X}_i^+} - \mathbf{X}_j^- \frac{\partial}{\partial \mathbf{X}_i^-}\right). \tag{12}$$

Among these are three antisymmetric tensors, each with three components: (i) L_{ij} [Eq. (9)], which corresponds to the physical angular momentum, (ii) ΔL_{ij} [Eq. (10)], which equals the difference of partial angular momenta $L_{ij}^{\lambda} - L_{ij}^{\rho}$), which is generally not conserved, and (iii) W_{ij} [Eq. (11)], with no immediately obvious physical meaning.

The symmetric part of the tensor $K_{\mu\nu}$ [Eq. (7)] contains the quadrupole tensor Q_{ij} [Eq. (12)], which can be decomposed into an irreducible second-rank tensor (under rotations), with five components and a scalar (with one component). The trace of Q_{ij} is the scalar

$$Q \equiv \sum_{i=1}^{3} Q_{ii} = \sum_{i=1}^{3} \mathbf{X}_i^+ \frac{\partial}{\partial \mathbf{X}_i^+} - \sum_{i=1}^{3} \mathbf{X}_i^- \frac{\partial}{\partial \mathbf{X}_i^-}, \tag{13}$$

which (obviously) commutes with the rotation generators L_{ij} and generates the so-called democracy transformations [42,46,47], which are continuous generalizations of permutations of three particles.

V. PERIODIC SOLUTIONS IN THE $1/R^2$ HYPERRADIAL POTENTIAL

Due to the superintegrability of this system, all periodic orbits in Efimov's hyperspherical $1/R^2$ potential can be integrated exactly. Efimov's condition of vanishing (total) orbital angular momentum $\mathbf{L} = 0$ simplifies the problem further. For this purpose we find the most appropriate variables.

A. The O(4) symmetry of L = 0 three-body motion

As Efimov's effect involves only states with vanishing (total) orbital angular momentum $\mathbf{L} = 0$, we use the fact that all such orbits (must) lie in a plane, e.g., the x-y plane with the z components of all position and velocity vectors being equal to zero. Thus the problem is reduced to a two-dimensional one, without loss of generality. This means that the O(6) symmetry turns into O(4) symmetry, which does not change the superintegrability of the problem, but simplifies the search for periodic orbits.

Therefore, we define the two-dimensional grand angular momentum tensor $K_{\mu\nu}$,

$$\begin{aligned} K_{\mu\nu} &= m(\mathbf{x}_\mu \dot{\mathbf{x}}_\nu - \mathbf{x}_\nu \dot{\mathbf{x}}_\mu) \\ &= (\mathbf{x}_\mu \mathbf{p}_\nu - \mathbf{x}_\nu \mathbf{p}_\mu), \end{aligned} \tag{14}$$

where $\boldsymbol{x}_\mu = (\boldsymbol{\rho}, \boldsymbol{\lambda})$, as a subset of the three-dimensional one (7) with $\mu, \nu = 1, 2, 3, 4$. In particular, $l_\rho \equiv K_{12}$ and $l_\lambda \equiv K_{34}$ generate SO(2) rotation of vectors $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$, respectively.

Next we introduce

$$\mathbf{M} = \tfrac{1}{2}(l_\rho + l_\lambda, K_{13} - K_{24}, K_{14} + K_{23}), \tag{15}$$

$$\mathbf{N} = \tfrac{1}{2}(l_\rho - l_\lambda, K_{13} + K_{24}, K_{14} - K_{23}). \tag{16}$$

Note that $\mathbf{M}$ and $\mathbf{N}$ commute and that each of them satisfies separate SO(3) commutation rules

$$\begin{aligned} [\mathbf{M}^i, \mathbf{M}^j] &= i\varepsilon^{ijk}\mathbf{M}^k, \\ [\mathbf{N}^i, \mathbf{N}^j] &= i\varepsilon^{ijk}\mathbf{N}^k, \end{aligned} \tag{17}$$

explicitly demonstrating that $o(4) = o(3) \oplus o(3)$. In other words, there are two quasi-three-dimensional hyperangular momentum vectors $\mathbf{M}$ and $\mathbf{N}$ that completely describe the four-dimensional antisymmetric tensor $K_{\mu\nu}$, very much like the electric $\mathbf{E}$ and magnetic field $\mathbf{B}$ completely describe the EM field tensor $F_{\mu\nu}$. In the limit when the total angular mo-

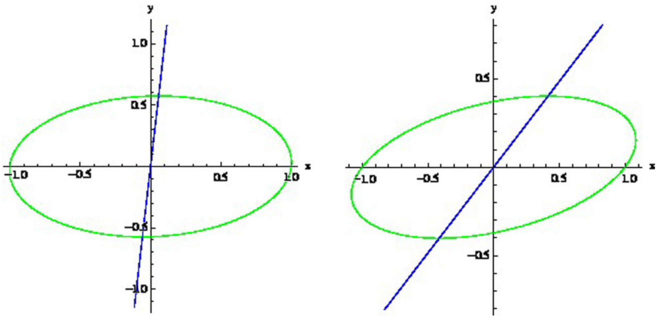

FIG. 1. Real-space trajectories of periodic three-body orbits with vanishing angular momentum in the Efimov potential. Two bodies move on the oval in opposite directions, through the two-body collisions, until they return to their initial positions, which is why only one color (green) can be seen in the oval. The third body moves back and forth on the straight (blue) line.

mentum vanishes $L = l_\rho + l_\lambda = 0$, the kinetic energy attains a new level of simplicity.

B. Shape sphere

We may relate the three scalar variables to the unit three-vector $\hat{\mathbf{n}}$ defined by the Cartesian components

$$\hat{\mathbf{n}} = \left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{R^2}, \frac{\lambda^2-\rho^2}{R^2}, \frac{2(\boldsymbol{\rho}\times\boldsymbol{\lambda})\cdot\boldsymbol{e}_z}{R^2} \right). \tag{18}$$

The domain of these three-body variables is a sphere with unit radius [48,49], as illustrated in Fig. 2. The sphere coordinates depend only on the shape of the triangle formed by the three bodies, not on the hyperradius R or on the orientation of the triangle in space. The equatorial circle corresponds to collinear three-body configurations (degenerate triangles). The three points shown in Fig. 2 correspond to two-body collisions, which are singularities in the potential. Three Euler meridians on the shape sphere are orthogonal to the equator and pass through one of the collision points and its corresponding Euler point that lies on the equator opposite to the collision point.

We will see that periodic three-body orbits with vanishing angular momentum form great circles on the shape sphere, including the equator (see Fig. 2). We may formulate the initial conditions in terms of any one of the well-known parametrizations of the shape sphere or hyperangles.

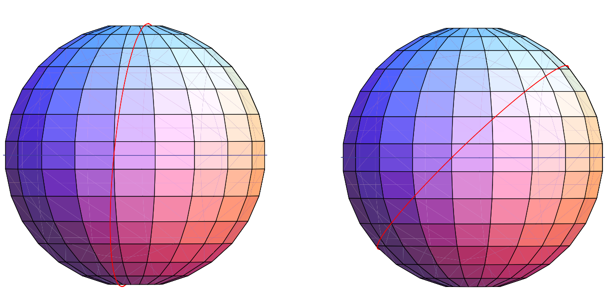

FIG. 2. Shape-space–sphere trajectories of periodic three-body orbits with vanishing angular momentum in the Efimov potential are great circles on the shape-space sphere.

C. Kinetic energy in terms of shape-spherical hyperangles

There are two standard sets of hyperangles: (i) the Delves [50] hyperangles χ and θ,

$$\rho = R\sin\chi, \quad \lambda = R\cos\chi \quad \text{with } 0 \leqslant \chi \leqslant \pi/2, \tag{19}$$

$$\boldsymbol{\rho}\cdot\boldsymbol{\lambda} = R^2\cos\chi\,\sin\chi\,\cos\theta, \tag{20}$$

leading to

$$\frac{\lambda^2-\rho^2}{R^2} = \cos(2\chi),$$

and (ii) the Smith-Iwai [44,45,48] hyperangles α and ϕ, $(\sin\alpha)^2 = 1 - (\frac{2\boldsymbol{\rho}\times\boldsymbol{\lambda}}{R^2})^2$ and $\tan\phi = (\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\rho^2-\lambda^2})$, which reveal the full S_3 permutation symmetry of the problem. The angle α does not change under permutations, so all permutation properties are encoded in the ϕ dependence.

The three-body kinetic energy with vanishing angular momentum $L = 0$ in two dimensions in terms of Delves angle χ is [51]

$$T = \frac{m}{2}\left[\dot{R}^2 + R^2\left(\dot{\chi}^2(t) + \frac{1}{4}\sin^2[2\chi(t)]\dot{\theta}^2(t)\right)\right].$$

If we redefine $\chi' = 2\chi$, then this assumes the standard spherical-angular form

$$T = \frac{m}{2}\left[\dot{R}^2 + \left(\frac{R}{2}\right)^2\{\dot{\chi}'^2(t) + \sin^2[\chi'(t)]\dot{\theta}^2(t)\}\right].$$

In terms of Iwai-Smith hyperangles,

$$T = \frac{m}{2}\left[\dot{R}^2 + \left(\frac{R}{2}\right)^2\{\dot{\alpha}(t)^2 + \sin^2[\alpha(t)]\dot{\phi}(t)^2\}\right].$$

This is equivalent to the kinetic energy of a free particle in spherical coordinates, on the shape sphere defined in Sec. V B.

D. Efimov's initial conditions

Efimov explicitly demanded [5] that all three pairs of particles have vanishing partial angular momenta $\mathbf{r}_{ij}\times\dot{\mathbf{r}}_{ij} = 0$, so we see that both L and ΔL must vanish. As both L and ΔL are conserved, due to the hyperradial nature of the three-body potential, we set

$$L = \mathbf{l}_\rho + \mathbf{l}_\lambda = m(\boldsymbol{\rho}\times\dot{\boldsymbol{\rho}} + \boldsymbol{\lambda}\times\dot{\boldsymbol{\lambda}}) = 0$$

and

$$\Delta L = \mathbf{l}_\rho - \mathbf{l}_\lambda = m(\boldsymbol{\rho}\times\dot{\boldsymbol{\rho}} - \boldsymbol{\lambda}\times\dot{\boldsymbol{\lambda}}) = 0$$

as an additional initial condition. That, together with the constraint (initial condition) $\dot{R} = 0$, i.e.,

$$R\dot{R} = \boldsymbol{\rho}\cdot\dot{\boldsymbol{\rho}} + \boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}} = 0,$$

leads to the initial conditions

$$\boldsymbol{\rho}_0 \neq 0, \quad \boldsymbol{\lambda}_0 = 0,$$

$$\dot{\boldsymbol{\rho}}_0 = 0, \quad \dot{\boldsymbol{\lambda}}_0 \neq 0.$$

The first set of these initial conditions is equivalent to the initial configuration being one of three Euler configurations, i.e., the midway collinear configuration.

The second set of initial conditions (the velocities) is equivalent to all three bodies' initial velocities $\dot{\mathbf{r}}_{0i}$, $i = 1, 2, 3$, being parallel with one planar vector $\mathbf{v}$. The absolute value of $\mathbf{v}$ is determined by the vanishing energy condition $E = \frac{1}{2}(2 + \frac{1}{4})\mathbf{v}^2 - V(R) = 0$, which leads to $|\mathbf{v}| = \frac{2}{3}$. Thus there is only one free parameter left: the angle of this vector with respect to the initial collinear configuration.

The choice of center-of-mass system ($\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0$) cuts down the number of independent variables to eight, as there are (only) two independent relative coordinate vectors and two corresponding velocities. The choice of vanishing angular momentum ($L = 0$) reduces this number down to seven.

We choose the so-called Euler initial configuration, the three bodies being collinear, say, on the x axis, with the distance between bodies 1 and 2 equaling two units and with body 3 at the midpoint between bodies 1 and 2. That sets the (initial state) hyperradius at $R = \sqrt{2}$ and fixes the potential energy at $V(R) = -1/2$, with the coupling constant g set equal to unity $g = 1$. As the hyperradius must remain constant during periodic motion, we are left with six independent variables. The conditions $L = 0$, $\dot{R} = 0$, and $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0$ put together lead to

$$\mathbf{v}_1 = \mathbf{v}_2 = -\tfrac{1}{2}\mathbf{v}_3,$$

thus implying that only one velocity two-vector is independent with Efimov's choice of initial conditions.

Finally, demanding $\dot{R} = 0$ leaves the system with two independent variables: the angle ϕ between the x axis and the velocity 2-vector v_1, and the overall size R. Due to the zero-energy condition $E = 0$, the size R of the system, which has already been set at $R = \sqrt{2}$ by our choice of initial positions, determines the value of the initial kinetic energy T as $T = -V(R)$, thus leaving the angle ϕ as the only free variable. This means that, in order to find periodic orbits passing through the Euler point, the only variable that can be varied in this subspace of initial conditions is the angle ϕ between the two components of the vector $\mathbf{v}_1 = (v_{x1}, v_{y1})$: $\tan\phi = \frac{v_{y1}}{v_{x1}}$.

E. Strictly periodic solutions

Two arbitrarily chosen trajectories, out of infinitely many periodic solutions in real space, are shown in Fig. 1 and on the shape sphere in Fig. 2; they are distinguished by the (arbitrary) value of the angle ϕ. Of course, for every orbit shown here, there are two other independent orbits that are cyclic permutations of the ones shown. There we can see that two of the particles always move on a common (quasielliptic) oval trajectory, including mutual two-body collisions, whereas the third one moves on the straight line midway between the first two and never gets close to either one. In other words, the motion is clearly separated into a two-body system and a third (spectator) body. As the initial inclination angle $\phi = \arctan(\frac{v_{y1}}{v_{x1}})$ decreases towards zero, the oval (quasiellipse) and the straight line tend closer to each other until they merge into the extreme collinear case, when all three particles interact pairwise with each other, including all possible two-body collisions.

Only the collinear trajectory is consistent with Efimov's qualitative description [5] of his state as follows: "Any particle of the pair can be picked up by the third particle to form another pair. This process of particle exchange may take place any number of times (and between any two particles), bringing about an effective three-body interaction." However, the same collinear trajectory has the shortest time before a two-body collision occurs and, moreover, as already described in the Introduction, in one spatial dimension the Efimov problem turns into the one-dimensional (Lieb-Liniger) three-body problem.

The periodic orbits on the shape sphere are great circles, with constant hyperangular velocities $\dot{\alpha}$ and $\dot{\phi}$ and constant hyperradius $R = R_0$. The period T equals $T = 8.88577$ with $R_0 = \sqrt{2}$ and coupling constant $g = 1$. However, this is just the motion of a free rigid rotor in some (abstract) three-dimensional space, a problem in quantum mechanics that was solved long ago [52].

F. Semiclassical quantization of periodic orbits

The periods of all these orbits are equal, $T = 8.88577$, which also equals the action S of all orbits, as the (constant) potential energy equals $1/2$,

$$S(T) = \int_0^T (K - V)dt = 2\int_0^T K dt = -2\int_0^T V(R)dt$$
$$= -2V(R_0)\int_0^T dt = -2V(R_0)T = T,$$

due to our initial condition $V(R_0) = -1/2$. Of course, the action remains invariant under scale transformations $S(R_0, T) \to S(\lambda R_0, \lambda^2 T)$, in accordance with the general scaling rule $S \to \lambda^{1-\alpha/2}S$, and independent of the periodic orbit's energy $E = 0$.[9] Next we note that Bhaduri *et al.* [23] obtained their discrete energy states by WKB quantizing the hyperradial motion [see Eq. (52) in [23]]. Expressing action as an integral over the hyperradius, we have

$$\Delta S_R = 2\int_{R_{\min}}^{R_{\max}} \sqrt{2m[E + V(R)] - \frac{(n_\phi \hbar)^2}{R^2}}dR = n_R h.$$

As shown in Sec. III A, hyperradial motion is (strictly) forbidden by the joint requirements of periodicity of motion and the (classical) dilation symmetry in the $V(R) = -1/R^2$ potential, so we have $R_{\min} = R_{\max}$ and this integral vanishes exactly. Therefore, any explanation of the discrete energy levels in the Efimov potential must be looked for elsewhere. This is a fundamental impasse to semiclassical quantization of a scale-invariant system. As already suggested in Sec. II and further elaborated in Sec. VI, one proposal is to introduce a cutoff at low values of the hyperradius, which does not appear to affect periodic orbits in this system, however, as they all have fixed hyperradii. Therefore, we extend our analysis to slowly shrinking quasiperiodic orbits.

[9]Recall the exact yet, in the case of $\alpha = 2$, seemingly indeterminate formula relating the action $S^{\alpha}_{\min} = (\frac{\alpha+2}{\alpha-2})ET$ to the orbit's energy E and period T.

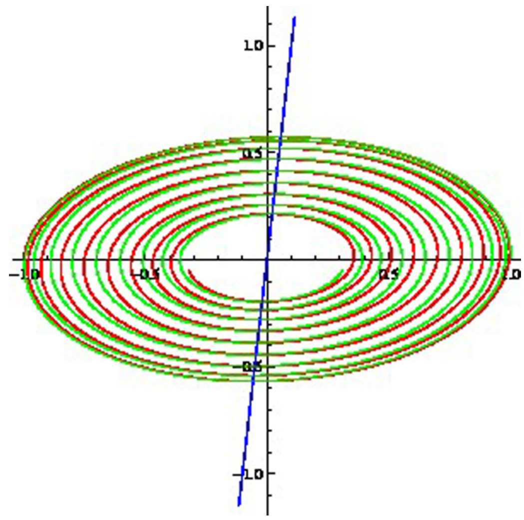

FIG. 3. Real-space trajectories of quasiperiodic adiabatically shrinking orbits in the Efimov potential with energy $E = -5 \times 10^{-4}$. Two bodies move on the inspirals (green and red), whereas the third body oscillates on the straight (blue) line with decreasing amplitude. The orbit was stopped arbitrarily at time $t - t_0 = 42$, for better clarity.

G. Adiabatically shrinking (and expanding) quasiperiodic orbits

As the hyperradial and hyperangular motions decouple in this potential (with or without cutoff), one may allow for some small negative value of energy (following Bhaduri *et al.*), with the understanding that such an initial condition necessarily leads to inspiraling of all three bodies and ultimately to the collapse of the system after a (finite) time interval $t - t_0 > \frac{G(t_0)}{2E}$. With appropriate initial conditions, the system might be quasiperiodic: The hyperangular motion on the shape sphere remains periodic (with its intrinsic period T), provided it is so at the initial time $t = 0$, whereas the hyperradius $R(t)$ shrinks towards collapse. The free-fall time τ of such a system,

$$\tau = \int_{R_{\min}}^{R_{\max}} \frac{dR}{\sqrt{\frac{2}{m}\left(E - \frac{K^2}{2mR^2} + gR^{-2}\right)}},$$

depends crucially on the energy value E, both directly in the integrand and through the upper integration bound (turning point) $R_{\max}$.[10] Here K^2 is the value of the (conserved) hyperangular momentum squared on the shape sphere. Thus, if the energy $|E| = |\frac{K^2}{2mR_{\max}^2} - gR_{\max}^{-2}|$ is small enough, the (initial) free-fall rate $\dot{R}$ is small, the motion is adiabatic, and the free-fall time τ may be (many) thousands of times longer than the intrinsic period T: $\tau \gg T$. For example, if $E = -5 \times 10^{-7}$ (in our units), the hyperradius shrinks by only about 5% over a time of 500 intrinsic periods, whereas for $E = -5 \times 10^{-5}$, the hyperradius shrinks by about 50% over a time extending over ten intrinsic periods (see Fig. 3). Such a system can be quantized semiclassically, following the outline set out by Bhaduri *et al.* [23], with corresponding results for the energy spectrum.

What happens to the system beyond the free-fall time is a sensitive function of the boundary conditions at the time of collapse, just as in the one-body 2D case (Sec. III B): The system may rebound along the same trajectory that it fell in or it may emerge in a randomly different geometrical configuration, i.e., at a random point in the shape space. In this sense, this system is random, and thus unpredictable, and its motion cannot be called strictly periodic, though the size (hyperradius) of the system undergoes periodic oscillations.

If one were to speculate about the physical consequences of these dynamics in a realistic situation, i.e., in the presence of EM interactions and/or surrounding gas of particles, then one might conclude that the three-body collision might be inelastic and therefore that the rebounding system may emerge with a smaller kinetic energy, leading thus to a deeper-lying Efimov state. In other words, the higher-lying larger Efimov state would decay, after a (classical) half-life equal to the free-fall time, into a deeper-lying, smaller one, until it reaches the ground state, which is stable.

Thus we may conclude that Efimov's states are faithfully reproduced by an infinite set of quasiperiodic, adiabatically shrinking and expanding three-body orbits, as described above. This semiclassical picture of a quantum phenomenon can be put to a test; see point (vii) in the following section.

VI. DISCUSSION

Several remarks are due now.

(i) It is well known [36–39,53] that the (nonrelativistic) quantum-mechanical motion of a single particle in the $-1/R^2$ potential in one and two spatial dimensions requires some kind of regularization in order to provide finite results. This (quantum-mechanical) regularization can be effected in different ways; a common one is to introduce a solid wall (billiard-ball) boundary condition at a finite distance from the center [36–39]. Indeed, Bhaduri *et al.* used precisely such a regularization of classical one-dimensional (hyperradial) motion in Ref. [23]. What they did not address was the question of what would such a quasi-one-dimensional boundary condition imply for three-body dynamics in the Efimov potential in three dimensions.

(ii) The above billiard-ball prescription for regularization is clear enough in the case of 1D motion of one or two particles. Its extension to three or more bodies moving classically in two or three dimensions leaves many open questions however. Thus, the problem of semiclassical quantization of three-body motion in the Efimov hyperradial potential boils down to finding and imposing boundary conditions that prevent the classical motion of the system from collapsing into three-body collisions (fall to center) and ensure the subsequent bounce (time-reversed motion), i.e., so as to ensure the existence of periodic orbits with a varying hyperradius in the Efimov potential.

(iii) Two-body (binary) collisions can be regularized in power-law potentials r^{α}, with $\alpha < 0$, so long as the power satisfies $\alpha > -2$ [41]. There are several regularization methods available for binary collisions in the Newtonian potential ($\alpha = -1$) [54,55]. However, in the $\alpha = -2$ strong potential in two

[10] Of course, in the $E \to 0$ limit, $R_{\max} \to \infty$ and the free-fall time goes to infinity $\tau \to \infty$.

dimensions even the two-body collision is not regularizable [41] by conventional (analytic continuation) means, e.g., by the Levi-Cività regularization [54].

Unconventional methods have been tried in two-body dynamics however. For example, an alternative regularization of the quantum mechanical problem, suggested in Ref. [56], is to add a stronger repulsive interaction, e.g. power law, acting at shorter distances; this procedure manifestly breaks the assumed scaling symmetry of the system. Moreover, Wu and Sprung [57] have studied, both numerically and analytically, two-dimensional classical orbits (of one or two bodies) in power-law potentials, as the power approaches the singular limits $\alpha \to -2$ (the strong potential) and $\alpha \to +\infty$ (the solid wall or billiard ball). This might lead one to believe that such methods can be extended to three bodies moving in three dimensions, but the necessary work has not yet been done.

(iv) The fall to the center or, equivalently, the triple collision ("der Dreierstoß" in Siegel's vocabulary [24]) singularity is not regularizable in power-law potentials r^{α}, with $\alpha < 0$, in general. Even the billiard-ball regularization of the classical EOM, which works for one- and two-body dynamics, cannot be unambiguously extended to three-body collisions [25], so the numerical solutions to three-body EOM cannot be continued in time beyond the three-body collision point. Thus, even though this system is integrable, in the Liouville sense of the word, it is numerically incalculable.

(v) One simple heuristic way to deal with the problem in point (iv) might be to stop the numerical calculations at the point of impact on the hard wall and to assume that after the impact, the temporal evolution of the system is just the time-reversed one of the preceding motion.[11] We call this assumption the elastic bounce from a hard wall, as if the Efimov three-body system were a solid object that could bounce without changing its shape or structure.[12]

(vi) Point (v) above is just a formal recipe that might be used to continue the solution beyond the time of impact on the hard wall, though physically many different scenarios may appear here, including (a) formation of a dimer plus monomer (which is explicitly forbidden in the original Efimov scenario however) or (b) scattering into a (semi)classical three-body state different from the initial one.

(vii) The dimer plus monomer scenario implies a finite lifetime of classical Efimov states, even *in vacuo*, which is just the free-fall time (or half-period of the Bhaduri *et al.* periodic orbit) in the $-1/R^2$ potential. The above scenario makes definite predictions about the scaling properties of this half-life τ or quantum mechanically half-width $\Gamma/2$, viz., $1/\tau = \Gamma/2$ scales as $\tau \to \lambda^2\tau$, i.e., $\Gamma \to \lambda^{-2}\Gamma$. Since the binding energy E_n scales in the same way as Γ, $E_n \to \lambda^{-2}E_n$, we conclude that the line shape $\tau_n E_n \simeq \frac{E_n}{\Gamma_n} = \text{const}$ of Efimov states (resonances) must be invariant under scaling transformations. Moreover, this shows that in the limit $n \to \infty$ Efimov states become stable: $\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \lambda^{2n}\tau_0 = \lim_{n\to\infty} (22.7)^{2n}\tau_0 = \infty$. Some quantum-mechanical estimates of the Efimov states' widths have been made in Refs. [26,27], which results are in qualitative agreement with our (semi)classical approach.[13] As the line shape of Efimov states has not yet been determined experimentally, this feature would provide a clear and definite test of our (semi)classical approach. If it passes this test, it will raise new questions about the quasistatic nature of excited Efimov states.

VII. CONCLUSION

In summary, we have studied the classical three-body dynamics in Efimov's potential. We have shown that this system without a short-distance cutoff is superintegrable. There are infinitely many periodic orbits, all of which have vanishing (binding) energy, which therefore do not allow semiclassical quantization. This is a consequence of the exact dilation-scaling symmetry of periodic motion in such a potential. Of course, Efimov states are a consequence of the quantum-mechanical breaking (anomaly) of the dilation-scaling symmetry, usually by means of a cutoff introduced into the quantum dynamics.

Once one formally introduces a cutoff into the classical dynamics as well, the dilational symmetry is explicitly broken and semiclassical quantization becomes possible. The primary drawback of the cutoff scheme is that the boundary conditions associated with the cutoff cannot be uniquely determined for three-body dynamics in three dimensions, as opposed to one-body dynamics, or three-body dynamics in one dimension. Indeed, a practical implementation of the short-hyperradius cutoff in three-body classical dynamics remains undefined. This is a fundamental problem of three-body physics in general [24,25], not merely of Efimov physics. However, strictly periodic orbits remain unaffected by the cutoff, so some other class of orbits must be the basis of semiclassical quantization.

We turned to adiabatically shrinking quasiperiodic orbits for this purpose and showed that they satisfy all requirements. Indeed, this scenario can be tested, by way of decay properties of Efimov states: Our adiabatically shrinking quasiperiodic orbits ultimately collapse to the size of a point, or of the cutoff, which entails finite lifetimes, or quan-

[11] This recipe is neither unique nor a good assumption; it demands that the c.m. variables behave differently from the internal ones.

[12] However, in view of Efimov's own description [5] of his three-body states as floppy, as well as in quantum calculations, this particular assumption seems rather inappropriate. There may exist other ways of continuing the time evolution after the impact of the hard wall (reversal of motion of two particles, with the third one continuing in the same direction), but we do not consider them here.

[13] "The widths of the deeper Efimov states are proportional to their binding energies, which behave asymptotically like Eq. (9). $E_T^{(n)} \to (e^{-2\pi/s_0})^{n-n_0}\hbar^2\kappa_*^2/m$ as $n \to +\infty$, with $a = \pm\infty$. This geometric increase in the widths of deeper Efimov states has been observed in calculations of the elastic scattering of atoms with deep dimers [39]," where Ref. [39] in [9] is our [26]. Of course, one ought to keep in mind the fact that the decay of Efimov states in Ref. [9] is enabled by introducing a complex three-body parameter $\theta_* \to \theta_* + i\eta_*$ and in Ref. [26] two-body potentials with deep bound states are considered. These deep bound states provide the decay channels for Efimov states. Both scenarios go beyond the Efimov effect as discussed in Ref. [2].

tum mechanically speaking half-widths. These semiclassical widths compare favorably with the full quantum-mechanical ones [26,27].

For most of the lifetime of a Efimov state, its size is close to the maximum, the collapse of the triangle subtended by three monomers occurring fairly suddenly. The narrowing of widths with increasing n indicates longer lifetimes and a general slowdown of Efimov dynamics of the higher-lying and consequently also spatially larger states. In the $n \to \infty$ limit, the Efimov states become absolutely static and grow to macroscopic sizes. In this limit, the quasiperiodic adiabatically shrinking orbits that correspond to this Efimov state would stop moving entirely.[14]

[14]This is not in conflict with Heisenberg's uncertainty principle because the distances are macroscopic.

One theoretical challenge for the future is to construct a viable set of boundary conditions for three bodies at short distances and then to implement them in a numerical calculation. Another one would be to understand the relation between the high-n states and the quantum-mechanical condensate of the surrounding Bose gas. Finally would be the challenge to find higher-n states in experiment and to study their spatial extent and temporal evolution or decay.

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