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# State-dependent motional squeezing of a trapped ion: Proposed method and applications

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We show that the motion of a cold trapped ion can be squeezed by modulating the intensity of a phase-stable optical lattice placed inside the trap. The method we propose is reversible (unitary) and state selective: it effectively implements a controlled-squeeze gate. This resource could be useful for quantum information processing with continuous variables. We show that the controlled-squeeze gate can prepare coherent superpositions of states which are squeezed along complementary quadratures. Furthermore, we show that these states, which we denote "$\mathcal{X}$ states," exhibit a high sensitivity to small displacements along two complementary quadratures, which makes them useful for quantum metrology.

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Cold trapped ions are one of the leading platforms for quantum simulations [1,2], quantum metrology [3–5], and quantum information processing [6–10]. In these devices, the preparation, manipulation, and control of quantum states of internal (spin) and motional degrees of freedom play a central role. In this context, historical benchmarks have been achieved, such as the preparation and detection of Schrödinger cat states [11] and of squeezed states [12] and the characterization of their decoherence [13]. Squeezing, an extremely valuable resource for quantum metrology [4,5,14] and information processing [15–17], has been generated in trapped ions using various methods. Here, we present an alternative one which has three main features: it is reversible, is state selective, and can generate large squeezings.

Before describing our idea we review some aspects of the existing methods. In a seminal paper [12], Wineland and coworkers demonstrated the generation of a squeezed state of motion by irradiating an ion with a pair of Raman beams. This scheme is a variation of an older idea [18] that was later expanded by Home and coworkers to generate and characterize families of squeezed sates [5,19]. These methods are based on the fact that squeezing is generated when an atom is placed in a potential modulated at twice the trapping frequency. This can be achieved with "traveling standing waves" [5,12] or aided by dissipation as a special kind of environmental engineering [19]. More recently Wineland and coworkers implemented a method [4] originally proposed in [20]. In this case, the squeezing is induced by a temporal modulation of the trapping potential. The procedure they use is reversible. The squeezing induced by a certain modulation can always be undone by applying a second, appropriately chosen, temporal driving. Using this tool, a small displacement was amplified by interposing it between a squeezing and an antisqueezing operation.

Here, we present a strategy that extends the above ones. The main idea is to place the ion in a valley (or a crest) of an optical lattice (OL) with a time-varying intensity. This generates a time-varying potential that depends on the internal state of the ion, allowing us to squeeze the ion's motion in a state-selective way. Implementing this method requires control of the absolute phase of the lasers forming the OL. The ability to do this with an accuracy of better than 2% of the lattice spacing was demonstrated recently [21] by actively stabilizing the relative phase of the OL by monitoring the ac-Stark shift that the same OL generates on the ion. In turn, the state dependence of the lattice potential can be obtained by a combination of standard OL techniques [22] and the idea of electronic shelving [23].

We show how to use this to construct a "control-squeeze" (C-Sqz) gate. Our method provides a tool for quantum information processing protocols with continuous variables [15] with ion traps. We also show how this gate can be used to prepare a class of non-Gaussian states defined as coherent superpositions of states with squeezing along complementary quadratures. We denote them "$\mathcal{X}$ states" because their Wigner function is positive in an $\mathcal{X}$-shaped region formed by two squeezed ellipses oriented at 90° from each other (these Wigner functions display significant oscillations between the positive-valued regions). As we show here, $\mathcal{X}$ states may be useful for quantum metrology, as they are highly sensitive to small displacements along pairs of complementary quadratures (such as position and momentum). This kind of state has been discussed theoretically [24,25]. Also, methods to generate them in ion traps by driving of multiple sidebands or by dissipative engineering have been proposed [26,27].

We begin by recalling that a squeezed state, $|\xi\rangle$, of a harmonic oscillator is such that $|\xi\rangle = S(\xi)|0\rangle$, where $|0\rangle$ is the ground state and $S(\xi)$ is the squeezing operator $S(\xi) = \exp((\xi^* a^2 - \xi a^{\dagger 2})/2)$. Here, $a^\dagger$ and $a$ are creation and annihilation operators satisfying $[a, a^\dagger] = 1$, and $\xi = re^{i\theta}$ defines the degree of squeezing $r$ and the quadrature $\theta$

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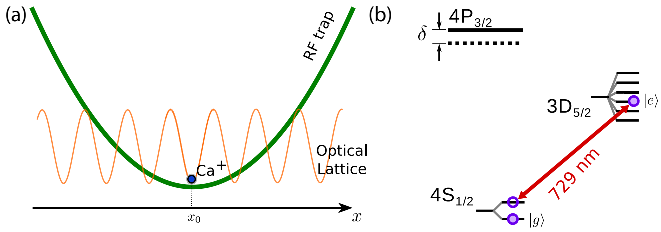


FIG. 1. (a) Optical lattice and harmonic trapping potential; (b) electronic level scheme for $^{40}Ca^+$. The phase of the OL is stabilized so that one of its minima coincides with the minimum of the trapping potential. The OL has $\lambda_{\rm OL} = 854$ nm and is visible only when the ion is in state $|e\rangle$. State-dependent squeezing is induced by modulation of the intensity of the OL.

along which the state is squeezed. The main features of a squeezing operation follow from the transformation law: $a' \equiv S^\dagger(\xi) a S(\xi) = \cosh(r) a - e^{i\theta}\sinh(r) a^\dagger$. For example, for $\theta = 0$, we have $(a' \pm a'^\dagger) = e^{\mp r}(a \pm a^\dagger)$. This shows that squeezing produces exponential stretching and exponential contraction along complementary quadratures. The expansion of $|\xi\rangle$ in terms of energy eigenstates $|n\rangle$ involves only even values of $n$ and reads

$$|\xi\rangle = \frac{1}{\sqrt{\cosh(r)}} \sum_{m\geqslant 0} \frac{\sqrt{2m!}}{m! 2^m} (-\tanh(r) e^{i\theta})^m \; |2m\rangle. \quad (1)$$

Squeezing can be generated by varying the frequency of a harmonic oscillator. We propose to squeeze the motion of a trapped ion by varying the effective trapping frequency with an optical lattice. The OL can be created by the interference of a single laser beam reflected back onto itself [22]. The basic ingredients, similar to the ones used in [21], are shown in Fig. 1. We analyze here the case of $^{40}Ca^+$ ions, whose relevant internal states are shown in Fig. 1, but our protocol is easily applicable to other ions. We use an optical qubit [28] and choose state $|g\rangle$ as one of the Zeeman sublevels of $^4S_{1/2}$ and state $|e\rangle$ as one of the states in the $^3D_{5/2}$ manifold. The OL will be generated with a laser detuned from the $^3D_{5/2} \leftrightarrow {}^4P_{3/2}$ transition, whose wavelength is close to $\lambda_{\rm SW} = 854$ nm. We consider typical detunings of a few GHz with respect to transitions with line widths of a few MHz. In this situation the field generates an ac-Stark shift mainly for state $|e\rangle$, because of its coupling to the $^4P_{3/2}$ manifold. Conversely, the transitions from $|g\rangle$ to other levels are far detuned, and as a consequence, this state remains mostly unaffected by the OL. Thus, the Hamiltonian is the sum of the contribution of the harmonic trap plus that of the ac-Stark shift generated by the OL [29]. It reads

$$H_{\rm OL} = \frac{p^2}{2m} + \frac{1}{2} m\omega_T^2 x^2 + |e\rangle\langle e| \otimes V_0 \sin^2(k_l x + \Phi). \quad (2)$$

Here, $\omega_T$ is the trapping frequency and $V_0 = \frac{\hbar\Omega_R^2}{4\delta}$ is the ratio between the Rabi frequency, $\Omega_R$, and the detuning $\delta$ ($k_l = 2\pi/\lambda$ is the lattice wave vector and $\Phi$ is a phase that determines the position of the lattice minima with respect to the trapping potential). As discussed above, one can actively lock the phase to $\Phi = 0$ and prepare a state well localized near $x = 0$, where both the OL and the trapping potential have a common minimum. By modulating the laser intensity (which determines $\Omega_R$) we introduce an explicit time dependence in the Hamiltonian [that is, $V_0$ can be transformed into $V_0(t)$].

We now analyze the evolution operators for the system assuming a harmonic driving of the form $V_0(t) = \hbar\epsilon(1 - \sin(\omega_d t - \theta))$. When the ion is in state $|g\rangle$, it interacts only with the harmonic trap and its Hamiltonian is $H_g^{(0)} = p^2/2m + m\omega_T^2 x^2/2$. On the other hand, when the state is $|e\rangle$ the ion sees the OL. For an initial state whose wave packet is concentrated near $x \approx 0$, the OL can be approximated by a quadratic potential and the Hamiltonian can be written as $H_e = H_e^{(0)} + H_e^{(\rm int)}$, where $H_e^{(0)} = p^2/2m + m\omega_e^2 x^2/2$ (with the rescaled trapping frequency $\omega_e = \sqrt{\omega_T^2 + 2k_l^2\hbar\epsilon/m}$) and $H_e^{(\rm int)} = -\hbar G \sin(\omega_d t - \theta) x^2/\sigma_e^2$, where the driving amplitude is $G = \epsilon k_l^2 \sigma_e^2$ and $\sigma_e = \sqrt{\hbar/m\omega_e}$ (this approximation is valid in the Lamb Dicke limit where $\sigma_e \ll \lambda_{\rm OL}$, and thus $\eta \equiv k_l^2\sigma_e^2 \ll 1$).

In parametric resonance, when $\omega_d = 2\omega_e$, squeezing is generated in a constructive way. To prove this, we write the Hamiltonian $H_e$ in the interaction picture with respect to $H_e^{(0)}$ (and in the rotating wave approximation) as $\tilde{H}_e = i\frac{\hbar G}{4}(a_e^2 e^{-i\theta} - a_e^{\dagger 2} e^{i\theta})$. Here $a_e$ is the canonical operator associated with $H_e^{(0)}$, which can be interchanged with $a_g$ [that of $H_g^{(0)}$] in the Lamb Dicke limit [since $a_e \approx a_g + O(\eta)$]. Then the evolution operator for a time $T$ is $\tilde{U}(T) = S(\frac{GT}{2} e^{i\theta})$, which would generate a squeezed state with a degree of squeezing that increases linearly with time. Clearly, this squeezing can be inverted by an identical modulation with a temporal phase $\pi - \theta$.

To benchmark our approximation we numerically [30] solved the Schrödinger equation considering the full nonlinearity of the OL potential and avoiding the rotating wave approximation invoked above, i.e., the Hamiltonian of Eq. (2). The results, shown in Fig. 2, agree with the above analysis: the overlap between the numerical state and the squeezed state with $\xi = Gt/2$ stays close to unity even for small variations of the phase $\Phi$ consistent with experimentally achievable errors (which can be as small as 2%). After nearly 100 driving periods, $r$ becomes of order unity for a modulation amplitude $\epsilon \approx \omega_T$ ($\epsilon > \omega_T$ was attained in [31] for an OL on an S-P transition in calcium).

Using the above results we can write the following expression for the full evolution operator [in the interaction picture associated with $H_g^{(0)}$],

$$U(T) = |g\rangle\langle g| \otimes I + |e\rangle\langle e| \otimes U_{ge}\, S(\xi_T), \quad (3)$$

where $\xi_T = \frac{GT}{2} e^{i(\theta)}$ [note that the operator $U_{ge} = U_g^\dagger(T) U_e(T)$ maps the interaction picture associated with $H_e^{(0)}$ into that of $H_g^{(0)}$]. This shows that by modulating the OL we implement a controlled-squeezing operation, which we denote C-Sqz$(r,\theta)$: the motional state does not change when the internal state is $|g\rangle$, while it is squeezed by $S(\xi)$ when the internal state is $|e\rangle$ (where $\xi = re^{i\theta}$).

Let us now show how to use C-Sqz$(r,\theta)$ to prepare a special class of non-Gaussian states. When the internal state is either $|e\rangle$ or $|g\rangle$, this operator is Gaussian. However, when combined with rotations of the internal state it can be used

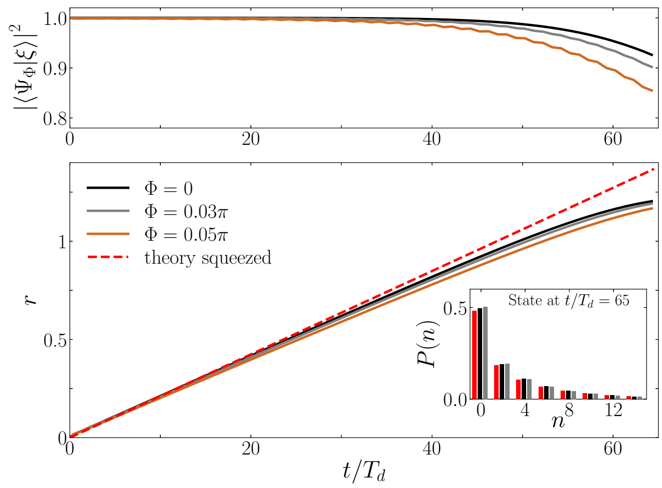


FIG. 2. The squeezing of an ion caused by the modulation of the intensity of an optical lattice (OL) of the form $V_0(t) = \hbar\omega_T[1 - \sin(2\omega_e t)]$. The relative position of the OL minimum and that of the trap is controlled by the absolute phase $\Phi$, which is locked to $\Phi = 0$ up to experimental errors whose effects are shown. The obtained state coincides with the one predicted by theory, which is $|\xi\rangle$ with $\xi = Gt/2$. Deviations are seen, as expected, for long times (when the nonlinearity of the OL potential becomes significant) or for large values of $\Phi$. Errors induce the decay of the overlap with the ideal state and deviations from the predicted phononic population $P(n)$.

to prepare highly non-Gaussian states, which are an essential requirement to achieve universality in quantum computation with continuous variables. We prepare even and odd $\mathcal{X}$ states, which are defined as the superposition of squeezed states $|\mathcal{X}_\pm\rangle = N_\pm(|\xi\rangle \pm |\xi e^{i\pi}\rangle)$, with the normalization constant $N_\pm = 1/\sqrt{2 \pm 2/\sqrt{\cosh(2r)}}$.

These states can be prepared using the following six-step protocol (we consider $\theta = 0$, i.e., $\xi = r$): (i) Prepare the ion in the motional ground state and in a balanced superposition of the internal state, $\frac{1}{\sqrt{2}}(|g\rangle + |e\rangle) \otimes |0\rangle$; (ii) apply a C-Sqz(r,0) and generate the state $\frac{1}{\sqrt{2}}(|g\rangle \otimes |0\rangle + |e\rangle \otimes U_{ge}|r\rangle)$; (iii) perform a $\pi$ rotation in the internal state to obtain $\frac{1}{\sqrt{2}}(|e\rangle \otimes |0\rangle + |g\rangle \otimes U_{ge}|r\rangle)$; (iv) apply C-Sqz$(r, \pi)$ to obtain $\frac{1}{\sqrt{2}}(|e\rangle \otimes U_{ge}|-r\rangle + |g\rangle \otimes U_{ge}|r\rangle)$; (v) perform a $\pi/2$ rotation in the internal state to obtain $\frac{1}{2}(\frac{1}{N_+}|e\rangle \otimes U_{ge}|\mathcal{X}_+\rangle + \frac{1}{N_-}|g\rangle \otimes U_{ge}|\mathcal{X}_-\rangle)$; and (vi) measure the internal state and obtain either $|e\rangle$ or $|g\rangle$. These two results appear, respectively, with probabilities $1/4N_\pm^2$. In each case, the motional state [in the interaction picture of $H_e^{(0)}$] is either $|\mathcal{X}_+\rangle$ or $|\mathcal{X}_-\rangle$. Note that for large $r = GT/2$ both states are equally likely but for smaller values of $r$ the even state (which is a superposition of $n = 0, 4, 8, \ldots$ states) is much more likely to be prepared than the odd state (which contains only $n = 2, 6, 10, \ldots$).

The even and odd $\mathcal{X}$ states have very interesting metrological properties. To visualize them, we analyze their Wigner function $W(\alpha)$. In fact, as the $\mathcal{X}$ states have well-defined parity, $W(\alpha)$ is proportional to the simpler characteristic function $W(\alpha) \propto C(2\alpha)$. The characteristic function is defined as $C_\pm(\alpha) = \langle\mathcal{X}_\pm|D(\alpha)|\mathcal{X}_\pm\rangle$, where $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is

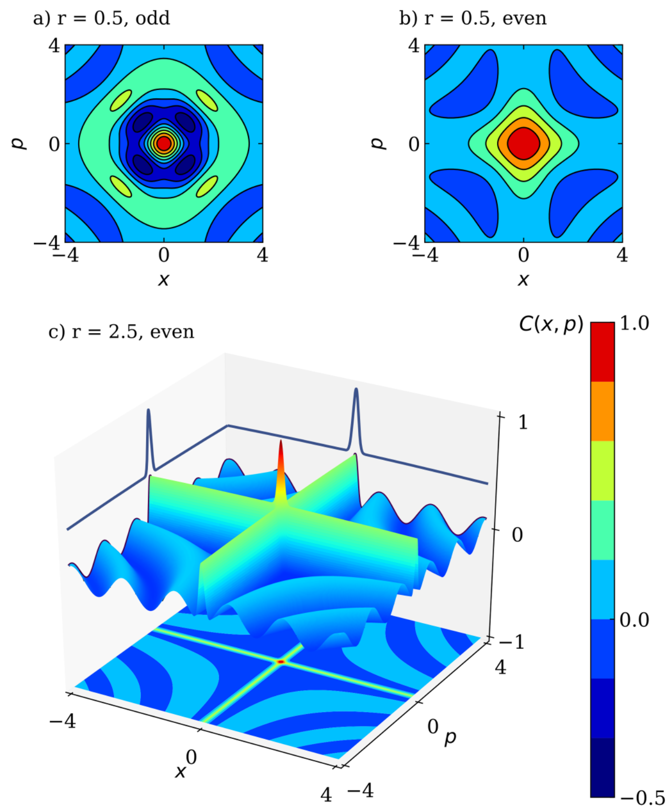


FIG. 3. Characteristic function for $\mathcal{X}$ states. Results for (a) odd and (b) even $\mathcal{X}$ states for low squeezing ($r = 0.5$) where the $\mathcal{X}$ shape is barely visible. For the odd state $\mathcal{X}_-$ oscillations close to the origin with negative values are seen. These make this state orthogonal to the ground state. For higher squeezing, as in (c), where $r = 2.5$, the $\mathcal{X}$-shaped region is clearly visible. Also, one sees that $C(x, p)$ rapidly decays close to the origin, where it becomes 0 for displacements along the diagonals with a magnitude that scales as $e^{-r}$.

a displacement operator. Using $\alpha = x + ip$, we found that $C(\alpha)$ can be written as

$$W_\pm(\alpha/2) = C_\pm(x, p) = 2N_\pm^2\left(e^{-\frac{(x^2+p^2)\cosh(2r)}{2}}\cosh\left(\frac{x^2 - p^2}{2}\sinh(2r)\right) \pm \frac{e^{-\frac{(x^2+p^2)}{2\cosh(2r)}}}{\sqrt{\cosh(2r)}}\cos\left(xp\tanh(2r)\right)\right). \quad (4)$$

The first term on the right-hand side is the sum of the two direct terms, while the second term brings about the interference and the oscillations. In Fig. 3 we display $C(\alpha)$ for $\mathcal{X}$ states with small and large squeezing. For large $r$ the result is simple to interpret: the $\mathcal{X}$ region becomes exponentially large and narrow, extending along the two main quadratures [with a high peak in the origin, where $C_\pm(0) = 1$]. The oscillations are oriented along hyperbolas located in the four quadrants.

The behavior of $|\mathcal{X}_-\rangle$ is remarkable: As this state is orthogonal to the ground state, $C_-(\alpha)$ decays very rapidly for small $|\alpha|$ and becomes negative inside the unit circle (which defines the Gaussian support of the ground state). Using Eq. (4) we find its zeros. These indicate the displacements that are required to transform $|\mathcal{X}_-\rangle$ into an orthogonal state. Evaluating

$C_-(x,p)$ on the diagonal lines defined by the equation $x^2 = p^2$, we find that $C_-(x,p) = 0$ iff $\sqrt{\cosh(2r)}e^{-\frac{x^2\sinh^2(2r)}{\cosh(2r)}} = \cos(x^2\tanh(2r))$. For large $r$ the solution to this equation is close to $x^2 \approx re^{-2r}$. This shows the extreme sensitivity of $|\mathcal{X}_-\rangle$ to small displacements along both main diagonals. On the contrary, for the $\mathcal{X}_+$ state, $C_+$ vanishes only for $x \approx 1$. Similarly, the behavior along the two main quadratures (where either $x = 0$ or $p = 0$) is such that when $x \approx e^{-r}$, $C(\alpha)$ rapidly decays to half its maximum value and stays constant (until large values of $x$ are reached). This is different from the behavior of an ordinary squeezed state, where there is a fast decay along one axis ($x$) and a slow decay along the other one.

Finally, we note that one could also conceive $\mathcal{X}$ states with more than a single ion. With two ions, for example, they are superpositions of two-mode states which are squeezed along complementary quadratures. To generate them we need a C-Sqz for two modes, which can be implemented by extending the previous idea. For example, we can trap two ions in a linear Paul trap and orient the OL so that it affects the motion along one of the transverse directions, illuminating both ions at once (the two ions are placed at common minima of the trapping and OL potentials). Then by beating the laser intensity with two frequencies that excite the parametric resonances of both normal modes we would generate state-dependent squeezing of the two modes at once. In particular, a two-mode squeezed state would be created when the two beating signals are dephased by $\pi$. This is a generalized C-Sqz gate for two modes that can be used to build a simple sequence of operations that would create a generalized $\mathcal{X}$ state (we omit this sequence and simply mention that such states are superpositions of two EPR states with complementary properties).

We have presented a method to create state-dependent squeezing and explored one application: the generation of displacement-sensitive non-Gaussian $\mathcal{X}$ states. Such tools are critical in the development of new techniques which will allow the use of the motional modes of trapped ions for metrological applications as well as for realizing quantum information protocols with continuous variables.

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