Universal relations for spin-orbit-coupled Fermi gases in two and three dimensions

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(Received 27 July 2019; revised manuscript received 14 March 2020; accepted 17 March 2020; published 22 April 2020)

We present a comprehensive derivation of a set of universal relations for spin-orbit-coupled Fermi gases in three or two dimensions, which follow from the short-range behavior of the two-body physics. Besides the adiabatic energy relations, the large-momentum distribution, the grand canonical potential, and pressure relation derived in our previous work for three-dimensional systems [Phys. Rev. Lett. **120**, 060408 (2018)], we further derive high-frequency tail of the radio-frequency spectroscopy and the short-range behavior of the pair correlation function. In addition, we also extend the derivation to two-dimensional systems with Rashba-type spin-orbit coupling. To simply demonstrate how the spin-orbit-coupling effect modifies the two-body short-range behavior, we solve the two-body problem in the sub-Hilbert space of zero center-of-mass momentum and zero total angular momentum, and we perturbatively take the spin-orbit-coupling effect into account at short distance, since the strength of the spin-orbit coupling should be much smaller than the corresponding scale of the finite range of interatomic interactions. The universal asymptotic forms of the two-body wave function at short distance are then derived, which are independent on the short-range details of interatomic potentials. We find that new scattering parameters emerge in the universal asymptotic forms, apart from the traditional *s*- and *p*-wave scattering length (volume) and effective ranges. This is a general and unique feature for spin-orbit-coupled systems. We show how these two-body parameters characterize the universal relations in the presence of spin-orbit coupling. This work probably sheds light on understanding the profound properties of the many-body quantum systems in the presence of the spin-orbit coupling.

DOI: 10.1103/PhysRevA.101.043616

I. INTRODUCTION

Understanding strongly interacting many-body systems is one of the most daunting challenges in modern physics. Owing to the development of the experimental technique, ultracold atomic gases acquire a high degree of control and tunability in interatomic interaction, geometry, purity, atomic species, and lattice constant (of optical lattices) [1–5]. To date, ultracold quantum gases have emerged as a versatile platform for exploring a broad variety of many-body phenomena as well as offering numerous examples of interesting many-body states [6–8]. Unlike conventional electric gases in condensed matter, atomic quantum gases are extremely dilute, and the mean distance between atoms is usually very large (on the order of μm), while the range of interatomic interactions is several orders smaller (on the order of several tens of nm). Therefore, the two-body correlations characterize the key properties of such many-body systems near scattering resonances, where the two-body interactions are simply described by the scattering length and become irrelevant to the specific form of interatomic potentials.

A set of universal relations, following from the short-range behavior of the two-body physics, govern some crucial features of ultracold atomic gases, and provide powerful constraints on the behavior of the system. Many of these relations were first derived by Shina Tan, such as the adiabatic energy relation, energy theorem, general virial theorem, and pressure relation [9–11]. Afterwards, more universal behaviors were obtained by others, such as the radio-frequency (rf) spectroscopy, photoassociation, static structure factors, and so on [12]. All these relations are characterized by the only universal quantity, named *contact*, and therefore known as the contact theory. During the past few years, the concept of contact theory was further generalized to higher-partial-wave interactions [13–20] as well as to low dimensions [21–29], and more contacts appear when additional two-body parameters are involved.

The reason why the contact theory is significantly important in ultracold atoms is attributed to its direct connection to the experimental measurements. Some of the universal relations were experimentally confirmed, involving various measurements of the contact itself. For two-component

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Fermi gases with s-wave interactions, D. S. Jin's group measured the contact according to three different methods, i.e., the momentum distribution, photoemission spectroscopy, and radio-frequency (rf) spectroscopy, and tested the adiabatic energy relation when the interatomic interaction was adiabatically swept [30]. The asymptotic behavior of the static structure factor at large momentum was confirmed by C. J. Vale's group, by using Bragg spectroscopy technique [31,32]. Recently, the temperature evolution of the contact was resolved independently by M. Zwierlein's group and C. J. Vale's group, especially the characteristic behavior of the contact across the superfluid transition [33,34]. For single-component Fermi gases with p-wave interactions, the feasibility of generalizing the contact theory for higher-partial-wave scatterings was confirmed experimentally by Thywissen's group [35], in which the anisotropic p-wave interaction was tuned according to the magnetic vector [36]. Nowadays, the contact gradually becomes one of fundamental concepts in ultracold atomic physics both theoretically and experimentally.

In the past decade, the realizations of the spin-orbit (SO) coupling in ultracold neutral atoms have sparked a great deal of interest [37–44]. It provides an ideal platform on which to study novel quantum phenomena resulted from SO coupling in a highly controllable and tunable way, such as topological insulators and superconductors [6,7], and (spin) Hall effect [45–47]. Nevertheless, it is still challenging to theoretically deal with the many-body correlations for SO-coupled systems. Unlike the situation in condensed matter, where the interactions between electrons are dominated by the long-range Coulomb potential, the intrinsic short-range feature of interatomic potentials is unchanged for neutral atoms even in the presence of SO coupling. The natural question may be raised, from the point of view of the contact theory, as to whether the two-body physics could provide crucial constraints on many-body behaviors of SO-coupled atomic systems. In addition, it was pointed out that although the short-range feature remains, the SO-coupling effect does modify the short-range behavior of the two-body wave function [48]. Therefore, the existence and exact forms of universal relations for SO-coupled atomic systems attract a great deal of attention. In [49], we preliminarily discussed some of the universal relations for three-dimensional (3D) Fermi gases in the presence of 3D isotropic SO coupling. We proposed a simple way to construct the short-range wave function, in which the SO coupling effect could be taken into account perturbatively. Since SO-coupling in general couples different partial waves of the two-body scatterings, additional contact parameters appear in universal relations. Before long, our theory was verified by different groups near s-wave resonances [50,51].

So far, the generalization of the contact theory in the presence of SO-coupling is mostly discussed in 3D, while the derivation of these universal relations is still elusive in two-dimensional (2D) systems. The short-range behavior of the two-body physics in 2D is different from that in 3D: the two-body wave function in 3D is power-law divergent, while one has to deal with the logarithmic divergence in 2D. From the point of view of the contact theory, different short-range correlations in two-body physics result in different forms of universal relations. Therefore, it requires a direct extension to 2D in the similar manner as in 3D in the presence of SO coupling.

The purpose of this article is to present a comprehensive derivation of universal relations for SO-coupled Fermi gases. Besides the adiabatic energy relations, the large-momentum distribution, the grand canonical potential and pressure relation derived in our previous work for 3D systems [49], we further derive high-frequency tail of the rf spectroscopy and the short-range behavior of the pair correlation function. Then we generalize the derivation of universal relations for 3D systems to 2D case with Rashba SO coupling in a similar way. For the convenience of the presentation, we still construct the short-range behavior of the two-body wave function in the sub-Hilbert space of zero center-of-mass (c.m.) momentum and zero total angular momentum as before, and then only s- and p-wave scatterings are coupled [49,52,53]. This simplification might be valid at extremely low temperature. However, when the temperature becomes higher, the contributions from nonzero c.m. momentum and nonzero total angular momentum channels come into the problem. We may expect more partial waves should be involved, which in turn introduce additional two-body parameters. Our results show that the SO coupling introduces new contacts and modifies the universal relations of many-body systems.

The remainder of this paper is organized as follows. In the next section, we present the derivations of the short-range behavior of two-body wave functions for SO-coupled Fermi gases in three and two dimensions, respectively. Subsequently, with the short-range behavior of the two-body wave functions in hands, we derive a set of universal relations for a 3D SO-coupled Fermi gases in Sec. III, and then generalize them to 2D SO-coupled Fermi gases in Sec. IV, including adiabatic energy relations, asymptotic behavior of the large-momentum distribution, the high-frequency behavior of the rf response, short-range behavior of the pair correlation function, grand canonical potential, and pressure relation. Finally, the main results are summarized in Sec. V.

II. UNIVERSAL SHORT-RANGE BEHAVIOR OF TWO-BODY WAVE FUNCTIONS

The ultracold atomic gases are dilute, while the range of interatomic potentials is extremely small. When two fermions get close enough to interact with each other, they usually far away from the others. If only these two-body correlations are taken into account, then some key properties of many-body systems are characterized by the short-range two-body physics, which is the basic idea of the contact theory. In this section, we are going to discuss the short-range behavior of two-body wave functions for 3D Fermi gases in the presence of 3D SO coupling and 2D Fermi gases in the presence of 2D SO coupling, respectively. Let us consider spin-half SO-coupled Fermi gases, and the Hamiltonian of a single fermion is modeled as

$$\hat{\mathcal{H}}_1 = \frac{\hbar^2\hat{\mathbf{k}}_1^2}{2M} + \frac{\hbar^2\lambda}{M}\hat{\chi} + \frac{\hbar^2\lambda^2}{2M}, \tag{1}$$

where $\hat{\mathbf{k}}_1 = -i\nabla$ is the single-particle momentum operator, M is the atomic mass, and $\hbar$ is the Planck's constant divided by 2π. Here, the SO coupling is described by the term $\hbar^2\lambda\hat{\chi}/M$

with the strength $\lambda > 0$, and $\hat{\chi}$ takes the isotropic form of $\hat{\mathbf{k}}_1 \cdot \hat{\boldsymbol{\sigma}}$ in 3D or the Rashba form of $\hat{\boldsymbol{\sigma}} \times \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{n}}$ in 2D [54], where $\hat{\boldsymbol{\sigma}}$ is the Pauli operator, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to the $(x-y)$ plane.

Provided that the c.m. momentum $\mathbf{K}$ and the total angular momentum $\mathbf{J}$ for two particles with SO coupling are conserved [52], we can conveniently deal with two-body problem in the subspace of $\mathbf{K}=0$ and $\mathbf{J}=0$. Hereupon, other partial-wave scatterings are avoided, and only s- and p-wave scatterings are involved in the problem [49,52,53]. Consequently, the Hamiltonian of two spin-half fermions can be written as

$$\hat{\mathcal{H}}_2 = \frac{\hbar^2\hat{\mathbf{k}}^2}{M} + \frac{\hbar^2\lambda}{M}\hat{Q}(\mathbf{r}) + \frac{\hbar^2\lambda^2}{M} + V(\mathbf{r}), \tag{2}$$

where $\hat{\mathbf{k}} = (\hat{\mathbf{k}}_2 - \hat{\mathbf{k}}_1)/2$ and $V(\mathbf{r})$, respectively, denote the momentum operator for the relative motion $\mathbf{r} = \mathbf{r_2} - \mathbf{r_1}$ and the short-range interatomic interaction with a finite range ϵ, $\hat{Q}(\mathbf{r}) = (\hat{\boldsymbol{\sigma}}_2 - \hat{\boldsymbol{\sigma}}_1)\cdot\hat{\mathbf{k}}$ in 3D or $\hat{Q}(\mathbf{r}) = (\hat{\boldsymbol{\sigma}}_2 - \hat{\boldsymbol{\sigma}}_1)\times\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}$ in 2D, and $\hat{\boldsymbol{\sigma}}_i$ labels the spin operator of the ith fermion. In the following, let us consider the two-body problems in the 3D systems with 3D SO coupling and 2D systems with 2D SO coupling, respectively.

A. For 3D systems with 3D SO coupling

In the subspace of $\mathbf{K}=0$ and $\mathbf{J}=0$, we may choose the common eigenstates of the total Hamiltonian $\hat{\mathcal{H}}_2$ and total angular momentum $\mathbf{J}(=0)$ as the basis of Hilbert space, which take the forms of [49,52,53]

$$\Omega_0(\hat{\mathbf{r}}) = Y_{00}(\hat{\mathbf{r}})|S\rangle, \tag{3}$$

$$\Omega_1(\hat{\mathbf{r}}) = -\frac{i}{\sqrt{3}}[Y_{1-1}(\hat{\mathbf{r}})|\uparrow\uparrow\rangle + Y_{11}(\hat{\mathbf{r}})|\downarrow\downarrow\rangle - Y_{10}(\hat{\mathbf{r}})|T\rangle], \tag{4}$$

where $Y_{lm}(\hat{\mathbf{r}})$ denotes the spherical harmonics with azimuthal quantum numbers (l,m), $\hat{\mathbf{r}} \equiv (\theta,\varphi)$ is the angular degree of freedom of the relative coordinate $\mathbf{r}$, $|S\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ indicates the singlet spin state with total spin $S=0$, and $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |T\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}\}$ indicate the three triplet states with total spin $S=1$. Then, in the basis of $\{\Omega_0(\hat{\mathbf{r}}), \Omega_1(\hat{\mathbf{r}})\}$, the two-body wave function can be generally written as

$$\Psi(\mathbf{r}) = \psi_0(r)\Omega_0(\hat{\mathbf{r}}) + \psi_1(r)\Omega_1(\hat{\mathbf{r}}), \tag{5}$$

where $\psi_i(r)\,(i=0,1)$ is the radial part of the wave function. Note that we here consider an isotropic p-wave interaction and the radial wave function $\psi_1(r)$ is identical for three scattering channels, i.e., $m=0,\pm1$.

Typically, in the low-energy scattering limit, the relative energy of two atoms as well as the SO-coupling strength can be treated as a perturbation, as long as the atoms get as close as the distance ϵ [38,39,49]. Therefore, we assume that the two-body wave function may take the form of the following ansatz as in Ref. [49],

$$\Psi(\mathbf{r}) \approx \phi(\mathbf{r}) + k^2 f(\mathbf{r}) - \lambda g(\mathbf{r}), \tag{6}$$

as the distance of two fermions approaches ϵ. Substituting the ansatz Eq. (6) into the Schrödinger equation $\hat{\mathcal{H}}_2\Psi(\mathbf{r}) = E\Psi(\mathbf{r})$, and comparing the corresponding coefficients of the term k^2 and λ on both sides, we obtain

$$\left[-\nabla^2 + \frac{MV(\mathbf{r})}{\hbar^2}\right]\phi(\mathbf{r}) = 0, \tag{7}$$

$$\left[-\nabla^2 + \frac{MV(\mathbf{r})}{\hbar^2}\right]f(\mathbf{r}) = \phi(\mathbf{r}), \tag{8}$$

$$\left[-\nabla^2 + \frac{MV(\mathbf{r})}{\hbar^2}\right]g(\mathbf{r}) = \hat{Q}(\mathbf{r})\phi(\mathbf{r}). \tag{9}$$

These coupled equations can easily be solved for $r > \epsilon$, and we obtain

$$\begin{aligned}\phi(\mathbf{r}) = {}& \alpha_0\left(\frac{1}{r} - \frac{1}{a_0}\right)\Omega_0(\hat{\mathbf{r}}) \\ &+ \alpha_1\left(\frac{1}{r^2} - \frac{1}{3a_1}r\right)\Omega_1(\hat{\mathbf{r}}) + O(r^2),\end{aligned} \tag{10}$$

$$\begin{aligned}f(\mathbf{r}) = {}& \alpha_0\left(\frac{1}{2}b_0 - \frac{1}{2}r\right)\Omega_0(\hat{\mathbf{r}}) \\ &+ \alpha_1\left(\frac{1}{2} + \frac{b_1}{6}r\right)\Omega_1(\hat{\mathbf{r}}) + O(r^2),\end{aligned} \tag{11}$$

$$g(\mathbf{r}) = -\alpha_1 u\Omega_0(\hat{\mathbf{r}}) - \alpha_0(1+vr)\Omega_1(\hat{\mathbf{r}}) + O(r^2). \tag{12}$$

Here, α_0 and α_1 denote two complex superposition coefficients, a_i and b_i are s-wave scattering length and effective range for $i=0$, and p-wave scattering volume and effective range for $i=1$, respectively. What makes sense is that, as we anticipate, two new scattering parameters u and v emerge in s- and p-wave channels, respectively. They characterize the modification to the short-range behavior of the two-body wave function because of SO coupling.

In the absence of SO coupling, if atoms are initially prepared near an s-wave resonance, then the contribution from the p-wave channel could be ignored, and we have $\alpha_1 \approx 0$. Naturally, the two-body wave function $\Psi(\mathbf{r})$ reduces to the known s-wave form of (up to a constant α_0),

$$\Psi(\mathbf{r}) = \left(\frac{1}{r} - \frac{1}{a_0} + \frac{b_0k^2}{2} - \frac{k^2}{2}r\right)\Omega_0(\hat{\mathbf{r}}) + O(r^2), \tag{13}$$

at short distance $r \gtrsim \epsilon$. Subsequently, when SO coupling is switched on near the s-wave resonance, a considerable p-wave contribution is involved, and the two-body wave function becomes

$$\begin{aligned}\Psi_s(\mathbf{r}) = {}& \left(\frac{1}{r} - \frac{1}{a_0} + \frac{b_0k^2}{2} - \frac{k^2}{2}r\right)\Omega_0(\hat{\mathbf{r}}) \\ &+ (1+vr)\lambda\Omega_1(\hat{\mathbf{r}}) + O(r^2),\end{aligned} \tag{14}$$

which recovers the modified Bethe-Peierls boundary condition of Ref. [48] by noticing $\Omega_0(\hat{\mathbf{r}}) = |S\rangle/\sqrt{4\pi}$ and $\Omega_1(\hat{\mathbf{r}}) = -i(\hat{\boldsymbol{\sigma}}_2 - \hat{\boldsymbol{\sigma}}_1)\cdot(\mathbf{r}/r)|S\rangle/\sqrt{16\pi}$. We can see that the parameter v characterizes the hybridization of the p-wave component into the s-wave scattering due to SO coupling. If atoms are initially prepared near a p-wave resonance without SO coupling, then the s-wave scattering could be ignored and we have $\alpha_0 \approx 0$. The two-body wave function $\Psi(\mathbf{r})$ takes the known p-wave form at short distance, i.e.,

$$\Psi(\mathbf{r}) = \left(\frac{1}{r^2} - \frac{1}{3a_1}r + \frac{k^2}{2} + \frac{b_1k^2}{6}r\right)\Omega_1(\hat{\mathbf{r}}) + O(\mathbf{r}^2). \quad (15)$$

In the presence of SO coupling near the p-wave resonance, an s-wave component is introduced, and the two-body wave function becomes

$$\Psi_p(\mathbf{r}) = \left[\frac{1}{r^2} + \frac{k^2}{2} + \left(-\frac{1}{3a_1} + \frac{b_1k^2}{6}\right)r\right]\Omega_1(\hat{\mathbf{r}}) + u\lambda\Omega_0(\hat{\mathbf{r}}) + O(\mathbf{r}^2) \quad (16)$$

at short distance. We can see that the parameter u describes the hybridization of the s-wave component into the p-wave scattering due to SO coupling. In general, both s- and p-wave scatterings exist between atoms in the absence of SO coupling. Therefore, when SO coupling is introduced, the two-body wave function is generally the arbitrary superposition of Eqs. (14) and (16) and can be written as

$$\Psi_{3D}(\mathbf{r}) = \alpha_0\left(\frac{1}{r} - \frac{1}{a_0} + \frac{b_0k^2}{2} + \frac{\alpha_1}{\alpha_0}u\lambda - \frac{k^2}{2}r\right)\Omega_0(\hat{\mathbf{r}}) + \alpha_1\left[\frac{1}{r^2} + \frac{k^2}{2} + \frac{\alpha_0}{\alpha_1}\lambda + \left(-\frac{1}{3a_1} + \frac{b_1k^2}{6} + \frac{\alpha_0}{\alpha_1}v\lambda\right)r\right]\Omega_1(\hat{\mathbf{r}}) + O(r^2) \quad (17)$$

at short distance $r \gtrsim \epsilon$. Equation (17) can be treated as the short-range boundary condition for two-body wave functions in 3D in the presence of 3D SO coupling, when both s- and p-wave interactions are considered.

B. For 2D systems with 2D SO coupling

Let us consider two spin-half fermions scattering in the x-y plane. We easily find that the total angular momentum $\mathbf{J}$ perpendicular to the x-y plane is conserved as well as the c.m. momentum $\mathbf{K}$ [55]. Therefore, we may still focus on the two-body problem in the subspace of $\mathbf{K} = \mathbf{0}$ and $\mathbf{J} = \mathbf{0}$, which is spanned by the following three orthogonal basis

$$\Omega_0(\varphi) = \frac{1}{\sqrt{2\pi}}|S\rangle, \quad (18)$$

$$\Omega_{-1}(\varphi) = \frac{e^{-i\varphi}}{\sqrt{2\pi}}|\uparrow\uparrow\rangle, \quad (19)$$

$$\Omega_1(\varphi) = \frac{e^{i\varphi}}{\sqrt{2\pi}}|\downarrow\downarrow\rangle, \quad (20)$$

where φ is the azimuthal angle of the relative coordinate $\mathbf{r}$. Then the two-body wave function can formally be expanded as

$$\Psi(\mathbf{r}) = \sum_{m=0,\pm1}\psi_m(r)\Omega_m(\varphi), \quad (21)$$

and $\psi_m(r)$ is the radial wave function. Analogously, the strength of SO coupling as well as the energy can be taken into account perturbatively at short distance. We assume that the two-body wave function has the form of the ansatz Eq. (6), and the corresponding functions to be determined can easily be solved out from the Schrödinger equation outside the range of the interatomic potential, i.e., $r \gtrsim \epsilon$. After straightforward algebra, we obtain

$$\phi(\mathbf{r}) = \alpha_0\left(\ln\frac{r}{2a_0} + \gamma\right)\Omega_0(\varphi) + \left(\frac{1}{r} - \frac{\pi}{4a_1}r\right)\sum_{m=\pm1}\alpha_m\Omega_m(\varphi) + O(r^2), \quad (22)$$

$$f(\mathbf{r}) = -\alpha_0\left(\frac{\pi}{4}b_0 + \frac{1}{4}r^2\ln\frac{r}{2a_0}\right)\Omega_0(\varphi) + \left(\frac{1-2\gamma}{4}r - \frac{1}{2}r\ln\frac{r}{2b_1}\right)\sum_{m=\pm1}\alpha_m\Omega_m(\varphi) + O(r^2), \quad (23)$$

$$g(\mathbf{r}) = -\left(\sum_{m=\pm1}\alpha_m\right)u\Omega_0(\varphi) - \alpha_0\left(vr + \frac{r}{\sqrt{2}}\ln\frac{r}{2b_1}\right)\sum_{m=\pm1}\Omega_m(\varphi) + O(r^2) \quad (24)$$

for $r \gtrsim \epsilon$, where γ is Euler's constant, α_m ($m = 0, \pm1$) is complex superposition coefficients, a_m and b_m are s-wave scattering length and effective range for $m = 0$, and p-wave scattering area and effective range for $|m| = 1$, respectively. Here, we have assumed that the p-wave interaction is isotropic and thus is the same in $m = \pm1$ channels, and applied the p-wave effective-range expansion of the scattering phase shift, i.e., $k^2\cot\delta_1 = -1/a_1 + 2k^2\ln(kb_1)/\pi$ [56]. We find that two new scattering parameters are similarly introduced, and they demonstrate the hybridization of s- and p-wave scattering in the presence of Rashba SOC in 2D. Finally, the asymptotic form of the two-body wave function at short distance can be written as

$$\Psi_{2D}(\mathbf{r}) = \alpha_0\left[\ln\frac{r}{2a_0} + \gamma - \frac{\pi}{4}b_0k^2 + \left(\sum_{m=\pm1}\frac{\alpha_m}{\alpha_0}\right)u\lambda - \frac{k^2}{4}r^2\ln\frac{r}{2a_0}\right]\Omega_0(\varphi) + \sum_{m=\pm1}\alpha_m\left[\frac{1}{r} + \left(-\frac{\pi}{4a_1} + \frac{1-2\gamma}{4}k^2 + \frac{\alpha_0}{\alpha_m}v\lambda\right)r + \left(-\frac{k^2}{2} + \frac{\alpha_0}{\alpha_m}\frac{\lambda}{\sqrt{2}}\right)r\ln\frac{r}{2b_1}\right]\Omega_m(\varphi) + O(r^2) \quad (25)$$

for $r \gtrsim \epsilon$. It is apparent that $\Psi_{2D}(\mathbf{r})$ naturally decouples to the s- and p-wave short-range boundary conditions in the absence of SO coupling. However, Rashba SO coupling mixes the s- and p-wave scatterings, and two new scattering parameters u and v are introduced. We should note that the short-range behaviors of the two-body wave function, i.e., Eqs. (17) and (25), are universal and does not depend on the specific form of interatomic potentials.

C. New two-body parameters for a spherical-square-well potential

As we have discussed, new two-body parameters need to be introduced in the presence of SO coupling. These new two-body parameters are independent from the well-known scattering length (volume) as well as the effective range, and should be determined by the specific form of the two-body interaction. To provide a clear picture for these new parameters, we take the spherical-square-well potential as an example to demonstrate how the new parameters are evaluated. For a 3D system, we assume that the two-body interaction has the following form:

$$V(\mathbf{r}) = \begin{cases} -V_0, & 0 \leqslant r \leqslant \epsilon, \\ 0, & r > \epsilon, \end{cases} \tag{26}$$

where ϵ is the range of the potential and $V_0 > 0$ is the depth of the well. Outside the potential, i.e., $r > \epsilon$, $\phi(\mathbf{r})$ has the form of Eq. (10). Inside the potential, using $V(\mathbf{r}) = -V_0$, we obtain

$$\phi(\mathbf{r}) = \alpha_0 c_0 j_0(\tilde{r}\sqrt{\tilde{V}_0})\Omega_0(\hat{\mathbf{r}}) + \alpha_1 c_1 j_1(\tilde{r}\sqrt{\tilde{V}_0})\Omega_1(\hat{\mathbf{r}}), \tag{27}$$

where $j_\nu(\cdot)$'s are the spherical Bessel function of the first kind, c_ν's are real coefficient, $\tilde{r} = r/\epsilon$, $\tilde{V}_0 = V_0/E_0$, and $E_0 = \hbar^2/M\epsilon^2$. The continuity conditions of $\phi(\mathbf{r})$ at $r = \epsilon$ as well as those of its first derivative provide four independent equations, which uniquely determine the s-wave scattering length a_0, p-wave scattering volume a_1, and c_1 and c_2. Then $\phi(\mathbf{r})$ is obtained for a spherical-square-well potential.

With $\phi(\mathbf{r})$ in hand, the new two-body parameters u and v can be calculated from Eq. (9). We have obtained $g(\mathbf{r})$ outside the potential, i.e., Eq. (12). Inside the potential, i.e., $0 \leqslant r < \epsilon$, we easily obtain

$$g(\mathbf{r}) = \frac{\alpha_1}{2\tilde{r}\tilde{V}_0}[-2c_1\tilde{r}\sqrt{\tilde{V}_0}\cos(\tilde{r}\sqrt{\tilde{V}_0}) + A_1\sin(\tilde{r}\sqrt{\tilde{V}_0})]\Omega_0(\hat{\mathbf{r}}) + \frac{\alpha_0}{2\tilde{r}\tilde{V}_0}\{[A_2\tilde{r}\tilde{V}_0 + c_0\tilde{r}\sqrt{\tilde{V}_0}\cos(2\tilde{r}\sqrt{\tilde{V}_0}) - 2c_0\sin(2\tilde{r}\sqrt{\tilde{V}_0})]j_1(\tilde{r}\sqrt{\tilde{V}_0}) + c_0[-2 + 2\tilde{r}^2\tilde{V}_0 + 2\cos(2\tilde{r}\sqrt{\tilde{V}_0}) + \tilde{r}\sqrt{\tilde{V}_0}\sin(2\tilde{r}\sqrt{\tilde{V}_0})]n_1(\tilde{r}\sqrt{\tilde{V}_0})\}\Omega_1(\hat{\mathbf{r}}), \tag{28}$$

and $n_\nu(\cdot)$ is the spherical Bessel function of the second kind. We find that there are totally four parameters to be determined, i.e., A_1, A_2, u, and v, which can be solved out from the continuity conditions of $g(\mathbf{r})$ as well as those of its first derivative at $r = \epsilon$. After some straightforward algebra, we obtain

$$u\epsilon = v\epsilon = \frac{\sqrt{\tilde{V}_0}}{\tan(\sqrt{\tilde{V}_0}) - \sqrt{\tilde{V}_0}}. \tag{29}$$

For a 2D system, we can also calculate the new two-body parameters introduced by SO coupling for a model potential following the similar procedure.

III. UNIVERSAL RELATIONS IN THE PRESENCE OF ISOTROPIC 3D SO COUPLING

In the previous section, we have discussed the two-body problem in the presence of SO coupling, and obtained the short-range behaviors of the two-body wave functions with zero c.m. momentum and zero total angular momentum. However, for a many-body system, the c.m. momentum of a pair of fermions as well as their total angular momentum is generally not conserved any longer. In this case, all the partial waves should be taken into account besides the s- and p-wave scatterings. Therefore, we may anticipate that additional new two-body parameters in high-partial-wave channels should be introduced besides u and v, then our perturbation method can still be generalized to many-body systems if more new two-body parameters are included. To simplify the discussion of Tan's universal relations of SO-coupled Fermi gases, we assume that only a few partial waves come into the problem, for example, only s- and p-wave scatterings are taken into account as discussed before. Then we are ready to consider Tan's universal relations of SO-coupled many-body systems, if only two-body correlations are taken into account. Owing to the short-range property of interactions between neutral atoms, when two fermions (i and j) get as close as the range of interatomic potentials, all the other atoms are usually far away. In this case, the many-body wave functions approximately take the forms of Eq. (17) in 3D systems with 3D SO coupling, when the fermions i and j approach to each other. We need to pay attention that the arbitrary superposition coefficient $\alpha_m(\mathbf{X})$ then becomes the functions of the c.m. coordinates of the pair (i, j) as well as those of the rest of the fermions, which we include into the variable $\mathbf{X}$. In the follows, we derive a set of universal relations for SO-coupled many-body systems by using Eq. (17) for 3D SO-coupled Fermi gases. These relations include adiabatic energy relations, the large-momentum behavior of the momentum distribution, the high-frequency tail of the rf spectroscopy, the short-range behavior of the pair correlation function, the grand canonical potential, and pressure relation. Let us consider a strongly interacting two-component Fermi gases with total atom number N. For simplicity, we consider the case with $b_0 \approx 0$ for broad s-wave resonances in the follows.

A. Adiabatic energy relations

To investigate how the energy varies with the two-body interaction, we consider two many-body wave functions Ψ and Ψ', which are corresponding to different interaction strengths, respectively. And they satisfy the Schrödinger equation with different energies

$$\sum_{i=1}^{N}\hat{\mathcal{H}}_1^{(i)}\Psi = E\Psi, \tag{30}$$

$$\sum_{i=1}^{N}\hat{\mathcal{H}}_1^{(i)}\Psi' = E'\Psi', \tag{31}$$

if there is not any pair of atoms within the range of the interaction, where $\hat{\mathcal{H}}_1^{(i)}$ denotes the single-atom Hamiltonian Eq. (1) for the ith fermion. By subtracting [31]$^* \times \Psi$ from $\Psi'^* \times$ [30], and integrating over the domain $\mathcal{D}_\epsilon$, the set of all configurations $(\mathbf{r}_i, \mathbf{r}_j)$ in which $r = |\mathbf{r}_i - \mathbf{r}_j| > \epsilon$, we arrive at

$$(E - E')\int_{\mathcal{D}_\epsilon}\prod_{i=1}^{N} d\mathbf{r}_i \Psi'^*\Psi = -\frac{\hbar^2}{M}\mathcal{N}\int_{r>\epsilon} d\mathbf{X}d\mathbf{r}\big[\Psi'^*\nabla_{\mathbf{r}}^2\Psi - \big(\nabla_{\mathbf{r}}^2\Psi'^*\big)\Psi\big] + \frac{\hbar^2\lambda}{M}\mathcal{N}\int_{r>\epsilon} d\mathbf{X}d\mathbf{r}[\Psi'^*(\hat{Q}\Psi) - (\hat{Q}\Psi')^*\Psi], \quad (32)$$

where $\mathcal{N} = N(N-1)/2$ is the number of all the possible ways to pair atom. Using the Gauss' theorem, the first term on the right-hand side (RHS) can be written as

$$-\frac{\hbar^2}{M}\mathcal{N}\int_{r>\epsilon} d\mathbf{X}d\mathbf{r}\big[\Psi'^*\nabla_{\mathbf{r}}^2\Psi - \big(\nabla_{\mathbf{r}}^2\Psi'^*\big)\Psi\big] = -\frac{\hbar^2}{M}\mathcal{N}\oint\!\!\oint_{r=\epsilon}[\Psi'^*\nabla_{\mathbf{r}}\Psi - (\nabla_{\mathbf{r}}\Psi'^*)\Psi]\cdot\hat{\mathbf{n}}d\mathcal{S} = \frac{\hbar^2\epsilon^2}{M}\mathcal{N}\sum_{i=0}^{1}\int d\mathbf{X}\left(\psi_i'^*\frac{\partial}{\partial r}\psi_i - \psi_i\frac{\partial}{\partial r}\psi_i'^*\right)_{r=\epsilon}, \quad (33)$$

where $\mathcal{S}$ is the boundary in which the distance between the two atoms in the pair (i, j) is ϵ with, $\hat{\mathbf{n}}$ is the direction normal to $\mathcal{S}$ but opposite to the radial direction, and ψ_0 (ψ_1) is the s-wave (p-wave) component of the radial two-body wave function. In addition, for the second term on the RHS of Eq. (32), we have

$$\hat{Q}(\mathbf{r})\Psi = -\frac{2}{r^2}\frac{\partial}{\partial r}(r^2\psi_1)\Omega_0(\hat{\mathbf{r}}) + 2\frac{\partial\psi_0}{\partial r}\Omega_1(\hat{\mathbf{r}}), \quad (34)$$

then it becomes

$$\frac{\hbar^2\lambda}{M}\mathcal{N}\int_{r>\epsilon} d\mathbf{X}d\mathbf{r}[\Psi'^*(\hat{Q}(\mathbf{r})\Psi) - (\hat{Q}(\mathbf{r})\Psi')^*\Psi] = \frac{2\lambda\hbar^2\epsilon^2}{M}\mathcal{N}\int d\mathbf{X}(\psi_0'^*\psi_1 - \psi_1'^*\psi_0)_{r=\epsilon}. \quad (35)$$

Combining Eqs. (32), (33), and (35), we obtain

$$(E - E')\int_{\mathcal{D}_\epsilon}\prod_{i=1}^{N} d\mathbf{r}_i\Psi'^*\Psi = \frac{\hbar^2\epsilon^2}{M}\mathcal{N}\sum_{i=0}^{1}\int d\mathbf{X}\left(\psi_i'^*\frac{\partial}{\partial r}\psi_i - \psi_i\frac{\partial}{\partial r}\psi_i'^*\right)_{r=\epsilon} + \frac{2\lambda\hbar^2\epsilon^2}{M}\mathcal{N}\int d\mathbf{X}\big(\psi_0'^*\psi_1 - \psi_1'^*\psi_0\big)_{r=\epsilon}. \quad (36)$$

Inserting the asymptotic form of the many-body wave function Eq. (17) into Eq. (36), and letting $E' \to E$ and $\Psi' \to \Psi$, we find

$$\delta E\cdot\int_{\mathcal{D}_\epsilon}\prod_{i=1}^{N} d\mathbf{r}_i|\Psi|^2 = -\frac{\hbar^2}{M}\big(\mathcal{I}_a^{(0)} - \lambda\mathcal{I}_\lambda\big)\delta a_0^{-1} - \frac{\hbar^2\mathcal{I}_a^{(1)}}{M}\delta a_1^{-1} + \frac{\mathcal{E}_1}{2}\delta b_1 + \frac{3\lambda\hbar^2}{2M}\mathcal{I}_\lambda\delta v - \frac{\lambda\hbar^2}{M}\left(2\lambda\mathcal{I}_a^{(1)} - \frac{1}{2}\mathcal{I}_\lambda\right)\delta u + \left(\frac{1}{\epsilon} + \frac{b_1}{2}\right)\mathcal{I}_a^{(1)}\delta E, \quad (37)$$

where

$$\mathcal{I}_a^{(m)} = \mathcal{N}\int d\mathbf{X}|\alpha_m(\mathbf{X})|^2, \quad (38)$$

$$\mathcal{E}_m = \mathcal{N}\int d\mathbf{X}\alpha_m^*(\mathbf{X})[E - \hat{T}(\mathbf{X})]\alpha_m(\mathbf{X}), \quad (39)$$

$$\mathcal{I}_\lambda = \mathcal{N}\int d\mathbf{X}\alpha_0^*(\mathbf{X})\alpha_1(\mathbf{X}) + \text{c.c.}, \quad (40)$$

$$\mathcal{E}_\lambda = \mathcal{N}\int d\mathbf{X}\alpha_0^*(\mathbf{X})[E - \hat{T}(\mathbf{X})]\alpha_1(\mathbf{X}) + \text{c.c.} \quad (41)$$

for $m = 0, 1$, and $\hat{T}(\mathbf{X})$ is the kinetic operator including the c.m. motion of the pair (i, j) and those of all the rest fermions. Using the normalization of the many-body wave function (see Appendix A)

$$\int_{\mathcal{D}_\epsilon}\prod_{i=1}^{N} d\mathbf{r}_i|\Psi|^2 = 1 + \left(\frac{1}{\epsilon} + \frac{b_1}{2}\right)\mathcal{I}_a^{(1)}, \quad (42)$$

we can further simplify Eq. (37) as

$$\delta E = -\frac{\hbar^2}{M}\big(\mathcal{I}_a^{(0)} - \lambda\mathcal{I}_\lambda\big)\delta a_0^{-1} - \frac{\hbar^2\mathcal{I}_a^{(1)}}{M}\delta a_1^{-1} + \frac{\mathcal{E}_1}{2}\delta b_1 + \frac{3\lambda\hbar^2\mathcal{I}_\lambda}{2M}\delta v + \frac{\lambda\hbar^2}{2M}\big(\mathcal{I}_\lambda - 4\lambda\mathcal{I}_a^{(1)}\big)\delta u, \quad (43)$$

which yields the following set of adiabatic energy relations

$$\frac{\partial E}{\partial a_0^{-1}} = -\frac{\hbar^2}{M}\big(\mathcal{I}_a^{(0)} - \lambda\mathcal{I}_\lambda\big), \quad (44)$$

$$\frac{\partial E}{\partial a_1^{-1}} = -\frac{\hbar^2\mathcal{I}_a^{(1)}}{M}, \quad (45)$$

$$\frac{\partial E}{\partial b_1} = \frac{\mathcal{E}_1}{2}, \quad (46)$$

$$\frac{\partial E}{\partial u} = \frac{\lambda\hbar^2}{2M}\big(\mathcal{I}_\lambda - 4\lambda\mathcal{I}_a^{(1)}\big), \quad (47)$$

$$\frac{\partial E}{\partial v} = \frac{3\lambda\hbar^2\mathcal{I}_\lambda}{2M}. \quad (48)$$

Interestingly, two additional new adiabatic energy relations appear, i.e., Eqs. (47) and (48), which originate from new scattering parameters introduced by SO coupling. These relations clearly describe the relationships between the macroscopic internal energy and the microscopic two-body scattering parameters for an SO-coupled many-body system.

B. Tail of the momentum distribution at large q

Let us then study the asymptotic behavior of the large momentum distribution for a many-body system with N

fermions. The momentum distribution of the ith fermion is defined as

$$n_i(\mathbf{q}) = \int \prod_{t\neq i} d\mathbf{r}_t |\tilde{\Psi}_i(\mathbf{q})|^2, \tag{49}$$

where $\tilde{\Psi}_i(\mathbf{q}) \equiv \int d\mathbf{r}_i \Psi_{3D} e^{-i\mathbf{q}\cdot\mathbf{r}_i}$, and then the total momentum distribution can be written as $n(\mathbf{q}) = \sum_{i=1}^{N} n_i(\mathbf{q})$. When two fermions (i, j) get close while all the other fermions are far away, we may write the many-body function Ψ_{3D} at $r = |\mathbf{r}_i - \mathbf{r}_j| \approx 0$ as the following ansatz:

$$\Psi_{3D}(\mathbf{X}, \mathbf{r}) = \left[\frac{\alpha_0(\mathbf{X})}{r} + \mathcal{B}_0(\mathbf{X}) + \mathcal{C}_0(\mathbf{X})r\right]\Omega_0(\hat{\mathbf{r}}) + \left[\frac{\alpha_1(\mathbf{X})}{r^2} + \mathcal{B}_1(\mathbf{X}) + \mathcal{C}_1(\mathbf{X})r\right]\Omega_1(\hat{\mathbf{r}}) + O(r^2), \tag{50}$$

where α_m, $\mathcal{B}_m$ and $\mathcal{C}_m$ $(m = 0, 1)$ are all regular functions. Comparing Eq. (17) with Eq. (50) at small r, we find

$$\mathcal{B}_0(\mathbf{X}) = -\frac{\alpha_0}{a_0} + \alpha_1 u\lambda, \tag{51}$$

$$\mathcal{B}_1(\mathbf{X}) = \frac{\alpha_1 k^2}{2} + \alpha_0\lambda, \tag{52}$$

$$\mathcal{C}_0(\mathbf{X}) = -\frac{\alpha_0 k^2}{2}, \tag{53}$$

$$\mathcal{C}_1(\mathbf{X}) = -\frac{\alpha_1}{3a_1} + \frac{\alpha_1 b_1 k^2}{6} + \alpha_0 v\lambda. \tag{54}$$

The asymptotic form of the momentum distribution at large $\mathbf{q}$ but still smaller than ϵ^{-1} is determined by the asymptotic behavior at short distance with respect to the two interacting fermions, then we obtain

$$\tilde{\Psi}_i(\mathbf{q}) \underset{q\to\infty}{\approx} \int d\mathbf{r}\Psi_{3D}(\mathbf{X}, \mathbf{r}\sim 0)e^{-i\mathbf{q}\cdot\mathbf{r}}. \tag{55}$$

With the help of $\nabla^2(r^{-1}) = -4\pi\delta(\mathbf{r})$, we have

$$f(q) \equiv \int d\mathbf{r}\frac{e^{-i\mathbf{q}\cdot\mathbf{r}}}{r} = \frac{4\pi}{q^2}, \tag{56}$$

so that

$$\int d\mathbf{r}\frac{\alpha_0(\mathbf{X})}{r}\Omega_0(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = \frac{4\pi}{q^2}\alpha_0(\mathbf{X})\Omega_0(\hat{\mathbf{q}}), \tag{57}$$

$$\int d\mathbf{r}\mathcal{B}_0(\mathbf{X})\Omega_0(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = 0, \tag{58}$$

$$\int d\mathbf{r}\mathcal{C}_0(\mathbf{X})r\Omega_0(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = -\frac{8\pi}{q^4}\mathcal{C}_0(\mathbf{X})\Omega_0(\hat{\mathbf{q}}), \tag{59}$$

$$\int d\mathbf{r}\frac{\alpha_1(\mathbf{X})}{r^2}\Omega_1(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = -i\frac{4\pi}{q}\alpha_1(\mathbf{X})\Omega_1(\hat{\mathbf{q}}), \tag{60}$$

$$\int d\mathbf{r}\mathcal{B}_1(\mathbf{X})\Omega_1(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = -i\frac{8\pi}{q^3}\mathcal{B}_1(\mathbf{X})\Omega_1(\hat{\mathbf{q}}), \tag{61}$$

$$\int d\mathbf{r}\mathcal{C}_1(\mathbf{X})r\Omega_1(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = 0. \tag{62}$$

Inserting Eqs. (57)–(62) into Eq. (55), and then into Eq. (49), we find that the total momentum distribution $n(\mathbf{q})$ at large $\mathbf{q}$ takes the form

$$n_{3D}(\mathbf{q}) \approx \mathcal{N}\int d\mathbf{X}\frac{32\pi^2\alpha_1\alpha_1^*\Omega_1(\hat{\mathbf{q}})\Omega_1^*(\hat{\mathbf{q}})}{q^2} + i\frac{32\pi^2[\alpha_0\alpha_1^*\Omega_0(\hat{\mathbf{q}})\Omega_1^*(\hat{\mathbf{q}}) - \alpha_0^*\alpha_1\Omega_0^*(\hat{\mathbf{q}})\Omega_1(\hat{\mathbf{q}})]}{q^3} + [32\pi^2\alpha_0\alpha_0^*\Omega_0(\hat{\mathbf{q}})\Omega_0^*(\hat{\mathbf{q}}) + 64\pi^2k^2\alpha_1\alpha_1^*\Omega_1(\hat{\mathbf{q}})\Omega_1^*(\hat{\mathbf{q}}) + 64\pi^2\lambda(\alpha_0\alpha_1^* + \alpha_0^*\alpha_1)\Omega_1(\hat{\mathbf{q}})\Omega_1^*(\hat{\mathbf{q}})]\frac{1}{q^4} + O(q^{-5}). \tag{63}$$

If we only show the solicitude for the dependence of the momentum distribution on the amplitude of $\mathbf{q}$, then we find that the odd-order terms of q^{-1} vanish after the integration over the direction of $\mathbf{q}$. Finally, we obtain

$$n_{3D}(q) = \frac{C_a^{(1)}}{q^2} + \left(C_a^{(0)} + C_b^{(1)} + \lambda\mathcal{P}_\lambda\right)\frac{1}{q^4} + O(q^{-6}), \tag{64}$$

where the contacts are defined as

$$C_a^{(m)} = 32\pi^2\mathcal{I}_a^{(m)},\ (m = 0, 1), \tag{65}$$

$$C_b^{(1)} = \frac{64\pi^2 M}{\hbar^2}\mathcal{E}_1, \tag{66}$$

$$\mathcal{P}_\lambda = 64\pi^2\mathcal{I}_\lambda. \tag{67}$$

With these definitions in hands, the adiabatic energy relations Eqs. (44)–(48) can alternatively be rewritten as

$$\frac{\partial E}{\partial a_0^{-1}} = -\frac{\hbar^2 C_a^{(0)}}{32\pi^2 M} + \lambda\frac{\hbar^2\mathcal{P}_\lambda}{64\pi^2 M}, \tag{68}$$

$$\frac{\partial E}{\partial a_1^{-1}} = -\frac{\hbar^2 C_a^{(1)}}{32\pi^2 M}, \tag{69}$$

$$\frac{\partial E}{\partial b_1} = \frac{\hbar^2 C_b^{(1)}}{128\pi^2 M}, \tag{70}$$

$$\frac{\partial E}{\partial u} = \lambda\left[\frac{\hbar^2\mathcal{P}_\lambda}{128\pi^2 M} - \lambda\frac{\hbar^2 C_a^{(1)}}{16\pi^2 M}\right], \tag{71}$$

$$\frac{\partial E}{\partial v} = \frac{3\lambda\hbar^2\mathcal{P}_\lambda}{128\pi^2 M}. \tag{72}$$

In the case of $\lambda = 0$, Eqs. (68), (69), and (70) simply reduce to the common form of the adiabatic energy relations for s- and p-wave interactions [10,17], with regard to the scattering length (or volume) as well as effective range. We should note that for the s-wave interaction, there is a difference of the factor 8π from the well-known form of adiabatic energy relations. This is because we include the spherical harmonics $Y_{00}(\hat{\mathbf{r}}) = 1/\sqrt{4\pi}$ in the s-partial wave function. Besides, an additional factor $1/2$ is introduced to keep the definition of contacts consistent with those in the tail of the momentum distribution at large $\mathbf{q}$. In the presence of SO coupling, two additional adiabatic energy relations appear, i.e., Eqs. (71) and (72), and a new contact $\mathcal{P}_\lambda$ is introduced.

C. The high-frequency tail of the rf spectroscopy

Next, we discuss the asymptotic behavior of the rf spectroscopy at high frequency. The basic ideal of the rf transition is as follows. For an atomic Fermi gas with two hyperfine states, denoted as $|\uparrow\rangle$ and $|\downarrow\rangle$, the rf field drives transitions between one of the hyperfine states (i.e., $|\downarrow\rangle$) and an empty hyperfine state $|3\rangle$ with a bare atomic hyperfine energy difference $\hbar\omega_{3\downarrow}$ due to the magnetic field splitting [57,58]. The universal scaling behavior at high frequency of the rf response of the system is governed by contacts. In this subsection, we are going to show how the contacts defined by the adiabatic energy relations characterize such high-frequency scalings of the rf transition in 3D Fermi gases with 3D SO coupling. Here, we will present a two-body derivation first, which may avoid complicated notations as much as possible, and the results can easily be generalized to many-body systems later. The rf field driving the spin-down particle to the state $|3\rangle$ is described by

$$\mathcal{H}_{\rm rf} = \gamma_{\rm rf}\sum_{\mathbf{k}}(e^{-i\omega t}c^\dagger_{3\mathbf{k}}c_{\downarrow\mathbf{k}} + e^{i\omega t}c^\dagger_{\downarrow\mathbf{k}}c_{3\mathbf{k}}), \tag{73}$$

where $\gamma_{\rm rf}$ denotes the strength of the rf drive, ω is the rf frequency, and $c^\dagger_{\sigma\mathbf{k}}$ and $c_{\sigma\mathbf{k}}$ are the creation and annihilation operators for fermions with the momentum $\mathbf{k}$ in the spin states $|\sigma\rangle$, respectively.

For any two-body state $|\Psi_{2b}\rangle$, we may write it in the momentum space as

$$|\Psi_{2b}\rangle = \sum_{\sigma_1\sigma_2}\sum_{\mathbf{k}_1\mathbf{k}_2}\tilde{\phi}_{\sigma_1\sigma_2}(\mathbf{k}_1,\mathbf{k}_2)c^\dagger_{\sigma_1\mathbf{k}_1}c^\dagger_{\sigma_2\mathbf{k}_2}|0\rangle, \tag{74}$$

where $\tilde{\phi}_{\sigma_1\sigma_2}(\mathbf{k}_1,\mathbf{k}_2)$ is the Fourier transform of $\phi_{\sigma_1\sigma_2}(\mathbf{r}_1,\mathbf{r}_2)\equiv\langle\mathbf{r}_1,\mathbf{r}_2;\sigma_1,\sigma_2|\Psi_{2b}\rangle$, i.e.,

$$\tilde{\phi}_{\sigma_1\sigma_2}(\mathbf{k}_1,\mathbf{k}_2) = \int d\mathbf{r}_1 d\mathbf{r}_2\phi_{\sigma_1\sigma_2}(\mathbf{r}_1,\mathbf{r}_2)e^{-i\mathbf{k}_1\cdot\mathbf{r}_1}e^{-i\mathbf{k}_2\cdot\mathbf{r}_2}, \tag{75}$$

and $\sigma_i=\uparrow,\downarrow$ denotes the spin of the ith particle. The specific form of $\tilde{\phi}_{\sigma_1\sigma_2}(\mathbf{k}_1,\mathbf{k}_2)$ can easily be obtained by using that of the two-body wave function $\langle\mathbf{r}_1,\mathbf{r}_2;\sigma_1,\sigma_2|\Psi_{2b}\rangle$ in the coordinate space, i.e., Eq. (5). Acting Eq. (73) onto Eq. (74), we obtain the two-body wave function after the rf transition,

$$\begin{aligned}&\mathcal{H}_{\rm rf}|\Psi_{2b}\rangle\\&= \gamma_{\rm rf}e^{-i\omega t}\sum_{\mathbf{k}_1\mathbf{k}_2}[\tilde{\phi}_{\downarrow\uparrow}(\mathbf{k}_1,\mathbf{k}_2)c^\dagger_{3\mathbf{k}_1}c^\dagger_{\uparrow\mathbf{k}_2} - \tilde{\phi}_{\uparrow\downarrow}(\mathbf{k}_1,\mathbf{k}_2)c^\dagger_{3\mathbf{k}_2}c^\dagger_{\uparrow\mathbf{k}_1}\\&\quad + \tilde{\phi}_{\downarrow\downarrow}(\mathbf{k}_1,\mathbf{k}_2)(c^\dagger_{3\mathbf{k}_1}c^\dagger_{\downarrow\mathbf{k}_2} - c^\dagger_{3\mathbf{k}_2}c^\dagger_{\downarrow\mathbf{k}_1})]|0\rangle.\end{aligned} \tag{76}$$

The physical meaning of Eq. (76) is apparent: after the rf transition, the atom with initial spin state $|\downarrow\rangle$ is driven to the empty spin state $|3\rangle$, while the other one remains in the spin state $|\uparrow\rangle$. Therefore, there are totally four possible final two-body states with, respectively, possibilities of $|\tilde{\phi}_{\downarrow\uparrow}|^2$, $|\tilde{\phi}_{\uparrow\downarrow}|^2$, $|\tilde{\phi}_{\downarrow\downarrow}|^2$, and $|\tilde{\phi}_{\downarrow\downarrow}|^2$. Taking all these final states into account, and according to the Fermi's golden rule [29], the two-body rf transition rate is therefore given by the Franck-Condon factor,

$$\begin{aligned}\Gamma_2(\omega) = \frac{2\pi\gamma_{\rm rf}^2}{\hbar}\sum_{\mathbf{k}_1\mathbf{k}_2}(|\tilde{\phi}_{\downarrow\uparrow}|^2 + |\tilde{\phi}_{\uparrow\downarrow}|^2 + 2|\tilde{\phi}_{\downarrow\downarrow}|^2)\\ \times\,\delta(\hbar\omega-\triangle E),\end{aligned} \tag{77}$$

where $\triangle E$ is the energy difference between the final and initial states, and takes the form of

$$\Delta E = \frac{\hbar^2k^2}{M} - \frac{\hbar^2q^2}{M} + \hbar\omega_{3\downarrow}, \tag{78}$$

where $\mathbf{k}=(\mathbf{k}_1-\mathbf{k}_2)/2$, $\hbar^2q^2/M$ is the relative energy of two fermions in the initial state, and $\omega_{3\downarrow}\equiv\omega_3-\omega_\downarrow$ is the bare hyperfine splitting between the spin states $|3\rangle$ and $|\downarrow\rangle$, and can be set to 0 without loss of generality. Now, we are interested in the asymptotic form of $\Gamma_2(\omega)$ at large ω but still small compared to $\hbar/M\epsilon^2$, which is determined by the short-range behavior when two fermions get as close as ϵ. Combining Eqs. (75) and (77), as well as the asymptotic form of the two-body wave function Eq. (17) at $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2\sim 0$, we finally obtain the asymptotic behavior of the rf response of 3D SO-coupled Fermi gases at large ω,

$$\Gamma_2(\omega) = \frac{M\gamma_{\rm rf}^2}{16\pi^2\hbar^3}\left[\frac{c_a^{(1)}}{(M\omega/\hbar)^{1/2}} + \frac{c_a^{(0)}+3c_b^{(1)}/4+\lambda p_\lambda}{(M\omega/\hbar)^{3/2}}\right], \tag{79}$$

where $c_a^{(0)}$, $c_a^{(1)}$, $c_b^{(1)}$, and p_λ are contacts for a two-body system with $\mathcal{N}=1$ in the definitions Eqs. (65)–(67).

For many-body systems, all possible $\mathcal{N}=N(N-1)/2$ pairs may contribute to the high-frequency tail of the rf spectroscopy, while high-order contributions from more than two fermions are ignored. Then we can generalize the above two-body picture to many-body systems by simply redefining the constant $\mathcal{N}$ into the contacts, and then obtain

$$\Gamma_N(\omega) = \frac{M\gamma_{\rm rf}^2}{16\pi^2\hbar^3}\left[\frac{C_a^{(1)}}{(M\omega/\hbar)^{1/2}} + \frac{C_a^{(0)}+3C_b^{(1)}/4+\lambda\mathcal{P}_\lambda}{(M\omega/\hbar)^{3/2}}\right], \tag{80}$$

where $C_a^{(0)}$, $C_a^{(1)}$, $C_b^{(1)}$, and $\mathcal{P}_\lambda$ are corresponding contacts for many-body systems. In the absence of SO coupling, Eq. (80) simply reduces to the ordinary asymptotic behaviors of the rf response for s- and p-wave interactions, respectively [13,59].

D. Pair correlation function at short distances

The physical meaning of pair correlation function $g_2(\mathbf{s}_1,\mathbf{s}_2)$ is apparent, which gives the probability of finding two fermions with one at the position $\mathbf{s}_1$ and the other one at the position $\mathbf{s}_2$ simultaneously. It is defined as $g_2(\mathbf{s}_1,\mathbf{s}_2)\equiv\langle\hat{\rho}(\mathbf{s}_1)\hat{\rho}(\mathbf{s}_2)\rangle$, here $\hat{\rho}(\mathbf{s})=\sum_i\delta(\mathbf{s}-\mathbf{r}_i)$ denotes the density operator at the position $\mathbf{s}$. Then we can formally write a pure many-body state $|\Psi\rangle$ of N fermions as [29]

$$\begin{aligned}g_2(\mathbf{s}_1,\mathbf{s}_2) &= \int d\mathbf{r}_1 d\mathbf{r}_2\cdots d\mathbf{r}_N\langle\Psi|\hat{\rho}(\mathbf{s}_1)\hat{\rho}(\mathbf{s}_2)|\Psi\rangle\\&= N(N-1)\int d\mathbf{X}'|\Psi(\mathbf{X},\mathbf{r})|^2,\end{aligned} \tag{81}$$

where $\mathbf{r}=\mathbf{s}_1-\mathbf{s}_2$ is relative coordinates of the pair fermions at positions $\mathbf{s}_1$ and $\mathbf{s}_2$, and $\mathbf{X}'$ denotes the degrees of freedom of all the other fermions. If we further integrate over the c.m. coordinate of the pair, then we can define the spatially integrated pair correlation function as

$$G_2(\mathbf{r}) \equiv N(N-1)\int d\mathbf{X}|\Psi(\mathbf{X},\mathbf{r})|^2, \tag{82}$$

and $\mathbf{X}$ includes the c.m. coordinate $\mathbf{R}=(\mathbf{s}_1+\mathbf{s}_2)/2$ of the pair besides $\mathbf{X}'$. Inserting the short-range form of many-body wave functions for SO coupled Fermi gases, i.e., Eq. (17) into Eq. (82), we find

$$\begin{aligned}G_2(\mathbf{r}) \approx{}& N(N-1)\int d\mathbf{X}\bigg\{\frac{\alpha_1\alpha_1^*\Omega_1\Omega_1^*}{r^4}\\&+\frac{\alpha_0^*\alpha_1\Omega_0^*\Omega_1+\alpha_0\alpha_1^*\Omega_0\Omega_1^*}{r^3}+\bigg[\alpha_0\alpha_0^*\Omega_0\Omega_0^*\\&+k^2\alpha_1\alpha_1^*\Omega_1\Omega_1^*+\lambda(\alpha_0^*\alpha_1+\alpha_0\alpha_1^*)\Omega_1^*\Omega_1\\&+\lambda u\alpha_1^*\alpha_1(\Omega_0^*\Omega_1+\Omega_0\Omega_1^*)\\&-\frac{\alpha_0^*\alpha_1\Omega_0^*\Omega_1+\alpha_0\alpha_1^*\Omega_0\Omega_1^*}{a_0}\bigg]\frac{1}{r^2}+O(r^{-1})\bigg\}.\end{aligned} \tag{83}$$

Further, if we are only care about the dependence of $G_2(\mathbf{r})$ on the amplitude of $r=|\mathbf{r}|$, then we can integrate over the direction of $\mathbf{r}$, and use the definitions of contacts Eqs. (65)–(67), then it yields

$$\begin{aligned}G_2(r)\approx{}&\frac{1}{16\pi^2}\bigg[\frac{C_a^{(1)}}{r^4}+\bigg(C_a^{(0)}+\frac{C_b^{(1)}}{2}+\lambda\frac{\mathcal{P}_\lambda}{2}\bigg)\frac{1}{r^2}\\&+\bigg(-\frac{2C_a^{(0)}}{a_0}-\frac{2C_a^{(1)}}{3a_1}+\frac{b_1C_b^{(1)}}{6}+\lambda(u+v)\frac{\mathcal{P}_\lambda}{2}\bigg)\frac{1}{r}\bigg],\end{aligned} \tag{84}$$

which reduces to the results in the absence of the SO coupling for s- and p-wave interactions, respectively [9,13,31,60,61].

E. Grand canonical potential and pressure relation

In this subsection, let us first consider the grand thermodynamic potential $\mathcal{J}$ for a homogeneous system, which takes the form of [62]

$$\mathcal{J}\equiv -PV = E-TS-\mu N, \tag{85}$$

where P, V, T, S, μ, N are, respectively, the pressure, volume, temperature, entropy, chemical potential, and total particle number. The grand canonical potential $\mathcal{J}$ is the function of V, T, S, and takes the following differential form:

$$d\mathcal{J}=-PdV-SdT-Nd\mu. \tag{86}$$

For the two-body microscopic parameters, we may evaluate their dimensions as $a_0\sim \text{Length}^1$, $a_1\sim\text{Length}^3$, $b_1\sim\text{Length}^{-1}$, $u\sim\text{Length}^{-1}$, and $v\sim\text{Length}^{-1}$. Therefore, there are basically following energy scales in the grand thermodynamic potential, i.e., k_BT, μ, $\hbar^2/MV^{2/3}$, $\hbar^2/Ma_0^2$, $\hbar^2/Ma_1^{2/3}$, $\hbar^2b_1^2/M$, $\hbar^2u^2/M$, $\hbar^2v^2/M$. Then we may express the thermodynamic potential $\mathcal{J}$ in the terms of a dimensionless function $\bar{\mathcal{J}}$ as [21,63]

$$\begin{aligned}&\mathcal{J}(V,T,\mu,a_0,a_1,b_1,u,v)\\&\quad=k_BT\bar{\mathcal{J}}\bigg(\frac{\hbar^2/MV^{2/3}}{k_BT},\frac{\mu}{k_BT},\frac{\hbar^2/Ma_0^2}{k_BT},\\&\qquad\frac{\hbar^2/Ma_1^{2/3}}{k_BT},\frac{\hbar^2b_1^2/M}{k_BT},\frac{\hbar^2u^2/M}{k_BT},\frac{\hbar^2v^2/M}{k_BT}\bigg).\end{aligned} \tag{87}$$

Consequently, the simple scaling law can be deduced as

$$\begin{aligned}&\mathcal{J}(\gamma^{-3/2}V,\gamma T,\gamma\mu,\gamma^{-1/2}a_0,\gamma^{-3/2}a_1,\gamma^{1/2}b_1,\gamma^{1/2}u,\gamma^{1/2}v)\\&\quad=\gamma\mathcal{J}(V,T,\mu,a_0,a_1,b_1,u,v).\end{aligned} \tag{88}$$

Keeping all other variables constant, the derivative of Eq. (88) with respect to γ at $\gamma=1$ simply yields

$$\begin{aligned}&\bigg(-\frac{3V}{2}\frac{\partial}{\partial V}+T\frac{\partial}{\partial T}+\mu\frac{\partial}{\partial\mu}-\frac{a_0}{2}\frac{\partial}{\partial a_0}\\&\quad-\frac{3a_1}{2}\frac{\partial}{\partial a_1}+\frac{b_1}{2}\frac{\partial}{\partial b_1}+\frac{u}{2}\frac{\partial}{\partial u}+\frac{v}{2}\frac{\partial}{\partial v}\bigg)\mathcal{J}=\mathcal{J}.\end{aligned} \tag{89}$$

Since

$$\mathcal{J}-T\frac{\partial\mathcal{J}}{\partial T}-\mu\frac{\partial\mathcal{J}}{\partial\mu}=\mathcal{J}+TS+\mu N=E, \tag{90}$$

and the variation of the grand thermodynamic potential $\delta\mathcal{J}$ with respect to the two-body parameters at fixed volume V, temperature T, and chemical potential μ is equal to that of the energy δE at fixed volume V, entropy S, and particle number N, i.e., $(\delta\mathcal{J})_{V,T,\mu}=(\delta E)_{V,S,N}$ and $V\frac{\partial\mathcal{J}}{\partial V}=\mathcal{J}$, we easily obtain from Eqs. (89) and (90)

$$-\frac{3}{2}\mathcal{J}-\frac{a_0}{2}\frac{\partial E}{\partial a_0}-\frac{3a_1}{2}\frac{\partial E}{\partial a_1}+\frac{b_1}{2}\frac{\partial E}{\partial b_1}+\frac{u}{2}\frac{\partial E}{\partial u}+\frac{v}{2}\frac{\partial E}{\partial v}=E. \tag{91}$$

Further, by using adiabatic energy relations, Eq. (91) becomes

$$\begin{aligned}\mathcal{J}={}&-\frac{2}{3}E-\frac{\hbar^2}{96\pi^2Ma_0}\bigg(C_a^{(0)}-\lambda\frac{\mathcal{P}_\lambda}{2}\bigg)\\&-\frac{\hbar^2C_a^{(1)}}{32\pi^2Ma_1}+\frac{\hbar^2b_1C_b^{(1)}}{384\pi^2M}\\&-\frac{\lambda u\hbar^2}{48\pi^2M}\bigg(\lambda C_a^{(1)}-\frac{\mathcal{P}_\lambda}{8}\bigg)+\frac{\lambda v\mathcal{P}_\lambda\hbar^2}{128\pi^2M}.\end{aligned} \tag{92}$$

Consequently, we obtain the pressure relation, i.e., $P=-\mathcal{J}/V$, which respectively reduces to the well-known results in the absence of the spin-orbit coupling

$$P=\frac{2E}{3V}+\frac{\hbar^2C_a^{(0)}}{96\pi^2MVa_0} \tag{93}$$

for s-wave interactions, which is consistent with the result of Refs. [11,61,64], and

$$P=\frac{2E}{3V}+\frac{\hbar^2C_a^{(1)}}{32\pi^2MVa_1}-\frac{b_1\hbar^2C_b^{(1)}}{384\pi^2MV} \tag{94}$$

for p-wave interactions, which is consistent with the result of Ref. [13].

IV. UNIVERSAL RELATIONS IN 2D SYSTEMS WITH RASHBA SO COUPLING

The derivation of the universal relations for 3D Fermi gases with 3D SO coupling can directly be generalized to those for 2D systems with 2D SO coupling. In this section, with the short-range form of the two-body wave function for 2D systems with 2D SO coupling in hands, i.e., Eq. (25), we are going to discuss Tan's universal relations for 2D Fermi gases with 2D SO coupling, by taking into account only two-body correlations.

A. Adiabatic energy relations

Let us consider how the energy of the SO-coupled system varies with the two-body interaction in 2D systems with 2D SO coupling. The two wave functions of a many-body system $\Psi(\mathbf{r})$ and $\Psi'(\mathbf{r})$, corresponding to different interatomic interaction strengths, satisfy the Schrödinger equation with different energies, i.e., formally as Eqs. (30) and (31). Analogously, by subtracting $[31]^* \times \Psi$ from $\Psi'^* \times [30]$, and integrating over the domain $\mathcal{D}_\epsilon$, the set of all configurations $(\mathbf{r}_i, \mathbf{r}_j)$ in which $r = |\mathbf{r}_i - \mathbf{r}_j| > \epsilon$, we obtain

$$\begin{aligned}(E - E')\int_{\mathcal{D}_\epsilon}\prod_{i=1}^{N} d\mathbf{r}_i \Psi'^*\Psi &= -\frac{\hbar^2}{M}\mathcal{N}\int_{r>\epsilon} d\mathbf{X}d\mathbf{r}\big[\Psi'^*\nabla^2_{\mathbf{r}}\Psi - \big(\nabla^2_{\mathbf{r}}\Psi'^*\big)\Psi\big] \\ &\quad + \frac{\hbar^2\lambda}{M}\mathcal{N}\int_{r>\epsilon} d\mathbf{X}d\mathbf{r}[\Psi'^*(\hat{Q}\Psi) - (\hat{Q}\Psi')^*\Psi], \end{aligned} \tag{95}$$

where $\mathcal{N} = N(N-1)/2$ is again the number of all the possible ways to pair atom. Using the Gauss' theorem, the first term on the RHS can be written as

$$\begin{aligned} &-\frac{\hbar^2}{M}\mathcal{N}\int_{r>\epsilon} d\mathbf{X}d\mathbf{r}\big[\Psi'^*\nabla^2_{\mathbf{r}}\Psi - \big(\nabla^2_{\mathbf{r}}\Psi'^*\big)\Psi\big] \\ &= -\frac{\hbar^2}{M}\mathcal{N}\oint_{r=\epsilon}[\Psi'^*\nabla_{\mathbf{r}}\Psi - (\nabla_{\mathbf{r}}\Psi'^*)\Psi]\cdot\hat{\mathbf{n}}d\mathcal{S}, \\ &= \frac{\hbar^2\epsilon}{M}\mathcal{N}\int d\mathbf{X}\sum_{m=0,\pm1}\left(\psi'^*_m\frac{\partial}{\partial r}\psi_m - \psi_m\frac{\partial}{\partial r}\psi'^*_m\right)_{r=\epsilon}, \end{aligned} \tag{96}$$

where $\mathcal{S}$ is the boundary of $\mathcal{D}_\epsilon$ that the distance between the two fermions in the pair (i, j) is ϵ, $\hat{\mathbf{n}}$ is the direction normal to $\mathcal{S}$, but is opposite to the radial direction, and ψ_0 ($\psi_{\pm1}$) is the s-wave (p-wave) component of the two-body wave function as defined in Eq. (21). Since

$$\hat{Q}(\mathbf{r})\Psi = \sum_{m=\pm1}\left[-\frac{\sqrt{2}}{r}\frac{\partial}{\partial r}(r\psi_m)\Omega_0(\hat{\mathbf{r}}) + \sqrt{2}\frac{\partial\psi_0}{\partial r}\Omega_m(\hat{\mathbf{r}})\right], \tag{97}$$

we find that the second term on the RHS of Eq. (95) can be written as

$$\begin{aligned} &\frac{\hbar^2\lambda}{M}\mathcal{N}\int_{r>\epsilon} d\mathbf{X}d\mathbf{r}[\Psi'^*(\hat{Q}(\mathbf{r})\Psi) - (\hat{Q}(\mathbf{r})\Psi')^*\Psi] \\ &= \frac{\sqrt{2}\lambda\hbar^2\epsilon}{M}\mathcal{N}\int d\mathbf{X}\sum_{m=\pm1}(\psi'^*_0\psi_m - \psi'^*_m\psi_0)_{r=\epsilon}. \end{aligned} \tag{98}$$

Combining Eqs. (95), (96), and (98), we have

$$(E - E')\int_{\mathcal{D}_\epsilon}\prod_{i=1}^{N} d\mathbf{r}_i\Psi'^*\Psi = \frac{\hbar^2\epsilon}{M}\mathcal{N}\int d\mathbf{X}\sum_{m=0,\pm1}\left(\psi'^*_m\frac{\partial}{\partial r}\psi_m - \psi_m\frac{\partial}{\partial r}\psi'^*_m\right)_{r=\epsilon} + \frac{\sqrt{2}\lambda\hbar^2\epsilon}{M}\mathcal{N}\int d\mathbf{X}\sum_{m=\pm1}\left(\psi'^*_0\psi_m - \psi'^*_m\psi_0\right)_{r=\epsilon}. \tag{99}$$

Inserting the asymptotic form of the many-body wave function Eq. (25) into Eq. (99), and letting $E' \to E$ and $\Psi' \to \Psi$, we arrive at

$$\begin{aligned}\delta E\cdot\int_{\mathcal{D}_\epsilon}\prod_{i=1}^{N} d\mathbf{r}_i|\Psi|^2 &= \frac{\hbar^2}{M}\left(\mathcal{I}_a^{(0)} + \sum_{m=\pm1}\frac{\sqrt{2}}{2}\lambda\mathcal{I}_\lambda^{(m)}\right)\delta\ln a_0 + \sum_{m=\pm1}\left\{-\frac{\pi\hbar^2\mathcal{I}_a^{(m)}}{2M}\delta a_1^{-1} + \left(\mathcal{E}_m - \frac{\lambda\hbar^2\mathcal{I}_\lambda^{(m)}}{\sqrt{2}M}\right)\delta\ln b_1\right. \\ &\quad \left. - \frac{\hbar^2}{M}\left[\left(\sqrt{2}\lambda^2\mathcal{I}_a^{(m)} + \frac{\lambda}{2}\mathcal{I}_\lambda^{(m)}\right) + \frac{\sqrt{2}\lambda^2}{2}\mathcal{I}_p\right]\delta u + \frac{\lambda\hbar^2}{M}\mathcal{I}_\lambda^{(m)}\delta v - \left(\ln\frac{\epsilon}{2b_1} + \gamma\right)\mathcal{I}_a^{(m)}\delta E\right\}, \end{aligned} \tag{100}$$

where

$$\mathcal{I}_a^{(m)} = \mathcal{N}\int d\mathbf{X}|\alpha_m|^2, \tag{101}$$

$$\mathcal{E}_m = \mathcal{N}\int d\mathbf{X}\alpha_m^*(E - \hat{T})\alpha_m \tag{102}$$

for $m = 0, \pm1$,

$$\mathcal{I}_\lambda^{(\pm1)} = \mathcal{N}\int d\mathbf{X}\alpha_0^*\alpha_{\pm1} + \text{c.c.}, \tag{103}$$

$$\mathcal{E}_\lambda^{(\pm1)} = \mathcal{N}\int d\mathbf{X}\alpha_0^*(E - \hat{T})\alpha_{\pm1} + \text{c.c.}, \tag{104}$$

$$\mathcal{I}_p = \mathcal{N}\int d\mathbf{X}\alpha_{-1}^*\alpha_1 + \text{c.c.}, \tag{105}$$

and $\hat{T}(\mathbf{X})$ is the kinetic operator including the c.m. motion of the pair as well as those of all the rest fermions. With the help of the normalization of the wave function (see Appendix B)

$$\int_{\mathcal{D}_\epsilon}\prod_{i=1}^{N}d\mathbf{r}_i|\Psi|^2 = 1-\sum_{m=\pm1}\left(\ln\frac{\epsilon}{2\mathrm{b}_1}+\gamma\right)\mathcal{I}_a^{(m)}, \tag{106}$$

we can further simplify Eq. (100) as

$$\delta E = \frac{\hbar^2}{M}\left(\mathcal{I}_a^{(0)}+\sum_{m=\pm1}\lambda\frac{\mathcal{I}_\lambda^{(m)}}{\sqrt{2}}\right)\delta\ln a_0+\sum_{m=\pm1}\left\{-\frac{\pi\hbar^2\mathcal{I}_a^{(m)}}{2M}\delta a_1^{-1}+\left(\mathcal{E}_m-\lambda\frac{\hbar^2\mathcal{I}_\lambda^{(m)}}{\sqrt{2}M}\right)\delta\ln b_1 - \lambda\frac{\hbar^2}{M}\left[\sqrt{2}\lambda\mathcal{I}_a^{(m)}+\frac{\mathcal{I}_\lambda^{(m)}}{2}+\lambda\frac{\mathcal{I}_p}{\sqrt{2}}\right]\delta u+\frac{\lambda\hbar^2}{M}\mathcal{I}_\lambda^{(m)}\delta v\right\}, \tag{107}$$

which characterizes how the energy of a 2D system with 2D SO coupling varies as the scattering parameters adiabatically change and yields the following set of adiabatic energy relations:

$$\frac{\partial E}{\partial\ln a_0} = \frac{\hbar^2}{M}\left(\mathcal{I}_a^{(0)}+\frac{\lambda}{\sqrt{2}}\sum_{m=\pm1}\mathcal{I}_\lambda^{(m)}\right), \tag{108}$$

$$\frac{\partial E}{\partial a_1^{-1}} = -\frac{\pi\hbar^2}{2M}\sum_{m=\pm1}\mathcal{I}_a^{(m)}, \tag{109}$$

$$\frac{\partial E}{\partial\ln b_1} = \sum_{m=\pm1}\left(\mathcal{E}_m-\lambda\frac{\hbar^2\mathcal{I}_\lambda^{(m)}}{\sqrt{2}M}\right), \tag{110}$$

$$\frac{\partial E}{\partial u} = \frac{-\hbar^2\lambda}{\sqrt{2}M}\sum_{m=\pm1}\left[\frac{\mathcal{I}_\lambda^{(m)}}{\sqrt{2}}+\lambda\left(2\mathcal{I}_a^{(m)}+\mathcal{I}_p\right)\right], \tag{111}$$

$$\frac{\partial E}{\partial v} = \frac{\hbar^2\lambda}{M}\sum_{m=\pm1}\mathcal{I}_\lambda^{(m)}. \tag{112}$$

Obviously, there are additional two new adiabatic energy relations appear, i.e., Eqs. (111) and (112), which originate from new scattering parameters introduced by SO coupling.

B. Tail of the momentum distribution at large q

In general, the momentum distribution at large q is determined by the short-range behavior of the many-body wave function when the fermions i and j are close. Similarly as in the 3D case, we can formally write the many-body wave function $\Psi_{2\mathrm{D}}$ at $\mathbf{r}\approx\mathbf{0}$ as the following ansatz:

$$\Psi_{2\mathrm{D}}(\mathbf{X},\mathbf{r}) = [\alpha_0\ln r+\mathcal{B}_0+\mathcal{C}_0r^2\ln r]\Omega_0(\hat{\mathbf{r}}) + \sum_m\left[\frac{\alpha_m}{r}+\mathcal{B}_mr\ln r+\mathcal{C}_mr\right]\Omega_m(\hat{\mathbf{r}})+O(r^2), \tag{113}$$

where α_j, $\mathcal{B}_j$, and $\mathcal{C}_j$ ($j=0,\pm1$) are all regular functions of $\mathbf{X}$. Comparing Eqs. (25) and (113) at small r, we find that

$$\mathcal{B}_0(\mathbf{X}) = \alpha_0(\gamma-\ln 2a_0)+\sum_{m=\pm1}\alpha_m\lambda u, \tag{114}$$

$$\mathcal{B}_m(\mathbf{X}) = -\frac{\alpha_mk^2}{2}+\lambda\frac{\alpha_0}{\sqrt{2}}, \tag{115}$$

$$\mathcal{C}_0(\mathbf{X}) = -\frac{\alpha_0k^2}{4}, \tag{116}$$

$$\mathcal{C}_m(\mathbf{X}) = \alpha_m\left(-\frac{\pi}{4a_1}+\frac{1-2\gamma}{4}k^2\right) + \alpha_0\lambda v+\left(\frac{\alpha_mk^2}{2}-\lambda\frac{\alpha_0}{\sqrt{2}}\right)\ln 2\mathrm{b}_1. \tag{117}$$

In the following, we derive the momentum distribution at large $\mathbf{q}$ but still smaller than ϵ^{-1}. Using the plane-wave expansion

$$e^{i\mathbf{q}\cdot\mathbf{r}} = \sqrt{2\pi}\sum_{m=0}^{\infty}\sum_{\sigma=\pm}\eta_mi^mJ_m(qr)e^{-i\sigma m\varphi_q}\Omega_m^{(\sigma)}(\varphi), \tag{118}$$

here $\eta_m = 1/2$ for $m=0$, $\eta_m=1$ for $m\geqslant 1$, and φ_q denotes the azimuthal angle of $\mathbf{q}$, we have

$$\int d\mathbf{r}\alpha_0\ln r\Omega_0(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = -\frac{2\pi}{q^2}\alpha_0\Omega_0(\hat{\mathbf{q}}), \tag{119}$$

$$\int d\mathbf{r}\mathcal{B}_0\Omega_0(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = 0, \tag{120}$$

$$\int d\mathbf{r}\mathcal{C}_0r^2\ln r\Omega_0(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = \frac{8\pi}{q^4}\mathcal{C}_0\Omega_0(\hat{\mathbf{q}}), \tag{121}$$

$$\int d\mathbf{r}\frac{\alpha_m}{r}\Omega_m(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = -i\frac{2\pi}{q}\alpha_m\Omega_m(\hat{\mathbf{q}}), \tag{122}$$

$$\int d\mathbf{r}\mathcal{B}_mr\ln r\Omega_m(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = i\frac{4\pi}{q^3}\mathcal{B}_m\Omega_m(\hat{\mathbf{q}}), \tag{123}$$

$$\int d\mathbf{r}\mathcal{C}_mr\Omega_m(\hat{\mathbf{r}})e^{-i\mathbf{q}\cdot\mathbf{r}} = 0, \tag{124}$$

where $\hat{\mathbf{q}}$ is the angular part of $\mathbf{q}$. Inserting Eqs. (119)–(124) into Eq. (49), we can obtain the total momentum distribution

$n_{2D}(\mathbf{q})$ at large $\mathbf{q}$ as

$$n_{2D}(\mathbf{q}) \approx \mathcal{N}\int d\mathbf{X}\sum_{m,m'}\alpha_m\alpha_{m'}^*\Omega_m(\hat{\mathbf{q}})\Omega_{m'}^*(\hat{\mathbf{q}})\frac{8\pi^2}{q^2} + i\sum_m[\alpha_0^*\alpha_m\Omega_0^*(\hat{\mathbf{q}})\Omega_m(\hat{\mathbf{q}}) - \alpha_0\alpha_m^*\Omega_0(\hat{\mathbf{q}})\Omega_m^*(\hat{\mathbf{q}})]\frac{8\pi^2}{q^3}$$
$$+\left\{\alpha_0\alpha_0^*\Omega_0(\hat{\mathbf{q}})\Omega_0^*(\hat{\mathbf{q}}) + \sum_{m,m'}[-\sqrt{2}\lambda(\alpha_0\alpha_m^*\Omega_{m'}(\hat{\mathbf{q}})\Omega_m^*(\hat{\mathbf{q}}) + \alpha_0^*\alpha_m\Omega_m(\hat{\mathbf{q}})\Omega_{m'}^*(\hat{\mathbf{q}})) + 2k^2\alpha_m\alpha_{m'}^*\Omega_m(\hat{\mathbf{q}})\Omega_{m'}^*(\hat{\mathbf{q}})]\right\}$$
$$\times\frac{8\pi^2}{q^4} + O(q^{-5}), \tag{125}$$

and the summations are over $m, m' = \pm1$. If we are only interested in the dependence of $n_{2D}(\mathbf{q})$ on the amplitude of $\mathbf{q}$, then the expression can further be simplified by integrating $n_{2D}(\mathbf{q})$ over the direction of $\mathbf{q}$, and all the odd-order terms of q^{-1} vanish. Finally, we arrive at

$$n_{2D}(q) = \frac{\sum_{m=\pm1}C_a^{(m)}}{q^2} + \left[C_a^{(0)} + \sum_{m=\pm1}\left(C_b^{(m)} - \lambda\mathcal{P}_\lambda^{(m)}\right)\right]\frac{1}{q^4} + O(q^{-6}), \tag{126}$$

where the contacts are defined as

$$C_a^{(j)} = 8\pi^2\mathcal{I}_a^{(j)} \tag{127}$$

for $j = 0, \pm1$, and

$$C_b^{(m)} = \frac{16\pi^2 M}{\hbar^2}\mathcal{E}_m, \tag{128}$$
$$\mathcal{P}_\lambda^{(m)} = 8\sqrt{2}\pi^2\mathcal{I}_\lambda^{(m)} \tag{129}$$

for $m = \pm1$. With these definitions in hand, the adiabatic energy relations Eqs. (108)–(112) can alternatively be written as

$$\frac{\partial E}{\partial \ln a_0} = \frac{\hbar^2}{8\pi^2 M}\left(C_a^{(0)} + \frac{\lambda}{2}\sum_{m=\pm1}\mathcal{P}_\lambda^{(m)}\right), \tag{130}$$
$$\frac{\partial E}{\partial a_1^{-1}} = -\frac{\hbar^2}{16\pi M}\sum_{m=\pm1}C_a^{(m)}, \tag{131}$$
$$\frac{\partial E}{\partial \ln b_1} = \frac{\hbar^2}{16\pi^2 M}\sum_{m=\pm1}\left(C_b^{(m)} - \lambda\mathcal{P}_\lambda^{(m)}\right), \tag{132}$$
$$\frac{\partial E}{\partial u} = -\frac{\hbar^2\lambda}{16\sqrt{2}\pi^2 M}\sum_{m=\pm1}\mathcal{P}_\lambda^{(m)}, \tag{133}$$
$$\frac{\partial E}{\partial v} = \frac{\hbar^2\lambda}{8\sqrt{2}\pi^2 M}\sum_{m=\pm1}\mathcal{P}_\lambda^{(m)}. \tag{134}$$

In the case of $\lambda = 0$, Eqs. (130), (131), and (132) simply reduce to the common form of the adiabatic energy relations for s- and p-wave interactions [24,29], with regard to the scattering length (or area) as well as effective range. And for the s-wave interaction, there is a difference of the factor 2π from the Ref. [24], which is because we include the angular part $1/\sqrt{2\pi}$ in the s-partial wave function. In addition, two new adiabatic energy relations, i.e., Eqs. (133) and (134), and new contacts $\mathcal{P}_\lambda^{(m)}$ appear due to SO coupling.

C. The high-frequency tail of the rf spectroscopy

We may carry out the analogous procedure as that in 3D systems with 3D SO coupling, and the two-body rf transition rate takes the form

$$\Gamma_2(\omega) = \frac{2\pi\gamma_{\rm rf}^2}{\hbar}\sum_{\mathbf{k}_1\mathbf{k}_2}(|\tilde{\phi}_{\uparrow\downarrow}|^2 + |\tilde{\phi}_{\downarrow\uparrow}|^2 + 2|\tilde{\phi}_{\downarrow\downarrow}|^2)\delta(\hbar\omega - \Delta E), \tag{135}$$

where

$$\tilde{\phi}_{\sigma_1\sigma_2}(\mathbf{k}_1,\mathbf{k}_2) = \int d\mathbf{r}_1 d\mathbf{r}_2\phi_{\sigma_1\sigma_2}(\mathbf{r}_1,\mathbf{r}_2)e^{-i\mathbf{k}_1\cdot\mathbf{r}_1}e^{-i\mathbf{k}_2\cdot\mathbf{r}_2}. \tag{136}$$

If we are only interested in the high-frequency tail of the transition rate, then we can use the asymptotic behavior of the two-body wave function for a 2D system with 2D SO coupling, i.e., Eq. (25). Combining with Eqs. (135) and (136), we obtain the two-body rf transition rate $\Gamma_2(\omega)$ as

$$\Gamma_2(\omega) = \frac{M\gamma_{\rm rf}^2}{4\pi\hbar^3}\left[\frac{c_a^{(1)}}{M\omega/\hbar} + \frac{c_a^{(0)}/2 + c_b^{(1)}/2 - \lambda p_\lambda^{(1)}}{(M\omega/\hbar)^2}\right], \tag{137}$$

where $c_a^{(0)}$, $c_a^{(1)}$, $c_b^{(1)}$, and $p_\lambda^{(1)}$ are contacts for a two-body system with $\mathcal{N} = 1$ in the definitions Eqs. (127)–(129).

For many-body systems, all $\mathcal{N} = N(N-1)/2$ pairs contribute to the transition rate. Similarly, we can redefining the constant $\mathcal{N}$ into the contacts, and then obtain

$$\Gamma_N(\omega) = \frac{M\gamma_{\rm rf}^2}{4\pi\hbar^3}\left[\frac{C_a^{(1)}}{M\omega/\hbar} + \frac{C_a^{(0)}/2 + C_b^{(1)}/2 - \lambda\mathcal{P}_\lambda^{(1)}}{(M\omega/\hbar)^2}\right], \tag{138}$$

where $C_a^{(0)}$, $C_a^{(1)}$, $C_b^{(1)}$, and $\mathcal{P}_\lambda^{(1)}$ are corresponding contacts for many-body systems. In the absence of SO coupling, Eq. (138)

simply reduces to the ordinary results for s- and p-wave interactions, respectively [29,65].

D. Pair correlation function at short distances

Let us then discuss the short-distance behavior of the pair correlation function for a 2D Fermi gases with 2D SO coupling. Inserting the asymptotic form of the many-body wave function at short distance, i.e., Eq. (25) into Eq. (82), we easily obtain spatially integrated pair correlation function $G_2(\mathbf{r})$. If we only care about the dependence of $G_2(\mathbf{r})$ on the amplitude of $r = |\mathbf{r}|$, then we may take the average of momentum distribution over the direction of $\mathbf{r}$ and obtain

$$G_2(r) \approx \frac{1}{4\pi^2}\left[\frac{\sum_{m=\pm1} C_a^{(m)}}{r^2} + C_a^{(0)}\left(\ln\frac{r}{2a_0}\right)^2 + \left(2\gamma C_a^{(0)} + \frac{\lambda u}{\sqrt{2}}\sum_{m=\pm1}\mathcal{P}_\lambda^{(m)}\right)\ln\frac{r}{2a_0} + \sum_{m=\pm1}\frac{1}{2}\left(-C_b^{(m)} + \lambda\mathcal{P}_\lambda^{(m)}\right)\ln\frac{r}{2b_1}\right]. \tag{139}$$

In the absence of SO coupling, Eq. (139) simply reduces to the ordinary results for s- and p-wave interactions, respectively [24,29].

E. Grand canonical potential and pressure relation

Similarly, according to the dimension analysis, we easily obtain

$$-\mathcal{J} - \frac{a_0}{2}\frac{\partial E}{\partial a_0} - a_1\frac{\partial E}{\partial a_1} - \frac{b_1}{2}\frac{\partial E}{\partial b_1} = E. \tag{140}$$

Further, by using adiabatic energy relations, Eq. (140) becomes

$$\mathcal{J} = -E - \frac{\hbar^2}{16\pi^2 M}\left(C_a^{(0)} + \frac{\lambda}{2}\sum_{m=\pm1}\mathcal{P}_\lambda^{(m)}\right) - \frac{\hbar^2}{16\pi M}\sum_{m=\pm1}\left[\frac{C_a^{(m)}}{a_1} + \frac{1}{2\pi}\left(C_b^{(m)} - \lambda\mathcal{P}_\lambda^{(m)}\right)\right]. \tag{141}$$

The pressure relation can be obtained by dividing both sides of Eq. (141) by $-V$, which respectively reduces to the results in the absence of SO coupling,

$$P = \frac{E}{V} + \frac{\hbar^2 C_a^{(0)}}{16\pi^2 MV}, \tag{142}$$

for s-wave interactions, which is consistent with the result of Ref. [22], and

$$P = \frac{E}{V} + \sum_{m=\pm1}\frac{\hbar^2}{16\pi MV}\left(\frac{C_a^{(m)}}{a_1} + \frac{C_b^{(m)}}{2\pi}\right) \tag{143}$$

for p-wave interactions, which is consistent with the result of Ref. [29].

V. CONCLUSIONS

We develop a perturbation method to construct the short-range form of a two-body problem in the presence of the spin-orbit coupling. For a two-body system, the center-of-mass momentum as well as the total angular momentum is a good quantum number. Then the simplest situation is that with zero center-of-mass momentum and zero total angular momentum, in which only s- and p-wave scatterings need to be taken into account. We find that two new microscopic scattering parameters in s- and p-wave channels appear because of spin-orbit coupling besides the conventional scattering length (volume) and effective range. The obtained short-range behaviors of two-body wave functions do not depend on the short-range details of interatomic potentials. Based on the constructed short-range form of two-body wave functions, we systematically study a set of universal relations for spin-orbit-coupled Fermi gases in three or two dimension, respectively. We find that new contacts need to be introduced in both three- and two-dimensional systems. However, due to different short-range behaviors of two-body wave functions for three- and two-dimensional systems, the specific forms of universal relations are distinct in different dimensions. As we anticipate, the universal relations for spin-orbit-coupled systems, such as the adiabatic energy relations, the large-momentum distributions, the high-frequency behavior of the radio-frequency responses, short-range behaviors of the pair correlation functions, grand canonical potentials, and pressure relations, are fully captured by the contacts defined.

In this work, we only consider a simplest situation of a two-body system with zero center-of-mass momentum and zero total angular momentum, in which only s- and p-wave scatterings are included. Our method can also be generalized to the case with nonzero center-of-mass momentum and nonzero total angular momentum. Then more partial waves should be involved, and we may anticipate that more new two-body parameters need to be introduced. For a many-body system consisting of fermions, the center-of-mass momentum of pairs is no longer conserved as well as the total angular momentum of pairs. Therefore, we cannot discuss the problem in the sub-Hilbert space with a specific center-of-mass momentum and total angular momentum of pairs. Then all the partial-wave scatterings of pairs should be taken into account. However, in the Bose-Einstein-condensation limit, the pairs are tightly bound and their center-of-mass motion is frozen in the zero-temperature limit. Then our discussion in the sub-Hilbert space with zero center-of-mass momentum and zero total angular momentum is still applicable. For a more general situation, such as that in the unitary limit, our results of this work need to be further developed, because all the partial waves should be taken into account theoretically. In experiments, many-body systems are usually prepared near a specific partial-wave resonance, i.e., s- or p-wave resonance. Then in the practical discussion, we may still theoretically focus on few partial waves [50,51].

ACKNOWLEDGMENTS

C.-X.Z. is supported by NSFC under Grant No.11947097. S.G.P. and K.J. are supported by NSFC under Grants No. 11974384, No. 11674358, No. 11474315, NKRDP (National Key Research and Development Program) under Grant No. 2016YFA0301503, and CAS under Grant No. YJKYYQ20170025. K.C. Wong Education Foundation (GJTD-2019-15).

APPENDIX A: NORMALIZATION OF THE WAVE FUNCTION FOR 3D SYSTEMS WITH 3D SO COUPLING

In this Appendix, we are going to derive $\int_{\mathcal{D}_\epsilon}\prod_{i=1}^N d\mathbf{r}_i|\Psi|^2$ for 3D many-body systems with 3D SO coupling. Let us consider two many-body wave functions Ψ' and Ψ, corresponding to different energies $\hbar^2k'^2/M$ and $\hbar^2k^2/M$, respectively. They should be orthogonal, i.e., $\int_{\mathcal{D}_\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'^*\Psi=0$, and therefore we have

$$\int_{r<\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'^*\Psi=-\int_{r>\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'^*\Psi. \tag{A1}$$

From the Schrödinger equation satisfied by Ψ' and Ψ outside the interaction potential, i.e., $r>\epsilon$, we easily obtain

$$\int_{r>\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'^*\Psi=\frac{\epsilon^2}{k^2-k'^2}\mathcal{N}\int d\mathbf{X}\int_{r=\epsilon}d\hat{\mathbf{r}}\left[\left(\Psi'^*\frac{\partial}{\partial r}\Psi-\Psi\frac{\partial}{\partial r}\Psi'^*\right)+\frac{\lambda}{2\pi}\left(\psi_0'^*\psi_1-\psi_1'^*\psi_0\right)\right]. \tag{A2}$$

In the presence of SO coupling, only s- and p-wave scatterings are involved in the subspace $\mathbf{K}=0$ and $\mathbf{J}=0$, and the wave function at short distance takes the form of Eq. (17). Using the asymptotic behavior of the wave function, we easily evaluate

$$\begin{aligned}\int_{r<\epsilon}\prod_{i=1}^N d\mathbf{r}_i|\Psi|^2&=-\lim_{k'\to k}\frac{1}{2}\left(\int_{r>\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'^*\Psi+\int_{r>\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'\Psi^*\right)\\&=-\mathcal{N}\int d\mathbf{X}\left\{\frac{|\alpha_1|^2}{\epsilon}+\frac{|\alpha_1|^2b_1}{2}\right\}=-\left(\frac{1}{\epsilon}+\frac{b_1}{2}\right)\mathcal{I}_a^{(1)},\end{aligned} \tag{A3}$$

which in turn yields

$$\int_{\mathcal{D}_\epsilon}\prod_{i=1}^N d\mathbf{r}_i|\Psi|^2=1+\left(\frac{1}{\epsilon}+\frac{b_1}{2}\right)\mathcal{I}_a^{(1)}. \tag{A4}$$

APPENDIX B: NORMALIZATION OF THE WAVE FUNCTION FOR 2D SYSTEM WITH 2D SO COUPLING

In this Appendix, we are going to derive $\int_{\mathcal{D}_\epsilon}\prod_{i=1}^N d\mathbf{r}_i|\Psi|^2$ for 2D many-body systems with 2D SO coupling. Let us consider two many-body wave functions Ψ' and Ψ, corresponding to different energies $\hbar^2k'^2/M$ and $\hbar^2k^2/M$, respectively. They should be orthogonal, i.e., $\int_{\mathcal{D}_\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'^*\Psi=0$, and therefore we have

$$\int_{r<\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'^*\Psi=-\int_{r>\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'^*\Psi. \tag{B1}$$

From the Schrödinger equation satisfied by Ψ' and Ψ outside the interaction potential, i.e., $r>\epsilon$, we easily obtain

$$\int_{r>\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'^*\Psi=\frac{\epsilon}{k^2-k'^2}\mathcal{N}\int d\mathbf{X}\int_{r=\epsilon}d\hat{\mathbf{r}}\left[\left(\Psi'^*\frac{\partial}{\partial r}\Psi-\Psi\frac{\partial}{\partial r}\Psi'^*\right)+\sum_{m=\pm1}\frac{\lambda}{\sqrt{2}\pi}\left(\psi_0'^*\psi_m-\psi_m'^*\psi_0\right)\right]. \tag{B2}$$

In the presence of SO coupling, only s- and p-wave scatterings are involved in the subspace $\mathbf{K}=0$ and $\mathbf{J}=0$, and the wave function at short distance takes the form of Eq. (25). Using the asymptotic behavior of the wave function, we easily evaluate

$$\begin{aligned}\int_{r<\epsilon}\prod_{i=1}^N d\mathbf{r}_i|\Psi|^2&=-\lim_{k'\to k}\frac{1}{2}\left(\int_{r>\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'^*\Psi+\int_{r>\epsilon}\prod_{i=1}^N d\mathbf{r}_i\Psi'\Psi^*\right)\\&=\mathcal{N}\int d\mathbf{X}\sum_{m=\pm1}\left(\ln\frac{\epsilon}{2\mathrm{b}_1}+\gamma\right)|\alpha_m|^2=\sum_{m=\pm1}\left(\ln\frac{\epsilon}{2\mathrm{b}_1}+\gamma\right)\mathcal{I}_a^{(m)},\end{aligned} \tag{B3}$$

which in turn yields

$$\int_{\mathcal{D}_\epsilon}\prod_{i=1}^N d\mathbf{r}_i|\Psi|^2=1-\sum_{m=\pm1}\left(\ln\frac{\epsilon}{2\mathrm{b}_1}+\gamma\right)\mathcal{I}_a^{(m)}. \tag{B4}$$

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