Expansion of the strongly interacting superfluid Fermi gas: Symmetries and self-similar regimes

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(Received 13 March 2019; revised manuscript received 9 February 2020; accepted 23 March 2020; published 16 April 2020)

We consider an expansion of the strongly interacting superfluid Fermi gas in a vacuum in the so-called unitary regime when the chemical potential $\mu \propto \hbar^2 n^{2/3}/m$, where n is the density of the Bose-Einstein condensate of Cooper pairs of fermionic atoms. At low temperatures $T \to 0$, such an expansion can be described in the framework of the Gross-Pitaevskii equation (GPE). For such a dependence of the chemical potential on the density, the GPE has additional symmetries, resulting in the existence of the virial theorem, connecting the mean size of the gas cloud and its Hamiltonian. It leads asymptotically at $t \to \infty$ to the gas cloud expansion, linearly growing in time. We study such asymptotics and reveal the perfect match between the quasiclassical self-similar solution and the asymptotic expansion of the noninteracting gas. This match is governed by the virial theorem, derived through utilizing the Talanov transformation, which was first obtained for the stationary self-focusing of light in media with a cubic nonlinearity due to the Kerr effect. In the quasiclassical limit, the equations of motion coincide with three-dimensional hydrodynamics for the perfect monatomic gas with $\gamma = 5/3$. Their self-similar solution describes, within the background of the gas expansion, the angular deformities of the gas shape in the framework of the Ermakov-Ray-Reid–type system.

DOI: 10.1103/PhysRevA.101.043612

I. INTRODUCTION

One of the key experiments in the discovery of the Bose-Einstein condensates of gaseous alkali-metal elements [1,2] was connected to the determination of the distribution function of Bose atoms during gas expansion in vacuum. This expansion had some interesting features similar to those in the expansion of inviscid gas in vacuum, i.e., of a perfect compressible fluid. In particular, in experiments [1], the gas expansion was accompanied by angular deformation of the cloud shape from a cigar form to a disk. The experiment ignited discussion about the quantum and classical origins of hydrodynamics [3]. In particular, it is worth noting that Refs. [3,4] obtained the first scaling time-dependent solutions for bosonic atoms in the hydrodynamic regime for the anisotropic trap; in addition, Ref. [3] found the spectrum of breathing modes of the oscillating type in the trapping potential. Later, the self-similar regimes were observed in the experiments of O'Hara *et al.* [5] for the anisotropic expansion of a strongly interacting degenerate Fermi gas of ^{6}Li atoms from the optical trap. The measurements of this group were performed by exploiting the Feshbach resonance [6,7], which allows one to change the scattering length a_s in a wide range from positive to negative values. In this range of a_s there is one special point, corresponding to the so-called unitary limit $(a_s k_F)^{-1} \to 0$ ($p_F = \hbar k_F$ is Fermi momentum), where exact and universal results can be obtained (see, e.g., [8,9] for a review). The reason for the universality of the unitary limit is connected to the fact that there is no other energy scale besides the Fermi energy ε_F at this point. Thus, the chemical potential μ in the unitary limit scales linearly with ε_F.

In hydrodynamic content, the first classical works on gas expansion into vacuum were performed by Ovsyannikov [10] and Dyson [11]. Both Dyson [11] and Nemchinov [12] predicted the appearance of anisotropy during gas expansion. Anisimov and Lysikov [13] constructed an exact self-similar axial-symmetric solution to the problem of the expansion of an ideal gas with an adiabatic index $\gamma = 5/3$. This solution describes anisotropic expansion, with a varying angular shape of the gas cloud, in particular, with the inversion of the initial cigar-shaped form to the disk and vice versa.

As we will show in this paper, the analogy between the hydrodynamic expansion of ordinary gases and the expansion of a quantum Fermi gas into vacuum is actually deeper than it might seem at first glance. At the quantitative level, the similarity between quantum and classical hydrodynamics is seen in the expansion of a strongly interacting Fermi gas of atoms [5].

In this paper we consider an expansion of the strongly interacting superfluid Fermi gas in a vacuum in the unitary regime when the chemical potential $\mu \propto \hbar^2 n^{2/3}/m$, where n is the density of the Bose-Einstein condensate of Cooper pairs of fermionic atoms assuming temperature $T \to 0$. Such expansion can be described in the framework of the Gross-Pitaevskii equation (GPE). Because of the chemical potential

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dependence on the density $\sim n^{2/3}$, the GPE has additional symmetries resulting in the virial theorem [14] connecting the mean size of the gas cloud and its Hamiltonian. It leads asymptotically at $t \to \infty$ to the linear-in-time expansion of the gas. We carefully study such asymptotics and reveal a perfect matching between the quasiclassical self-similar solution and the asymptotic expansion of the noninteracting gas. It is worth noting that the quasiclassical time-dependent GPE in the unitary limit represents, after simple rescaling, the hydrodynamic equations for potential flows of the classical gas with the adiabatic constant $\gamma = 5/3$. Note that all symmetries of the GPE in this case remain in the quasiclassical approximation, which allows one to construct more easily the Anisimov-Lysikov solution [13] by using the virial theorem and the Ermakov integral.

The paper is organized as follows. In Sec. II we discuss the problem concerning symmetries of the GPE in the unitary limit and how these symmetries are connected to those found for the nonlinear Schrödinger equation (NLSE) in the critical case when the virial theorem can be applied for the description of the stationary self-focusing of light in media with the Kerr nonlinearity [14] and symmetries in the case of an ideal monatomic gas. Section III mainly deals with the self-similar solution of the anisotropic type of quasiclassical GPE in the unitary limit for expansion of the Fermi superfluid gas. This solution describes the angular deformations of the gas shape within the background of the gas expansion. In Sec. IV we discuss to what extent the analytical results obtained in the previous sections are related to the experimental data. In Sec. V we summarize the results of the paper.

II. SYMMETRIES AND INTEGRALS OF MOTION

First of all, we recall that the topic of gas expansion was very popular in the hydrodynamic content. It worth noting that the application of these studies ranged from astrophysics [15] to laser-matter interaction [16].

Anisimov and Lysikov [13] discovered a very interesting phenomenon connected with the nonlinear angular deformation of the gas cloud during its expansion. Such behavior directly follows from their remarkable solution for an ideal monatomic gas (see also [16–18] and references therein). This result, as was pointed out by Dzyaloshinskii [19], represents a consequence of the symmetry which is well known in quantum mechanics for motion of a nonrelativistic particle in the potential $V(r) = \beta/r^2$. This symmetry, independent of the sign of β, is dilatation of both spatial coordinates and time for which $\mathbf{r} \to \alpha\mathbf{r}$ and $t \to \alpha^2 t$, where α is a scaling parameter. Indeed, such symmetry was exploited for the first time by Ermakov in [20] to construct solutions for some mechanical systems including the motion of a particle in the potential which is a combination of the oscillator potential and $V(r) = \beta/r^2$. Just this symmetry helps one find an exact solution in the quantum case also (see, e.g., [21,22]).

Subsequently, Ray and Reid [23] rediscovered the Ermakov results. Now all such equations are commonly called Ermakov-Ray-Reid systems (see, e.g., [24] and references therein). As we will show in this paper, this additional symmetry for the GPE takes place for both attractive and repulsive interactions (in optical content, corresponding to focusing and defocusing nonlinearities). Note that, in quantum mechanics (see [25]), for the attractive potential $V < 0$, with constant $|\beta|$ larger than some critical value ($=\hbar^2/8m$), the quantum falling of a particle with mass m into the center is possible, which can be understood as collapse. Moreover, this falling becomes more quasiclassical as the particle approaches the center. In the case of the (cubic) Gross-Pitaevskii equation [26,27], which can be applied for the description of the nonlinear dynamics of the Bose condensate for dilute gases, the kinetic energy has the same scaling as in the usual quantum mechanics, i.e., proportional to α^{-2}. The nonlinear interaction term in the GPE, due to the s scattering, has a scaling proportional to α^{-d} which appears from the conservation of the total number of particles $N = \int |\psi|^2 d\mathbf{r}$, with d the space dimension and ψ the wave function of the Bose condensate. Thus, at $d = 2$ only, the situation is analogous to that in the quantum mechanics for potentials $V(r) = \beta/r^2$. This is a very special case, as it was first demonstrated by Vlasov *et al.* [14] for the two-dimensional (2D) nonlinear Schrödinger equation, for which the so-called virial relation is valid,

$$m\frac{d^2}{dt^2}\int r^2|\psi|^2 d\mathbf{r} = 4H, \tag{1}$$

where the Hamiltonian H in the case of the GPE for the Bose condensate has the form

$$H = \int \left[\frac{\hbar^2}{2m}|\nabla\psi|^2 + g|\psi|^4\right] d\mathbf{r}.$$

Here the coupling coefficient $g = 4\pi\hbar^2 a_s/m$, with a_s the scattering length and m the particle mass. It is necessary to emphasize that the virial relation (1) is valid for any sign of g. The only restriction follows from the requirement of convergence of the integrals in (1). It is worth noting that in classical mechanics the virial theorem establishes the ratio between mean values of the total kinetic and potential energies. The simplest way to derive this theorem is calculation of the second time derivative of a moment of inertia [this results in a virial relation like Eq. (1)] and then averaging it in time. Further, we will call the relation (1) the virial theorem.

In this paper we consider another example of the same symmetry, when the generalized Gross-Pitaevskii equation [8] can be applied for description of the strongly interacting Fermi gas in the superfluid phase at $T = 0$ [28],

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2(2m)}\Delta\psi + \mu(n)\psi, \tag{2}$$

where ψ is the wave function of the Bose condensate of fermion pairs, m is a fermion mass ($2m$ is a mass of a fermion pair), and μ is the chemical potential. In the unitary limit [when $(k_F a_s)^{-1} \to 0$] the chemical potential reads (see, e.g., [8])

$$\mu(n) = 2(1+\beta)\varepsilon_F, \tag{3}$$

where the universal interaction parameter $\beta = -0.63$, in accordance with [29,30], and the local Fermi energy

$$\varepsilon_F = \frac{\hbar^2}{2m}(6\pi^2 n)^{2/3}.$$

Here $n = |\psi|^2$ is the concentration of fermionic pairs. Below we will normalize the density n by its initial maximum value n_0, the inverse time t^{-1} by $\frac{\hbar}{2m} n_0^{2/3}$, and the coordinate r by $n_0^{-1/3}$. In these new (dimensionless) units Eq. (2) reads

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\Delta\psi + \mu(n)\psi, \tag{4}$$

where

$$\mu(n) = 2(1+\beta)(6\pi^2 n)^{2/3}. \tag{5}$$

Choosing the standard ansatz for the ψ function, $\psi = \sqrt{n(r,t)}\exp[i\varphi(r,t)]$, and separating then real and imaginary parts in (2), we get the system of continuity and Euler (eikonal) equations

$$\frac{\partial n}{\partial t} + (\nabla\cdot n\nabla\varphi) = 0, \tag{6}$$

$$\frac{\partial\varphi}{\partial t} + \left[\mu(n) + \frac{(\nabla\varphi)^2}{2} + T_{\rm QP}\right] = 0, \tag{7}$$

where $\mathbf{v} = \nabla\varphi$ has the meaning of velocity. Here we used the condition of the absence of vortices $\nabla\times\mathbf{v} = 0$.

The term in Eq. (7) represents the quantum pressure given by

$$T_{\rm QP} = -\frac{\Delta\sqrt{n}}{2\sqrt{n}}. \tag{8}$$

Throughout most of the present paper we will neglect this term and will discuss its possible role in the last two sections. Neglecting quantum pressure corresponds to the quasiclassical (or eikonal) approximation (called also the time-dependent Thomas-Fermi approximation), which assumes more rapid space and time variations of phase (larger phase gradients and time derivatives) in comparison with the space and time variations of the modulus of the ψ function in Eq. (4).

It is important to emphasize that the generalized Gross-Pitaevskii equation (4) coincides with the NLSE widely used in nonlinear optics and plasma physics. It is convenient to exclude in (5) the factor $2(1+\beta)(6\pi^2)^{2/3}$ by simple rescaling of the density n,

$$2(1+\beta)(6\pi^2 n)^{2/3} \to \tfrac{5}{3} n^{2/3},$$

so that Eq. (4) takes the standard form accepted for the NLSE,

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\Delta\psi - (\upsilon+1)|\psi|^{2\upsilon}\psi = 0, \tag{9}$$

with the exponent $\upsilon = 2/3$. This equation can be written in the Hamiltonian form

$$i\frac{\partial\psi}{\partial t} = \frac{\delta H}{\delta\psi^*},$$

where the Hamiltonian

$$H = \int\left[\frac{1}{2}|\nabla\psi|^2 + |\psi|^{2(\upsilon+1)}\right]d\mathbf{r}, \tag{10}$$

with the first term coinciding with the total kinetic energy and the second one responsible for nonlinear interaction of the repulsion type. After applying the transformation $\psi = \sqrt{n(r,t)}\exp[i\varphi(r,t)]$, equations for the density n and phase φ remain of the Hamiltonian form

$$\frac{\partial n}{\partial t} = \frac{\delta H}{\delta\varphi}, \quad \frac{\partial\varphi}{\partial t} = -\frac{\delta H}{\delta n}, \tag{11}$$

where the Hamiltonian coincides with (10). In terms of n and φ, H takes the form

$$H = \int\left[\frac{n(\nabla\varphi)^2}{2} + \frac{(\nabla\sqrt{n})^2}{2} + n^{\upsilon+1}\right]d\mathbf{r}.$$

The Hamiltonian equations of motion (11) are the same equations (6) and (7) transformed under simple rescaling; n and φ in this case play the role of canonically conjugated quantities.

The second term in H is responsible for the quantum pressure in Eq. (7). In the quasiclassical limit (the Thomas-Fermi approximation), this term becomes small and can be neglected so that we arrive at the hydrodynamic equations for potential flow of monatomic gas with a specific heat ratio (adiabatic index) $\gamma = 5/3$ ($\upsilon = 2/3$). This γ is remarkable for both the NLSE and its quasiclassical limit. It turns out that the equations of motion in this case have two additional symmetries. The first symmetry forms a dilatation group of the scaling type. However, for the NLSE (9) such symmetry appears as a result of the conservation of the total number of particles N so that at $d = 3$ only the nonlinear potential $\sim|\psi|^{4/3}$ in Eq. (9) has the same scaling as the Laplace operator Δ. At $d = 2$ such symmetry takes place for the nonlinear potential $\sim|\psi|^2$ (in this case the NLSE describes the stationary self-focusing of light in a medium with the Kerr nonlinearity). In the general case the dilatation symmetry arises at $\nu = 2/d$ (see, for instance, [31,32]). The second symmetry of conformal type was first found by Talanov for the cubic NLSE at $d = 2$ [33] and is called now the Talanov transformations. In optical content these are the lens transformations well known in linear optics.

These symmetries are of the Noether type and generate two additional integrals of motion. They can be obtained from the virial theorem (1) (first obtained for the 2D cubic NLSE in [14]), after integrating twice in time (dimensionless variables):

$$\int r^2|\psi|^2 d\mathbf{r} = 2Ht^2 + C_1 t + C_2. \tag{12}$$

Hence we get asymptotically at $t\to\infty$, independently of C_1 and C_2,

$$\int r^2|\psi|^2 d\mathbf{r} \to 2Ht^2.$$

Therefore, the mean size (indeed, the rms) of the gas cloud varies at large t linearly in time,

$$\langle r^2\rangle^{1/2} \propto t\sqrt{2H/N}. \tag{13}$$

This result is very important since it perfectly matches the quasiclassical solution in Eq. (13) with the linearly varying solutions for the noninteracting particles (ballistic expansion) (see, e.g., [28]).

It should be emphasized that the virial theorem (1) is the exact result; it can be applied in particular in the quasiclassical limit also when the quantum pressure term in H is eliminated. The latter corresponds to the classical monatomic gas expansion. In this case Anisimov and Lysikov [13] constructed exact axial-symmetric self-similar solution based in fact on the existence of two integrals of motion C_1 and C_2 (see the next section). This solution describes the gas expansion in time in correspondence with (12) with nonlinear angular deformation of the gas shape.

III. SELF-SIMILAR QUASICLASSICAL SOLUTION

Note that the quasiclassical limit of Eq. (9) or its equivalent [Eqs. (6) and (7)] corresponds to neglecting the quantum pressure term so that we arrive at the gas dynamical system consisting of the continuity equation (6) and the equation for φ,

$$\frac{\partial \varphi}{\partial t} + \frac{(\nabla \varphi)^2}{2} + \frac{5}{3} n^{2/3} = 0. \tag{14}$$

Let us search for a solution of these equations in the self-similar form (see, e.g., [13,34–36])

$$n = \frac{1}{a_x a_y a_z} f\left(\frac{x}{a_x}, \frac{y}{a_y}, \frac{z}{a_z}\right), \tag{15}$$

assuming that the three scaling parameters a_x, a_y, and a_z are functions of time. Note that the ansatz (15) conserves the total number of particles.

Then the continuity equation (6) admits integration resulting in the expression for the phase φ,

$$\varphi = \varphi_0(t) + \sum_l \frac{\dot{a}_l a_l}{2} \xi_l^2, \tag{16}$$

where the function $\varphi_0(t)$ can be found after substitution in the eikonal equation and $\xi_x = x/a_x$, $\xi_y = y/a_y$, and $\xi_z = z/a_z$ are the self-similar variables. Substitution of (16) into the eikonal equation yields that

$$\ddot{a}_x a_x = \ddot{a}_y a_y = \ddot{a}_z a_z = \frac{\lambda}{(a_x a_y a_z)^{2/3}}, \tag{17}$$

where λ is an arbitrary positive constant which is determined from the initial condition. For $f(\xi)$ we have

$$f(\xi) = \left[1 - \frac{3\lambda}{10}\xi^2\right]^{3/2}. \tag{18}$$

We will assume that the initial density is also defined from the Thomas-Fermi approximation. In the presence of a harmonic trap, at the stationary state we have the equilibrium condition

$$\mu(n) = \mu(n_0) - m \sum \omega_i^2 x_i^2,$$

where $\mu(n)$ is given by (5). Recall that, because of pairing in this expression, $2m$ appears instead of m. This gives the initial density distribution

$$n = n_0 \left[1 - \frac{m\omega_m^2 n_0^{-2/3}}{\mu(n_0)} \sum \xi_i^2\right]^{3/2},$$

where $\omega_m = \max(\omega_i)$ and $a_i(0) = \omega_m/\omega_i$. This profile matches precisely with the self-similar solution (15) and (18) at $t = 0$. Hence we have that

$$\lambda = \frac{10 m\omega_m^2 n_0^{-2/3}}{3\mu(n_0)},$$

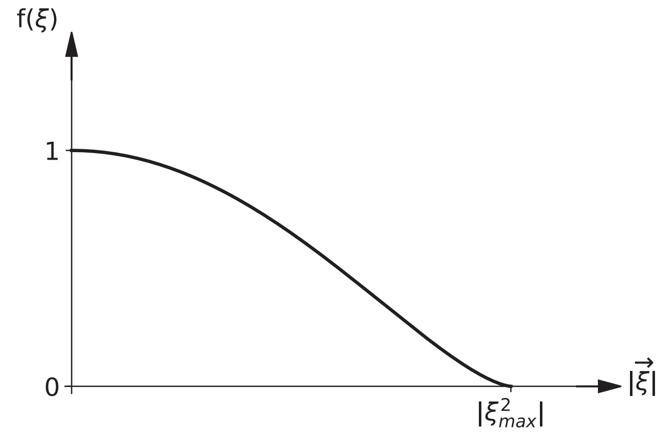

FIG. 1. Behavior of the density factor $f(\xi)$ (arbitrary units).

or in terms of N,

$$\lambda = \frac{5}{6}\left(\frac{\pi^2}{N}\right)^{2/3}\left(\frac{\omega_m^3}{\omega_x\omega_y\omega_z}\right)^{2/3}.$$

The function $f(\xi)$ [Eq. (18)] is spherically symmetric with respect to ξ. It varies from 1 at $\xi = 0$ up to zero at $\xi_{\max} = \sqrt{10/3\lambda}$; above $\xi_{\max}$ the density n is equal to zero (see Fig. 1). In accordance with (17), the dynamics of the three scaling parameters $a_i(t)$ $(i = 1, 2, 3)$ is described by the Newton equation for the motion of a particle

$$\ddot{a}_i = -\frac{\partial U}{\partial a_i}, \tag{19}$$

where the potential

$$U = \frac{3\lambda}{2(a_x a_y a_z)^{2/3}}. \tag{20}$$

It is worth noting that at the point $\xi = \xi_{\max}$ the obtained quasiclassical solution given by (15)–(19) breaks down, which follows from estimation of the quantum pressure term which becomes infinitely large. In this case, $\xi = \xi_{\max}$ plays the role of a reflection point in the usual quasiclassical approximation in quantum mechanics. This means that in the vicinity of $\Delta\xi$ around $\xi = \xi_{\max}$ one needs to match the constructed solution at $\xi < \xi_{\max}$ (inner region) with that at $\xi > \xi_{\max}$ (outer region), where we should neglect nonlinearity in the NLSE (free Schrödinger equation). This problem was discussed in [34] for the strong collapse regime in the supercritical NLSE with $d = 3$ and $\nu = 1$. In the given case, the matching problem can be also resolved if $\Delta\xi = |\xi - \xi_{\max}| \ll \xi_{\max}$. However, the given problem is more complicated because of the time-dependent anisotropy for the quasiclassical solutions (15)–(19) (see the following sections). Nevertheless, we can understand qualitatively the behavior of these solutions in the transient region. Solutions near $\xi = \xi_{\max}$ have to oscillate in space and time. The nature of such oscillations is due to diffraction. These oscillations are analogous to Newton rings in optics. This question is beyond the scope of the present paper.

We should notice also that near $\xi = \xi_{\max}$ the unitary limit is not also applicable because k_F becomes infinitely large. Near this point, however, the nonlinear term is small and we again return to the same matching problem as in the previous case.

A. Virial theorem for the scaling parameters

It is more or less evident that Newton equations (19) have to have the same symmetry properties as the original NLSE (9), which can be easily verified. Note first that for Eq. (19) the energy integral is written in the standard form

$$E = \frac{1}{2}\sum_{i=1,2,3} \dot{a}_i^2 + \frac{3\lambda}{2(a_x a_y a_z)^{2/3}}.$$

Second, for Eq. (19), by direct calculation it is possible to get the virial theorem (1), written in terms of a_i. For $\sum a_i^2$ we have

$$\frac{d^2}{dt^2}\sum_i a_i^2 = 2\sum_i\left[\left(\frac{da_i}{dt}\right)^2 + a_i\frac{d^2a_i}{dt^2}\right].$$

Then substitution of (19) into this relation gives finally

$$\frac{d^2}{dt^2}\sum_i a_i^2 = 2\sum_i\left(\frac{da_i}{dt}\right)^2 + \frac{6\lambda}{(a_x a_y a_z)^{2/3}} = 4E,$$

which coincides with the virial identity (1). Its twice integration gives two integrals C_1 and C_2:

$$\sum_i a_i^2 = 2Et^2 + C_1 t + C_2. \tag{21}$$

Hence

$$C_1 = \frac{d}{dt}\sum_i a_i^2 - 4Et, \tag{22}$$

$$C_2 = \sum_i a_i^2 - 2Et^2 - C_1 t. \tag{23}$$

In the isotropic (spherically symmetric) case when $a_x = a_y = a_z \equiv a$ the equations of motion transform into one equation

$$\ddot{a} = \frac{\lambda}{a^3}, \tag{24}$$

with the energy $E = \frac{3}{2}(\dot{a}^2 + \frac{\lambda}{a^2})$ and $3a^2 = 2Et^2 + C_1 t + C_2$. From the latter relation we immediately have that the gas cloud expands asymptotically in the radial direction at $t \to \infty$ with constant velocity

$$v_\infty = \sqrt{2E/3} \tag{25}$$

(ballistic regime). This result is in agreement with the virial theorem (13).

If we change the sign of the potential in Eq. (24) then we get the falling of the particle on the potential center, which, as known in quantum mechanics, becomes more quasiclassical while approaching the center (see [25]).

For the expansion of a noninteracting gas from a harmonic potential

$$\sqrt{\langle x_i^2\rangle} \propto (\sqrt{\hbar\omega_i/2m})t$$

and $v_\infty = \text{const}$, in agreement with our intuitive considerations and with Eq. (25) as well. (Let us recall that for quasi-2D disk-shaped traps the trapping frequency $\omega_z \gg \omega_x \simeq \omega_y$.) Thus, we have almost perfect matching of ballistic results for a noninteracting gas and quasiclassical results derived for a strongly interacting Fermi gas in the eikonal approximation.

It is worth noting that in the virial relation (21), besides total energy E there are two more integrals of motion C_1 and C_2. In principle, if $C_1 > 0$ then the solution with $a^2 \propto C_1 t$ is possible for some intermediate times, but not initially. The regime with $a \propto (t_0 - t)^{1/2}$ is typical for weak self-similar collapse (see [34,37]).

B. Anisotropic self-similar solution

The simplest anisotropic case corresponds to the cylindrically symmetric expansion and is governed by the scaling parameters $a_x = a_y = a/\sqrt{2}$ and $a_z = b$. For $a \gg b$ we have the case of an initially disk-shaped cloud, while for $b \gg a$ we are effectively in the cigar-shaped limit. An isotropic limit obviously corresponds to $b = a/\sqrt{2}$.

In the anisotropic cylindrically symmetric case Eq. (17) reads

$$\ddot{a} = -\frac{\partial U}{\partial a}, \quad \ddot{b} = -\frac{\partial U}{\partial b}, \tag{26}$$

with the initial ratio $b/a|_{t=0} = \omega_\perp/\sqrt{2}\omega_z$. The effective potential in accordance with Eq. (20) is given by

$$U = \frac{3\lambda}{2(a^2 b/2)^{2/3}}.$$

Note that this system belongs to the so-called Ermakov type of equations [20]. These equations describe the motion of two degrees of freedom and therefore to integrate this system it is enough to have two autonomous integrals of motion which should be in involution. In our case, however, we have three integrals of motion. The first one is the total energy

$$E = \frac{1}{2}(\dot{a}^2 + \dot{b}^2) + \frac{3\lambda}{2(a^2 b/2)^{2/3}}. \tag{27}$$

The second and third integrals are two constants C_1 and C_2 which appear in (12), while the double integration over time of the virial identity (1),

$$\frac{d^2}{dt^2}(a^2 + b^2) = 4E. \tag{28}$$

The integrals (22) and (23), however, are not autonomous; they contain an explicit dependence on time and therefore cannot provide a complete integration of the system. As we will see, only their combination defines the needed integral of motion for the Ermakov type of equations.

Let us introduce now the polar coordinates for a and b,

$$a = r\cos\Phi, \quad b = r\sin\Phi.$$

In these variables the virial theorem (28) acquires the evident form

$$\frac{d^2}{dt^2}r^2 = 4E,$$

where the total energy E, in accordance with (27), is

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\Phi}^2) + \frac{3\lambda}{2^{1/3}r^2(\cos^2\Phi\sin\Phi)^{2/3}}, \tag{29}$$

and correspondingly

$$r^2 = 2Et^2 + C_1 t + C_2, \quad C_1 = \frac{d}{dt}r^2 - 4Et. \tag{30}$$

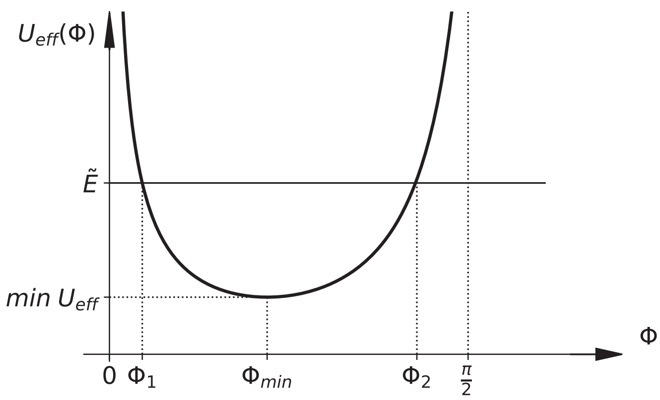

FIG. 2. Effective potential $U_{\text{eff}}(\Phi)$.

Multiplying now (29) by r^2 and using the relations (30), simple calculations give that the combination

$$\tilde{E} = Er^2 - \tfrac{1}{2}r^2\dot{r}^2 = EC_2 - C_1^2/8$$

is a constant (the Ermakov integral). As a result, we arrive at a conservation law for the new "energy"

$$\tilde{E} = \frac{1}{2}\left(\frac{d\Phi}{d\tau}\right)^2 + U_{\text{eff}}(\Phi), \tag{31}$$

with new time τ,

$$d\tau = \frac{dt}{r^2}, \tag{32}$$

where $\tau = \int_0^t \frac{dt'}{2E(t')^2+C_1t'+C_2}$, with

$$U_{\text{eff}}(\Phi) = \frac{3\lambda}{2^{1/3}(\cos^2\Phi\sin\Phi)^{2/3}} \tag{33}$$

playing the role of potential energy. It is always positive and goes to infinity for $\Phi \to 0$ and $\Phi \to \pi/2$. The minimum of $U_{\text{eff}}(\Phi) = 9\lambda/2$ corresponds to the isotropic case when $\sin\Phi_{\text{min}} = 1/\sqrt{3}$. Graphically, the effective potential $U_{\text{eff}}(\Phi)$ is shown in Fig. 2.

The new time τ [Eq. (32)] can be easily expressed through t,

$$\sqrt{2\tilde{E}}\tau = \arctan\frac{\sqrt{2E}(t+t_0)}{\chi} - \arctan\frac{\sqrt{2E}t_0}{\chi},$$

where $\chi^2 = \tilde{E}/E$ and $t_0 = \frac{C_1}{4E}$ so that $\tau = 0$ at $t = 0$. If the initial velocity is equal to zero (which is typical for experiment) the constant $C_1 = 0$ and

$$\sqrt{2\tilde{E}}\tau = \arctan\frac{\sqrt{2\tilde{E}}t}{C_2}.$$

In this case, asymptotically at $t \to \infty$,

$$\tau \to \tau_\infty = \frac{\pi}{2\sqrt{2\tilde{E}}}. \tag{34}$$

The trajectory $\Phi(\tau)$ is defined from integration of (31),

$$\tau = \int \frac{d\Phi}{\sqrt{2[\tilde{E} - U_{\text{eff}}(\Phi)]}}.$$

Hence the τ period of the oscillations in the potential $U_{\text{eff}}(\Phi)$ [Eq. (33)] is expressed through the integral

$$T = 2\int_{\Phi^{(-)}}^{\Phi^{(+)}} \frac{d\Phi}{\sqrt{2[\tilde{E} - U_{\text{eff}}(\Phi)]}},$$

where $\Phi^{(\pm)}$ are roots of the equation $\tilde{E} = U_{\text{eff}}(\Phi)$ (reflection points). This integral is expressed via elliptic integrals of the third order (see [13]). At large value of $\tilde{E}$ oscillations are almost independent of the details of $U_{\text{eff}}(\Phi)$. Asymptotically, in this case the angular velocity $\frac{d\Phi}{d\tau} \to \pm\sqrt{2\tilde{E}}$ and the τ period $T \to \pi/\sqrt{2\tilde{E}}$, namely, in this limit, T exceeds two times τ_∞ [Eq. (34)]. Notice also that the dependence $T(\tilde{E})$ is monotonic for the given potential $U_{\text{eff}}(\Phi)$ with the maximum corresponding to the potential minimum. This means that in the real experiment (which we will discuss in the next section), in the better case it is possible to observe only half of such an oscillation t_{osc}. It is important to note that a return to the initial shape is impossible in this case. In the quasiclassical regime, the gas shape behavior will be different for cigar- and disk-shaped initial conditions. For example, in the cigar-shaped case we start at fixed $\tilde{E}$ from the left reflection point of the potential $U_{\text{eff}}(\Phi)$; in the disk-shaped case we start from the right reflection point. Therefore, the shape forms will coincide only for intermediate moments of time, far from the initial reflection points. We should take into account that at fixed $\tilde{E}$ starting from any reflection point we cannot reach its opposite reflection point. It should be emphasized that the solution presented here was first obtained by Anisimov and Lysikov [13] for expansion of an ideal gas with $\gamma = 5/3$.

C. General anisotropic case

In the general anisotropic case, when all the scaling parameters are different $a_x \neq a_y \neq a_z$ it is convenient to introduce the spherical coordinates (r, θ, and φ) where the total energy acquires the form

$$E = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\varphi}{dt}\right)^2\right] + \frac{3\lambda}{2^{1/3}r^2}\frac{1}{(\sin^2\theta\cos\theta\sin 2\varphi)^{2/3}}.$$

Correspondingly introducing again the Ermakov reduced energy $\tilde{E}$, which is a sequence of the dilatation symmetry, and new time τ, following the same prescriptions as in the preceding section, we get

$$\tilde{E} = C_2E - \frac{1}{8}C_1^2 = \left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{dt}\right)^2 + U_{\text{eff}}, \tag{35}$$

where the effective potential is now

$$U_{\text{eff}} = \frac{3\lambda}{2^{1/3}(\sin^2\theta\cos\theta\sin 2\varphi)^{2/3}}. \tag{36}$$

Thus, we arrive at the system for two degrees of freedom. As it was pointed out in the preceding section, the integral (35) is a consequence of the scaling symmetry, but for integration of the system it is not enough. As it was shown by Gaffet [38], this system indeed has one additional integral (besides $\tilde{E}$) which follows from the Painlevé test. The existence of

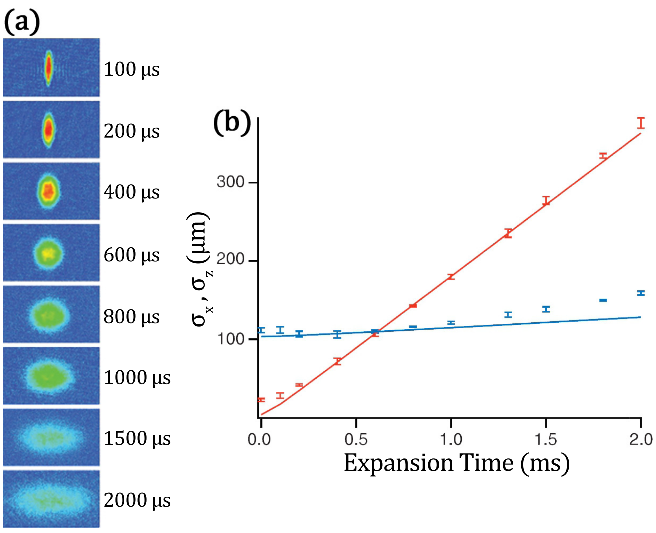

FIG. 3. (a) Images of a strongly interacting Fermi gas, which expands starting from the cigar shape. The expansion time is noted by each image. (b) Thomas-Fermi radii along the transverse (σ_x, red) and longitudinal (σ_z, blue) directions vs the expansion time. The markers are the data. The curves are the self-similar-expansion model without adjustable parameters. (Figure has been taken from [5].)

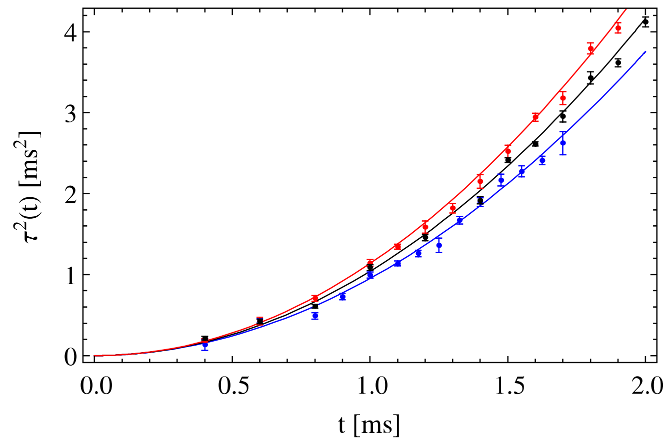

FIG. 4. Data are experimental values $\tau^2(t) \equiv m[\langle \mathbf{r}^2 \rangle - \langle \mathbf{r}^2 \rangle_{t=0}]/\langle \mathbf{r} \cdot \nabla U \rangle_{t=0}$ measured for a strongly interacting normal Fermi gas after expansion for time t, initially trapped in potential $U(\mathbf{r})$. The black curve is the expansion law (37). Black markers correspond to the gas on-resonance, $1/k_F a_s \approx 0$, red and blue markers correspond to $1/k_F a_s \simeq 0.59$ and -0.61, respectively, and the solid curves are the results of a calculation without free parameters [40].

these two integrals of motion guarantees complete integration of this system. As in the previous limit, motion in the potential (36) remains its nonlinear quasioscillatory character.

IV. DISCUSSION OF EXPERIMENTAL DATA AND COMPARISON WITH OBTAINED RESULTS

The self-similar expansion of a strongly interacting Fermi gas from a cigar-shaped trap was observed in [5]. The images of the expanding gas are shown in Fig. 3(a). The transverse size grows rapidly, while the longitudinal size is nearly stationary, with a weak growth. In Fig. 3(b) one may see qualitative agreement between the time behavior of the gas expanding shape and the self-similar-expansion model represented by Eqs. (15) and (26). The cloud images are changing in Fig. 3(a) from almost ellipsoid and significantly stretched along the z axis (exposition $t = 100\ \mu$s) to the almost spherical shape (at $t = 600\ \mu$s) and finally from the spherical shape to the ellipsoid stretched now in the direction perpendicular to z. The total time of the observation is 2000 μs, which can be taken as a half period (or less) of the angular shape oscillations $t \leqslant t_{\text{osc}}/2$, in accordance with the results of the preceding section. The frequency ratio (and thus the anisotropy ratio up to a factor $\sqrt{2}$) in the experiments [5] was initially rather large, (around 30) which follows from the Thomas-Fermi estimation.

The small deviation of the data from the self-similar behavior in Fig. 3(b) was attributed [39] to the contribution of quantum pressure (8) to the hydrodynamic model (6) and (7). That T_{QP} term is neglected to obtain a self-similar solution of Eqs. (15) and (26). This difference, however, may be explained even without the quantum pressure, by possible deviation of the equation of state from the $\mu \propto n^{2/3}$ dependence (3) since in experiment the interaction parameter is not tuned exactly on-resonance $1/k_F a_s = 0$, with the estimate $1/k_F a_s \simeq -0.14$ [5].

Exactly on-resonance, the mean-square cloud size $\langle \mathbf{r}^2 \rangle$ is found [40] to evolve as

$$\langle \mathbf{r}^2 \rangle = \langle \mathbf{r}^2 \rangle_{t=0} + \frac{t^2}{m} \langle \mathbf{r} \cdot \nabla U(\mathbf{r}) \rangle_{t=0}, \tag{37}$$

where $U(\mathbf{r})$ is the initial trapping potential. The expansion law (37) was obtained within the Thomas-Fermi approximation and coincides with the quasiclassical dependence of $\langle \mathbf{r}^2 \rangle$ [Eq. (21)] in the unitary limit for $C_1 = 0$ (or equivalently for initial velocity equal zero). It should be emphasized that, according to (21), $\langle \mathbf{r}^2 \rangle$ indeed depends linearly on energy E, which was verified in experiments [40].

Note that the expansion law (37) is indeed the same as for the ideal gas and coincides with the virial theorem (1). Note that the expansion law is obtained for both zero- and finite-temperature gas with an equation of state $P = \frac{2}{3}\mathcal{E}$, where P is the pressure and $\mathcal{E}$ is the energy density, while the relation $\mu \propto n^{2/3}$ is a particular case of this state that corresponds to the isentropic regime for $\gamma = 5/3$. The equation of state $P = \frac{2}{3}\mathcal{E}$ and expansion law (37) are consequences of the resonant interaction with $1/k_F a_s = 0$. Away from resonance, the expansion laws differ slightly from each other, which is seen in the measurements displayed in Fig. 4. Nevertheless, these laws for different parameters $1/k_F a_s$ have the same parabolic dependence on t. It should be emphasized that during the expansion the interaction parameter $1/k_F a_s$ changes due to a drop in gas density. When the parameter values fall outside the interval $(-1, 1)$, the quantum effects become less significant and the gas expansion approaches the law for a classical monatomic gas, which coincides, however, with (37). For these reasons we guess that expansions with $1/k_F a_s \simeq 0.59$ and -0.61 (red and blue curves, respectively, in Fig. 4) correspond to a normal Fermi gas.

V. CONCLUSION

We have demonstrated that the symmetry for the GPE in the unitary limit, describing a strongly interacting superfluid Fermi gas, results in the virial theorem (1). As a consequence, independently of the ratio between the quantum pressure and chemical potential while the Fermi superfluid gas expands, the rms size of the gas cloud scales linearly with time asymptotically. so the expansion velocity tends to the constant value $v_\infty = (2H/N)^{1/2}$.

For the description of the expansion of the strongly interacting superfluid Fermi gas we have applied the self-similar quasiclassical theory. For large timescales the theory matches quite well with simple ballistic ansatz and also with the initial quasiclassical distribution of the trapped gas. This self-similar solution is a consequence of the scaling symmetry of the Ermakov type. In the unitary limit, when both kinetic and potential energies scale linearly with the Fermi energy, our quasiclassical solution for the superfluid quantum gas coincides with the Anisimov-Lysikov solution [13] for classical gas expansion in the isentropic regime. This anisotropic solution describes the nonlinear deformations of the cloud shape during the self-similar gas expansion. For the initial condition in the cigar-shaped form this solution demonstrates successively all the stages of gas expansion, starting from the distribution extended along the cigar axis, bypassing the spherically symmetrical one, and ending with the distribution turned at angle $\pi/2$ with respect to the initial cigar form. Such behavior was first observed in experiments [5]. For the initial distribution in the form of a quasi-2D disk, all stages of the expansion are inverse to those for the initial distribution in the cigar form.

It should be emphasized that the solutions developed in this paper are based on a quasiclassical theory which, in the leading order, does not differ from the hydrodynamics of an ideal gas with the adiabatic exponent $\gamma = 5/3$. The difference between a quantum gas and a classical one in the problem of gas expansion into vacuum consists in taking into account quantum pressure, the inclusion of which leads to the appearance of density oscillations in both time and space at the boundary of the expanding cloud. These oscillations are of the diffraction type and have the same nature as Newton rings in optics.

ACKNOWLEDGMENTS

The work of E.A.K. was supported by the Russian Science Foundation through Grant No. 19-72-30028. M.Y.K and A.V.T. are grateful for support from the Russian Science Foundation under Grant No. 18-12-00002.

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