Subspace stabilization analysis for a class of non-Markovian open quantum systems

Shikun Zhang [1,*] Kun Liu, [1,†] Daoyi Dong, [2] Xiaoxue Feng, [1] and Feng Pan [1,‡]
[1] *School of Automation, Beijing Institute of Technology, Beijing 100081, China*
[2] *School of Engineering and Information Technology, University of New South Wales, Caberra ACT 2600, Australia*

(Received 19 February 2019; accepted 24 March 2020; published 22 April 2020)

Studied in this article is non-Markovian open quantum systems parametrized by Hamiltonian H, coupling operator L, and memory kernel function γ, which is a proper candidate for describing the dynamics of various solid-state quantum information processing devices. We look into the subspace stabilization problem of the system from the perspective of dynamical systems and control. The problem translates itself into finding analytic conditions that characterize invariant and attractive subspaces. Necessary and sufficient conditions are found for subspace invariance based on algebraic computations, and sufficient conditions are derived for subspace attractivity by applying a double integral Lyapunov functional. Mathematical proof is given for those conditions and a numerical example is provided to illustrate the theoretical result.

DOI: 10.1103/PhysRevA.101.042327

I. INTRODUCTION

Human beings are now in a century when we can not only observe and describe quantum systems, but also alter and control them so as to harness their power unparalleled by classical resources. A promising application lies in quantum information processing (QIP), where exponentially faster computation and provably safer communication are possible to be realized [1]. In the recent decade, effective QIP devices have been known including silicon photonic crystals [2], trapped ions [3], and superconducting quantum circuits [4].

"Quantum information" in the digital world must be represented by, stored in, and manipulated through actual physical systems, whose states evolve according to the laws of quantum mechanics and even quantum field theory. Therefore, rigorously analyzing and actively tuning the dynamics of those systems are among the fundamental building blocks of quantum information engineering. This coincides with the basic objective of systems and control science, which is to predict the evolution of dynamical systems and make them behave in the way we desire. As a result, quantum control (cybernetics) [5–7], born at the intersection of quantum physics, control science, and applied mathematics, becomes a useful tool to achieve successful QIP and other quantum engineering applications.

In this work, we take an in-depth look into the subspace stabilization problem which lies in the realm of systems and control theory and finds applications in a wide range of QIP problems, e.g., initialization of qubit, generation of entangled states, and realization of decoherence-free quantum information. This problem was first studied in [8], where it was analyzed in the framework of subspace invariance and attractivity. The authors in [8] presented a set of algebraic conditions that characterize invariant and attractive subspaces. Moreover, in [9], sufficient and necessary conditions were derived for invariance and attractivity as opposed to mostly necessary conditions in [8]. As subsequent works, the authors in [10] constructively designed system parameters (H, L) to stabilize generic quantum states, and Ref. [11] introduced a computable algorithm to verify those previously proposed conditions and analyzed the speed of convergence.

The existing results on the subspace stabilization problem, to date, mainly cover Lindblad systems [12]. Among the several assumptions that lead to the Lindblad master equation lies the Markovian assumption, which requires that environmental correlations be sufficiently short compared with the system's characteristic timescale. This results in a memoryless, or in other words, Markovian, system where information only flows in one direction. Yet this assumption does not apply to all scenarios. For instance, the modeling of mesoscopic quantum circuits, where field propagation time delay and nonclassical input states are considered, often sees the breakdown of the Markovian assumption [13]. It is thereby natural to extend the analysis of subspace stabilization into the non-Markovian regime.

In the recent decade, non-Markovian quantum systems have attracted increasing interest from the academia. A large amount of work has been done on deriving proper mathematical models, defining and measuring non-Markovianity, and analyzing complete positivity (see [14] for an excellent review). However, very few results have addressed the properties of system dynamics given a non-Markovian master equation, which is a topic of major focus for systems and control theorists. Here we study the subspace stabilization problem for non-Markovian quantum systems as an investigation of quantum dynamics with memory and for achieving QIP tasks on physical devices with significant non-Markovian effects.

The master equation on which our work is based was derived in [15] for non-Markovian input-output networks. It applies to atomlike structures in radiation fields; for example, the

*daxiayusuozhang@126.com
†Corresponding author: kunliubit@bit.edu.cn
‡Corresponding author: andropanfeng@126.com

superconducting circuit and microwave system. The resulting equation is a time-convolutional one where the derivative of current state depends on all history states and environmental interactions, as opposed to its Markovian (Lindblad) counterpart where only the present state matters. The mathematical object behind time-convolutional non-Markovian equations is the integro-differential system (see [16]).

The rest of the article is organized as follows. In Sec. II, we introduce the non-Markovian master equation to be studied and define the scope of system parameters to our interest. This is followed by Sec. III, where the definition of invariant subspaces is given and its *iff* conditions are provided and proved. Section IV presents the definition and sufficient conditions of subspace attractivity, and Sec. V gives an example of a three-level system followed by numerical simulation. The article is concluded in Sec. VI, which summarizes the work and suggests future directions. Moreover, Sec. VI covers some important physical implications which are worth deeper investigation.

II. NON-MARKOVIAN SYSTEM MODEL

In this article, we study non-Markovian open quantum systems described by the following time-convolutional master equation, which was derived in [15] by applying the Born approximation.

$$\dot{\rho} = -i[H,\rho] + \int_0^t \{\gamma^*(t-\tau)[L\rho(\tau), L_H^\dagger(\tau-t)] + \gamma(t-\tau)[L_H(\tau-t), \rho(\tau)L^\dagger]\}d\tau, \tag{1}$$

where

$$L_H(t) = e^{iHt}Le^{-iHt}. \tag{2}$$

There are three parameters in the system model. The Hermitian operator H stands for system Hamiltonian, which generates internal dynamics for the system. Meanwhile, L represents the coupling operator, which describes the interaction interface between the quantum system and its environment. Finally, the memory kernel function $\gamma(t)$ demonstrates the non-Markovianity of the system by weighing the influence of all history system-environment interactions.

Physically, this is a type of perturbative non-Markovian master equation for a localized quantum system interacting with its environment. As opposed to Markovian systems driven by quantum white noise [17] which has a flat spectrum, non-Markovian systems are driven by colored noise whose spectral density varies with frequency [18]. This distinction is what results in a memory kernel function in master equation (1), which is derived under certain assumptions for colored noise perturbed systems.

In terms of microwave-superconducting circuit systems [13,15], the input field (modeled as white noise) does not interact directly with the central superconducting qubit (localized system), but with an intermediate bath which transforms it into colored noise. As complicated as the structure of the bath might be, as long as we can pin down its spectrum, the dynamics of the qubit can be approximately described by (1). Therefore, the study of dynamics described by (1) is instructive to the manipulation of quantum computation units with a similar mechanism.

For the sake of simplicity, only real, continuous, and finite kernel functions are considered in this work. It is also assumed that $\gamma(0) \neq 0$ and $\gamma \in L^1[0,\infty)$. More restrictions on γ may need to be considered to guarantee complete positivity of the non-Markovian master equation. However, deriving such conditions is beyond the scope of this article. In fact, complete positivity has been proven in the case of Lorentz spectrum quantum noises (exponentially decaying memory kernels) [15], which indicates that completely positive dynamics can be induced by a set of kernel functions that subsumes the exponential family. Therefore, we make a further assumption that γ belongs to this set.

It is worthwhile noticing the linkage between this master equation and the well-known Lindblad master equation:

$$\dot{\rho} = -i[H,\rho] + L\rho L^\dagger - \tfrac{1}{2}L^\dagger L\rho - \tfrac{1}{2}\rho L^\dagger L. \tag{3}$$

It is intuitively convenient to view (3) as a "limit" when the finite memory kernel function $\gamma(t)$ is squeezed higher and higher around the origin, yielding Dirac $\delta(t)$ and a Markovian master equation.

Given that the open system evolves under (1), its subspace stabilization problem is divided into invariance and attractivity analysis, which will be discussed separately in the following sections.

III. SUBSPACE INVARIANCE

This section involves the first half of the subspace stabilization problem, subspace invariance. We give a definition of invariant subspaces and present necessary and sufficient conditions that characterize them.

Let $\mathcal{H}_I$ be a finite dimensional Hilbert space, and $\mathcal{D}(\mathcal{H}_I)$ be the set of all semipositive, trace-one, Hermitian linear bounded operators on $\mathcal{H}_I$ (density matrices), which forms the state space for quantum system (1). The Hilbert space admits the following decomposition:

$$\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R, \tag{4}$$

where $\mathcal{H}_S = \text{span}\{|\varphi_j^S\rangle\}_{j=0}^m$ and $\mathcal{H}_R = \text{span}\{|\psi_k^R\rangle\}_{k=0}^n$. All basis vectors are orthonormal. According to this subspace decomposition, each operator in (1) has a block matrix representation given this set of bases. We denote those matrices as follows:

$$H = \begin{pmatrix} H_S & H_P \\ H_Q & H_R \end{pmatrix}, \quad L = \begin{pmatrix} L_S & L_P \\ L_Q & L_R \end{pmatrix},$$

$$\rho(t) = \begin{pmatrix} \rho_S(t) & \rho_P(t) \\ \rho_Q(t) & \rho_R(t) \end{pmatrix},$$

$$L_H(t) = \begin{pmatrix} L_H^S(t) & L_H^P(t) \\ L_H^Q(t) & L_H^R(t) \end{pmatrix}.$$

The Hermiticity of H and ρ implies that $H_Q = H_P^\dagger$ and $\rho_Q(t) = \rho_P^\dagger(t)$.

We now define what an invariant subspace is. It can be verified that our definition is equivalent to that in [8] and [9]. However, we simplify the narration in those works by suppressing the notion of quantum subsystems.

Definition 1 (Subspace invariance). Let the quantum system evolve under (1). Hilbert space $\mathcal{H}_S$ is an invariant subspace if the following condition is satisfied:

$$\text{if}\qquad \rho(0)=\begin{pmatrix}\rho_S^0 & 0\\ 0 & 0\end{pmatrix},\quad \forall\rho_S^0\in\mathcal{D}(\mathcal{H}_S),$$

$$\text{then}\qquad \rho(t)=\begin{pmatrix}\rho_S(t) & 0\\ 0 & 0\end{pmatrix},\quad \forall t\geqslant 0.$$

The following lemma completely characterizes invariant subspaces.

Lemma 1 (Subspace invariance). The following conditions (i), (ii), and (iii) are necessary and sufficient for $\mathcal{H}_S$ to be an invariant subspace.

(i)

$$H=\begin{pmatrix}H_S & 0\\ 0 & H_R\end{pmatrix};$$

(ii)

$$L=\begin{pmatrix}L_S & L_P\\ 0 & L_R\end{pmatrix};$$

(iii) Denote by $\rho_S(t;\rho_S^0)$ the trajectory, with initial value ρ_S^0, which satisfies the following integro-differential equation

$$\dot\rho_S=-i[H_S,\rho_S]+\int_0^t\gamma^*(t-\tau)\big[L_S\rho_S(\tau),L_H^{S\dagger}(\tau-t)\big]+(\text{H.c.})d\tau,\tag{5}$$

where

$$L_H^S(t)=e^{iH_St}L_Se^{-iH_St}.\tag{6}$$

Then, $\forall\rho_S^0\in\mathcal{D}(\mathcal{H}_S)$,

$$\int_0^t\gamma(t-\tau)\rho_S(\tau;\rho_S^0)L_S^\dagger L_H^P(\tau-t)d\tau=0.\tag{7}$$

Proof. (Necessity). Suppose $\mathcal{H}_S$ is an invariant subspace. Then, we have the following relationship according to Definition 1:

$$\rho(t)=\begin{pmatrix}\rho_S\big(t;\rho_S^0\big) & 0\\ 0 & 0\end{pmatrix},\quad\forall t\geqslant 0,\quad\forall\rho_S^0\in\mathcal{D}(\mathcal{H}_S);$$

$$\dot\rho(t)=\begin{pmatrix}\dot\rho_S\big(t;\rho_S^0\big) & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}S(t) & P(t)\\ Q(t) & R(t)\end{pmatrix},\quad\forall t\geqslant 0.$$

Hermiticity of the state density matrix and its derivative imply that $Q(t)=P^\dagger(t)$. We proceed to compute explicitly the S, P, and R blocks.

$$S(t)=-i[H_S,\rho_S]+\int_0^t\big\{\gamma^*(t-\tau)\big[L_S\rho_S(\tau),L_H^{S\dagger}(\tau-t)\big]-L_H^{Q\dagger}(\tau-t)L_Q\rho_S(\tau)\big\}+(\text{H.c.})d\tau,\tag{8}$$

$$P(t)=i\rho_SH_P+\int_0^t\gamma^*(t-\tau)L_S\rho_S(\tau)L_H^{Q\dagger}(\tau-t)+\gamma(t-\tau)\big\{L_H^S(\tau-t)\rho_S(\tau)L_Q^\dagger-\rho_S(\tau)\big[L_S^\dagger L_H^P(\tau-t)+L_Q^\dagger L_H^R(\tau-t)\big]\big\}d\tau,\tag{9}$$

$$R(t)=\int_0^t\gamma^*(t-\tau)L_Q\rho_S(\tau)L_H^{Q\dagger}(\tau-t)+(\text{H.c.})d\tau.\tag{10}$$

Since $R(t)\equiv 0$, then $\dot R(t)\equiv 0$, and $\dot R(0)=0$. Changing the integration variable yields

$$R(t)=\int_0^t\gamma^*(\tau)L_Q\rho_S(t-\tau)L_H^Q(-\tau)+(\text{H.c.})d\tau,\tag{11}$$

$$\dot R(t)=\int_0^t\gamma^*(\tau)L_Q\partial_t\rho_S(t-\tau)L_H^Q(-\tau)+(\text{H.c.})d\tau+\gamma^*(t)L_Q\rho_S^0L_H^{Q\dagger}(-t)+\text{H.c.},\tag{12}$$

$$\dot R(0)=[\gamma^*(0)+\gamma(0)]L_Q\rho_S^0L_Q^\dagger=0,\quad\forall\rho_S^0\in\mathcal{D}(\mathcal{H}_S).\tag{13}$$

Therefore, $L_Q=0$. The S and P blocks are thus reduced to

$$S(t)=-i[H_S,\rho_S]+\int_0^t\gamma^*(t-\tau)\big[L_S\rho_S(\tau),L_H^{S\dagger}(\tau-t)\big]+(\text{H.c.})d\tau,\tag{14}$$

$$P(t)=i\rho_SH_P+\int_0^t\gamma^*(t-\tau)L_S\rho_S(\tau)L_H^{Q\dagger}(\tau-t)-\gamma(t-\tau)\rho_S(\tau)L_S^\dagger L_H^P(\tau-t)d\tau.\tag{15}$$

Moreover, since $P(0)=i\rho_S^0H_P=0$, the arbitrariness of ρ_S^0 indicates that $H_P=0$. It follows that H must have a block diagonal structure, thus leading to the explicit form of $L_H(t)$:

$$L_H(t)=\begin{pmatrix}e^{iH_St}L_Se^{-iH_St} & e^{iH_St}L_Pe^{-iH_Rt}\\ 0 & e^{iH_Rt}L_Re^{-iH_Rt}\end{pmatrix}.\tag{16}$$

This structure implies that $L_H^Q(t)=0$, which further reduces the P block to

$$P(t)=-\int_0^t\gamma(t-\tau)\rho_S(\tau,\rho_S^0)L_S^\dagger L_H^P(\tau-t)d\tau\equiv 0.\tag{17}$$

Necessity is thus proved.

Sufficiency. Suppose that conditions (i), (ii), and (iii) are satisfied. Direct calculation yields the following integro-differential equations for subblocks of the state density matrix:

$$\dot\rho_S(t)=-i[H_S,\rho_S]+\int_0^t\gamma^*(t-\tau)\big\{\big[L_S\rho_S(\tau)+L_P\rho_P^\dagger(\tau),L_H^{S\dagger}(\tau-t)\big]+[L_S\rho_P(\tau)+L_P\rho_R(\tau)]L_H^{P\dagger}(\tau-t)\big\}+(\text{H.c.})d\tau,\tag{18}$$

$$\dot{\rho}_P(t) = -i(H_S\rho_P - \rho_P H_R) + \int_0^t \gamma^*(t-\tau)\{[L_S\rho_P(\tau) + L_P\rho_R(\tau)]L_H^{R\dagger}(\tau-t) - L_H^{S\dagger}(\tau-t)[L_S\rho_P(\tau) + L_P\rho_R(\tau)]\}$$
$$+\gamma(t-\tau)\{[L_H^S(\tau-t)\rho_P(\tau) + L_H^P(\tau-t)\rho_R(\tau)]L_R^\dagger - [\rho_S(\tau)L_S^\dagger + \rho_R(\tau)L_P^\dagger]L_H^P(\tau-t) - \rho_R(\tau)L_R^\dagger L_H^R(\tau-t)\}d\tau, \quad (19)$$

and

$$\dot{\rho}_R(t) = -i[H_R, \rho_R] + \int_0^t \gamma^*(t-\tau)\{[L_R\rho_R(\tau), L_H^{R\dagger}(\tau-t)] - L_H^{P\dagger}(\tau-t)[L_S\rho_P(\tau) + L_P\rho_R(\tau)]\} + (\text{H.c.})d\tau. \quad (20)$$

It suffices to verify that $\rho(t;\rho_S^0)$, $\rho_P(t)\equiv 0$, and $\rho_R(t)\equiv 0$ are solutions of (18), (19), and (20). It is clear that (18) and (20) are satisfied, while (19) leads to

$$-\int_0^t \gamma(t-\tau)\rho_S(\tau;\rho_S^0)L_S^\dagger L_H^P(\tau-t)d\tau = 0, \quad (21)$$

which is satisfied because of condition (iii). This completes the proof of sufficiency. ■

Remark 1. Lemma 1 can be readily generalized to non-Markovian quantum systems with multiple environmental coupling operators (multiple noise input channels). The master equation is expressed as

$$\dot{\rho} = -i[H,\rho] + \sum_{k=1}^n \int_0^t \{\gamma^*(t-\tau)[L_k\rho(\tau), L_{H,k}^\dagger(\tau-t)] + \gamma(t-\tau)[L_{H,k}(\tau-t), \rho(\tau)L_k^\dagger]\}d\tau, \quad (22)$$

where

$$L_{H,k}(t) = e^{iHt}L_k e^{-iHt}, \quad k = 1,2,\ldots,n. \quad (23)$$

We list the *iff* conditions (i′), (ii′), and (iii′) for subspace invariance regarding (22) without proving them, since the proof can be completed by following exactly the same procedure as that of Theorem 1.

(i′) = (i)

(ii′)

$$L_k = \begin{pmatrix} L_{S,k} & L_{P,k} \\ 0 & L_{R,k} \end{pmatrix}, \quad k = 1,2,\ldots,n;$$

(iii′) Denote by $\rho_S(t;\rho_S^0)$ the trajectory, with initial value ρ_S^0, which satisfies the following integro-differential equation

$$\dot{\rho}_S = -i[H_S,\rho_S] + \sum_{k=1}^n \int_0^t \gamma^*(t-\tau)[L_{S,k}\rho_S(\tau), L_{H,k}^{S\dagger}(\tau-t)] + (\text{H.c.})d\tau, \quad (24)$$

where

$$L_{H,k}^S(t) = e^{iH_St}L_{S,k}e^{-iH_St}, \quad k = 1,2,\ldots,n. \quad (25)$$

Then, $\forall \rho_S^0 \in \mathcal{D}(\mathcal{H}_S)$,

$$\int_0^t \gamma(t-\tau)\rho_S(\tau;\rho_S^0)\sum_{k=1}^n L_{S,k}^\dagger L_{H,k}^P(\tau-t)d\tau = 0, \quad (26)$$

where

$$L_{H,k}^P(t) = e^{iH_St}L_{P,k}e^{-iH_Rt}, \quad k = 1,2,\ldots,n. \quad (27)$$

Although the conditions given in Lemma 1 are necessary and sufficient, condition (iii) may be difficult to verify for systems with high dimensions. Therefore, some useful necessary (not sufficient) and sufficient (not necessary) conditions are provided as theorems of this section.

Theorem 1. Consider the following conditions (iv), (v), and (vi):

(iv) $L_S^\dagger L_P = 0$;

(v) $[L_S^\dagger, H_S] = 0$;

(vi) $H_S L_P = L_P H_R$.

(iv) is necessary for $\mathcal{H}_S$ to be invariant. (i), (ii), (iv), and (v) are sufficient for subspace invariance. Moreover, (i), (ii), (iv), and (vi) are also sufficient.

Proof. We begin by showing that (iv) is necessary.

Calculating the derivative of $P(t)$ at $t = 0$ yields

$$\dot{P}(0) = -\gamma(0)\rho_S^0 L_S^\dagger L_P = 0,$$

which holds for arbitrary ρ_S^0. This implies that $L_S^\dagger L_P = 0$.

For the sufficiency of (i), (ii), (iv), and (v), we prove that (iv) and (v) lead to (iii).

This is clear since

$$L_S^\dagger L_H^P(\tau-t) = L_S^\dagger e^{iH_S(\tau-t)}L_P e^{-iH_R(\tau-t)}$$
$$= e^{iH_S(\tau-t)}L_S^\dagger L_P e^{-iH_R(\tau-t)}$$
$$= 0.$$

Similarly, it can also be proven that (iv) and (vi) lead to (iii). This follows from the fact $L_S^\dagger L_H^P(\tau-t) = L_S^\dagger L_P = 0$ under conditions (iv) and (vi).

Thus we end the proof of Theorem 1. ■

At this point, it should be noted that the conditions for subspace invariance regarding the Markovian case (Lindblad equation) [8] are $L_Q = 0$ and $iH_P - 1/2L_S^\dagger L_P = 0$, where a single noise input channel is considered for simplicity. These conditions are apparently weaker than that in our non-Markovian case (see Lemma 1), consistent with the fact that stronger assumptions are required to derive Lindblad master equations in the first place [12]. Technically, by replacing $\gamma(t)$ in (15) with $\delta(t)$ and substituting $L_Q = 0$ (a condition required in both cases), we have $P(0) = \rho_S(0)(iH_P - 1/2L_S^\dagger L_P)$. The requirement that $P(0) = 0$ and arbitrariness of $\rho_S(0)$ together yield the Markovian condition.

After defining and characterizing invariant subspaces, it can be seen that each of them determines an "invariant set" in $\mathcal{D}(\mathcal{H}_I)$, which is the set of density matrices that are "compressed" within the top left S block. If the initial state locates in this set, all future states will remain in it as long as the system evolves under (1). Invariant subspaces thus correspond to preserved quantum information: initial quantum states supported by them will not decohere into the maximally mixed state. Moreover, they pave the way for subspace attractivity, which will be discussed in the next section.

IV. SUBSPACE ATTRACTIVITY

Building on the analysis of subspace invariance in the previous section, we proceed to define and characterize attractive subspaces. It can also be checked that this is equivalent to the definition in [8].

Definition 2 (Subspace attractivity). Let $\rho(t)$ evolve under (1). If

$$\lim_{t\to+\infty}\left[\rho(t)-\begin{pmatrix}\rho_S(t) & 0\\ 0 & 0\end{pmatrix}\right]=0$$

for all initial states in $\mathcal{D}(\mathcal{H}_I)$, and $\mathcal{H}_S$ is invariant, then $\mathcal{H}_S$ is said to be an attractive subspace.

It is straightforward from this definition that an attractive subspace $\mathcal{H}_S$ is related to an invariant and attractive set of density matrices.

Suppose $[L_S^\dagger, H_S]=0$. Then (20) reduces to the following equation considering real γ functions.

$$\begin{aligned}\dot\rho_R(t) &= -i[H_R,\rho_R]+\int_0^t \gamma(t-\tau)\{[L_R\rho_R(\tau),L_H^{R\dagger}(\tau-t)]\\ &\quad+[L_H^R(\tau-t),\rho_R(\tau)L_R^\dagger]\}d\tau+\int_0^t\gamma(t-\tau)\\ &\quad\times[-L_H^{P\dagger}(\tau-t)L_P\rho_R(\tau)-\rho_R(\tau)L_P^\dagger L_H^P(\tau-t)]d\tau.\end{aligned} \tag{28}$$

This implies that the evolution of ρ_R is independent, as opposed to (20), where it also relies on ρ_P.

We cast (28) into superoperator form:

$$\dot\rho_R=\mathcal{A}\rho_R+\int_0^t\mathcal{B}(t-\tau)\rho_R(\tau)d\tau+\int_0^t\mathcal{K}(t-\tau)\rho_R(\tau)d\tau, \tag{29}$$

where

$$\begin{aligned}\mathcal{A}[\cdot]&=-i[H_R,\cdot],\\ \mathcal{B}(t)[\cdot]&=\gamma(t)\{[L_R\cdot,L_H^{R\dagger}(-t)]+[L_H^R(-t),\cdot L_R^\dagger]\},\\ \mathcal{K}(t)[\cdot]&=-\gamma(t)[L_H^{P\dagger}(-t)L_P\cdot+\cdot L_P^\dagger L_H^P(-t)].\end{aligned}$$

Before presenting the main theorem of this section, we shall first prove a useful lemma.

Lemma 2. Let $f(t)$ be a continuously differentiable function on $[0,\infty)$, and $f(t)\geqslant 0$. If $\dot f(t)\leqslant\phi(t)$, where $\phi(t)\geqslant 0$ and $\phi\in L^1[0,\infty)$, then $f(t)$ must have a finite limit when t tends to infinity.

Proof. We first prove that the statement is correct when $\dot f(t)$ has a finite number of zero points.

Let $t_{\max}$ be the largest zero point. On $(t_{\max},\infty)$, $\dot f(t)$ must either remain negative or positive. If it remains negative, then $f(t)$ must have a limit since it is descending and lower bounded by 0 on $(t_{\max},\infty)$. If it remains positive, consider the following inequalities.

$$\begin{aligned}f(t)&=f(0)+\int_0^t\dot f(s)ds\\ &\leqslant f(0)+\int_0^t\phi(s)ds\\ &\leqslant f(0)+\int_0^\infty\phi(s)ds.\end{aligned} \tag{30}$$

Because $\phi\in L^1[0,\infty)$, $f(t)$ is upper bounded. It thus has a limit since it is increasing on $(t_{\max},\infty)$.

We then proceed to consider the case where $\dot f(t)$ has an infinite number of zero points. The statement can be proved by contradiction. Suppose that $f(t)$ has no limits. Then we have

$$U=\overline{\lim_{t\to+\infty}}f(t)>\underline{\lim_{t\to+\infty}}f(t)=L.$$

Denote the sequence of peaks by $\{u_n\}_{n=1}^\infty$, and the sequence of valleys by $\{l_n\}_{n=1}^\infty$. The definition of limit superior and limit inferior implies that

$$U=\overline{\lim_{t\to+\infty}}f(t)=\overline{\lim_{n\to+\infty}}u_n,$$

$$L=\underline{\lim_{t\to+\infty}}f(t)=\underline{\lim_{n\to+\infty}}l_n.$$

The equivalent definition of superior and inferior limits indicates that there exist

$$\{u_{n_k}\}_{k=1}^\infty\subset\{u_n\}_{n=1}^\infty,\quad \lim_{k\to+\infty}u_{n_k}=U;$$
$$\{l_{n_k}\}_{k=1}^\infty\subset\{l_n\}_{n=1}^\infty,\quad \lim_{k\to+\infty}l_{n_k}=L.$$

We also have that $\forall t,s>0, t\geqslant s$,

$$f(t)-f(s)=\int_s^t\dot f(\tau)d\tau\leqslant\int_s^t\phi(\tau)d\tau.$$

Since $\phi\in L^1[0,\infty)$, $f(t)-f(s)$ tends to 0 when t and s tend to infinity. However, if we pick $\{u_{n_{k_i}}\}_{i=1}^\infty\subset\{u_{n_k}\}_{k=1}^\infty$, s.t. $l_{n_i}\leqslant u_{n_{k_i}}, \forall i\in N^+$, we have

$$\lim_{i\to+\infty}[f(u_{n_{k_i}})-f(l_{n_i})]=U-L>0.$$

This results in a contradiction. Therefore, $f(t)$ must have a limit. ■

We are now in the position to present the main result on subspace attractivity.

Theorem 2 (Subspace attractivity). If $[L_S^\dagger,H_S]=0$, and there exists a negative number α, s.t.

$$-2\gamma(0)L_P^\dagger L_P+\int_t^\infty\|\Omega(\tau,t)\|d\tau\, I\leqslant\alpha I$$

for all $t\geqslant 0$, then $\mathcal{H}_S$ is an attractive subspace. $\Omega(\cdot,\cdot)$ is a two-variable superoperator expressed as

$$\begin{aligned}\Omega(t,s)&=\mathcal{K}(t-s)-\partial_s\mathcal{K}(t-s)-\mathcal{K}(t-s)\mathcal{A}\\ &\quad-\int_s^t\mathcal{K}(t-u)[\mathcal{K}(u-s)+\mathcal{B}(u-s)]du,\end{aligned} \tag{31}$$

and $\|\cdot\|$ denotes the norm of superoperators on the Banach space of all Hermitian matrices.

Proof. Since $\rho(t)$ is always positive, it suffices to show that the origin is asymptotically stable (all solutions tend to 0) for (28) and (29). The variation of parameters technique of integro-differential equations (see [16]) allows us to swap (29) into an equivalent equation:

$$\dot{\sigma} = \mathcal{N}\sigma + \int_0^t \mathcal{L}(t,s)\sigma(s)ds + \mathcal{K}(t)\sigma_0, \tag{32}$$

where

$$\mathcal{N} = \mathcal{A} - \mathcal{K}(0),$$

and

$$\mathcal{L} = \Omega(t,s) + \mathcal{B}(t,s).$$

Consider the following Lyapunov functional:

$$V[t,\sigma(\cdot)] = \mathrm{tr}(\sigma) + \int_0^t \int_t^\infty \|\Omega(\tau,s)\|d\tau\, \mathrm{tr}[\sigma(s)]ds. \tag{33}$$

Taking the derivative of this functional with respect to t yields

$$\begin{aligned}\dot{V}[t,\sigma(\cdot)] &= \mathrm{tr}\{[\mathcal{A}-\mathcal{K}(0)]\sigma\} + \mathrm{tr}\left[\int_0^t \Omega(t,s)\sigma(s)ds\right]\\ &\quad + \mathrm{tr}\left[\int_0^t \mathcal{B}(t,s)\sigma(s)ds\right] + \mathrm{tr}[\mathcal{K}(t)\sigma_0]\\ &\quad - \int_0^t \|\Omega(t,s)\|\mathrm{tr}[\sigma(s)]ds\\ &\quad + \int_t^\infty \|\Omega(\tau,t)\|d\tau\, \mathrm{tr}(\sigma).\end{aligned} \tag{34}$$

Using the fact that $\mathrm{tr}(\mathcal{A}[\cdot]) = 0$ and $\mathrm{tr}\{\mathcal{B}(t,s)[\cdot]\} = 0$, and applying the norm inequality we obtain

$$\begin{aligned}\dot{V}[t,\sigma(\cdot)] &\leqslant -\mathrm{tr}[\mathcal{K}(0)\sigma] + \mathrm{tr}\left[\int_0^t \|\Omega(t,s)\|\sigma(s)ds\right]\\ &\quad + \mathrm{tr}[\mathcal{K}(t)\sigma_0] - \int_0^t \|\Omega(t,s)\|\mathrm{tr}[\sigma(s)]ds\\ &\quad + \int_t^\infty \|\Omega(\tau,t)\|d\tau\, \mathrm{tr}(\sigma)\\ &= \mathrm{tr}\left(\left[-2\gamma(0)L_P^\dagger L_P + \int_t^\infty \|\Omega(\tau,t)\|d\tau\, I\right]\sigma\right)\\ &\quad + \mathrm{tr}[\mathcal{K}(t)\sigma_0].\end{aligned}$$

The negative definiteness of

$$-2\gamma(0)L_P^\dagger L_P + \int_t^\infty \|\Omega(\tau,t)\|d\tau\, I$$

implies that $\dot{V}[t,\sigma(\cdot)] \leqslant \mathrm{tr}[\mathcal{K}(t)\sigma_0]$, which is a scalar function in $L^1[0,\infty)$ because $\gamma \in L^1[0,\infty)$ and all other time-dependent terms are oscillatory and bounded.

Therefore, by applying Lemma 2, we know that the Lyapunov functional (32) must have a finite limit when t tends to infinity. The natural boundedness of density matrices implies that the second derivative of V with respect to t is also bounded. Barbalat's lemma thus tells us that $\dot{V}$ tends to zero. This leads to the fact that

$$\lim_{t\to+\infty} \mathrm{tr}\left(\left[-2\gamma(0)L_P^\dagger L_P + \int_t^\infty \|\Omega(\tau,t)\|d\tau\, I\right]\sigma\right) = 0$$

because $\mathrm{tr}[\mathcal{K}(t)\sigma_0]$ tends to 0. The upper bounded negative definiteness again says that $\sigma(t) \to 0$, which completes the proof. ■

Remark 2. Theorem 2 can also be generalized to the case of multiple noise operators (22). We present without proof the sufficient conditions (a) and (b) for subspace attractivity regarding the invariant subspace $\mathcal{H}_S$ of (22).

(a) $[L_{S,k}^\dagger, H_S] = 0,\ k = 1, 2, \ldots, n.$

(b) The matrix

$$-2\gamma(0)\sum_{k=1}^n L_{P,k}^\dagger L_{P,k} + \int_t^\infty \|\tilde{\Omega}(\tau,t)\|d\tau\, I$$

is negative definite for all $t \geqslant 0$. $\tilde{\Omega}(\cdot,\cdot)$ is a two-variable superoperator expressed as

$$\begin{aligned}\tilde{\Omega}(t,s) &= \tilde{\mathcal{K}}(t-s) - \partial_s\tilde{\mathcal{K}}(t-s) - \tilde{\mathcal{K}}(t-s)\mathcal{A}\\ &\quad - \int_s^t \tilde{\mathcal{K}}(t-u)[\tilde{\mathcal{K}}(u-s) + \tilde{\mathcal{B}}(u-s)]du,\end{aligned} \tag{35}$$

where

$$\tilde{\mathcal{B}}(t)[\cdot] = \gamma(t)\sum_{k=1}^n \{[L_{R,k}\cdot, L_{H,k}^{R\dagger}(-t)] + [L_{H,k}^R(-t), \cdot L_{R,k}^\dagger]\},$$

$$\tilde{\mathcal{K}}(t)[\cdot] = -\gamma(t)\sum_{k=1}^n [L_{H,k}^{P\dagger}(-t)L_{P,k}\cdot + \cdot L_{P,k}^\dagger L_{H,k}^P(-t)].$$

Remark 3. In an attractive subspace, the invariant set determined by the subspace is autonomously stabilized for all initial states. This is interesting for QIP applications, where quantum information may be manipulated and free from decoherence. If the attractive subspace is only one-dimensional, then the set shrinks to a single pure state that spans the subspace. Tasks such as qubit initialization, cooling, and entanglement generation can be realized if we choose a proper subspace decomposition.

V. NUMERICAL EXAMPLE AND SIMULATION

In this section, an example with numerical simulation is presented to illustrate the results.

Although standard qubits are theoretically two-level quantum systems, it is inevitable that when realizing them with actual physical systems, those with higher level excitations are encountered [19–24]. To perform computation designed for two-level qubits, we need to truncate higher levels, or in other words, avoid unwanted excitations. This objective is naturally achieved if the subspace that supports the qubit states is constructed to be attractive. The problem is now translated into finding system parameters that satisfy the conditions proposed in previous sections.

We consider a three-level quantum system evolving under (1) as the physical realization of a qubit. If the following parameters could be experimentally designed (we have set $\hbar = 1$), quantum information will be preserved in the qubit subspace. The kernel function is $\gamma(t) = \kappa^2 e^{-3\kappa t}$, where

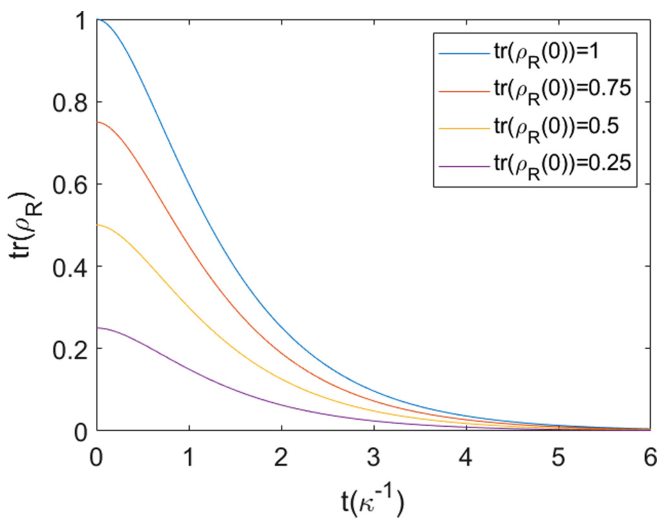

FIG. 1. Four different initial values for $\mathrm{tr}(\rho_R)$: 1, 0.75, 0.5, and 0.25 are chosen. Simulation results show that $\mathrm{tr}(\rho_R)$ vanishes as time elapses, demonstrating subspace attractivity. κ carries the unit of hertz.

$\kappa \gg 1$, and

$$H = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The S block corresponds to the 2×2 block on the top left. The corresponding subspace $\mathcal{H}_S$ is thus two-dimensional. Its attractivity is demonstrated via simulation. It can be verified directly that the matrices satisfy sufficient conditions for invariance proposed in Sec. III. Direct calculation yields

$$\begin{aligned} &-2\gamma(0)L_P^\dagger L_P + \int_t^\infty \|\Omega(\tau,t)\| d\tau \\ &\leqslant -2\kappa^2 + \kappa^2 \int_0^\infty e^{-3\kappa u}|-2+6\kappa-4\kappa^2 u|du \\ &\leqslant -2\kappa^2 + \kappa^2\left(2-\frac{2}{9\kappa}\right) \\ &= -\frac{2}{9}\kappa \\ &< 0. \end{aligned}$$

Therefore, sufficient conditions for attractivity are also met. We plot $\mathrm{tr}(\rho_R)$ with respect to time in Fig. 1 for four different initial values. Figure 1 shows that the values of $\mathrm{tr}(\rho_R)$ converge to 0, meaning that $\mathcal{H}_S$ is attractive.

VI. CONCLUSION AND DISCUSSION

We have made an analysis of subspace invariance and attractivity for a class of non-Markovian quantum systems. The results may be useful to understand asymptotic dynamical properties of non-Markovian quantum systems. In future research, other non-Markovian models with potential QIP applications will be investigated. It is also worthwhile investigating whether non-Markovian quantum systems may have other undefined dynamical properties compared with their Markovian counterpart, which makes future studies much more intriguing.

Moreover, the work in this article inspires us that it is worth further investigating the physicality of the two master equation models of Lindblad equations and memory kernel (finite, nonzero, and continuous at $t = 0$) master equations. This inspiration comes from the following observation. If Dirac $\delta(t)$ is viewed as a "limit" of a sequence of finite memory kernel functions $\gamma_n(t)$ with increasingly heavier distribution around the origin, the condition $iH_P - 1/2L_S^\dagger L_P = 0$ is clearly not sufficient for subspace invariance in terms of systems involving each member of the sequence. It is sufficient, however, for the limiting Markovian case, thus presenting a "discontinuity" despite the correctness of [8] and our work. It can also be verified that $iH_P - 1/2L_S^\dagger L_P = 0$ holds even when considering the invariance of Lindblad equations under transformations $L = L + \alpha I$ and $H = H - i/2(\alpha^* L - \alpha L^\dagger)$ for any complex number α [12].

Therefore, the above-mentioned discontinuity suggests that at least one of the two master equation models might be unphysical. It could be that Lindblad equations correspond to an unphysical limit, or that all physical open systems are described by memory kernels that diverge at $t = 0$, or even that both equations are unphysical.

It is beyond the scope of this article to investigate which model (or both) is unphysical, but we believe that the physical implication itself should as least be of equal importance to other results in this article. The physical intuition that non-Markovian systems should yield consistent results with Markovian systems is worth further examination. Also, the physicality issue should be clarified before one designs control protocols for open quantum systems based on these mathematical models.

ACKNOWLEDGMENTS

This work is supported by National Natural Science Foundation (NNSF) of China under Grants No. 61603040, No. 61828303, No. 61873034, and No. 61433003. It is partially supported by the Australian Research Council's Discovery Projects Funding Scheme under Project DP190101566, Beijing Natural Science Foundation under Grant No. 4182057, and Yunnan Provincial Science and Technology Department Key Research Program (Engineering) (2018BA070).

[1] M. A. Nielsen and I. L. Chuang, in *Quantum Computation and Quantum Information*, 10th anniversary ed. (Cambridge University Press, Cambridge, UK, 2010), pp. 1–59.

[2] A. H. Atabaki, S. Moazeni, F. Pavanello, H. Gevorgyan, J. Notaros, L. Alloatti, M. T. Wade, C. Sun, S. A. Kruger, H. Meng *et al.*, Nature (London) **556**, 349 (2018).

[3] M. Bock, P. Eich, S. Kucera, M. Kreis, A. Lenhard, C. Becher, and J. Eschner, Nat. Commun. **9**, 1998 (2018).

[4] S. Shankar, M. Hatridge, Z. Leghtas, K. M. Sliwa, A. Narla, U. Vool, S. M. Girvin, L. Frunzio, M. Mirrahimi, and M. H. Devoret, Nature (London) **504**, 419 (2013).

[5] D. Dong and I. R. Petersen, IET Control Theory Appl. **4**, 2651 (2010).

[6] C. Altafini and F. Ticozzi, IEEE Trans. Autom. Control **57**, 1898 (2012).

[7] J. Zhang, Y.-X. Liu, R. B. Wu, K. Jacobs, and F. Nori, Phys. Rep. **679**, 1 (2017).

[8] F. Ticozzi and L. Viola, IEEE Trans. Autom. Control **53**, 2048 (2008).

[9] F. Ticozzi and L. Viola, Automatica **45**, 2002 (2009).

[10] F. Ticozzi, S. G. Schirmer, and X. Wang, IEEE Trans. Autom. Control **55**, 2901 (2010).

[11] F. Ticozzi, R. Lucchese, P. Cappellaro, and L. Viola, IEEE Trans. Autom. Control **57**, 1931 (2012).

[12] H. P. Breuer and F. Petruccione, in *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2002), pp. 117–124.

[13] J. Combes, J. Kerckhoff, and M. Sarovar, Adv. Phys.: X **2**, 784 (2016).

[14] I. de Vega and D. Alonso, Rev. Mod. Phys. **89**, 015001 (2017).

[15] J. Zhang, Y.-X. Liu, R.-B. Wu, K. Jacobs, and F. Nori, Phys. Rev. A **87**, 032117 (2013).

[16] V. Lakshmikantham, *Theory of Integro-Differential Equations*, Stability and Control (CRC Press, Boca Raton, FL, 1995).

[17] C. W. Gardiner and M. J. Collett, Phys. Rev. A **31**, 3761 (1985).

[18] S. Xue, M. R. Hush, and I. R. Petersen, IEEE Trans. Control Syst. Technol. **25**, 1552 (2017).

[19] R. Bianchetti, S. Filipp, M. Baur, J. M. Fink, C. Lang, L. Steffen, M. Boissonneault, A. Blais, and A. Wallraff, Phys. Rev. Lett. **105**, 223601 (2010).

[20] M. B. Kenmoe and L. C. Fai, Phys. Rev. B **94**, 125101 (2016).

[21] P. Pechukas, Phys. Rev. Lett. **73**, 1060 (1994).

[22] Y. Pan and T. Nguyen, IEEE Trans. Autom. Control **62**, 4625 (2017).

[23] B. Vacchini, Phys. Rev. Lett. **117**, 230401 (2016).

[24] S.-B. Xue, R.-B. Wu, W.-M. Zhang, J. Zhang, C.-W. Li, and T.-J. Tarn, Phys. Rev. A **86**, 052304 (2012).