Detecting a logarithmic nonlinearity in the Schrödinger equation using Bose-Einstein condensates

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We study the effect of a logarithmic nonlinearity in the Schrödinger equation (SE) on the dynamics of a freely expanding Bose-Einstein condensate (BEC). The logarithmic nonlinearity was one of the first proposed nonlinear extensions to the SE which emphasized the conservation of important physical properties of the linear theory, e.g., the separability of noninteracting states. Using this separability, we incorporate it into the description of a BEC obeying a logarithmic Gross-Pitaevskii equation. We investigate the dynamics of such BECs by using variational and numerical methods and find that, by using experimental techniques like δ-kick collimation, experiments with extended free-fall times as available on microgravity platforms could be able to lower the bound on the strength of the logarithmic nonlinearity by at least one order of magnitude.

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I. INTRODUCTION

Quantum theory is the most fundamental theory in physics and on the elementary level, all types of matter, as well as radiation, have to be described by it. To this day, the time evolution of quantum systems as predicted by the Schrödinger equation (SE) has been confirmed in many experiments [1,2]. Nonetheless, whether the SE can be regarded as a complete description or rather a linearized approximation of a more general theory is still an open question since, despite its great success, quantum theory has a few unresolved problems, e.g., its missing connection to general relativity and the measurement problem.

However, it is not obvious how to modify the SE in order to tackle these problems. Employed modifications are for example the generalization of the uncertainty relation [3–5], the addition of nonlinear [6] and stochastic [7] terms or higher derivatives [8] to the Schrödinger equation. Such modifications can be the result of a theory of quantum gravity in its low-energy limit (see, e.g., Ref. [9]), or may result from attempts to find a solution to the measurement problem [10]. Current research focuses more on stochastic nonlinear terms that describe the wave-function collapse or decoherence, which can be either induced spontaneously or by gravity [11,12]. A deterministic nonlinear time evolution of the wave function is nowadays rarely considered, and then usually by the inclusion of a semiclassical description of gravity as in the Schrödinger-Newton equation [13]. This can be mainly contributed to the fact that the most prominent generalization of a nonlinear SE, as described by Weinberg [6], leads to problems of locality if extended to the case of multiple entangled particles [14–16]. Even though it is conjectured that this can be concluded for all nonlinear deterministic extensions [17,18], there have also been subsequent attempts to extend existing deterministic nonlinear SEs to the case of multiple particles that do not violate locality [19,20] or have disputed the apparent violation of relativity by nonlinear quantum mechanics at all [16,21].

Two special cases of nonlinearity have received wider attention. One is the first formulation of a fundamentally nonlinear SE which emphasized the necessary separability of noninteracting states: the logarithmic Schrödinger equation (LogSE) proposed by Bialynicki-Birula and Mycielski [22]. The nonlinear time evolution is described by

$$i\hbar\frac{\partial}{\partial t}\Psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V - b\ln\alpha|\Psi|^2\right)\Psi. \tag{1}$$

The LogSE comprises terms of the ordinary SE and the nonlinear term $-b\ln\alpha|\Psi|^2$, where α is a physically irrelevant real constant (as it only leads to a global energy shift) of dimension L^3 and b is the strength of the logarithmic nonlinearity in units of energy. It keeps important properties of the linear SE, e.g., conservation of probability and norm, invariance under permutations, Galileo transforms, and more. Most importantly, it guarantees that the time evolution of product states can be separated for all times.

The second case is the Gross-Pitaevskii equation

$$i\hbar\frac{\partial}{\partial t}\Psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V + g|\Psi|^2\right)\Psi, \tag{2}$$

which describes the dynamics of a Bose-Einstein condensate (BEC). Its nonlinearity is just an effective description of scattering processes taking place in the degenerate quantum gas. As macroscopically sized quantum objects, BECs have proven to be suitable candidates for testing fundamental physics, whether it be the Schrödinger equation, general relativity, and their possible intersections [23–25]. These dedicated tests are important because they broaden the domain of application of quantum theory or might give hints about what direction to follow in search for deviations. This is especially true with regard to recent developments in long-fall-time experiments

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of BECs in microgravity [26–28] on sounding rockets [29] and in space [30–32].

In this article we study deviations due to a possible fundamental logarithmic nonlinearity in the time evolution of Bose-Einstein Condensates (BECs).

The current upper bound on the strength of the logarithmic nonlinearity stems from neutron optical diffraction experiments [33] in which a neutron wave packet's lateral evolution of $|\Psi(\vec{r},t)|^2$ diffracted on a straight edge was observed. They found that $b \leqslant 3.3e-15\,\text{eV}$.

The article is structured as follows: In Sec. II we derive an equation describing a BEC at zero temperature including the logarithmic nonlinearity. In Sec. III we investigate the dynamics of the proposed equations describing the BEC. For this, we use numerical simulations of feasible experiments that could be able to establish new upper bounds on the nonlinearity's strength. In Sec. IV we analyze possible shortcomings of our investigation and propose experiments with their respective error budgets.

II. LOGARITHMIC GROSS-PITAEVSKII EQUATION

The logarithmic nonlinearity can be incorporated into the theoretical description of a BEC by starting with basic assumptions as made in many textbook examples on Bose-Einstein condensation.

We assume that the condensates contains a large number $N \gg 1$ of bosons, so that we can approximate the field operator as a wave function $\hat{\psi}(\{\vec{r}\},t) \approx \psi(\{\vec{r}\},t)$, where the set of position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ is written as $\{\vec{r}\}$. Furthermore, we assume that all N bosons occupy the same ground state ψ and are not correlated with each other. Thus, we can write the wave function in a Hartree ansatz as $\Psi(\{\vec{r}\},t) \approx \bigotimes_{i=1}^{N} \psi(\vec{r}_i,t)$ with $\int |\Psi(\{\vec{r}\},t)|^2 d\{\vec{r}\} = N$. In the low-energy limit, the scatter process of particle i with particle j in the dilute gas ($\rho \ll a^{-3}$, where a is the s-scatter length) is described by the approximated binary interaction potential $V(\vec{r}_i,\vec{r}_j) = g\delta(\vec{r}_i - \vec{r}_j)$, with $g = \frac{4\pi\hbar^2 a}{m}$.

The Hartree ansatz enables us to take advantage of the separability property of the logarithmic nonlinearity. Using this, we obtain

$$b \ln |\Psi(\{\vec{r}\},t)|^2 = b \sum_{i=1}^{N} \ln |\psi(\vec{r}_i,t)|^2. \tag{3}$$

This makes the logarithmic nonlinearity part of the single-particle Hamiltonian. The Hamiltonian is then

$$H = \sum_{i=1}^{N} \left[-\frac{\hbar^2 \nabla_{\vec{r}_i}^2}{2m} + V(\vec{r}_i,t) - b \ln |\psi(\vec{r}_i,t)|^2 \right] + g \sum_{i<j}^{N} \delta(\vec{r}_i - \vec{r}_j). \tag{4}$$

Imposing the stationary condition

$$\delta \int L dt = 0 = \delta \left[\int \frac{i\hbar}{2} \left(\Psi^\dagger \frac{\partial}{\partial t} \Psi - \Psi \frac{\partial}{\partial t} \Psi^\dagger \right) d\vec{r} dt + \int E dt \right], \tag{5}$$

where $E = \langle \Psi | H | \Psi \rangle$, one can derive the equation governing the time evolution which is

$$i\hbar \frac{\partial \Psi(\vec{r},t)}{\partial t} = \frac{\delta E}{\delta \Psi^\dagger} \tag{6}$$

$$= \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r},t) \right] \Psi(\vec{r},t) + [-b \ln |\Psi(\vec{r},t)|^2 + g |\Psi(\vec{r},t)|^2] \Psi(\vec{r},t). \tag{7}$$

We absorbed N into the wave function, $\psi \rightarrow \sqrt{N}\Psi$, and approximate $N^2 \approx N(N-1)$ during calculation of the energy functional E. The resulting equation (7) is constituent of the usual linear kinetic- and potential-energy operators, the nonlinear terms from the Gross-Pitaevskii interaction, and logarithmic nonlinearity. We hence refer to it as the logarithmic Gross-Pitaevskii equation (LogGPE).

The LogGPE preserves all properties associated with density-dependent nonlinearities such as conservation of probability and invariance under permutation. The separability is lost due to the interaction term. Note that any nonlinearity, which is homogeneous (as required by Weinberg) or otherwise obeys the separability condition, can be incorporated in the same way as done here.

Variational solutions of the logarithmic Gross-Pitaevskii equation

Finding analytical solutions to nonlinear problems is rather difficult, therefore we only discuss the dynamics using variational methods and the system's Lagrangian.

We can use Eq. (5) to obtain an approximated result for the time evolution of a logarithmic BEC by making an ansatz for the wave function as described in Refs. [34,35]. We will assume a Gaussian-shaped wave function

$$\Psi(x,y,z,t) = \prod_{\eta=x,y,z} \frac{1}{\sqrt{2\pi\sigma_\eta^2(t)}} \times \exp\left(-\frac{(\eta - \eta_0(t))^2}{4\sigma_\eta^2(t)} + i\eta\alpha_\eta(t) + i\eta^2\beta_\eta(t) \right), \tag{8}$$

with phase terms α_η and β_η, which are proportional to the average velocity and inverse radius of curvature, respectively. The Gaussian envelope is chosen because it keeps its shape in the linear limit and is the soliton solution of the LogSE. Using this ansatz for the wave function and setting $V(\vec{r},t) = m/2(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$, one can solve the corresponding Euler-Lagrange equations for the Lagrangian $L(t,\ \eta_0(t),\ \alpha_\eta(t),\ \beta_\eta(t))$. From the resulting equations we obtain a set of coupled ordinary differential equations describing the time evolution of the Gaussian trial function's width under the influence of dispersion, the harmonic potential, and both nonlinearities:

$$\frac{\partial^2}{\partial t^2}\sigma_x(t) = \frac{\hbar^2}{4m^2}\sigma_x^{-3}(t) - \omega_x^2\sigma_x(t) + \frac{\hbar^2 Na}{4m^2\sqrt{\pi}}\sigma_x^{-2}(t)\sigma_y^{-1}(t)\sigma_z^{-1}(t) - \frac{b}{m}\sigma_x^{-1}(t). \tag{9}$$

The equations for $\sigma_y(t)$ and $\sigma_z(t)$ can be obtained by cyclic permutation $x \to y \to z \to x$. From Eq. (9) we see that the coupling between spatial dimensions stems from the Gross-Pitaevskii interaction. The logarithmic nonlinearity does not induce a coupling between different spatial dimensions due to its separability and the Hartree ansatz we made. The applied approximations, namely, binary interaction potential and separable wave function, require us to search for deviations in regimes where the gas is dilute and atomic interactions weak. In addition, the nonlinearity's contribution to the overall energy of a trapped BEC is very small in the presence of two-body interactions. Experiments searching for deviations with trapped BECs are hence not suitable, even though many features, e.g., instability conditions, are altered by the logarithmic nonlinearity. Therefore, we investigate deviations in the ballistic expansion of BECs.

III. DYNAMICS OF A LOGARITHMIC BOSE-EINSTEIN CONDENSATE

In this section we study the dynamics of a freely expanding logarithmic BEC released off a trap under typical experimental parameters. We look at the time evolution of the BEC's width after release of a spherically symmetric trap [so that $\sigma_x(t) = \sigma_y(t) = \sigma_z(t) = \sigma(t)$] and show how the width as a function of time differs due to the logarithmic nonlinearity's influence. This idea is very much alongside the lines of the last experimental tests using neutron-optical diffraction experiments in 1981 [33]. Then we discuss the application of δ-kick collimation (DKC) to magnify these effects and follow up with an error estimation of the discussed experiment.

A. Free expansion of a spherically symmetric Bose-Einstein condensate

When the LogSE was first proposed it turned out especially appealing because it allowed for Gaussian-shaped, solitonic solutions with width $\sigma_{\rm eq}$. If the wave function is of width $\sigma_{\rm eq}$, linear dispersion and nonlinear self-interaction are in equilibrium and $|\Psi(\vec{r},t)|^2$ does not change its appearance. In contrast with the linear time evolution, it leads to a spatial confinement of the wave function. Thus, we are searching for an unexpected narrowing of the matter wave packet. For the spherically symmetric case, the three equations of (9) reduce to

$$\frac{\partial^2}{\partial t^2}\sigma(t) = \frac{\hbar^2}{4m^2}\sigma^{-3}(t) - \omega^2\sigma(t) + \frac{\hbar^2 Na}{4m^2\sqrt{\pi}}\sigma^{-4}(t) - \frac{b}{m}\sigma^{-1}(t). \tag{10}$$

To explore the logarithmic nonlinearity's influence on a BEC, we numerically integrate Eq. (10) in order to obtain the width's time evolution $\sigma(t)$. We employ typical experimental parameters of a ^{87}Rb condensate [$N = 5 \times 10^4$, $a = 90a_0$, $\sigma(0) = 2.5\ \mu$m, $\dot{\sigma}(0) = 0$] and a free propagation time of 1 s. The left graph of Fig. 1 depicts $\sigma(t)$ of this BEC for different values of b.

One can see a narrowing of the BEC's width over time compared with the usual GP-energy-driven expansion due to the logarithmic nonlinearity. The blue (dashed) line corresponds to the current known upper bound $b < 3.3 \times 10^{-15}$ eV. After 1 s of free propagation it differs by $\approx$20 μm compared with the linear case ($b = 0$). The dotted (green, orange, red) lines depict the expansion for larger values of b. We chose b arbitrarily so that one can see the spatial confinement of the wave function taking place. One can see that there exists a maximum width $\sigma_{\max} = \max \sigma(t)$ which is, as we will show, not only a function of b, but also the initial conditions of the BEC [$\sigma(0)$, $\dot{\sigma}(0)$, N, and a].

The logarithmic BEC's maximum width can be calculated by using the energy functional $\langle\Psi|H|\Psi\rangle$. Inserting the Gaussian ansatz from Eq. (8),the energy per particle is

$$\frac{E(\sigma(t))}{N} = \frac{3}{2}m\dot{\sigma}^2(t) + \frac{3\hbar^2}{8m\sigma^2(t)} + \frac{3}{2}m\omega^2\sigma^2(t) + \frac{\hbar^2 Na}{4\sqrt{\pi}m\sigma^3(t)} + 3b\ln\sigma(t) + C, \tag{11}$$

where the time derivative joined the equation via the relation $m\dot{\sigma}(t) = -2\hbar\beta(t)\sigma(t)$, which is obtained by solving the Euler-Lagrange equations, the center-of-mass motion was set $\alpha_\eta(t) = 0$, and C is an arbitrary constant. As the BEC is released ($\omega = 0$), the width immediately starts to increase. Hence, energy contributions with $\sigma(t)$ in the denominator become smaller. Due to its conservation, the initial energy transfers to the $\dot{\sigma}(t)^2$-dependent term and, in the case of $b = 0$, reaches a constant value in the far field. If $b \neq 0$, the energy of the logarithmic constituent is increasing and will ultimately counter the dispersion. This leads to a contraction until, again, the pressure of Heisenberg uncertainty and two-particle interactions lead to an expansion.

Asserting that the initial energy $E(\sigma(0))$ completely transfers into the logarithmic energy contribution (at which point the BEC can only contract), we find the formula for the maximum value of $\sigma(t)$ is

$$\sigma_{\max} = \max \sigma(t) = \sigma(0)\exp\chi, \tag{12}$$

where

$$\chi = \frac{1}{3b}\left(\frac{3}{2}m\dot{\sigma}^2(0) + \frac{3\hbar^2}{8m\sigma^2(0)} + \frac{\hbar^2 Na}{4\sqrt{\pi}m\sigma^3(0)}\right). \tag{13}$$

This maximum width is indicated in the left graph of Fig. 1 by horizontal lines for the two examples for which the confinement is visible. The dimensionless parameter χ is the ratio of initial energy to b.

The right graph of Fig. 1 depicts the contour lines of the difference in width of linear and nonlinear free expansion $\sigma(t;\, b = 0) - \sigma(t;\, b = 3.3 \times 10^{-15}\ \text{eV})$ for different χ. Since b is set, we change the initial width $\sigma(0)$ for variation of χ. The values for N, a, and $\dot{\sigma}(t)$ are the same as in the simulations of Fig. 1 (left). We see that, in the regime of large χ, smaller absolute deviations of $\sigma(t)$ are found. This is due to the prevalence of interaction and kinetic energy in the system. In the vicinity of $\chi \approx 1$, we find the strongest deviations. This is the region in which we find equilibrium of logarithmic self-interaction and dispersion which results in solitonic behavior of the logarithmic BEC. When $\chi \ll 1$, the initial energy of the system consists mainly of logarithmic self-interaction energy, and the initial width is the upper bound $\sigma_{\max} \approx \sigma(0)$. In these cases the BEC does not even disperse but immediately starts

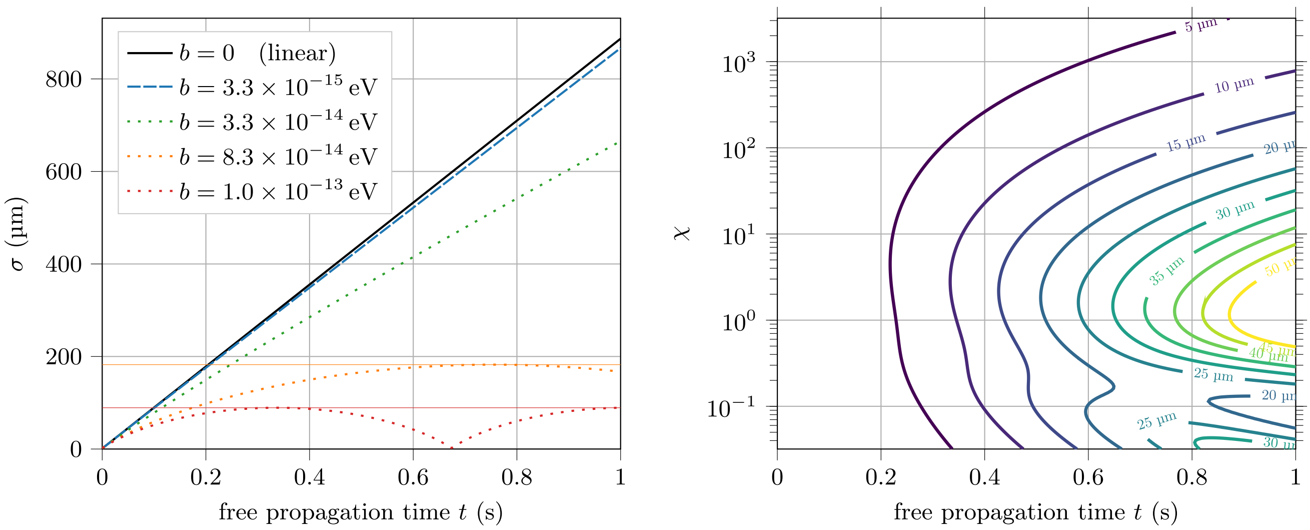

FIG. 1. (left) Numerically simulated time evolution of a spherically symmetric BEC's width $\sigma(t)$ during free propagation for different values of b. The blue (dashed) curve represents the evolution under the current known upper bound on b. The two horizontal lines show the calculated maximum width from Eq. (12) for the two examples for which spatial confinement is visible. (right) Contour lines depicting the difference in width of linear and nonlinear expansion $\sigma(t;\, b = 0) - \sigma(t;\, b = 3.3 \times 10^{-15}\ \text{eV})$ for different χ [see Eq. (13)]. The parameter χ has been varied by changing the initial width $\sigma(0)$ while keeping $N = 5 \times 10^4$, $a = 90a_0$, and $\dot{\sigma}(0) = 0$. For example, $\chi = 10^3$ corresponds to roughly 1 μm and $\chi = 10^{-1}$ to 30 μm initial width of the simulated BEC.

to contract until the contraction is countered by the resulting repulsive two-body interactions.

As shown, decreasing the initial kinetic and GP interaction energy is advantageous when searching for deviations due to a logarithmic nonlinearity. One can think of many different ways of achieving this, for example, preparing BECs with less density (smaller atom number N or larger initial width $\sigma(0)$, however this might not be favorable because high densities are an important factor in Bose-Einstein condensation), smaller a (BEC species with inherently smaller scattering length or via tuning of Feshbach resonances), and removing the BEC's kinetic energy after the initial expansion via external potentials which slow down the expansion.

The latter can be achieved, for example, by using optical or magnetic δ-kick collimation (DKC) techniques, in which a trapping potential is shortly switched on in order to counter the BEC's dispersion [36]. This magnetic or optical lens can ultimately be used to generate a collimated (minimally spreading) matter beam with low interaction energy.

B. Enhancing the logarithmic nonlinearity's effects by using δ-kick collimation

By using DKC we are able to diminish the effects due to the high initial interaction energy and decrease χ which, in turn, magnifies the logarithmic nonlinearity's effects on the BEC's dynamics. This has been similarly used in order to infer bounds on collapse models from observation of collimated cold atom clouds [37,38].

For the simulation of free expansion experiments with DKC, we choose the same parameters as in the previously discussed simulations of Sec. III A. The only difference being that, after a time $t_{\text{DK}} = 10$ ms, a harmonic DKC pulse is applied. The angular frequency ω of the harmonic potential [see Eq. (9)] is set to a value that leads to $\beta = \dot{\sigma} = 0$ (infinite radius of curvature) at the end of the DKC sequence. The DKC pulse is so short ($\Delta t = 10\ \mu$s) that we approximate it as a thin lens. This means that, in order to collimate the matter beam, $\omega^2 \Delta t \approx 1/t_{\text{DK}}$ [37]. The simulated time evolution of a BEC's width $\sigma(t)$ subjected to this DKC pulse is shown in the left graph of Fig. 2.

We see that the collimation of the BEC does indeed increase the deviations if compared with the free expansion in Fig. 1. The value of χ is on the order of unity for all examples shown. The evolution under the current upper bound on b (blue, dashed line) already differs by $\approx$60 μm after 1 s of free propagation, instead of previously $\approx$20 μm without DKC. The simulated expansions for larger values for b (dotted, green, orange, and red line) show the previously described behavior in the case of $\chi \ll 1$, where the BEC's maximum width is given by the prepared width using the DQK pulse.

The difference in width, $\sigma(t; b = 0) - \sigma(t; b)$, for these experiments is shown in the left graph of Fig. 2. It becomes apparent that current experiments of BECs in microgravity could be able to lower the bound on b, since BECs have already been generated with multiple seconds of free-fall time [26–29]. For example, the width differs by 5 μm after 1 s of free propagation, with a value of b one magnitude lower than the current limit.

C. Error estimation

We now give a brief error estimation of the proposed experiment.

According to the GPE, the width's rate of change during free expansion of a BEC reaches a constant value [see

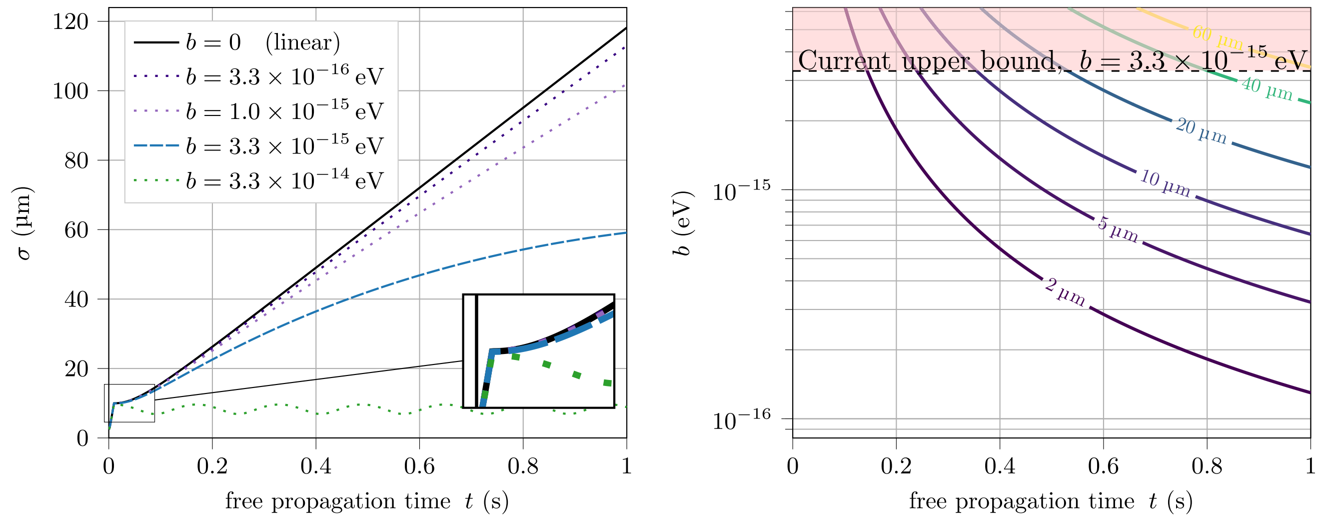

FIG. 2. (left) Numerically simulated time evolution of $\sigma(t)$ with same parameter as in Fig. 1 but with application of a DKC pulse after 10 ms of free propagation. The inset plot shows the collimation process of the matter beam. (right) Contour lines depicting the difference in width $\sigma(t;b=0)-\sigma(t;b)$ for different b of the same experiment using DKC as described for the left graph.

Eq. (11)] in the far-field $t \gg m\sigma^2(0)/\hbar$ (also $t \gg t_{\mathrm{DK}}$, so we set $t_{\mathrm{DK}}=0$ for convenient notation). The rate of expansion is

$$\dot{\sigma}^2(t \gg m\sigma^2(0)/\hbar) = \dot{\sigma}^2(0)+\frac{\hbar^2}{4m^2}\sigma^{-2}(0)+\frac{\hbar^2 Na}{6\sqrt{\pi}m^2}\sigma^{-3}(0)$$
$$= \dot{\sigma}_{\mathrm{R}}^2+\dot{\sigma}_{\mathrm{HU}}^2+\dot{\sigma}_{\mathrm{GP}}^2, \tag{14}$$

where the indices R, HU, and GP indicate the individual contributions due to residual rate of change, Heisenberg uncertainty, and the Gross-Pitaevskii interaction, respectively. We can interpret our experiment as a test of this linear expansion in the far-field. Using formula (14), we can estimate an error on the measurement of $\sigma(t)$ by acknowledging uncertainties of N, a, $\sigma(0)$, and $\dot{\sigma}(0)$. The relative error $\delta_{\dot{\sigma}(t)}=\Delta_{\dot{\sigma}(t)}/\dot{\sigma}(t)$ of the BEC's width can be estimated as

$$\delta_{\dot{\sigma}(t)} = \frac{1}{\dot{\sigma}_{\mathrm{R}}^2+\dot{\sigma}_{\mathrm{HU}}^2+\dot{\sigma}_{\mathrm{GP}}^2} \times\sqrt{\dot{\sigma}_{\mathrm{R}}^4\delta_{\dot{\sigma}(0)}^2+\left(\frac{3}{2}\dot{\sigma}_{\mathrm{GP}}^2+\dot{\sigma}_{\mathrm{HU}}^2\right)^2\delta_{\sigma(0)}^2+\frac{1}{4}\dot{\sigma}_{\mathrm{GP}}^4(\delta_a^2+\delta_N^2)}, \tag{15}$$

where δ_a, δ_N, $\delta_{\dot{\sigma}(0)}$, and $\delta_{\sigma(0)}$ are the relative errors of a, N, $\dot{\sigma}(0)$, and $\sigma(0)$ respectively. The errors of N and a are on equal footings in terms of influence on $\delta_{\sigma(t)}$. However, we see that $\delta_{\sigma(0)}$ has an especially strong impact on the overall error. This is due to the cubic impact of $\sigma(0)$ on the interaction energy and the quadratic dependence of the quantum pressure.

For example, the total estimated error $\Delta_{\dot{\sigma}(t)}$ for the simulated experiment in Fig. 2 is smaller than 0.1 $\mu\mathrm{m\,s}^{-1}$ if all relative errors are of 1%. We see that 60 μm is approximately the expected deviation in width after one second of free expansion with the current known bound. Even with a 20% relative error in N, a, and $\sigma(0)$, the error is 35 $\mu\mathrm{m\,s}^{-1}$ and, thus, after one second of free propagation, smaller then the expected deviation.

In addition to the exact characterization of the BEC's initial state which, in theory, could determine $\sigma(t)$, there are many environmental factors which could give rise to similar effects. For example, a inhomogeneous magnetic field B might lead (in first order) to an effective harmonic potential of frequency ω_B that could lead to a narrowing. In the case of the Gaussian-shaped wave function, the logarithmic nonlinearity can basically be seen as a time-dependent harmonic potential. One can calculate that the effect of a logarithmic nonlinearity and a parasitic harmonic potential are distinguishable if $\omega_B < b/\hbar \approx 1\ \mathrm{s}^{-1}$, with the current upper bound on b. This should be achievable if the BEC is prepared in a magnetic insensitive state, such that only quadratic Zeeman effects would need to be accounted for.

Furthermore, it would be important to distinguish a possible fundamental nonlinearity from an effective, nonlinear, logarithmic dynamic, for which there exist different ideas of how those could arise [39–43].

IV. SUMMARY AND DISCUSSION

In this work the feasibility of tests for a logarithmic nonlinearity in the SE using Bose-Einstein condensates was examined. Approximating the wave function in a Hartree ansatz, we proposed the logarithmic Gross-Pitaevskii equation which describes BECs governed by the LogSE. We analyzed the free expansion of the logarithmic BEC with several hundreds of milliseconds of free-fall time. As these timescales become progressively more accessible [26–32], new tests of the Schrödinger equation and the resulting wave nature of matter can yield strengthened confirmation for the exactness of quantum mechanics. However, under typical experimental parameters, the dynamics induced by two-body interactions largely outweigh those due to a possible nonlinearity. Therefore, we proposed using optical or magnetic potentials to collimate the condensate to obtain a coherent-matter wave which would propagate as described by the SE. In contrast with the linear free expansion, the LogSE spatially confines a wave packet. This effect can be measured by looking for a narrowing of the BEC's density distribution in the far-field. We have shown that available free-expansion experiments can be used to determine new bounds on b at least one order of magnitude below the current known value.

It should be noted that the procedure we followed in order to derive the LogGPE can be applied for every nonlinearity which respects the separability condition. There are several additional nonlinearities, usually proposed for dissipative and diffusion processes in open quantum systems, which have this property [39–43] and could therefore be incorporated. This is especially helpful since, as mentioned in Sec. III C, it would be necessary to distinguish processes due to residual interactions with the environment from fundamental nonlinearities.

The search for nonlinearities in the SE is of fundamental importance. Since Weinberg's proposal of a general nonlinear framework [6] and its experimental testings [44], searches for deterministic deviations have, to our knowledge, completely ceased. This is indeed interesting because, since the writings of the first articles showing the possibility of superluminal signaling [14–16] there have been several publications at least questioning its consequences [16,18–21,45].

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