


Dynamics of the mean-field-interacting quantum kicked rotor

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We study the dynamics of the many-body atomic kicked rotor with interactions at the mean-field level, governed by the Gross-Pitaevskii equation. We show that dynamical localization is destroyed by the interaction and replaced by a subdiffusive behavior. In contrast to results previously obtained from a simplified version of the Gross-Pitaevskii equation, the subdiffusive exponent does not appear to be universal. By studying the phase of the mean-field wave function, we propose an alternative approximation that describes correctly the dynamics at experimentally relevant times close to the start of subdiffusion, while preserving the reduced computational cost of the former approximation.

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I. INTRODUCTION

Ultracold quantum gases are versatile tools to simulate a great variety of condensed-matter systems [1]. In many cases, the use of *equivalent models*, i.e., alternative models that can be mapped onto a desired system, constitutes a promising route, at lower experimental or computational expense, to a better understanding of the underlying physics. A well-known illustration is Floquet engineering, which relies on the careful design of time-periodic systems whose stroboscopic evolution is governed by an effective static Hamiltonian featuring the desired properties [2,3]. Floquet engineering allows mimicking to a high degree of details the physics of quantum disordered systems. In this context, the atom-optics realization [4] of the quantum kicked rotor (QKR) [5,6] has proven to be an almost ideal quantum simulator [7]. The associated dynamics is known to display dynamical localization, which is the analog of Anderson localization in momentum space [8], and the QKR can, in fact, be rigorously mapped onto a one-dimensional (1D) Anderson model [9]. Moreover, by adding to the kick amplitude a temporal dependence made of $d - 1$ frequencies incommensurate with the kick frequency T_1^{-1} , one obtains the quasiperiodic kicked rotor which maps to an Anderson model in d dimensions [10–12]. Thanks to its experimental and conceptual simplicity, the QKR has been widely used to investigate Anderson-like physics experimentally: observation of Anderson transition [11], characterization of critical properties [12–15], critical dimension localization [16], or other universality classes [17].

A challenging question concerns many-body effects on Anderson localization [18,19]. This question arises in a wide class of condensed-matter systems, from disordered superconductors [20,21] to superfluid ^4He in porous media [22]. In addition, the interplay between disorder and interactions is known to give rise to nontrivial collective behavior, possibly underlying complicated many-body phase transitions [23–30]. In the more restricted frame of bosonic systems

in the mean-field interacting regime where interactions are simply taken into account by a quadratic nonlinearity in the Gross-Pitaevskii equation (GPE) [31], previous studies of the mean-field Anderson model have predicted that the localized regime should be replaced by a subdiffusive phase [32–35]. In three dimensions (3D), however, the full numerical simulation of the mean-field Anderson model implies a very heavy computational cost. Hence, the QKR could constitute a promising equivalent model to circumvent this issue; yet, it is still unclear whether its equivalence with the corresponding Anderson model holds in the presence of interactions since the latter are local in position space whereas localization occurs in momentum space.

The mean-field interacting bosonic QKR is modeled by a GPE [36], see Eq. (1) below. The nonlinear term in GPE prompts the appearance of *quasiclassical* chaotic behaviors, i.e., chaos related to a sensitivity to initial conditions, a subject that also attracted much interest in various kinds of degenerated quantum-gas systems both kicked [37–40] and nonkicked [41–47]. For the interacting quantum kicked rotor, the dynamics described by the GPE has been studied in various regimes [36,48–53]. However, a systematic study of the effects of the interplay between the interactions and the kicking strength on the long-term dynamics of the exact GPE is thus far lacking due to strong numerical instabilities and computer resource costs. For these reasons, the long-time dynamics of the interacting QKR has, instead, mostly been studied with an uncontrolled approximation [that we dub the *local momentum approximation* (LMA), see below] [33,35,36,54]. The latter predicts a subdiffusion of the kinetic energy $\langle p^2(t) \rangle \propto t^\alpha$ ($\alpha \sim 0.3$ – 0.4), thus, implying a destruction of dynamical localization, which corresponds to $\alpha = 0$. Furthermore, the exponent α is similar to that found in the 1D interacting Anderson model at the mean-field level (the equivalence between the QKR and the Anderson model being preserved in the frame of the LMA) and is expected to be universal.

There is, thus, a need to study the *exact* GPE of the interacting QKR to ensure that this subdiffusion is, indeed, present and to test the universality of the subdiffusion exponent. In this paper, we study the asymptotic behavior of the GPE at very long times (up to 10^5 kicks) and show that, although our data are compatible with subdiffusion, they are not described by a universal exponent. We offer an improved approximate dynamics which is numerically shown to be a better approximation of the full GPE approximation for experimentally accessible times (about 10^3 kicks).

II. THE NONLINEAR QUANTUM KICKED ROTOR

We consider a degenerate boson gas in a ring of circumference L_{\parallel} , tightly confined in the transverse direction with a characteristic energy scale $\hbar\omega_{\perp}$ and transverse dimension $L_{\perp} \ll L_{\parallel}$ such that this energy is much larger than any other energy scale in the system. In such conditions, the dynamics of the transverse degrees of freedom is effectively frozen, and the longitudinal dynamics is essentially one dimensional. The mean-field bosonic QKR wave-function $\psi(x, t)$ is governed by the GPE,

$$i\hbar \partial_t \psi = -\hbar^2 \frac{\partial_x^2 \psi}{2} + g|\psi|^2 \psi + K \cos(x) \sum_{n \in \mathbb{Z}} \delta(t - n) \psi, \quad (1)$$

where K is the kick amplitude, proportional to the optical potential created by a pulsed standing wave of wave-number k_L , time is expressed in units of the interval T_1 between two kicks, lengths are in units of $(2k_L)^{-1}$, and the effective Planck constant is $\hbar k = 4\hbar k_L^2 T_1 / M$ with M as the atom mass [12]. The dimensionless 1D nonlinear coupling constant in such units is given by

$$g = \pi \hbar^2 k_L a \frac{\hbar}{\omega_R M L_{\perp}^2} N = \frac{\hbar^2}{2} k_L a \frac{\omega_{\perp}}{\omega_R} N, \quad (2)$$

where the last equality is valid for a harmonic transverse confinement, see, for instance, Ref. [55], a is the (3D) s -wave scattering length of the contact interaction, N is the number of atoms, and $\omega_R = \hbar k_L^2 / (2M)$ is the recoil frequency. With typical values for potassium atoms, one obtains $g = 1$ for $L_{\perp} = 5 \mu\text{m}$ (or, equivalently, $\omega_{\perp} / 2\pi = 62 \text{ Hz}$) and $N \sim 1600$.

We use periodic boundary conditions $\psi(L_{\parallel}, t) = \psi(0, t)$ where $L_{\parallel} = 2\pi$ is the system size and normalization $\int_0^{L_{\parallel}} |\psi(x, t)|^2 dx = 1$. Because of this boundary condition, the spectrum of the momentum operator $p = i\hbar \partial / \partial x$ is discrete, $n\hbar k$ with n as an integer so that the momentum space wave-function $\hat{\psi}(p)$ is given by the Fourier series,

$$\psi(x) = \frac{1}{\sqrt{L_{\parallel}}} \sum_n \hat{\psi}(n\hbar k) e^{inx}, \quad (3)$$

with normalization $\sum_n |\hat{\psi}(n\hbar k)|^2 = 1$. The initial state is chosen to be δ peaked in momentum space $\hat{\psi}(p, t=0) = \delta_{p, p_0}$, and results are averaged over the initial momentum p_0 .

We solve Eq. (1) numerically by a real-time propagation. Two classes of unconditionally stable numerical methods can be used for this type of problem: a Crank-Nicolson scheme

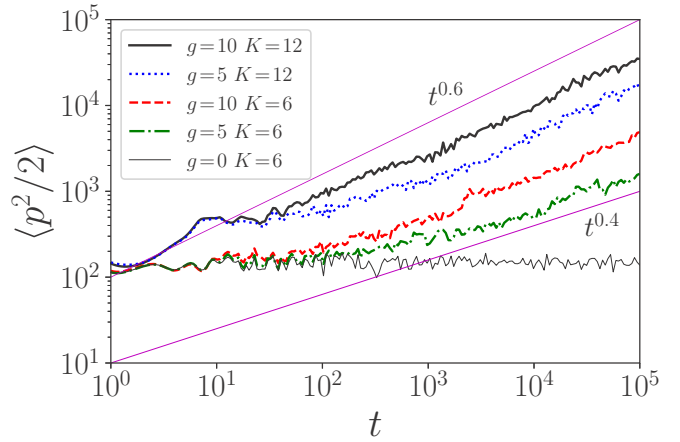


FIG. 1. Evolution of the kinetic energy in the GPE QKR for different combinations of g and K and with $\hbar k = 2.89$. The long-time dynamics reveals a universal subdiffusion with a nonuniversal subdiffusive exponent—the lines $t^{0.4}$ and $t^{0.6}$ are guides for the eye. The lowest curve is for the noninteracting kicked rotor $g = 0$ and displays dynamical localization at long times.

and a split-step method. Because of the high oscillations in space of the solution that occur as time evolves, both methods start to develop instabilities after some time, and one has to scale carefully the time step with the final time. Our numerical experiments indicate that such instabilities start developing earlier with the Crank-Nicolson method than with the split-step method. However, the split-step method also presents numerical instabilities, and long-time calculations demand higher-order split-step methods to be used. For all the results presented here, we used a second-order split-step method with a time-step $\Delta t = 10^{-4}$, which prevents numerical instability in the time range of $[0, 10^6]$ (verified by studying the convergence as a function of the time step).

The resulting evolution of the kinetic energy is displayed in Fig. 1, for various combinations of g and K (with $\hbar k = 2.89$). The curves are averaged over 10 different choices of $p_0 \in [\hbar k, 10\hbar k]$. Although the short-time dynamics is interaction independent and well described by the noninteracting QKR, the long-time one features subdiffusive behavior with a nonuniversal exponent α . As observed in the top panel of Fig. 2, this exponent generically falls in the range of $[0.4-0.8]$ depending on the values of the parameters. We estimate the uncertainty of the numerically fitted exponents to be on the order of 0.1 for each set of parameters due to the choice of the time window of the fits (typically, $t \in [10^4, 10^5]$) and to the noise of the kinetic energy, which is smaller than the range of α . There exist in the literature models predicting values for the subdiffusive exponent, relying on the nonlinearity-generated chaotic behavior, in both the nonlinear QKR and the nonlinear Anderson model [32,33,36]. Reference [36] predicts $\alpha = 2/5$ with an argument based on the Chirikov resonance overlapping criterion [5]. Using a somewhat different argument based on weak chaos, Ref. [33] found $\alpha = 1/3$. Due to the apparent nonuniversality of the subdiffusive exponent, both predictions can match numerical curves.

We, thus, confirm the existence of a subdiffusive behavior of the kinetic energy for the exact solution of the GPE but

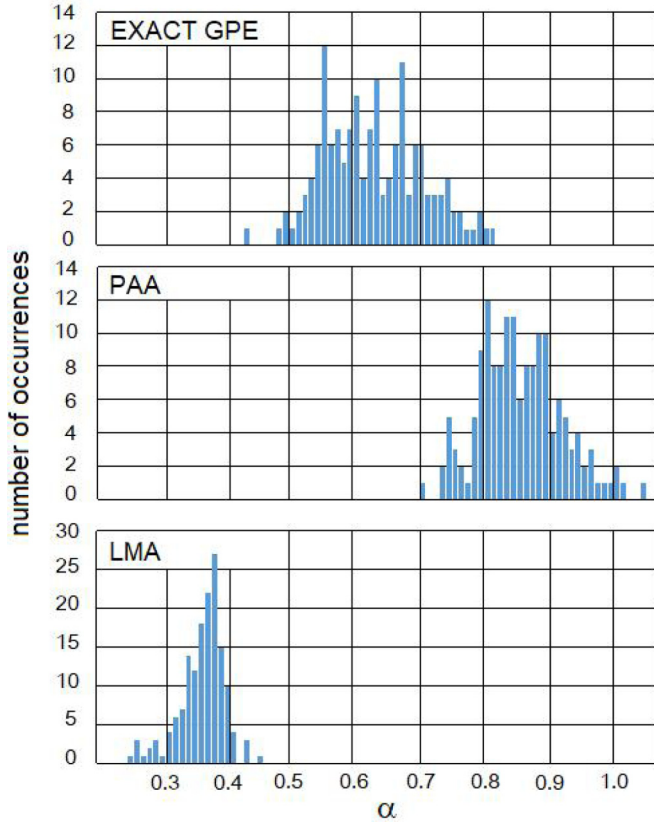


FIG. 2. Statistics of the subdiffusive exponents found for the three models studied in this paper (top panel: exact GPE; middle: phase-averaging approximation (PAA); bottom: LMA, see Sec. III). Each histogram is obtained by varying the parameters g and K in $[0, 20]$, and extracting for each parameter set a subdiffusive exponent by an algebraic fit of the kinetic energy vs time—i.e., a linear regression in a double logarithmic scale—in the asymptotic regime ($t \in [10^4, 10^5]$ for all models). The uncertainty on the exponent is about 0.1 for each model. The dispersion in each set of results suggests a nonuniversality of the subdiffusive exponent. The disagreement between the LMA and the PAA with the exact GPE shows their inability to capture correctly the asymptotic subdiffusive regime.

could not confirm the universality of the corresponding exponent. We could not find any obvious pattern in the dependence of α on the parameters k , K , and g , and a better theoretical

understanding of this dependence is beyond the scope of this paper.

III. ALTERNATIVE MODELS WITH APPROXIMATED DYNAMICS

In momentum space, the interaction term is nonlocal and given by the Fourier transform of $|\psi|^2\psi$, namely,

$$F(p, \hat{\psi}) \equiv \frac{1}{2\pi} \sum_{p_1, p_2} \hat{\psi}^*(p_1) \hat{\psi}(p_2) \hat{\psi}(p + p_1 - p_2). \quad (4)$$

In Ref. [36], a simplification was introduced neglecting all “off-diagonal” contributions, thus, making “by hand” the interaction local in momentum space, which amounts to replacing it by

$$F_{\text{LMA}}(p, \hat{\psi}) = \frac{\gamma}{2\pi} |\hat{\psi}(p)|^2 \hat{\psi}(p), \quad (5)$$

where γ could be a function of p that cannot be easily determined and that has been arbitrarily set to 1 in Ref. [36], a choice that we kept in this paper. Although not rigorously justified, this LMA significantly reduces the cost of the numerical integration since the resulting GPE can be integrated in momentum space between consecutive kicks. Moreover, the simplified equation evokes a mean-field interacting Anderson model translated in the momentum space. A rationale for the LMA is to suppose that the evolution makes the phases of the different $\hat{\psi}(p, t)$'s uncorrelated for large enough momentum differences so that the integrals in Eq. (4) are dominated by the contributions of $p_1 \simeq p_2 \simeq p$. Finding a rigorous derivation of the LMA is not obvious, but this heuristic argument can be used to construct an improved approximation, which we now describe.

Writing the wave function in the amplitude-phase representation $\hat{\psi}(p) = A(p)e^{i\phi(p)}$, our starting point is the numerical observation that, in the presence of kicks, $\phi(p)$ obtained from the full GPE solution appears to be a uniformly randomly distributed function of p . In particular, we observe numerically that replacing the phases $\phi(p)$ by independent random phases distributed uniformly in $[0, 2\pi]$, and, thus, $\hat{\psi}(p) \rightarrow \hat{\psi}_{\text{rand}}(p)$, does not change the (modulus of the) interaction functional $|F(p, \hat{\psi})| \simeq |F(p, \hat{\psi}_{\text{rand}})|$.

Denoting the average over random $\phi(p)$ realizations by an overline, we have $\overline{F(p, \hat{\psi})} = 0$ because it only involves three uncorrelated phases. For the correlations $\overline{F(p, \hat{\psi})F^*(p', \hat{\psi})}$, we have

$$\begin{aligned} \overline{F(p, \hat{\psi})F^*(p', \hat{\psi})} &= \frac{1}{(2\pi)^2} \sum_{p_1, p_2, p'_1, p'_2} A(p_1)A(p_2)A(p + p_1 - p_2)A(p'_1)A(p'_2)A(p' + p'_1 - p'_2) \\ &\quad \times \overline{\exp\{-i[\phi(p_1) - \phi(p_2) - \phi(p + p_1 - p_2) - \phi(p'_1) + \phi(p'_2) + \phi(p' + p'_1 - p'_2)]\}}. \end{aligned} \quad (6)$$

Using the independence of the phases, a straightforward calculation gives

$$\overline{F(p, \hat{\psi})F^*(p', \hat{\psi})} = \frac{\delta_{p, p'}}{(2\pi)^2} \left(4A(p)^2 + 2 \sum_{p_1, p_2} A(p_1)^2 A(p_2)^2 A(p + p_1 - p_2)^2 \right). \quad (7)$$

Furthermore, a numerical study of the phase of $F(p, \hat{\psi})$ shows that it also appears to be uniformly distributed and a calculation similar to that above shows that $\overline{F(p, \hat{\psi})\hat{\psi}^*(p')} = \frac{\delta_{p, p'}}{\pi} A(p)^2$. Therefore, the phase of $F(p, \hat{\psi})$ is strongly correlated to that of $\hat{\psi}(p)$ and can, thus, be replaced by $\phi(p)$.

This suggests the following ‘‘phase-averaging approximation’’ (PAA) of $F(p, \hat{\psi})$,

$$F_{\text{PAA}}(p, \hat{\psi}) = \frac{e^{i\phi(p)}}{2\pi} \sqrt{4A(p)^2 + 2 \sum_{p_1, p_2} A(p_1)^2 A(p_2)^2 A(p + p_1 - p_2)^2}. \quad (8)$$

The agreement between $|F_{\text{PAA}}(p, \hat{\psi})|$ and $|F(p, \hat{\psi})|$ for a typical $\hat{\psi}$ is very good as shown in Fig. 3.

Within the PAA, the GPE evolution of $\hat{\psi}(p, t)$ between kicks is replaced by

$$i\bar{k} \partial_t \hat{\psi} = \frac{p^2}{2} \hat{\psi} + g \frac{|F_{\text{PAA}}(p, \hat{\psi})|}{|\hat{\psi}|} \hat{\psi}. \quad (9)$$

The advantage of the PAA is that, as in the LMA, the amplitude of $\hat{\psi}(p)$ is conserved dynamically between the kicks, implying that $|F_{\text{PAA}}(p, \hat{\psi})|/|\hat{\psi}|$ is a constant of motion between kicks, making the integration trivial [56],

$$\hat{\psi}(p, t+1) = \exp \left[-i \left(\frac{p^2}{2} + g \frac{|F_{\text{PAA}}(p, \hat{\psi})|}{|\hat{\psi}(p)|} \right) \right] \hat{\psi}(p, t). \quad (10)$$

Figure 4 shows the dynamics obtained from the GPE, LMA, and PAA, as well as the noninteracting case. Although the short-time dynamics is interaction independent (and, thus, approximation independent), the LMA fails to capture the regime where the interactions start to be relevant, whereas this is well achieved by the PAA. This improved approximation is a faithful description of the exact GPE dynamics for the typical time range of $[0, 1000]$ accessible to state-of-the-art experiments (see Sec. IV). In particular, it captures well the timescale at which the interactions start to matter, contrary to the LMA. As already explained above and illustrated in Fig. 2, at longer, experimentally inaccessible, times, however, both the LMA and the PAA disagree with the exact integration of the GPE, probably due to the nonlinear correlations

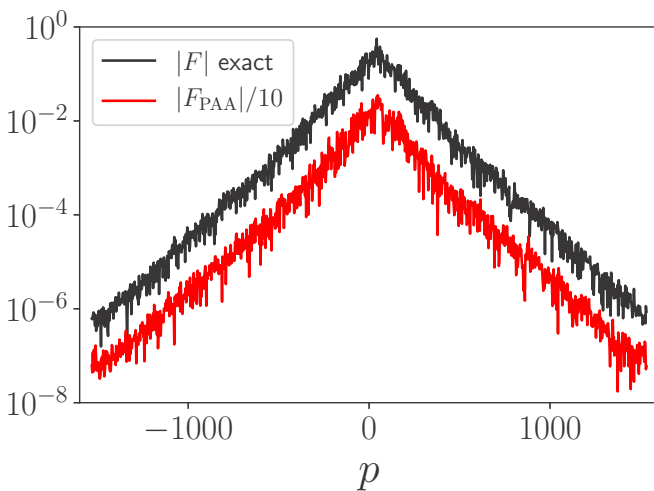


FIG. 3. $|F(p, \hat{\psi})|$, Eq. (4), computed from a solution $\hat{\psi}$ of the GPE QKR after 200 kicks with parameters $K = 12$, $g = 12$, and $\bar{k} = 2.89$ (top black curve), and its approximation $|F_{\text{PAA}}(p, \hat{\psi})|$ (bottom black curve), on a logarithmic scale. The latter curve has been shifted downwards by one decade for clarity. The agreement is very good, validating the PAA, see Eq. (8).

between the phases which start to build up. Note that we have extracted the subdiffusive exponents of the LMA and PAA, shown in Fig. 2, from the long-term dynamics in the typical time-window $t \in [10^4, 10^5]$ (data shown in the inset of Fig. 4). In particular, we recall again that none of these approximations are able to successfully capture the correct subdiffusive exponents for the GPE as is clear from Fig. 2.

IV. CAN SUBDIFFUSION BE OBSERVED EXPERIMENTALLY?

Both the interacting kicked rotor and the analogous Anderson model [32] are predicted to display a subdiffusive behavior at long times. Subdiffusion has, indeed, been experimentally observed in the closely related Aubry-André model with interactions [43]. The main limitations for observing subdiffusion in the nonlinear kicked rotor are threefold: First, quantum dynamics is limited by decoherence effects, such as spontaneous emission or stochastic fluctuations of the optical potential; second, free cold atoms fall under gravity and exit the interaction region; third, kicks induce a strong diffusion in real space that very quickly makes the atomic gas so diluted that interactions become negligible.

Decoherence can be reduced by increasing the laser-atom detuning and by carefully crafting the laser beams forming the kicking potential. Recently, an increase from a limit of ~ 200 to 1000 kicks has been obtained, allowing for the observation of the two-dimensional Anderson-like localization [16], and this limit might be increased in the near future. As shown in Fig. 1, subdiffusion can yet be observed at a few hundred kicks for a large enough nonlinearity $g \sim 10$. The use of Feshbach

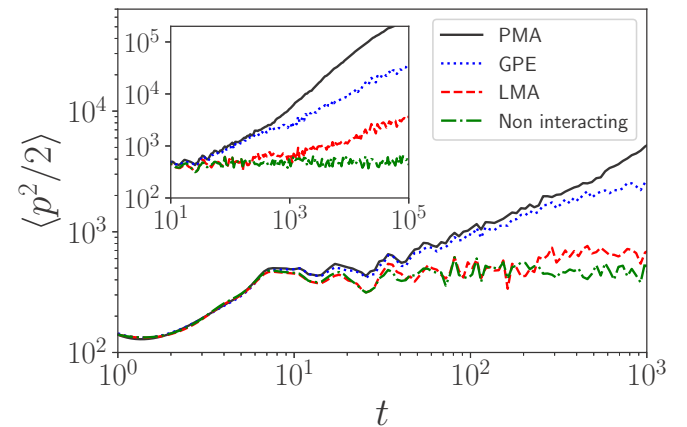


FIG. 4. Evolution of the kinetic energy as computed from the following dynamics (top to bottom): PAA, GPE, LMA and noninteracting. The parameters here are $g = 10$, $K = 12$, and $\bar{k} = 2.89$. The agreement between the exact dynamics governed by the GPE and the PAA approximation is very good up to $t = 10^3$. At a longer time, the inset shows significant deviations.

resonances allows one to easily reach even higher values of g [see Eq. (2) and the discussion following it].

It is now possible to create and exploit sophisticated atom traps in which a Bose-Einstein condensate can be created or inserted. A particularly interesting geometry is a ring trap created by Laguerre-Gauss modes. Such a ring trap can keep atoms for quite long times, allowing for a large number of kicks. Inside the ring, atoms can freely evolve in the azimuthal direction realizing an “atom rotor,” which corresponds to the original version of the kicked rotor; if the radial confinement energy $\hbar\omega_{\perp}$ is large compared to the other energy scales, the dynamics is effectively 1D. Moreover, Laguerre-Gauss modes display an azimuthal dependence $\exp(-im\varphi)$ where φ is the azimuthal angle. Hence, by combining two modes with azimuthal “quantum numbers” m and $-m$, one creates an azimuthal modulation of the form $\cos(2m\varphi)$ that can be pulsed to transfer angular momentum to the atoms, thus, realizing a genuine kicked rotor. Finally, if the ring is initially homogeneously filled with the condensate, no spatial dilution will occur as the kicks are applied. It is worth noting that the ring geometry generates a kicked rotor with true periodic boundary conditions.

We can, thus, conclude that the observation of subdiffusion is within reach of state-of-the-art experimental setups.

V. CONCLUSION

The study of the exact GPE dynamics has confirmed the breakdown of dynamical localization observed within the

LMA but invalidated the universality of the subdiffusion exponent α . Moreover, we have introduced an alternative approximation to the GPE, the PAA, which is better justified than the LMA, yet computationally as advantageous, and allows for a good description of the dynamics up to relatively long times where interactions do play a non-negligible role. However, it does not give the correct range of the subdiffusive exponent. The understanding of the breakdown of the PAA at longer times is left for future work. Explaining the variations of the nonuniversal exponent α in the exact GPE is also a very interesting line of research.

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- [1] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, *Rev. Mod. Phys.* **80**, 885 (2008).
 - [2] N. Goldman, G. Juzelinis, P. Öhberg, and I. B. Spielman, Light-induced gauge fields for ultracold atoms, *Rep. Prog. Phys.* **77**, 126401 (2014).
 - [3] A. Eckardt and E. Anisimovas, High-frequency approximation for periodically driven quantum systems from a Floquet-space perspective, *New J. Phys* **17**, 093039 (2015).
 - [4] F. L. Moore, J. C. Robinson, C. F. Bharucha, B. Sundaram, and M. G. Raizen, Atom Optics Realization of the Quantum Rotor, *Phys. Rev. Lett.* **75**, 4598 (1995).
 - [5] B. V. Chirikov, A universal instability of many-dimensional oscillator systems, *Phys. Rep.* **52**, 263 (1979).
 - [6] G. Casati, B. V. Chirikov, J. Ford, and F. M. Izrailev, in *Stochastic Behavior in Classical and Quantum Systems*, edited by G. Casati and J. Ford (Springer-Verlag, Berlin, 1979), Vol. 93, p. 334.
 - [7] C. Hainaut, A. Rançon, I. Manai, J. F. Clément, P. Szriftgiser, D. Delande, J. C. Garreau, and R. Chicireanu, Experimental realization of an ideal Floquet disordered system, *New J. Phys* **21**, 035008 (2019).
 - [8] P. W. Anderson, Absence of diffusion in certain random lattices, *Phys. Rev.* **109**, 1492 (1958).
 - [9] S. Fishman, D. R. Grempel, and R. E. Prange, Chaos, Quantum Recurrences, and Anderson Localization, *Phys. Rev. Lett.* **49**, 509 (1982).
 - [10] G. Casati, I. Guarneri, and D. L. Shepelyansky, Anderson Transition in a One-Dimensional System with Three Incommensurate Frequencies, *Phys. Rev. Lett.* **62**, 345 (1989).
 - [11] J. Chabé, G. Lemarié, B. Grémaud, D. Delande, P. Szriftgiser, and J. C. Garreau, Experimental Observation of the Anderson Metal-Insulator Transition with Atomic Matter Waves, *Phys. Rev. Lett.* **101**, 255702 (2008).
 - [12] G. Lemarié, J. Chabé, P. Szriftgiser, J. C. Garreau, B. Grémaud, and D. Delande, Observation of the Anderson metal-insulator transition with atomic matter waves: Theory and experiment, *Phys. Rev. A* **80**, 043626 (2009).
 - [13] G. Lemarié, H. Lignier, D. Delande, P. Szriftgiser, and J. C. Garreau, Critical State of the Anderson Transition: Between a Metal and an Insulator, *Phys. Rev. Lett.* **105**, 090601 (2010).
 - [14] M. Lopez, J.-F. Clément, P. Szriftgiser, J. C. Garreau, and D. Delande, Experimental Test of Universality of the Anderson Transition, *Phys. Rev. Lett.* **108**, 095701 (2012).
 - [15] M. Lopez, J.-F. Clément, G. Lemarié, D. Delande, P. Szriftgiser, and J. C. Garreau, Phase diagram of the anisotropic Anderson transition with the atomic kicked rotor: Theory and experiment, *New J. Phys* **15**, 065013 (2013).
 - [16] I. Manai, J.-F. Clément, R. Chicireanu, C. Hainaut, J. C. Garreau, P. Szriftgiser, and D. Delande, Experimental Observation of Two-Dimensional Anderson Localization with the Atomic Kicked Rotor, *Phys. Rev. Lett.* **115**, 240603 (2015).
 - [17] C. Hainaut, I. Manai, J.-F. Clément, J. C. Garreau, P. Szriftgiser, G. Lemarié, N. Cherroret, D. Delande, and R. Chicireanu, Controlling symmetry and localization with an artificial gauge field in a disordered quantum system, *Nat. Commun.* **9**, 1382 (2018).
 - [18] P. W. Anderson, Local moments and localized states, *Rev. Mod. Phys.* **50**, 191 (1978).

- [19] L. Sanchez-Palencia and M. Lewenstein, Disordered quantum gases under control, *Nat. Phys.* **6**, 87 (2010).
- [20] A. T. Fiory and A. F. Hebard, Electron Mobility, Conductivity, and Superconductivity near the Metal-Insulator Transition, *Phys. Rev. Lett.* **52**, 2057 (1984).
- [21] D. Belitz and T. R. Kirkpatrick, The Anderson-Mott transition, *Rev. Mod. Phys.* **66**, 261 (1994).
- [22] B. C. Crooker, B. Hebral, E. N. Smith, Y. Takano, and J. D. Reppy, Superfluidity in a Dilute Bose Gas, *Phys. Rev. Lett.* **51**, 666 (1983).
- [23] I. L. Aleiner, B. L. Altshuler, and G. V. Shlyapnikov, A finite-temperature phase transition for disordered weakly interacting bosons in one dimension, *Nat. Phys.* **6**, 900 (2010).
- [24] D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states, *Ann. Phys. (NY)* **321**, 1126 (2006).
- [25] P. Lugan, D. Clément, P. Bouyer, A. Aspect, and L. Sanchez-Palencia, Anderson Localization of Bogolyubov Quasiparticles in Interacting Bose-Einstein Condensates, *Phys. Rev. Lett.* **99**, 180402 (2007).
- [26] P. Lugan and L. Sanchez-Palencia, Localization of Bogoliubov quasiparticles in interacting Bose gases with correlated disorder, *Phys. Rev. A* **84**, 013612 (2011).
- [27] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, One dimensional bosons: From condensed matter systems to ultracold gases, *Rev. Mod. Phys.* **83**, 1405 (2011).
- [28] S. Lellouch and L. Sanchez-Palencia, Localization transition in weakly interacting Bose superfluids in one-dimensional quasiperiodic lattices, *Phys. Rev. A* **90**, 061602(R) (2014).
- [29] S. Lellouch, L.-K. Lim, and L. Sanchez-Palencia, Propagation of collective pair excitations in disordered Bose superfluids, *Phys. Rev. A* **92**, 043611 (2015).
- [30] E. Altman, Y. Kafri, A. Polkovnikov, and G. Refael, Superfluid-insulator transition of disordered bosons in one dimension, *Phys. Rev. B* **81**, 174528 (2010).
- [31] F. Dalfovo, S. Giorgini, L. Pitaevskii, and S. Stringari, Theory of Bose-Einstein condensation in trapped gases, *Rev. Mod. Phys.* **71**, 463 (1999).
- [32] A. S. Pikovsky and D. L. Shepelyansky, Destruction of Anderson Localization by a Weak Nonlinearity, *Phys. Rev. Lett.* **100**, 094101 (2008).
- [33] S. Flach, D. O. Krimer, and C. Skokos, Universal Spreading of Wave Packets in Disordered Nonlinear Systems, *Phys. Rev. Lett.* **102**, 024101 (2009).
- [34] T. V. Lapyteva, J. D. Bodyfelt, D. O. Krimer, C. Skokos, and S. Flach, The crossover from strong to weak chaos for nonlinear waves in disordered systems, *Europhys. Lett.* **91**, 30001 (2010).
- [35] N. Cherroret, B. Vermersch, J. C. Garreau, and D. Delande, How Nonlinear Interactions Challenge the Three-Dimensional Anderson Transition, *Phys. Rev. Lett.* **112**, 170603 (2014).
- [36] D. L. Shepelyansky, Delocalization of Quantum Chaos by Weak Nonlinearity, *Phys. Rev. Lett.* **70**, 1787 (1993).
- [37] S. A. Gardiner, D. Jaksch, R. Dum, J. I. Cirac, and P. Zoller, Nonlinear matter wave dynamics with a chaotic potential, *Phys. Rev. A* **62**, 023612 (2000).
- [38] V. S. Filho, A. Gammal, T. Frederico, and L. Tomio, Chaos in collapsing Bose-condensate gas, *Phys. Rev. A* **62**, 033605 (2000).
- [39] M. L. Chiofalo, Routes to chaos for driven Bose-Einstein condensates: From fast to slow crossing regimes, *Phys. Lett. A* **300**, 470 (2002).
- [40] A. Trombettoni and A. Smerzi, Discrete Solitons and Breathers with Dilute Bose-Einstein Condensates, *Phys. Rev. Lett.* **86**, 2353 (2001).
- [41] Q. Thommen, J. C. Garreau, and V. Zehnlé, Classical Chaos with Bose-Einstein Condensates in Tilted Optical Lattices, *Phys. Rev. Lett.* **91**, 210405 (2003).
- [42] G. P. Berman, F. Borgonovi, F. M. Izrailev, and A. Smerzi, Irregular Dynamics in a One-Dimensional Bose System, *Phys. Rev. Lett.* **92**, 030404 (2004).
- [43] E. Lucioni, B. Deissler, L. Tanzi, G. Roati, M. Zaccanti, M. Modugno, M. Larcher, F. Dalfovo, M. Inguscio, and G. Modugno, Observation of Subdiffusion in a Disordered Interacting System, *Phys. Rev. Lett.* **106**, 230403 (2011).
- [44] L. Fallani, L. De Sarlo, J. E. Lye, M. Modugno, R. Saers, C. Fort, and M. Inguscio, Observation of Dynamical Instability for a Bose-Einstein Condensate in a Moving 1D Optical Lattice, *Phys. Rev. Lett.* **93**, 140406 (2004).
- [45] C. Weiss and N. Teichmann, Differences between Mean-Field Dynamics and N-Particle Quantum Dynamics as a Signature of Entanglement, *Phys. Rev. Lett.* **100**, 140408 (2008).
- [46] Y. Kagan, A. E. Muryshev, and G. V. Shlyapnikov, Collapse and Bose-Einstein Condensation in a Trapped Bose Gas with Negative Scattering Length, *Phys. Rev. Lett.* **81**, 933 (1998).
- [47] A. R. R. Carvalho, R. L. de Matos Filho, and L. Davidovich, Environmental effects in the quantum-classical transition for the delta-kicked harmonic oscillator, *Phys. Rev. E* **70**, 026211 (2004).
- [48] F. Benvenuto, G. Casati, A. S. Pikovsky, and D. L. Shepelyansky, Manifestations of classical and quantum chaos in nonlinear wave propagation, *Phys. Rev. A* **44**, R3423 (1991).
- [49] S. Wimberger, M. Sadgrove, S. Parkins, and R. Leonhardt, Experimental verification of a one-parameter scaling law for the quantum and “classical” resonances of the atom-optics kicked rotor, *Phys. Rev. A* **71**, 053404 (2005).
- [50] T. S. Monteiro, A. Rançon, and J. Ruostekoski, Nonlinear Resonances in δ -Kicked Bose-Einstein Condensates, *Phys. Rev. Lett.* **102**, 014102 (2009).
- [51] G. Gligoric, J. D. Bodyfelt, and S. Flach, Interactions destroy dynamical localization with strong and weak chaos, *Europhys. Lett.* **96**, 30004 (2011).
- [52] I. Vakulchyk, M. V. Fistul, and S. Flach, Wave Packet Spreading with Disordered Nonlinear Discrete-Time Quantum Walks, *Phys. Rev. Lett.* **122**, 040501 (2019).
- [53] D. L. Shepelyansky, Kolmogorov turbulence, Anderson localization and KAM integrability, *Eur. Phys. J. B* **85**, 199 (2020).
- [54] L. Rebuzzini, S. Wimberger, and R. Artuso, Delocalized and resonant quantum transport in nonlinear generalizations of the kicked rotor model, *Phys. Rev. E* **71**, 036220 (2005).
- [55] T. J. Alexander, K. Heenan, M. Salerno, and E. A. Ostrovskaya, Dynamics of matter-wave solitons in harmonic traps with flashing optical lattices, *Phys. Rev. A* **85**, 063626 (2012).
- [56] Although $|F_{\text{PAA}}(p, \hat{\psi})|$ is typically broader than $|\hat{\psi}(p)|$, making $|F_{\text{PAA}}(p, \hat{\psi})|/|\hat{\psi}(p)| \gg 1$ for $|\hat{\psi}(p)| \ll 1$, this is not a problem in the numerics since the phase of $\hat{\psi}(p)$ does not matter for the dynamics in this regime.