Spectral invariance and scaling law for nonstationary optical fields

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We develop a scaling law for a class of statistically nonstationary scalar optical fields, which ensures spectral invariance on their propagation into the far zone of a planar source. The invariance involves the constraint that the normalized far-zone spectrum must be the same in every direction of observation, as well as equal to the normalized area-averaged source spectrum. Thus, it additionally represents an extension of the earlier work by Wolf on stationary fields [Phys. Rev. Lett. **56**, 1370 (1986)] that assumed the normalized source spectrum as independent of position. We present examples of both nonstationary and stationary fields that satisfy the scaling law and extended spectral invariance.

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I. INTRODUCTION

Wolf's scaling law for spectral invariance [1] is one of the cornerstone results in classical coherence theory, prompting a flood of follow-up research [2–8]. This law establishes a condition under which the normalized far-zone spectrum of a scalar optical field emanating from a planar source is the same in all directions of observation while additionally being equal to the normalized spectrum at the plane of a quasihomogeneous source that is taken to be point-wise constant. It was soon found both theoretically and experimentally that although many common sources obey the scaling law, it is not difficult to create sources that violate it; in the latter case noticeable spectral shifts can occur [9–12].

Up to now, the scaling law has been studied only in the context of statistically stationary fields. The purpose of the present work is to extend it beyond the stationary case by considering a class of nonstationary (pulsed or nonpulsed) fields which covers, e.g., sources which are spatially quasi-homogeneous at every frequency.

Our starting point is slightly more general than that of Wolf [1]. We retain the requirement that the normalized far-zone spectrum must be directionally invariant. However, we allow the source spectrum to vary with position in such a way that the normalized source-averaged spectrum is equal to the normalized far-zone spectrum. Such an assumption is physically appropriate, for example, when considering radiation from the sun (with sunspots, corona, flares, etc.) and other natural or manmade sources that spectrally exhibit spatial variations. In the stationary case this extended starting point leads exactly to the original functional form of the scaling law.

The paper is organized as follows. We begin in Sec. II with expressions that relate the far-zone spectrum with the source-plane correlation function of a scalar optical field. The main result of this paper, i.e., the spectral scaling law for nonstationary fields, is derived in Sec. III. Some analytical examples of pulsed and stationary fields that satisfy our scaling law are presented in Sec. IV. Finally, the main conclusions are summarized and some discussion of the results is provided in Sec. V.

II. FAR-ZONE SPECTRUM

Considering the geometry of Fig. 1, we denote the scalar field at point $\boldsymbol{\rho} = (x, y)$ in the source plane $z = 0$ by $E(\boldsymbol{\rho}; \omega)$, where ω is the angular frequency. An arbitrary point at a distance r in the far zone is denoted by $\mathbf{r} = r\hat{\mathbf{s}}$, with $\hat{\mathbf{s}} = (\boldsymbol{\sigma}, s_z)$ being a unit direction vector and $\boldsymbol{\sigma} = (s_x, s_y)$ its transverse component. Throughout the work we assume that the field is propagating in vacuum. The far field in direction $\hat{\mathbf{s}}$ is of the form (see Ref. [13], Sec. 3.2.2)

$$E^{(\infty)}(\hat{\mathbf{s}}; \omega) = -i2\pi s_z \frac{\omega}{c} A\left(\frac{\omega}{c}\boldsymbol{\sigma}; \omega\right) \frac{\exp(i\omega r/c)}{r}. \quad (1)$$

In Eq. (1), c is the vacuum speed of light,

$$A(\boldsymbol{\kappa}; \omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} E(\boldsymbol{\rho}; \omega) \exp(-i\boldsymbol{\kappa} \cdot \boldsymbol{\rho}) d^2\rho \quad (2)$$

is the angular spectrum of the source-plane field $E(\boldsymbol{\rho}; \omega)$, and $\boldsymbol{\kappa} = (k_x, k_y)$ is the spatial-frequency vector.

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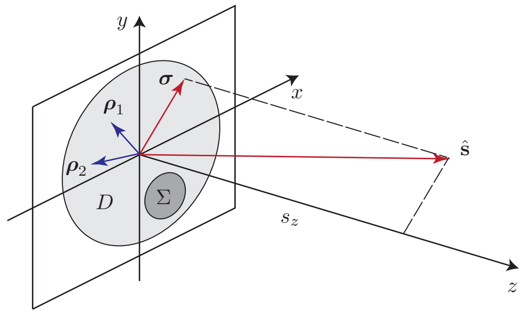

FIG. 1. Notation related to propagation of a partially coherent field into the far zone. Here D represents the effective source area, and Σ is the effective coherence area of the source.

The two-frequency cross-spectral density (CSD) function of a nonstationary optical field at the source plane, which describes correlations between the field values at spatial points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ and (angular) frequencies ω_1 and ω_2, is defined as

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) = \langle E^*(\boldsymbol{\rho}_1; \omega_1) E(\boldsymbol{\rho}_2; \omega_2)\rangle. \quad (3)$$

Here $E(\boldsymbol{\rho}; \omega)$ is a single field realization, the brackets denote ensemble averaging, and the asterisk represents complex conjugation. The source-plane spectral density is given by $S(\boldsymbol{\rho}; \omega) = W(\boldsymbol{\rho}, \boldsymbol{\rho}; \omega, \omega)$, while the far-zone spectral density is defined as

$$S^{(\infty)}(\hat{\mathbf{s}}; \omega) = \langle E^{(\infty)*}(\hat{\mathbf{s}}; \omega) E^{(\infty)}(\hat{\mathbf{s}}; \omega)\rangle. \quad (4)$$

On inserting from Eq. (1) into Eq. (4) and using Eqs. (2) and (3), we readily find that

$$S^{(\infty)}(\hat{\mathbf{s}}; \omega) = \left(\frac{2\pi s_z}{r}\right)^2 \left(\frac{\omega}{c}\right)^2 T\left(\frac{\omega}{c}\boldsymbol{\sigma}, \frac{\omega}{c}\boldsymbol{\sigma}; \omega, \omega\right), \quad (5)$$

where

$$T(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2; \omega_1, \omega_2) = \frac{1}{(2\pi)^4} \iint_{-\infty}^{\infty} W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) \times \exp[i(\boldsymbol{\kappa}_1 \cdot \boldsymbol{\rho}_1 - \boldsymbol{\kappa}_2 \cdot \boldsymbol{\rho}_2)] d^2\rho_1 d^2\rho_2 \quad (6)$$

is the two-frequency angular correlation function. The normalized spectrum in the far zone,

$$s^{(\infty)}(\hat{\mathbf{s}}; \omega) = \frac{S^{(\infty)}(\hat{\mathbf{s}}; \omega)}{\int_0^\infty S^{(\infty)}(\hat{\mathbf{s}}; \omega) d\omega}, \quad (7)$$

is then given by

$$s^{(\infty)}(\hat{\mathbf{s}}; \omega) = \frac{(\omega/c)^2 T(\omega\boldsymbol{\sigma}/c, \omega\boldsymbol{\sigma}/c; \omega, \omega)}{\int_0^\infty (\omega/c)^2 T(\omega\boldsymbol{\sigma}/c, \omega\boldsymbol{\sigma}/c; \omega, \omega) d\omega}. \quad (8)$$

Some comments are in order at this point.

First, only the diagonal ($\omega_1 = \omega_2 = \omega$) element of the source two-frequency CSD function $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2)$ appears in the expressions [cf., Eqs. (6) and (8)], since only the frequency component ω of the source-plane field contributes to the far-zone spectrum at frequency ω. This is naturally true regardless of whether the field is nonstationary or stationary, and therefore the expressions in the case of stationary fields similarly involve the usual cross-spectral density function $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega)$ at frequency ω [1]. Secondly, one may reasonably expect that there may be a multitude of two-frequency correlation functions $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2)$ with the same diagonal element $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega, \omega)$, leading to a rich variety of potential two-frequency solutions. This is not the case with $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega)$ of stationary fields, which only has a single frequency variable.

III. SPECTRAL SCALING LAW FOR NONSTATIONARY FIELDS

In what follows, we consider nonstationary fields at the source plane that have two-frequency CSDs of the form

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) = [S(\bar{\boldsymbol{\rho}}; \omega_1) S(\bar{\boldsymbol{\rho}}; \omega_2)]^{1/2} \times g(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2), \quad (9)$$

where $g(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) = g^*(\boldsymbol{\rho}_2, \boldsymbol{\rho}_1; \omega_2, \omega_1)$ is a Hermitian function that satisfies a boundary condition

$$g(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega, \omega) = \nu(\Delta\boldsymbol{\rho}; \omega), \quad (10)$$

but is, in general, not equal to the degree of coherence,

$$\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) = \frac{W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2)}{[S(\boldsymbol{\rho}_1; \omega_1) S(\boldsymbol{\rho}_2; \omega_2)]^{1/2}}. \quad (11)$$

In Eqs. (9) and (10), $\bar{\boldsymbol{\rho}} = (\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)/2$ and $\Delta\boldsymbol{\rho} = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$ are average and difference spatial coordinates at the source plane, and $\nu(\Delta\boldsymbol{\rho}; \omega)$ has the property $\nu(0; \omega) = 1$. We proceed to show that all fields having these characteristics with $\nu(\Delta\boldsymbol{\rho}; \omega)$ obeying the scaling law satisfy the spectral invariance. An example field obeying Eqs. (9) and (10) (and the scaling law) can be constructed as an incoherent superposition of Hermite-Gaussian field modes, as will be shown in Sec. IV. We remark that the CSD function constituted by Eqs. (9) and (10) may not be the most general one but it is the largest we have found, and it includes several different fields that are realizable in practice, as we demonstrate below.

For example, the above CSD form covers nonstationary fields which are spatially quasihomogeneous at all frequencies. The complex degree of coherence for this source type can be found by inserting Eq. (9) to Eq. (11), yielding

$$\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) = \frac{[S(\bar{\boldsymbol{\rho}}; \omega_1) S(\bar{\boldsymbol{\rho}}; \omega_2)]^{1/2} g(\bar{\boldsymbol{\rho}}, \Delta\boldsymbol{\rho}; \omega_1, \omega_2)}{[S(\bar{\boldsymbol{\rho}} - \Delta\boldsymbol{\rho}/2; \omega_1) S(\bar{\boldsymbol{\rho}} + \Delta\boldsymbol{\rho}/2; \omega_2)]^{1/2}}, \quad (12)$$

where

$$g(\bar{\boldsymbol{\rho}}, \Delta\boldsymbol{\rho}; \omega_1, \omega_2) = g(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2). \quad (13)$$

Expressions (9)–(13) imply that the source field does not necessarily need to be spatially quasihomogeneous. If, however, the spectral density at the source plane satisfies the conditions

$$S(\bar{\boldsymbol{\rho}} \pm \Delta\boldsymbol{\rho}/2; \omega) \approx S(\bar{\boldsymbol{\rho}}; \omega), \quad (14)$$

we have $\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) \approx g(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2)$. Further, $\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega, \omega) \approx \nu(\Delta\boldsymbol{\rho}; \omega)$, which means that the field is spatially quasihomogeneous at any single frequency ω (though not necessarily for an arbitrary pair of frequencies ω_1 and ω_2).

Inserting from Eq. (9) into Eq. (6), we find that the angular self-correlation function is of the form

$$T(\boldsymbol{\kappa}, \boldsymbol{\kappa}; \omega, \omega) = (2\pi)^{-2} S^{(\mathrm{int})}(\omega) \tilde{\nu}(\boldsymbol{\kappa}; \omega), \quad (15)$$

where

$$S^{(\mathrm{int})}(\omega) = \int_{-\infty}^{\infty} S(\bar{\boldsymbol{\rho}};\omega)d^2\bar{\rho} \tag{16}$$

is the source-integrated spectral density, and

$$\tilde{\nu}(\boldsymbol{\kappa};\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \nu(\Delta\boldsymbol{\rho};\omega)\exp(-i\boldsymbol{\kappa}\cdot\Delta\boldsymbol{\rho})d^2\Delta\rho. \tag{17}$$

It then follows from Eq. (8) that

$$s^{(\infty)}(\hat{\mathbf{s}};\omega) = \frac{(\omega/c)^2 S^{(\mathrm{int})}(\omega)\tilde{\nu}(\omega\boldsymbol{\sigma}/c;\omega)}{\int_0^\infty (\omega/c)^2 S^{(\mathrm{int})}(\omega)\tilde{\nu}(\omega\boldsymbol{\sigma}/c;\omega)d\omega}, \tag{18}$$

and the condition that the far-zone spectrum is the same in all directions $\hat{\mathbf{s}}$ is thus satisfied, at least if the factorization rule

$$\tilde{\nu}(\omega\boldsymbol{\sigma}/c;\omega) = F(\omega)\tilde{H}(\boldsymbol{\sigma}) \tag{19}$$

holds. Using this rule and the inverse of Eq. (17), we find that

$$\nu(\Delta\boldsymbol{\rho};\omega) = \left(\frac{\omega}{c}\right)^2 F(\omega)H\left(\frac{\omega}{c}\Delta\boldsymbol{\rho}\right), \tag{20}$$

where

$$H\left(\frac{\omega}{c}\Delta\boldsymbol{\rho}\right) = \int_{-\infty}^{\infty}\tilde{H}(\boldsymbol{\sigma})\exp\left(i\frac{\omega}{c}\boldsymbol{\sigma}\cdot\Delta\boldsymbol{\rho}\right)d^2\sigma. \tag{21}$$

Since $\nu(0;\omega) = 1$, we have $F(\omega) = [(\omega/c)^2 H(0)]^{-1}$, and hence

$$\nu(\Delta\boldsymbol{\rho};\omega) = \frac{H(\omega\Delta\boldsymbol{\rho}/c)}{H(0)}. \tag{22}$$

We refer to this expression as the scaling law of nonstationary scalar optical fields. When this condition holds, the normalized far-zone spectrum given by Eq. (18) reduces to

$$s^{(\infty)}(\hat{\mathbf{s}};\omega) = \frac{S^{(\mathrm{int})}(\omega)}{\int_0^\infty S^{(\mathrm{int})}(\omega)d\omega} = \bar{s}(\omega), \tag{23}$$

where $\bar{s}(\omega)$ represents the normalized source-averaged spectral density. Therefore, if the source CSD has the form of Eqs. (9) and (10) and the scaling law of Eq. (22) is satisfied, the normalized spectrum in any direction in the far zone is the same as the normalized spatially averaged source-plane spectrum, i.e., the field exhibits spectral invariance.

It is important to recognize that the scaling law (22) sets constraints only on the boundary value of the function $g(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega_1,\omega_2)$ in Eq. (9). In addition, only the Hermitian condition

$$g^*(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega_1,\omega_2) = g(\boldsymbol{\rho}_2,\boldsymbol{\rho}_1;\omega_2,\omega_1) \tag{24}$$

needs to be satisfied. Thus, the set of allowed functions $g(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega_2,\omega_1)$ is apparently very large. For example, all (Hermitian) choices of the functional form

$$g(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega_1,\omega_2) = f[(\omega_2\boldsymbol{\rho}_2 - \omega_1\boldsymbol{\rho}_1)/c] \tag{25}$$

are acceptable. A specific example of a field expressible by Eqs. (9) and (10) and which satisfies the scaling law without necessarily being quasihomogeneous will be introduced in Sec. IV B 2.

We note that since the spectral invariance and scaling-law analysis from Eq. (15) onwards involves only the frequency component ω, the same result—that is, Eq. (22)—applies to stationary fields alike. By starting from the stationary-field CSD of the form

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega) = S(\bar{\boldsymbol{\rho}};\omega)\nu(\Delta\boldsymbol{\rho};\omega), \tag{26}$$

we find that $\nu(\Delta\boldsymbol{\rho};\omega)$ must satisfy Eq. (22) in order to obey spectral invariance. If the field is quasihomogeneous, i.e., if the approximations in Eq. (14) can be made, we have $\nu(\Delta\boldsymbol{\rho};\omega) \approx \mu(\Delta\boldsymbol{\rho};\omega)$. This is precisely Wolf's original scaling law [1].

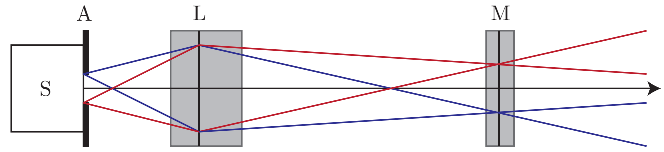

FIG. 2. Schematic illustration of generation of a pulsed secondary light source that obeys the scaling law. Here S is a thermal source, A is an aperture, L is an achromatic lens, and M is a temporal intensity modulator.

IV. EXAMPLES

A. Image of thermal source

A simple example of a partially coherent pulsed source that follows the scaling law is presented in Fig. 2. The aperture A (width ≫ wavelength of light) illuminated by a stationary thermal source S represents an essentially spatially incoherent planar source. When this source is imaged by an achromatic lens L (such as a microscope objective), a quasihomogeneous secondary source is generated whose spatial coherence properties depend on the size of the aperture A and the numerical aperture of the lens L (see, e.g., Ref. [14]) and satisfies the original scaling law [1]. A temporal intensity modulator M (such as an electro-optic modulator) inserted in the image plane converts the stationary secondary source into a pulsed source without affecting the spatial coherence, thereby generating a nonstationary source that satisfies the scaling law.

B. Superpositions of Hermite-Gaussian fields

Let us consider fully coherent, spatially normalized Hermite-Gaussian (HG) fields that are of the separable form

$$\psi_{mn}(x,y;\omega) = \psi_m(x;\omega)\psi_n(y;\omega), \tag{27}$$

where m and n are non-negative integers and

$$\psi_m(x;\omega) = [S_0(\omega)]^{1/4}\frac{(2/\pi)^{1/4}}{\sqrt{2^m m! w_{0x}(\omega)}} \times H_m\left[\frac{\sqrt{2}x}{w_{0x}(\omega)}\right]\exp\left[-\frac{x^2}{w_{0x}^2(\omega)}\right]. \tag{28}$$

The function $\psi_n(y;\omega)$ has a similar form as Eq. (28), with m and x replaced by n and y, respectively.

Here $w_{0x}(\omega)$ and $w_{0y}(\omega)$ define the transverse scales of the field at frequency ω in the x and y directions. In the case of $w_{0j}(\omega) = \sqrt{\omega_0/\omega}w_{0j}$ [15,16], $j = x, y$, the above HG modes are the transverse modes (at the waist) of spherical-mirror resonators and hence of significant practical importance. The spectral density of a fully coherent HG mode is given by

$$S_{mn}(x,y;\omega) = |\psi_{mn}(x,y;\omega)|^2 = |\psi_m(x;\omega)|^2|\psi_n(y;\omega)|^2. \quad (29)$$

Let us, in particular, choose

$$w_{0j}(\omega) = \frac{\omega_0}{\omega} w_{0j}, \quad j = x, y, \quad (30)$$

where w_{0x} and w_{0y} are the scale factors at a reference frequency $\omega = \omega_0$. With this choice all HG modes obey spectral invariance, as we will see next. In fact, this appears to be the only possible frequency dependence of the width parameters that ensures spectral invariance. In addition, no physical or mathematical principle exists that forbids the existence of these fields. Potentially they could be realized by the methods of [19]. Nonetheless, it is straightforward to show that the frequency dependence of Eq. (30) is obtained in the far zone if the waist-plane mode width is taken frequency independent. Equation (28) then becomes

$$\psi_m(x;\omega) = [S_0(\omega)]^{1/4}\frac{(2/\pi)^{1/4}}{\sqrt{2^m m! w_{0x}}}\sqrt{\frac{\omega}{\omega_0}} \times H_m\left(\frac{\omega}{\omega_0}\frac{\sqrt{2}x}{w_{0x}}\right)\exp\left[-\left(\frac{\omega}{\omega_0}\right)^2\frac{x^2}{w_{0x}^2}\right] \quad (31)$$

and correspondingly for $\psi_n(y;\omega)$. We call the field mode isotropic if $w_{0x} = w_{0y}$ and $m = n$, otherwise anisotropic.

In view of Eqs. (29)–(31), the spectral density of a single HG field mode takes the form

$$S_{mn}(x,y;\omega) = \frac{S_0(\omega)}{\pi w_{0x}w_{0y}2^{m+n-1}m!n!}\left(\frac{\omega}{\omega_0}\right)^2 \times H_m^2\left(\frac{\omega}{\omega_0}\frac{\sqrt{2}x}{w_{0x}}\right)H_n^2\left(\frac{\omega}{\omega_0}\frac{\sqrt{2}y}{w_{0y}}\right) \times \exp\left[-2\left(\frac{\omega}{\omega_0}\right)^2\left(\frac{x^2}{w_{0x}^2}+\frac{y^2}{w_{0y}^2}\right)\right]. \quad (32)$$

The source-averaged spectrum of a single HG mode can be evaluated using the integral formula of Eq. (63) listed in the Appendix. We straightforwardly obtain

$$S_{mn}^{(\mathrm{int})}(\omega) = \int_{-\infty}^{\infty} S_{mn}(x,y;\omega)dxdy = S_0(\omega). \quad (33)$$

Hence the normalized source-averaged spectral density,

$$\bar{s}_{mn}(\omega) = \frac{S_{mn}^{(\mathrm{int})}(\omega)}{\int_0^\infty S_{mn}^{(\mathrm{int})}(\omega)d\omega} = \frac{S_0(\omega)}{\int_0^\infty S_0(\omega)d\omega}, \quad (34)$$

is the same for every HG mode.

Let us next consider radiation into the far zone. Using the integral formula of Eq. (64) and the analog of Eq. (2) we find that the angular spectrum of a HG field mode is given by

$$A_{mn}(k_x,k_y;\omega) = A_m(k_x;\omega)A_n(k_y;\omega), \quad (35)$$

where

$$A_m(k_x;\omega) = [S_0(\omega)]^{1/4}\frac{(-i)^m}{2\pi}\left(\frac{2}{\pi}\right)^{1/4}\left(\frac{\pi w_{0x}}{2^m m!}\frac{\omega_0}{\omega}\right)^{1/2} \times H_m\left(\frac{1}{\sqrt{2}}\frac{\omega_0}{\omega}w_{0x}k_x\right)\exp\left[-\frac{1}{4}\left(\frac{\omega_0}{\omega}\right)^2 w_{0x}^2k_x^2\right], \quad (36)$$

and $A_n(k_y;\omega)$ is again of the same form with m and x replaced by n and y, respectively. Using Eqs. (1) and (4) we find that the far-zone spectrum of a single HG mode is

$$S_{mn}^{(\infty)}(\hat{\mathbf{s}};\omega) = \left(\frac{2\pi s_z}{r}\right)^2\left(\frac{\omega}{c}\right)^2 \times \left|A_m\left(\frac{\omega}{c}s_x;\omega\right)\right|^2\left|A_n\left(\frac{\omega}{c}s_y;\omega\right)\right|^2. \quad (37)$$

On inserting from Eq. (36), with substitutions $k_j = (\omega/c)s_j$, $j = x, y$, we have

$$S_{mn}^{(\infty)}(\hat{\mathbf{s}};\omega) = S_0(\omega)\left(\frac{k_0 s_z}{r}\right)^2\frac{(-1)^{m+n}w_{0x}w_{0y}}{\pi 2^{m+n+1}m!n!} \times H_m^2\left(\frac{1}{\sqrt{2}}k_0w_{0x}s_x\right)H_n^2\left(\frac{1}{\sqrt{2}}k_0w_{0y}s_y\right) \times \exp\left[-\frac{1}{2}k_0^2\left(w_{0x}^2s_x^2+w_{0y}^2s_y^2\right)\right], \quad (38)$$

where $k_0 = \omega_0/c$. Hence, the normalized spectrum in the far zone, given by Eq. (7), is

$$s_{mn}^{(\infty)}(\hat{\mathbf{s}};\omega) = \frac{S_0(\omega)}{\int_0^\infty S_0(\omega)d\omega}. \quad (39)$$

This result is independent of direction, the same for all m and n, and in view of Eq. (34) is also equal to the source-averaged spectrum.

Since the normalized spectrum of all field modes is of the direction-independent form of Eq. (39), the same holds true for their incoherent superpositions, regardless of whether pulsed or stationary fields are considered. This spectrum is also the same as the source-averaged spectrum of any such superposition, and therefore our requirements for spectral invariance are fulfilled. Next we consider in more detail specific HG-mode superpositions in the context of both stationary and nonstationary fields.

1. Stationary fields

Let us first consider incoherent superpositions of HG modes in the stationary case. To this end we represent the (single-frequency) CSD in the form $W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2;\omega) = W(x_1,x_2;\omega)W(y_1,y_2;\omega)$, where

$$W(x_1,x_2;\omega) = \sum_{m=0}^{\infty} c_m\psi_m^*(x_1;\omega)\psi_m(x_2;\omega), \quad (40)$$

and a similar expansion applies to $W(y_1,y_2;\omega)$. We assume that the modal functions are given by Eq. (31) and consider the isotropic case $w_{0x} = w_{0y} = w_0$ for brevity of notation.

Additionally, we choose the modal weights as

$$c_m = w_0\sqrt{\frac{2\pi}{\beta}}\frac{1}{1+1/\beta}\left(\frac{1-\beta}{1+\beta}\right)^m, \tag{41}$$

where β is a constant with values in the range $0 \leqslant \beta \leqslant 1$. The above choice for c_m is motivated by the fact that it leads to the Gaussian Schell-model fields [16–18].

We may now proceed in analogy with Ref. [16], where a different class of partially coherent Gaussian fields (so-called isodiffracting Gaussian Schell-model beams) was considered. On inserting from Eqs. (31) and (41) into Eq. (40), we arrive at an expression for the CSD, which can be evaluated using the generating-function formula of Eq. (65) with

$$t = \frac{1}{2}\frac{1-\beta}{1+\beta}. \tag{42}$$

Performing the calculation results in

$$W(x_1, x_2; \omega) = \frac{\omega}{\omega_0}[S_0(\omega)]^{1/2}\exp\left(-\frac{1+\beta^2}{2\beta}\frac{\omega^2}{\omega_0^2}\frac{x_1^2+x_2^2}{w_0^2}\right) \times \exp\left(\frac{1-\beta^2}{\beta}\frac{\omega^2}{\omega_0^2}\frac{x_1x_2}{w_0^2}\right). \tag{43}$$

On combining this result with a corresponding solution for $W(y_1, y_2; \omega)$, we can represent the CSD in the Schell-model form,

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega) = \sqrt{S(\boldsymbol{\rho}_1;\omega)S(\boldsymbol{\rho}_2;\omega)}\,\mu(\Delta\boldsymbol{\rho};\omega), \tag{44}$$

where the spectral density is given by

$$S(\boldsymbol{\rho};\omega) = S_0(\omega)\left(\frac{\omega}{\omega_0}\right)^2\exp\left[-\left(\frac{\omega}{\omega_0}\right)^2\frac{2\rho^2}{w^2}\right], \tag{45}$$

and the complex degree of spectral coherence is

$$\mu(\Delta\boldsymbol{\rho};\omega) = \exp\left[-\left(\frac{\omega}{\omega_0}\right)^2\frac{\Delta\rho^2}{2\sigma^2}\right]. \tag{46}$$

Here the parameters

$$w = \frac{w_0}{\sqrt{\beta}} \tag{47}$$

and

$$\sigma = \frac{\sqrt{\beta}}{\sqrt{1-\beta^2}}w_0 \tag{48}$$

represent the beam width and coherence width of the entire partially coherent field, respectively. In view of Eqs. (47) and (48), we have an explicit connection

$$\beta = \left(1+\frac{w^2}{\sigma^2}\right)^{-1/2} \tag{49}$$

between these beam parameters and the constant β.

Alternatively, we can express the CSD in the form of Eq. (26). After some manipulation we find that

$$\nu(\Delta\boldsymbol{\rho};\omega) = \exp\left[-\left(\frac{\omega}{\omega_0}\right)^2\frac{\Delta\rho^2}{2w^2\beta^2}\right], \tag{50}$$

which satisfies Eq. (22). Hence, we have a class of stationary fields that satisfies the scaling law and upholds extended spectral invariance but may possess any degree of spatial coherence.

In the quasihomogeneous limit ($\sigma \ll w$) the parameter β is approximately given by $\beta = \sigma/w$. In this limit the right-hand side of Eq. (50) takes on the form of the right-hand side of Eq. (46), and thus the function $\nu(\Delta\boldsymbol{\rho};\omega)$ reduces to the spectral complex degree of coherence $\mu(\Delta\boldsymbol{\rho};\omega)$. It should be noted that the source spectrum given by Eq. (45) varies with position also in the quasihomogeneous case (as was allowed by our initial assumptions).

2. *Nonstationary fields*

In the case of nonstationary fields, we write the separable two-frequency CSD of an incoherent superposition of HG modes in the form $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) = W(x_1, x_2; \omega_1, \omega_2)W(y_1, y_2; \omega_1, \omega_2)$, where

$$W(x_1, x_2; \omega_1, \omega_2) = \sum_{m=0}^{\infty} c_m \psi_m^*(x_1;\omega_1)\psi_m(x_2;\omega_2), \tag{51}$$

and a similar expansion holds for $W(y_1, y_2; \omega_1, \omega_2)$. We again choose the weight factors as in Eq. (41). By applying Eqs. (65) and (42) and simplifying, we arrive at

$$W(x_1, x_2; \omega_1, \omega_2) = \left(\frac{\omega_1}{\omega_0}\frac{\omega_2}{\omega_0}\right)^{1/2}[S_0(\omega_1)S_0(\omega_2)]^{1/4} \times \exp\left(-\frac{1+\beta^2}{2\beta}\frac{\omega_1^2x_1^2+\omega_2^2x_2^2}{\omega_0^2w_0^2}\right) \times \exp\left(\frac{1-\beta^2}{\beta}\frac{\omega_1\omega_2}{\omega_0^2}\frac{x_1x_2}{w_0^2}\right). \tag{52}$$

We can write the full CSD in the form

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) = \sqrt{S(\boldsymbol{\rho}_1;\omega_1)S(\boldsymbol{\rho}_2;\omega_2)} \times \mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2), \tag{53}$$

where the spectral density is still given by Eq. (45), and the complex degree of spectral coherence has the form

$$\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) = \exp\left[-\frac{(\omega_1\boldsymbol{\rho}_1 - \omega_2\boldsymbol{\rho}_2)^2}{2\omega_0^2\sigma^2}\right]. \tag{54}$$

The CSD defined by Eqs. (45), (53), and (54) obviously reduces to Schell-model form (spatially) when $\omega_1 = \omega_2 = \omega$ but not if $\omega_1 \neq \omega_2$. In addition, the field becomes spatially quasihomogeneous at any single frequency ω when $\sigma \ll w$.

The CSD in Eq. (52) can also be expressed in the form of Eq. (9). To this end, it is convenient to revert to average and difference spatial coordinates, and also use average and difference frequency coordinates $\bar{\omega} = \frac{1}{2}(\omega_1+\omega_2)$ and $\Delta\omega = \omega_2 - \omega_1$. In doing so, Eq. (54) becomes

$$\mu(\bar{\boldsymbol{\rho}}, \Delta\boldsymbol{\rho}; \bar{\omega}, \Delta\omega) = \exp\left[-\frac{(\Delta\omega\bar{\boldsymbol{\rho}} + \bar{\omega}\Delta\boldsymbol{\rho})^2}{2\omega_0^2\sigma^2}\right], \tag{55}$$

and the full CSD takes the form

$$W(\bar{\boldsymbol{\rho}}, \Delta\boldsymbol{\rho}; \bar{\omega}, \Delta\omega) = \left[\left(\frac{\bar{\omega}}{\omega_0}\right)^2 - \frac{1}{4}\left(\frac{\Delta\omega}{\omega_0}\right)^2\right][S_0(\omega_1)S_0(\omega_2)]^{1/2} \exp\left[-\left(\frac{\bar{\omega}}{\omega_0}\right)^2 \frac{2\bar{\rho}^2}{w^2}\right] \exp\left[-\left(\frac{\Delta\omega}{\omega_0}\right)^2 \frac{\bar{\rho}^2}{2w^2\beta^2}\right]$$
$$\times \exp\left[-\left(\frac{\bar{\omega}}{\omega_0}\right)^2 \frac{\Delta\rho^2}{2w^2\beta^2}\right] \exp\left[-\left(\frac{\Delta\omega}{\omega_0}\right)^2 \frac{\Delta\rho^2}{8w^2}\right] \exp\left(-\frac{\bar{\omega}\Delta\omega}{\omega_0^2}\frac{1+\beta^2}{\beta^2}\frac{\bar{\boldsymbol{\rho}}\cdot\Delta\boldsymbol{\rho}}{w^2}\right). \quad (56)$$

If we represent this in the form of Eq. (9), we find that

$$g(\bar{\boldsymbol{\rho}}, \Delta\boldsymbol{\rho}; \bar{\omega}, \Delta\omega) = \exp\left[-\left(\frac{\Delta\omega}{\omega_0}\right)^2\left(\frac{1}{\beta^2} - 1\right)\frac{\bar{\rho}^2}{2w^2}\right] \exp\left[-\left(\frac{\bar{\omega}}{\omega_0}\right)^2 \frac{\Delta\rho^2}{2w^2\beta^2}\right] \exp\left(-\frac{\bar{\omega}\Delta\omega}{\omega_0^2}\frac{1+\beta^2}{\beta^2}\frac{\bar{\boldsymbol{\rho}}\cdot\Delta\boldsymbol{\rho}}{w^2}\right)$$
$$\times \exp\left[-\left(\frac{\Delta\omega}{\omega_0}\right)^2 \frac{\Delta\rho^2}{8w^2}\right]. \quad (57)$$

The first exponential in Eq. (57) can be simplified sightly by using $1/\beta^2 - 1 = w^2/\sigma^2$. In particular,

$$g(\bar{\boldsymbol{\rho}}, \Delta\boldsymbol{\rho}; \omega, 0) = \exp\left[-\left(\frac{\omega}{\omega_0}\right)^2 \frac{\Delta\rho^2}{2w^2\beta^2}\right], \quad (58)$$

which obviously satisfies the boundary condition of Eq. (10) and the scaling law of Eq. (22) for nonstationary fields. In the quasihomogeneous case, $\beta \approx \sigma/w \ll 1$, Eq. (57) can be cast into the form

$$g(\bar{\boldsymbol{\rho}}, \Delta\boldsymbol{\rho}; \bar{\omega}, \Delta\omega) = \exp\left[-\frac{(\Delta\omega\bar{\boldsymbol{\rho}} + \bar{\omega}\Delta\boldsymbol{\rho})^2}{2\omega_0^2\sigma^2}\right] \exp\left[-\left(\frac{\Delta\omega}{\omega_0}\right)^2 \frac{\Delta\rho^2}{8w^2}\right]. \quad (59)$$

In this case the second exponential in Eq. (59) is effectively constant (unity) within the coherence area and we therefore have $g(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2) \approx \mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \omega_1, \omega_2)$. In particular, if $\omega_1 = \omega_2$, we get

$$\mu(\Delta\boldsymbol{\rho}; \omega) = \exp\left[-\left(\frac{\omega}{\omega_0}\right)^2 \frac{\Delta\rho^2}{2\sigma^2}\right], \quad (60)$$

which again satisfies Eqs. (10) and (22).

Some interpretations of the example fields presented above may help to emphasize their significance. In Sec. IV B 1, while dealing with stationary fields we constructed the CSD as an incoherent superposition of fully spatially coherent HG modes with a particular frequency-dependent scale factor given by Eq. (30). The result is a spatially partially coherent (but spectrally incoherent) field that satisfies our extended definition of spectral invariance. In Sec. IV B 2 we employed an incoherent superposition of fully spatially and spectrally coherent modal fields to construct the (two-frequency) CSD, which also exhibits the property of spectral invariance. Because of the intricate space-frequency coupling that is evident from, e.g., Eq. (55), the superposition is no longer a spectrally fully coherent field. In particular, the spectral behavior of the spatial degree of self-correlation,

$$\mu(\bar{\boldsymbol{\rho}}, 0; \bar{\omega}, \Delta\omega) = \exp\left[-\left(\frac{\Delta\omega}{\omega_0}\right)^2 \frac{\bar{\rho}^2}{2\sigma^2}\right], \quad (61)$$

depends on both the spatial positions and the frequency difference. This quantity equals unity only at the axial point $\bar{\boldsymbol{\rho}} = 0$, but its value reduces at off-axis positions because the partial spatial coherence implies partial spectral coherence via space-frequency coupling. This also leads to partial temporal coherence of the superposed field, even though each modal field is fully temporally coherent.

Finally, the spatial distribution of the spectral self-correlation function,

$$\mu(\bar{\boldsymbol{\rho}}, \Delta\boldsymbol{\rho}; \bar{\omega}, 0) = \exp\left[-\left(\frac{\bar{\omega}}{\omega_0}\right)^2 \frac{\Delta\rho^2}{2\sigma^2}\right], \quad (62)$$

depends on both the observation frequency and the spatial coordinate difference. Thus, it is spatially of the Schell-model form and, remarkably, formally similar to the (single-frequency) degree of spatial coherence of the corresponding stationary field [cf., Eq. (46)].

V. DISCUSSION AND CONCLUSIONS

In the present study we considered a spectral invariance of nonstationary (pulsed or nonpulsed) fields and extended the scaling law beyond stationary fields. This was accomplished by considering a class of nonstationary fields and requiring that the normalized far-zone spectrum is the same in all directions and equal to the source-averaged spectrum.

Specific examples of both stationary and pulsed fields that satisfy our scaling law were presented. These fields were constructed as incoherent superpositions of fully coherent Hermite-Gaussian (HG) modal fields. Since the HG fields are natural modes of oscillation of spherical-mirror resonators, one may ask whether multimode pulsed fields from such resonators (of, e.g., femtosecond lasers) satisfy the scaling law. The answer is negative. In order to satisfy the scaling law, the frequency dependence of the beam-waist dimensions was required to satisfy Eq. (30). However, HG modes from spherical-mirror resonators are isodiffracting with a different frequency dependence: $w_{0j}(\omega) = \sqrt{\omega_0/\omega}w_{0j}$, $j = x, y$. Therefore, none of their superpositions can satisfy the scaling law, even in the quasihomogeneous case [16]. It may be possible to employ optical systems with tailored chromatic properties to transform pulsed fields from spherical-mirror resonators into a form that satisfies the scaling law. The design of such compensating optical systems would follow procedures analogous to the design of achromatic Fourier-transform lenses (see, e.g., Refs. [19–21]). However, this is a subject of further study.

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APPENDIX

In this Appendix, we list some formulas used to derive the results in the main text: the integral formulas

$$\int_{-\infty}^{\infty} H_m^2(x)\exp(-x^2)dx = 2^m m!\sqrt{\pi} \tag{A1}$$

and

$$\int_{-\infty}^{\infty} H_m(\sqrt{2a}x)\exp(-ax^2 \pm ibx)dx = (\pm i)^m\sqrt{\frac{\pi}{a}}H_m\left(\frac{b}{\sqrt{2a}}\right)\exp\left(-\frac{b^2}{4a}\right) \tag{A2}$$

are given in Ref. [22], pages 811–812. The sum formula

$$\sum_{m=0}^{\infty} H_m(x)H_m(y)\frac{t^m}{m!} = \frac{1}{\sqrt{1-4t^2}}\exp\left[-\frac{4t^2(x^2+y^2)-4xyt}{1-4t^2}\right] \tag{A3}$$

can be found in Ref. [23] on page 194.

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