# Binary mixture of Bose-Einstein condensates in a double-well potential: Berry phase and two-mode entanglement

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A binary mixture of Bose-Einstein condensate structures exhibits an incredible richness in terms of holding different kinds of phases. Depending on the ratio of the inter- and intra-atomic interactions, the transition from the mixed to separated phase, which is also known as the miscibility-immiscibility transition, has been reported in different setups. Here, we describe such a quantum phase transition (QPT) in an effective Hamiltonian approach by applying the Holstein-Primakoff transformation in the limit of a large number of particles. We demonstrate that a nontrivial geometric phase near the critical coupling is present, which confirms the connection between the Berry phase and QPT. We also show that, by using the spin form of the Hillery and Zubairy criterion, a two-mode entanglement accompanies this transition in the limit of a large, but not infinite, number of particles.

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## I. INTRODUCTION

In recent years, there has been growing interest in studying two-component quantum fluids. Phase mixing and separation of the two components due to the relative strength of interspecies and intraspecies interactions open the gate to investigating a variety of research topics, including the dynamical phase transitions [1–3], the production of dipolar molecules [4], two-mode entanglement [5,6], and the macroscopic quantum self-trapping [7,8]. The phase-separation phenomenon was first observed in $^3$He-$^4$He mixtures [9]. Later, it was reported in different Bose-Einstein condensate (BEC) structures [10–17]. The ability of the resonant control of two-body interactions via Feshbach resonances makes these structures attractive for practical applications.

The observation of *controllable* phase separation of a binary BEC was reported by using Feshbach resonance at hyperfine levels [10–12] and in the isotopes [13] of rubidium atoms. Additionally, similar results were also obtained by using different kinds of atoms [14–17]. Theoretical investigations of these structures have shown that the relative interaction strength and number of particles play a crucial role in characterizing the density profiles. A two-species Bose-Hubbard (BH) model, in the limit of a weakly interacting gas, is widely used because it can perform a fully analytic derivation [18]. The characterization of the self-trapping [19], entanglement [20,21], the dynamical phase transition [22], etc., was done within the BH model.

In this work, we theoretically study the collective behaviors of a two-species BEC trapped in a double-well potential. While the dynamical properties of these structures have been extensively studied through the mean-field approach [1,23] and the well-described Bose-Hubbard model [5,24], which analyze the low excitations using the Holstein-Primakoff (HP) transformation [25,26], the analytical descriptions reside mainly in the mixed-phase solutions. The motivation of the present paper is to obtain simple analytical solutions to describe the system in each of its mixed and separated phases, which can serve to investigate the quantum properties of such mixtures. In that respect, we study the Berry phase and bipartite entanglement through the miscibility-immiscibility transition.

In many-body systems, the quantum phase transition (QPT) can be observed when the crossing between ground and excited states takes place. It is known that such level crossings generate singularities in the Hilbert space, and it is therefore natural to expect the reflections of such behaviors in the wave function. The Berry phase can capture these points, and its connection to QPT has been studied in different models [27–29]. Here, we obtain a nontrivial geometric phase by encircling the critical point and observe that with an increasing number of particles, the transition in the value of the Berry phase becomes sharper around the critical point, and in the thermodynamic limit, $N \to \infty$, there appears a steplike behavior. Having the ability to adiabatically control the interatomic interaction strength between two species [12,13] makes this result valuable for Berry phase-related applications, and it also provides a tool to detect criticality in the presence of QPT.

Besides fundamental interest, the model studied here also offers the possibility to test bipartite entanglement from macroscopic observables. In large systems, it was shown that entanglement can be inferred from collective spin measurements [30,31], and experimental observations using this method have been reported between two spatially separated atomic ensembles [32–38] and between the spins of atoms in optical lattices [39,40]. Entanglement characterization of similar models is, in general, done by using von Neumann entropy [41]. Here, we analyze such phenomena through the

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miscibility-immiscibility transition by adopting the *spin form* of the two-mode entanglement witness [42], originally introduced for two-mode entanglement in [43]. We observe that the criterion witnesses the entanglement onset in the separated phase. Unlike the Berry phase, entanglement decays faster with an increasing number of particles and/or interspecies interaction strength. We find that this is due to the vanishing effective coupling term, which is responsible for two-mode squeezing.

This paper is organized as follows. In Sec. II, we introduce the model of a two-species BEC trapped in a double-well potential and derive the effective Hamiltonians and associated ground-state wave functions of the mixed and separated phases. The appearance of the nontrivial geometric phase around the critical coupling is observed in Sec. III. In Sec. IV, we discuss the formation of bipartite entanglement by anticipating the spin form of the Hillery and Zubairy criterion. A summary appears in Sec. V.

## II. THE MODEL

We consider two-species ($a$ and $b$) condensate mixtures trapped in a double-well potential with a large number of particles $N_{a(b)} \gg 1$ in the low-excitation limit. By assuming the trap frequency $\omega$ of each local potential is much larger than the interactions among the atoms $N_i g_i$, i.e., $\omega \gg N_i g_i$, with $i = a, b$, we construct the Hamiltonian with the two-mode approximation [5,44,45], which is given by [1]

$$\hat{\mathcal{H}} = \frac{t_a}{2}(\hat{a}_L^\dagger \hat{a}_R + \hat{a}_R^\dagger \hat{a}_L) + \frac{t_b}{2}(\hat{b}_L^\dagger \hat{b}_R + \hat{b}_R^\dagger \hat{b}_L) + \frac{g_a}{2}[(\hat{a}_L^\dagger \hat{a}_L)^2 + (\hat{a}_R^\dagger \hat{a}_R)^2] + \frac{g_b}{2}[(\hat{b}_L^\dagger \hat{b}_L)^2 + (\hat{b}_R^\dagger \hat{b}_R)^2] + g_{ab}(\hat{a}_L^\dagger \hat{a}_L \hat{b}_L^\dagger \hat{b}_L + \hat{a}_R^\dagger \hat{a}_R \hat{b}_R^\dagger \hat{b}_R). \tag{1}$$

Here, $\hat{a}_j^\dagger$ ($\hat{a}_j$) and $\hat{b}_j^\dagger$ ($\hat{b}_j$) are the creation (annihilation) operators of species $a$ and $b$, respectively, that reside in the $j$th well, $j = L, R$. The parameters $t_a$ and $t_b$ describe the coupling (tunneling) between two wells, and $g_{a(b)}$ and $g_{ab}$ stand for intraspecies and interspecies interaction strengths, respectively, which are explicitly given by $g_{\alpha\beta} = 2\pi\hbar^2 A_{\alpha\beta}/m_{\alpha\beta} \int |\phi_\alpha|^2 |\phi_\beta|^2 dr$ [23]. Here, $A_{\alpha\beta}$ is the $s$-wave scattering length between atoms, and $m_{\alpha\beta}$ is the reduced mass, where we denote $g_\alpha = g_{\alpha\alpha}$ and $\alpha, \beta = a, b$.

The analysis of the Hamiltonian in Eq. (1) can be simplified by the introduction of the angular momentum operators for each species as [46]

$$\begin{aligned} \hat{J}_{\alpha x} &= (\hat{\alpha}_L^\dagger \hat{\alpha}_L - \hat{\alpha}_R^\dagger \hat{\alpha}_R)/2, \\ \hat{J}_{\alpha y} &= (\hat{\alpha}_L^\dagger \hat{\alpha}_R - \hat{\alpha}_R^\dagger \hat{\alpha}_L)/2i, \\ \hat{J}_{\alpha z} &= (\hat{\alpha}_L^\dagger \hat{\alpha}_R + \hat{\alpha}_R^\dagger \hat{\alpha}_L)/2, \end{aligned} \tag{2}$$

where $\alpha = a, b$. These operators obey the usual angular momentum commutation relations: $[\hat{J}_\alpha^+, \hat{J}_\alpha^-] = 2\hat{J}_{\alpha z}$ and $[\hat{J}_\alpha^\pm, \hat{J}_{\alpha z}] = \mp \hat{J}_\alpha^\pm$, where $\hat{J}_\alpha^\pm = \hat{J}_{\alpha x} \pm i\hat{J}_{\alpha y}$. Inserting these definitions into Eq. (1), the Hamiltonian can be rewritten as [5]

$$\hat{\mathcal{H}} = \sum_{\alpha=a,b} \{t_\alpha \hat{J}_{\alpha z} + g_\alpha \hat{J}_{\alpha x}^2\} + 2g_{ab}\hat{J}_{ax}\hat{J}_{bx}. \tag{3}$$

In the limit of a large number of particles, one can make use of the HP representation of the angular momentum operators. In this representation, the operators defined in Eq. (2) can be written in terms of a bosonic mode in the following way [25,26]:

$$\begin{aligned} \hat{J}_\alpha^+ &= \hat{c}_\alpha^\dagger \sqrt{N_\alpha - \hat{c}_\alpha^\dagger \hat{c}_\alpha} \quad \hat{J}_\alpha^- = \sqrt{N_\alpha - \hat{c}_\alpha^\dagger \hat{c}_\alpha}\,\hat{c}_\alpha, \\ \hat{J}_{\alpha z} &= \hat{c}_\alpha^\dagger \hat{c}_\alpha - j_\alpha, \end{aligned} \tag{4}$$

where $j_\alpha = N_\alpha/2$, $\alpha = a, b$, and $\hat{c}_\alpha$ is the standard bosonic operator, having the commutator $[\hat{c}_\alpha, \hat{c}_{\alpha'}^\dagger] = \delta_{\alpha,\alpha'}$. Next, we apply the HP transformation and show that the Hamiltonian of the two-spin system in Eq. (3) can be written in terms of two coupled oscillators [47,48]. To do this, we insert Eq. (4) into Eq. (3) and obtain

$$\begin{aligned} \hat{\mathcal{H}} = &\sum_{\alpha=a,b} \left\{ t_\alpha(\hat{c}_\alpha^\dagger \hat{c}_\alpha - N_\alpha/2) + \frac{g_\alpha j_\alpha}{2} \times \left( \hat{c}_\alpha^\dagger \sqrt{1 - \frac{\hat{c}_\alpha^\dagger \hat{c}_\alpha}{N_\alpha}} + \sqrt{1 - \frac{\hat{c}_\alpha^\dagger \hat{c}_\alpha}{N_\alpha}}\,\hat{c}_\alpha \right)^2 \right\} \\ &+ g_{ab}\sqrt{j_a j_b} \left( \hat{c}_a^\dagger \sqrt{1 - \frac{\hat{c}_a^\dagger \hat{c}_a}{N_a}} + \sqrt{1 - \frac{\hat{c}_a^\dagger \hat{c}_a}{N_a}}\,\hat{c}_a \right) \\ &\times \left( \hat{c}_b^\dagger \sqrt{1 - \frac{\hat{c}_b^\dagger \hat{c}_b}{N_b}} + \sqrt{1 - \frac{\hat{c}_b^\dagger \hat{c}_b}{N_b}}\,\hat{c}_b \right). \end{aligned} \tag{5}$$

In the thermodynamic limit, $N_\alpha \to \infty$, one can obtain the effective Hamiltonian as

$$\hat{\mathcal{H}}^{(1)} = \sum_{\alpha=a,b} \left\{ t_\alpha(\hat{c}_\alpha^\dagger \hat{c}_\alpha - N_\alpha/2) + \frac{g_\alpha j_\alpha}{2}(\hat{c}_\alpha^\dagger + \hat{c}_\alpha)^2 \right\} + g_{ab}\sqrt{j_a j_b}(\hat{c}_a^\dagger + \hat{c}_a)(\hat{c}_b^\dagger + \hat{c}_b), \tag{6}$$

which is analogous to that of two coupled oscillators. By defining the position and the momentum operators

$$\hat{x}_a = \frac{1}{\sqrt{2}}(\hat{c}_a^\dagger + \hat{c}_a), \quad \hat{p}_a = i\frac{1}{\sqrt{2}}(\hat{c}_a^\dagger - \hat{c}_a), \tag{7}$$

$$\hat{x}_b = \frac{1}{\sqrt{2}}(\hat{c}_b^\dagger + \hat{c}_b), \quad \hat{p}_b = i\frac{1}{\sqrt{2}}(\hat{c}_b^\dagger - \hat{c}_b), \tag{8}$$

we rewrite the effective Hamiltonian as

$$\hat{\mathcal{H}}^{(1)} = \sum_{\alpha=a,b} \left\{ \frac{t_\alpha}{2}(\hat{p}_\alpha^2 + \hat{x}_\alpha^2) + \tilde{g}_\alpha \hat{x}_\alpha^2 \right\} + 2\tilde{g}_{ab}\hat{x}_a\hat{x}_b, \tag{9}$$

where $\tilde{g}_\alpha = g_\alpha j_\alpha$ and $\tilde{g}_{ab} = g_{ab}\sqrt{j_a j_b}$. It is then straightforward to solve the resulting normal-mode frequencies of the coupled oscillation, which is given by

$$\epsilon_\pm^{(1)} = \sqrt{\omega_+ \pm \sqrt{\omega_-^2 + 4\tilde{g}_{ab}t_a t_b}}, \tag{10}$$

where $\omega_\pm = \tilde{g}_a t_a \pm \tilde{g}_b t_b + (t_a^2 \pm t_b^2)/2$. Crucially, one can see that the normal-mode frequencies can have complex values depending on the interaction strengths, and the critical interspecies coupling strength $g_{ab}^*$ can be given as

$$g_{ab}^* \equiv \frac{1}{2}\sqrt{\left(\frac{t_a}{j_a}+2g_a\right)\left(\frac{t_b}{j_b}+2g_b\right)}. \tag{11}$$

When the interspecies coupling strength exceeds this value, the system becomes unstable. Therefore, the condition $g_{ab} > g_{ab}^*$ ($g_{ab} < g_{ab}^*$) for the immiscibility (miscibility) of the two species is satisfied. In the thermodynamic limit, $j_\alpha \to \infty$, it reduces to the well-known criticality (e.g., $g_{ab}^* = \sqrt{g_a g_b}$) for the phase separation of two component BECs [49]. Depending on the ratio of the intraspecies ($g_{a(b)}$) and interspecies ($g_{ab}$) interaction strengths, different types of phases have been extensively studied theoretically [2,50,51], and such phases are also seen experimentally [12,16].

The main motivation in this work is to find a solution above a critical point where the phase transition occurs. Such a phase transition is well known in Dicke-type models, in which one can describe the interaction of a single-mode quantized field with an ensemble of $N$ two-level atoms. It was shown that above a critical coupling strength these systems can undergo quantum phase transition [52–54], from the normal to *superradiant phase*. Here, we adopt this method [54,55] to characterize the phase transition between mixed and separated phases.

To find a solution in a region $g_{ab} > g_{ab}^*$, we displace the bosonic operators as

$$\hat{c}_a^\dagger = \hat{d}_a^\dagger \pm \sqrt{N_a\beta_a}, \qquad \hat{c}_b^\dagger = \hat{d}_b^\dagger \mp \sqrt{N_b\beta_b}. \tag{12}$$

In the following, we shall just consider the displacements as $\hat{c}_a^\dagger = \hat{d}_a^\dagger + \sqrt{N_a\beta_a}$ and $\hat{c}_b^\dagger = \hat{d}_b^\dagger - \sqrt{N_b\beta_b}$. If we insert these definitions into Eq. (5) and eliminate the first-order term in the boson operators, we can find the amounts of displacement of each mode by solving

$$\frac{t_a}{j_a} + 2\left(g_a - g_{ab}\sqrt{\frac{\beta_b(1-\beta_b)}{\beta_a(1-\beta_a)}}\right)(1-2\beta_a) = 0, \tag{13}$$

$$\frac{t_b}{j_b} + 2\left(g_b - g_{ab}\sqrt{\frac{\beta_a(1-\beta_a)}{\beta_b(1-\beta_b)}}\right)(1-2\beta_b) = 0. \tag{14}$$

The resulting effective Hamiltonian can be given by

$$\hat{\mathcal{H}}^{(2)} = \sum_{\alpha=a,b}\{\omega_\alpha \hat{d}_\alpha^\dagger \hat{d}_\alpha + \kappa_\alpha(\hat{d}_\alpha^\dagger + \hat{d}_\alpha)^2\} + \lambda(\hat{d}_a^\dagger + \hat{d}_a)(\hat{d}_b^\dagger + \hat{d}_b), \tag{15}$$

where we consider the boson operators up to the second order, and the parameters can be found as

$$\omega_\alpha = t_\alpha + 2g_{ab}j_{\overline{\alpha}}\sqrt{\beta_a\beta_b}\sqrt{\frac{1-\beta_{\overline{\alpha}}}{1-\beta_\alpha}}, \tag{16}$$

$$\kappa_\alpha = \omega_\alpha - t_\alpha + g_\alpha j_\alpha \frac{6\beta_\alpha(\beta_\alpha - 1)+1}{2(1-\beta_\alpha)}, \tag{17}$$

$$\lambda = g_{ab}\sqrt{j_a j_b}\frac{(1-2\beta_a)(1-2\beta_b)}{\sqrt{(1-\beta_a)(1-\beta_b)}}, \tag{18}$$

where $\overline{a} = b$ and $\overline{b} = a$. Before proceeding, let us check these parameters in the limits of $g_{ab} \leqslant g_{ab}^*$ and $g_{ab} \gg g_{ab}^*$. For the case $g_{ab} \leqslant g_{ab}^*$ the displacement parameters have zero value,

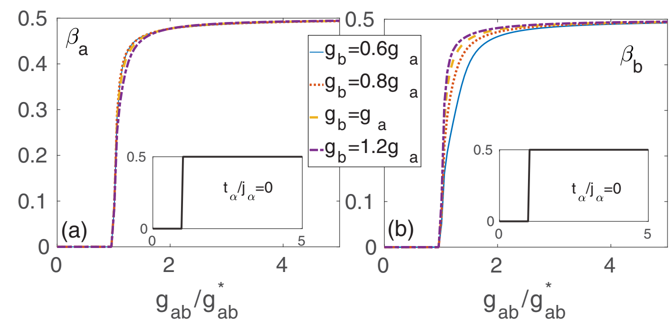


FIG. 1. The displacement amounts (a) $\beta_a$ and (b) $\beta_b$ as a function of coupling strength $g_{ab}$ for different values of $g_b$. Here, we use equal tunneling amplitudes, $t_\alpha/j_\alpha = 0.1g_a$, and in the insets, we consider the infinite number of particles case and take $t_\alpha/j_\alpha = 0$.

i.e., $\beta_\alpha = 0$, and the Hamiltonian in Eq. (15) reduces to the one given for the mixed phase in Eq. (6). When $g_{ab} \gg g_{ab}^*$, the displacement parameters take a single value, i.e., $\beta_\alpha = 0.5$, as seen in Fig. 1. In this limit, two oscillators become uncoupled, $\lambda = 0$, and by increasing the interaction strength of the interspecies, it contributes to only the *effective* strengths of the intraspecies, $\kappa_\alpha$, and tunneling, $\omega_\alpha$, coefficients, which makes the Hamiltonian $\hat{\mathcal{H}}^{(2)}$ stable for all $g_{ab}$ values. To see this, we find the new eigenfrequencies by moving to a position-momentum representation defined by

$$\hat{X}_a = \frac{1}{\sqrt{2}}(\hat{d}_a^\dagger + \hat{d}_a), \qquad \hat{P}_a = i\frac{1}{\sqrt{2}}(\hat{d}_a^\dagger - \hat{d}_a), \tag{19}$$

$$\hat{X}_b = \frac{1}{\sqrt{2}}(\hat{d}_b^\dagger + \hat{d}_b), \qquad \hat{P}_b = i\frac{1}{\sqrt{2}}(\hat{d}_b^\dagger - \hat{d}_b), \tag{20}$$

where one can see the relation between the coordinates: $\hat{X}_\alpha = \hat{x}_\alpha - \chi_\alpha\sqrt{2N_\alpha\beta_\alpha}$, $\chi_{a,b} = \pm 1$. By following the same steps that were done for $\hat{\mathcal{H}}^{(1)}$ in Eq. (6), we can find the corresponding oscillator energies for $\hat{\mathcal{H}}^{(2)}$ as

$$\epsilon_\pm^{(2)} = \frac{1}{2}\left[\tilde{\omega}_a^2 + \tilde{\omega}_b^2 \pm \sqrt{\left(\tilde{\omega}_a^2 - \tilde{\omega}_b^2\right)^2 + 16\lambda^2\omega_a\omega_b}\right]^{1/2}, \tag{21}$$

where $\tilde{\omega}_\alpha^2 = \omega_\alpha(\omega_\alpha + 4\kappa_\alpha)$. The new excitation energy, $\epsilon_-^{(2)}$, is real over the whole parameter space, and hence, $\hat{\mathcal{H}}^{(2)}$ describes the system in the phase-separation region.

Next, we give the ground-state wave functions of the mixed and separated phases. Since the two effective Hamiltonians are obtained in the form of two coupled harmonic oscillators, their wave functions can be found as the product of harmonic oscillator eigenfunctions, which can be given by

$$\psi_{\mathrm{GS}}^{(j)}(q_{ja}, q_{jb}) = \sqrt{\frac{m\Omega}{\pi}}G_1(q_{ja}, q_{jb})G_2(q_{ja}, q_{jb}), \tag{22}$$

where $j = 1, 2$ stand for the solutions of the mixed and separated phases, respectively, with $q_{1_\alpha} = x_\alpha$, $q_{2_\alpha} = X_\alpha$, and $G_1(q_{ja}, q_{jb})$ and $G_2(q_{ja}, q_{jb})$ represent Gaussian functions defined by

$$G_1(q_{ja}, q_{jb}) = e^{\frac{m\Omega}{2}[\xi(q_{ja}C - q_{jb}S)^2]}, \tag{23}$$

$$G_2(q_{ja}, q_{jb}) = e^{\frac{m\Omega}{2}[\xi^{-1}(q_{ja}S + q_{jb}C)^2]}, \tag{24}$$

with parameters

$$\xi = \frac{c_a + c_b + \sqrt{(c_a - c_b)^2 + 4\lambda^2}}{2K}, \tag{25}$$

$$K = \sqrt{c_a c_b - \lambda/2}, \quad c_\alpha = (\omega_\alpha + 4\kappa_\alpha)\sqrt{\omega_{\bar{\alpha}}/\omega_\alpha}, \tag{26}$$

$$C = \cos(\phi), \quad S = \sin(\phi), \quad \tan(2\phi) = \frac{2\lambda}{c_a - c_b}, \tag{27}$$

$$\Omega = \sqrt{K/m}, \qquad m = 1/\sqrt{\omega_a \omega_b}, \tag{28}$$

where we denote $\bar{a} = b$ and $\bar{b} = a$. We define the parameters above for only $\psi_{\rm GS}^{(2)}$; by inserting $\beta_\alpha = 0$ into the these parameters one can obtain the desired solution for $\psi_{\rm GS}^{(1)}$.

In the following sections, having derived the effective Hamiltonians and associated ground states that describe mixed and separated phases, we investigate the quantum features of this kind of phase transition in terms of the geometric phase and bipartite entanglement.

## III. GEOMETRIC PHASE

In this section, we demonstrate that by encircling the critical point in parameter space, where the miscibility-immiscibility transition occurs, a nontrivial Berry phase can be obtained for the system considered in this work. Let us start by introducing the collective angular momentum operators after displacement operation is done. In the limit of a large number of particles, they can be found as [see the Appendix]

$$\hat{J}_{\alpha x} \cong \chi_\alpha \sqrt{N_\alpha \beta_\alpha (1 - \beta_\alpha)} + \frac{1 - 2\beta_\alpha}{\sqrt{2(1 - \beta_\alpha)}} \hat{X}_\alpha, \tag{29}$$

$$\hat{J}_{\alpha y} \cong -\sqrt{\frac{1 - \beta_\alpha}{2}} \hat{P}_\alpha, \tag{30}$$

$$\hat{J}_{\alpha z} \cong N_\alpha(\beta_\alpha - 1/2) + 2\chi_\alpha \sqrt{N_\alpha \beta_\alpha} \hat{X}_\alpha + \frac{1}{2}\left(\hat{P}_\alpha^2 + \hat{X}_\alpha^2 - 1\right), \tag{31}$$

where $\chi_a = 1$, $\chi_b = -1$, and $\hat{X}_\alpha$ and $\hat{P}_\alpha$ are given in Eqs. (7) and (8). Here, we consider terms up to $(1/N)$th order in the expansion. In the ground state, $\langle \hat{J}_{\alpha y} \rangle = 0$, and the main contribution to the expectation values of $\hat{J}_{\alpha x}$ and $\hat{J}_{\alpha z}$ comes from the first terms in Eqs. (29) and (31), respectively. Thus, we can safely neglect the other terms in the thermodynamic limit and obtain

$$\frac{\langle \hat{J}_{\alpha z} \rangle}{N_\alpha} = \begin{cases} -0.5, & g_{ab} \leqslant g_{ab}^*, \\ (\beta_\alpha - 0.5), & g_{ab} > g_{ab}^*, \end{cases} \tag{32}$$

and

$$\frac{\langle \hat{J}_{\alpha x} \rangle}{\sqrt{N_\alpha}} = \begin{cases} 0, & g_{ab} \leqslant g_{ab}^*, \\ \chi_\alpha \sqrt{\beta_\alpha (1 - \beta_\alpha)}, & g_{ab} > g_{ab}^*, \end{cases} \tag{33}$$

in which one can clearly observe that above $g_{ab}^*$ there is a macroscopic excitation for each one. We introduce a time-dependent unitary transformation $U(\phi(t)) = e^{-i\phi(t)\hat{J}_z}$, where $\hat{J}_z = \hat{J}_{az} + \hat{J}_{bz}$ and $\phi(t)$ is the slowly time varying parameter, which can be defined in an experiment by constructing an adiabatic loop for the interatomic interaction strength between two species via Feshbach resonances [12,13]. When this phase, $\phi(t)$, is varied between 0 and $2\pi$, a state in phase space will encircle the critical point. Then, the Berry phase can be defined in the ground state as [29]

$$\gamma = i \int_0^{2\pi} d\phi \langle \psi | U^\dagger(\phi) \frac{d}{d\phi} U(\phi) | \psi \rangle = 2\pi \langle \hat{J}_z \rangle, \tag{34}$$

where $|\psi\rangle$ is the time-independent ground-state wave function. If we insert Eq. (32) into Eq. (34), the total scaled Berry phase of the system can be defined as

$$\frac{\tilde{\gamma}}{2\pi} = \begin{cases} 0, & g_{ab} \leqslant g_{ab}^*, \\ \beta_a + \beta_b, & g_{ab} > g_{ab}^*, \end{cases} \tag{35}$$

where we use $\tilde{\gamma} = 1 + \gamma/N$, with $N_a = N_b = N$. In Fig. 2, we demonstrate the scaled Berry phase of the system as a function of coupling strength $g_{ab}$ for finite and infinite numbers of particles. As shown, in the large-particle limit, the scaled Berry phase has a zero value for $g_{ab} \leqslant g_{ab}^*$ and above $g_{ab}^*$ increases with increasing coupling strength, and its derivative becomes discontinuous at the critical value $g_{ab}^*$. Interestingly, in the thermodynamic limit, there is a steplike transition, which can be seen in the inset of Fig. 2(a). This is due to solutions of Eqs. (13) and (14). When $t_\alpha/j_\alpha = 0$, one can see that there is a single solution for each displacement, i.e., $\beta_\alpha = 0.5$.

It is also possible to obtain a nontrivial Berry phase for each species if we define the unitary transformation as $U(\phi(t)) = e^{-i\phi(t)\hat{J}_{\alpha z}}$, and we can obtain $\gamma_\alpha = 2\pi\langle \hat{J}_{\alpha z} \rangle$. It can be read from Eq. (32) that above the critical coupling strength $g_{ab}^*$ there is a finite atomic inversion for each species. This illustrates the fact that each species has also a nontrivial Berry phase.

As shown above, increasing the number of particles creates a sharper transition. Increasing interspecies coupling leads to the higher value of the Berry phase. This scenario, however, is not the same for bipartite entanglement. In the following section, we discuss this in more detail.

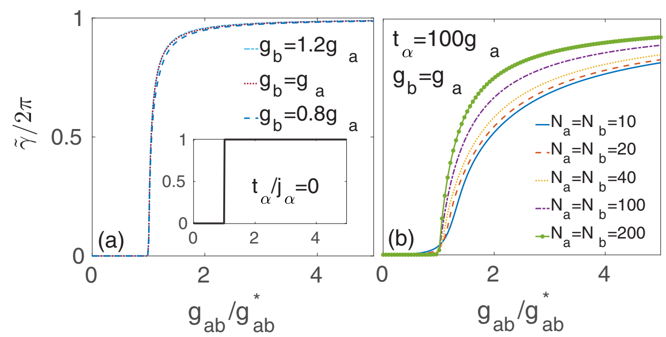


FIG. 2. The scaled Berry phase of the system for (a) a large and (b) small number of particles and as a function of coupling strength $g_{ab}$. Here, we use equal tunneling amplitudes for (a), $t_\alpha/j_\alpha = 0.1g_a$, and an equal number of particles, $N_a = N_b = 10^4$. In the inset, we take $t_\alpha/j_\alpha = 0$ (or, equivalently, $j_\alpha = \infty$). For a finite number of particles (b), we use $t_\alpha = 100g_a$ and obtain the results by solving the eigenstates of Eq. (3) numerically.

## IV. BIPARTITE ENTANGLEMENT

Detecting the entanglement in an ensemble system by accessing the individual particles is not practical. To obtain a solution in such structures, the global parameters, such as total

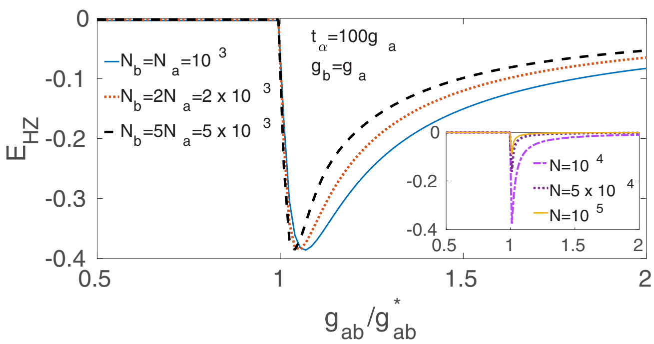


FIG. 3. Hillery-Zubairy entanglement criterion [Eq. (36)] as a function of coupling strength $g_{ab}$. $E_{\text{HZ}} < 0$ witnesses the presence of entanglement between the two collective spins. Here, we use $t_\alpha = 100 g_a$ and $g_b = g_a$. In the inset, we take $N_a = N_b = N$ and observe that entanglement decays faster as $N$ increases.

spin, are used instead [32,33]. By having such observables in a system, it becomes possible to quantify entanglement [31,56–59]. For example, in a recent experiments [36–38], bipartite entanglement was reported for ultracold atomic BECs by measuring collective spins.

There are several practical criteria [30,31,43] for the detection of entanglement. These methods are, in general, sufficient, but not necessary. Depending on the structure in a given system, some criteria work better. For instance, in our recent work [48], we compared different types of criteria when the system exhibits a quantum phase transition, where we observed that criteria based on bilinear products of spins can witness the entanglement of two strongly interacting ensembles for a small number of particles, which the criteria of first-order spin fails to detect.

In this section, we anticipate the spin form of the criterion derived by Hillery and Zubairy [43] to investigate bipartite entanglement. We first introduce the inequality for the detection of entanglement based upon the effective local spin operators (29)–(31), which is given by [42]

$$E_{\text{HZ}} = \langle \hat{J}_a^+ \hat{J}_a^- \hat{J}_b^+ \hat{J}_b^- \rangle - |\langle \hat{J}_a^+ \hat{J}_b^- \rangle|^2 < 0, \tag{36}$$

where $E_{\text{HZ}} < 0$ witnesses the presence of entanglement between the two collective spins. It is important to note that a complete analysis of the derivation of the criterion is beyond the scope of the present paper, and we point the reader to Refs. [42,60,61] for more details.

In Fig. 3, we demonstrate the results of Eq. (36) as a function of interspecies coupling strength $g_{ab}$ for various numbers of particles. When the interspecies coupling strength exceeds the critical value, a transition also appears in the entanglement. Unlike the Berry phase, entanglement decays at larger values of $g_{ab}$ and/or $N$. This is due to the effective coupling strength $\lambda$ in Eq. (15), which can be considered the source of bipartite entanglement. The explanation of this behavior is as follows. When the interatomic coupling strength increases, $g_{ab} \gg g_{ab}^*$, the value of the displacement parameters approaches the single value, i.e., $\beta_\alpha \to 0.5$, where effective coupling vanishes [see Eq. (18)]. A similar story is valid for an increasing number of particles. As $N \to \infty$, $\beta_\alpha \to 0.5$ (see Fig. 1). This can be observed in the inset of Fig. 3.

In Ref. [5], by using the criterion based on quadrature squeezing [62], it was shown that two-mode entanglement can be present in the stable region ($g_{ab} < g_{ab}^*$). Ng *et al.* observed that higher entanglement can be obtained as the system becomes closer to the critical point. Here, however, $E_{\text{HZ}}$ starts to capture entanglement around the critical value, $g_{ab} \gtrsim g_{ab}^*$ (see Fig. 3). This shows that the criteria obtained from the squeezing of the spin noise and the ones via the squeezing of the bilinear products of the spin are successful in different inseparability regimes [59]. Experimental testing of these criteria, however, is possible in a system of the two-component BEC by adiabatically changing the interatomic interaction strength around the critical value.

## V. SUMMARY

In summary, we have investigated theoretically the ground-state properties of the two-component Bose-Einstein condensate trapped in a double-well potential. We observed that the system can undergo a QPT at a critical coupling strength. We obtained the effective Hamiltonians and associated ground-state wave functions to describe the system in each of its mixed and separated phases. The nontrivial geometric phase is found near the critical coupling in the limit of small and large numbers of particles, where we observe a steplike transition in the thermodynamic limit. The accuracy of the model is confirmed by comparison with the exact numerical solution in the limit of a small number of particles limit (see Fig. 2). We also anticipated the spin form of the Hillery and Zubairy criterion to quantify entanglement across a QPT. It was observed that the entanglement witness $E_{\text{HZ}}$ decays with increasing the interspecies interaction strength and/or number of particles. The tunable interactions between the two species via Feshbach resonances make the model a promising simulator for this kind of structure and can find potential in the area of quantum communication [63] and quantum sensing [64].


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## APPENDIX

Here, we show the derivations of Eqs. (29) and (30). By inserting displaced operators defined in Eq. (12) into Eq. (4), they become

$$\hat{J}_\alpha^+ = (\hat{d}_\alpha^\dagger + \chi_\alpha \sqrt{N_\alpha \beta_\alpha}) \sqrt{1-\beta_\alpha} \sqrt{1-\xi_\alpha}, \tag{A1}$$

$$\xi_\alpha \equiv \left(1 - \frac{\hat{d}_\alpha^\dagger \hat{d}_\alpha + \chi_\alpha \sqrt{N_\alpha \beta_\alpha}}{N_\alpha (1-\beta_\alpha)}\right). \tag{A2}$$

After expanding the last term, $\sqrt{1-\xi_\alpha}$, in Eq. (A1) up to the $(1/N)$th order, one can arrive at

$$\hat{J}_\alpha^+ \approx \sqrt{1-\beta_\alpha}\left[(\hat{d}_\alpha^\dagger + \sqrt{N_1\beta_\alpha}) - \frac{\beta_\alpha}{1-\beta_\alpha}\frac{\hat{X}_\alpha}{\sqrt{2}}\right]. \quad \text{(A3)}$$

Similarly, one can derive the lowering component, $\hat{J}_\alpha^- = (\hat{J}_\alpha^+)^\dagger$. By using the definitions $\hat{J}_{\alpha x} = (\hat{J}_\alpha^+ + \hat{J}_\alpha^-)/2$ and $\hat{J}_{\alpha y} = (\hat{J}_\alpha^+ - \hat{J}_\alpha^-)/2i$, we derive Eqs. (29) and (30) as

$$\hat{J}_{\alpha x} \cong \chi_\alpha\sqrt{N_\alpha\beta_\alpha(1-\beta_\alpha)} + \frac{1-2\beta_\alpha}{\sqrt{2(1-\beta_\alpha)}}\hat{X}_\alpha, \quad \text{(A4)}$$

$$\hat{J}_{\alpha y} \cong -\sqrt{\frac{1-\beta_\alpha}{2}}\hat{P}_\alpha. \quad \text{(A5)}$$

It is straightforward to obtain the $z$ component of the angular momentum operator by inserting Eqs. (12), (7), and (8) into the definition of $\hat{J}_{\alpha z}$ given in Eq. (4).


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