Supersymmetry shielding the scaling symmetry of conformal quantum mechanics

J. V. S. Scursulim,* A. A. Lima,† U. Camara da Silva,‡ and G. M. Sotkov§
Department of Physics, Federal University of Espírito Santo, 29075-910 Vitória, Espírito Santo, Brazil

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Renormalization of the inverse square potential usually breaks its classical conformal invariance. In a strongly attractive potential, the scaling symmetry is broken to a discrete subgroup while, in a strongly repulsive potential, it is preserved at quantum level. In the intermediate, weak-medium range of the coupling, an anomalous length scale appears due to a flow of the renormalization group away from a critical point. We show that potentials with couplings in the strongly repulsive and in the weak-medium ranges can be related by a dynamical supersymmetry. Imposing supersymmetry invariance unifies these two ranges and fixes the anomalous scale to zero, thus restoring the continuous scaling symmetry.

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I. INTRODUCTION

The Hamiltonian of conformal quantum mechanics [1],

$$H = \frac{1}{2m}p^2 + V(x), \quad V(x) = \frac{\hbar^2}{2m}\frac{\alpha}{x^2}, \tag{1}$$

being singular, is infamously subtle [2,3]. Yet, it appears in a cornucopia of physical problems, including the Efimov effect of nuclear three-body scattering and its generalizations in condensed-matter theory [4,5], mathematical physics [6–9], quantum field theory near the horizons of black holes [10–12], fluctuations in gauge–gravity duality [13–16], the AdS_2-CFT_1 correspondence [17], and still other phenomena. Such a breadth of applications could be seen as a reflection of scaling invariance, which can appear as a (usually asymptotic) symmetry in various situations.

Scale invariance of (1) is a consequence of the homogeneous transformation of H under a rescaling,

$$t \to \rho^2 t, \quad x \to \rho x, \tag{2}$$

with $\rho > 0$. Since the momentum transforms as $p \to \rho^{-1}p$, the Hamiltonian has a definite dimension, $H \to \rho^{-2}H$. Thus, (2) is a symmetry of the stationary Schrödinger equation,

$$\left[-\frac{d^2}{dx^2} + \frac{\alpha}{x^2}\right]\psi(x) = \frac{2m}{\hbar^2}E\,\psi(x), \quad x > 0, \tag{3}$$

where $\psi(x) \to \tilde{\psi}(\rho x) = \rho^{-1/2}\psi(x)$, preserving the probability $dx|\psi(x)|^2$, and the energy changes as $E \to \rho^{-2}E$ [18]. But the symmetry can be broken at the level of the quantum states, as the regularization of the singularity of $V(x)$ may introduce "anomalous" length scales.

Breaking of scale invariance depends subtly on the value of the adimensional coupling α. Naïvely, there are two qualitatively distinct possibilities: either $\alpha > 0$ and the potential is repulsive, or $\alpha < 0$ and the potential is attractive. The latter is evidently problematic because the singularity at $x = 0$ makes the question of whether the particle can "fall to the center" nontrivial [19]. Naïveté is due precisely to the singularity: the Hamiltonian (1) is not self-adjoint, and physical results require a self-adjoint extension [18,20,21]. Constructing these extensions turns out to be completely equivalent to a renormalization procedure. Strictly, the singularity of $V(x)$ at $x = 0$ should be considered an effect of inadvertently extending the problem too much into the realm of some unknown short-distance physics. Once regarding a singular potential such as (1) an effective theory valid only at long distances, the singular vicinity of $x = 0$ requires a renormalization procedure, to which observables at large x should be insensitive [22–29].

The renormalized theory depends not simply on whether α is positive or negative, but on *three* qualitatively different regimes:

$$\text{strongly repulsive: } \alpha \in \left[\tfrac{3}{4}, \infty\right), \tag{4a}$$

$$\text{weak medium: } \alpha \in \left[-\tfrac{1}{4}, \tfrac{3}{4}\right), \tag{4b}$$

$$\text{strongly attractive: } \alpha \in \left(-\infty, -\tfrac{1}{4}\right). \tag{4c}$$

In the strongly repulsive range (4a), the renormalized solutions are scale invariant, while in the strongly attractive range (4c), scale invariance is broken into a discrete subgroup, and conformality is lost after a phase transitions of the Berezinskii-Kosterlitz-Thouless (BKT) type happens at $\alpha = -\frac{1}{4}$ [26]. In the weak-medium range (4b), renormalization introduces the anomalous scale L, and the continuous family of self-adjoint extensions of (1) corresponds to the renormalization group (RG) flow between two conformal fixed points where $L = 0$ and $L = \infty$. Therefore, in the weak-medium range, for finite L, conformality is *also* lost by "dimensional transmutation" [30–32]. Nevertheless, since there is still the possibility of restoring continuous scaling symmetry by choosing one of the fixed points of the RG flow, we call the entire range of $\alpha \in [-\frac{1}{4}, \infty)$ the continuous-scaling phase, in contrast with the discrete-scaling phase of $\alpha < -\frac{1}{4}$.

The objective of the present paper is to show that the continuous-scaling phase has a somewhat disguised

*josevictor.s.scursulim@gmail.com
†andrealves.fis@gmail.com
‡ulyssescamara@gmail.com
§gsotkov@gmail.com

TABLE I. Qualitatively different ranges of the coupling: α versus ν.

Str. attractive	Weak medium	Str. repulsive
$-\infty < \alpha < -\frac{1}{4}$	$-\frac{1}{4} \leqslant \alpha < \frac{3}{4}$	$\frac{3}{4} \leqslant \alpha < \infty$
$-\infty < \nu^2 < 0$	$0 \leqslant \nu^2 < 1$	$1 \leqslant \nu^2 < \infty$

symmetry that unifies the strongly repulsive and the weak-medium ranges: a supersymmetry (SUSY) of the inverse square potential. This is *not* an extension of the 1D conformal algebra, such as the ones which have been considered, e.g., in the context of holography of black holes [33]. Rather, it is a dynamical symmetry of the energy spectrum due to the factorization [34] of the Hamiltonian (1) into two different products $H_+ = \frac{\hbar^2}{2m} Q^\dagger Q$ and $H_- = \frac{\hbar^2}{2m} Q Q^\dagger$, where

$$Q = \frac{d}{dx} + \frac{\sqrt{2m}}{\hbar} W(x) \quad \text{and} \quad \frac{\sqrt{2m}}{\hbar} W(x) = -\frac{\nu + \frac{1}{2}}{x},$$

producing a pair of inverse square potentials $V_+(x)$ and $V_-(x)$ with different couplings $\alpha_\pm$, determined by $\alpha = \nu^2 - \frac{1}{4}$, with $\nu_- = \nu_+ + 1$. Our main observation is that consistency with this SUSY forces the anomalous scale in the weak-medium range to *vanish*, restoring conformal symmetry over the whole continuous-scaling phase. In the discrete-scaling phase, the SUSY construction leads to inverse-square potentials with complex couplings so, in this sense, it ceases to be a symmetry of (1).

In Sec. II, we review the renormalization procedure for the inverse square potential, and how the anomalous scale appears in the weak-medium coupling. In Sec. III, we review the basic aspects of SUSY quantum mechanics, and show our main result. In Sec. IV, we present a collection of examples and show how our construction can be generalized to supersymmetric potentials which are only asymptotically like (1). We conclude with a brief discussion.

II. CONFORMAL SYMMETRY AND RENORMALIZATION OF THE INVERSE SQUARE POTENTIAL

For $E > 0$, the general solution of (1) is

$$\psi_{\nu,k}(x) = A_{\nu,k}\, \sqrt{x} J_\nu(kx) + B_{\nu,k}\, \sqrt{x} N_\nu(kx),$$
$$\nu \equiv \sqrt{\alpha + 1/4}, \quad k \equiv \sqrt{2mE/\hbar^2}. \tag{5}$$

with $A_{\nu,k}$ and $B_{\nu,k}$ integration constants. The three ranges of α translate to ν as in Table I. We most often use the index ν instead of α. For now, consider $\nu \geqslant 0$ and leave the discussion of the strongly attractive range, characterized by an imaginary $\nu = i\tilde{\nu}$, for later in this section.

Physical wave functions must be *normalizable* in the vicinity of the singular point:

$$\lim_{x_0 \to 0} \int^{x_0} dx\, |\psi_{\nu;k}(x)|^2 < \infty. \tag{6}$$

The first solution in (5) is always square integrable at $x = 0$, since $J_\nu(kx) \sim (kx)^\nu$. The second solution goes as $N_\nu(kx) \sim (kx)^{-\nu}$ for $kx \ll 1$, so its norm diverges for $\nu \geqslant 1$ and, therefore, in the strongly repulsive range normalizability fixes

$$B_{\nu,k} = 0 \quad \text{if} \quad \nu \geqslant 1 \quad \text{(strongly repulsive)}. \tag{7}$$

Hence the wave function is determined uniquely (A_ν just fixes the norm). On the other hand, in the weak-medium range, $0 \leqslant \nu < 1$, *both* solutions in (5) are normalizable, so both constants A_ν and B_ν are arbitrary: the wave function is not uniquely fixed.

A. Renormalization and the anomalous scale

In any case, the singularity of $V(x)$ at the origin should be seen as the effect of using an effective theory outside its range of validity. Physical consistency can be obtained with a renormalization procedure: First, we define a regularized potential which is well-behaved at the origin; then we impose that physics at large distances should be insensitive to this regularization. Essentially, these steps have all been presented elsewhere, cf., e.g., Refs. [22,27–29], but we outline them now for completeness and for fixing our notation. First define the regularized potential:

$$\frac{2m}{\hbar^2} V_R(x) = \begin{cases} \alpha/x^2, & x > R \\ -\lambda/R^2, & x < R. \end{cases} \tag{8}$$

The short-distance cutoff scale R is much smaller than the only length scale of the system, i.e., $kR \ll 1$, and we impose the Dirichlet condition $\psi(0) = 0$. A Neumann boundary condition would give equivalent results. Also, the use of a square well near the origin is just a convenient choice: other regularizations (e.g., a Dirac delta function) give equivalent results, as it should be [27].

The regularization parameters λ and R must be related to each other in such a way that the long-distance properties of the system (including the coupling α), are insensitive to a sliding of the cutoff. The relation $\lambda(R)$ must be a property of the theory, hence it must be the same at any particular energy. We take as a reference the "ground-state" (see, however, Sec. III 2) solution with $E = 0$, which in the $x > R$ region is simply

$$\psi_{\nu,0}(x) = A_{\nu,0}\, x^{\frac{1}{2}+\nu} + B_{\nu,0}\, x^{\frac{1}{2}-\nu}. \tag{9}$$

It is clear that a length scale L, defined by

$$L^\nu \equiv \varepsilon B_{\nu,0}/A_{\nu,0}, \quad \varepsilon \equiv \text{Sign}[B_{\nu,0}/A_{\nu,0}] = \pm 1, \tag{10}$$

appears intrinsically into the solutions if both $A_{\nu,0}, B_{\nu,0} \neq 0$. In the regularized region, where $x < R$, the solution is $C_R \sin(\sqrt{\lambda} x/R)$, and imposing continuity of the logarithmic derivative $\psi'(x)/\psi(x)$ across the divide $x = R$ results in

$$\gamma(R) \equiv \sqrt{|\lambda|} \cot \sqrt{|\lambda|} - \frac{1}{2} = \nu \left[\frac{1 - \varepsilon (L/R)^{2\nu}}{1 + \varepsilon (L/R)^{2\nu}} \right]. \tag{11}$$

We do the same for a solution with $E > 0$, given by (5) for $x > R$ and by $C_\nu \sin \varkappa x$ for $x < R$, where $\varkappa \equiv \sqrt{\lambda + (kR)^2}/R$. Using the leading asymptotic forms of the Bessel functions for arguments $kR \ll 1$, we find [27]

$$\gamma(R) = \nu \left[\frac{1 + \frac{B_\nu}{A_\nu} \frac{\nu [\Gamma(\nu)]^2}{\pi} \left(\frac{kR}{2}\right)^{-2\nu}}{1 - \frac{B_\nu}{A_\nu} \frac{\nu [\Gamma(\nu)]^2}{\pi} \left(\frac{kR}{2}\right)^{-2\nu}} \right]. \tag{12}$$

Now we can combine Eqs. (12) and (11) to obtain an explicit relation between the constants in terms of the anomalous scale L,

$$\frac{B_{\nu,k}}{A_{\nu,k}} = -\frac{\varepsilon\pi}{\nu[\Gamma(\nu)]^2}\left(\frac{kL}{2}\right)^{2\nu}. \tag{13}$$

Remarkably, Eq. (13) does not depend on R, which can be taken to zero, so we end up with a family of renormalized wave functions given by (5), parameterized by L according to (13). Thus renormalization introduces a quantum anomalous scale, and conformal symmetry is spontaneously broken—a "dimensional transmutation" [30–32] happens.

If $\nu = 0$, the regularized zero-energy wave function is

$$\psi_{0,0}^{(R)}(x) = \begin{cases} \sqrt{x}\,[1 + c\ln(x/L_0)], & x > R \\ C_0 \sin(\sqrt{\lambda}x/R), & x < R. \end{cases} \tag{14}$$

The anomalous scale, which we denote by L_0, here appears together with a dimensionless constant c. Imposing continuity of ψ'/ψ at $x = R$ leads to a qualitatively different coupling,

$$\gamma(R) = \frac{c}{1 + c\ln(R/L_0)} \quad (\nu = 0), \tag{15}$$

and combining (15) with (11) we find the relation between the integration constants, to be compared with (13),

$$\frac{A_{0,k}}{B_{0,k}} = -\frac{2}{\pi}\ln\left(e^{-\frac{1}{c}+\mathscr{C}_E} kL_0\right), \tag{16}$$

where $\mathscr{C}_E \approx 0.577$ is Euler's constant.

B. RG flows

The running of the regularization coupling $\gamma(R)$ can be seen as a RG flow for the interaction [22,23,25–29]. In fact, $\gamma(R)$ given in (11) is the solution of the RG equation defined by the beta function:

$$\beta_\gamma \equiv \frac{d\gamma}{d\ln(R/L)} = -(\gamma^2 - \nu^2). \tag{17}$$

Its zeros define the "ultraviolet" and "infrared" fixed points

$$\{\gamma_{\rm UV} = \nu\ ;\ L = 0\} \quad \text{and} \quad \{\gamma_{\rm IR} = -\nu\ ;\ L = \infty\}, \tag{18}$$

where the coupling is independent of the cutoff and there is no anomalous scale. In the strongly attractive regime (4c), there are no zeros of β_γ, as $\nu = i\tilde{\nu}$ becomes imaginary; then the RG becomes cyclic [35]. This phase has a discrete energy spectrum $E_n \sim e^{-2\pi n/\tilde{\nu}}$, which is unbounded below and has an accumulation point near $E = 0$ [18]. Conformal symmetry is lost, but a discrete subgroup is preserved: (2) remains valid only for a discrete set of scaling parameters $\rho = e^{\frac{\pi}{\tilde{\nu}}n}$. We give some details of this process in the Appendix. Now, we concentrate on the case of real ν. Thinking of ν as an external parameter (corresponding to, say, a temperature) we can consider what happens when it sweeps the interval $\nu \geqslant 0$, spanning the weak-medium and the strongly repulsive regimes.

In the strongly repulsive regime ($\nu \geqslant 1$), the condition (7) fixes the scale $L = 0$, restricting the physical theory to the UV fixed point in (18). There is no RG flow and scaling invariance is unbroken.

Lowering ν, we enter the weak-medium range ($0 \leqslant \nu < 1$). The situation complicates considerably. After imposing (13), every theory with finite L is equally physical, since L is not required to vanish by normalizability. Thus, the theory can leave the UV fixed point, with two possible fates of the RG flow: it can go to the IR fixed point (18), or it could develop a massive limit. This latter case is subtle and less studied,[1] so let us make a brief description. The massive limit appears if the sign $\varepsilon = -1$ in (10). Then, from Eq. (11) we see that the cutoff R is restricted to the range $R \in (0, L)$,[2] since the function $\gamma(R)$ diverges at the finite scale $R = L$, the hallmark of a "massive flow.' This flow is associated with a bound state: if $E < 0$, the solution of (3) which is square integrable at $x = \infty$ is a modified Bessel function:

$$\psi_{\nu,\kappa}(x) = C\sqrt{\kappa x}K_\nu(\kappa x), \quad \kappa \equiv \sqrt{-2mE/\hbar^2}.$$

Continuity with the regularized region gives ($\kappa R \ll 1$)

$$\kappa = 2\left[-\frac{1}{\pi}\varepsilon\nu[\Gamma(\nu)]^2\sin\pi\nu\right]^{\frac{1}{2\nu}} L^{-1}, \quad 0 < \nu < 1. \tag{19}$$

If $\varepsilon = -1$, there is a bound state with energy $E \sim -(\hbar^2/m)L^{-2}$. It is not surprising to find a bound state because the regularized potential contains a well near the origin.[3] But it is surprising that the state actually does not depend on R, so it persists even in the limit $R \to 0$. Actually, it can be paradoxical because it is also independent on the value of α, so the bound state exists even if the potential is repulsive ($\alpha > 0$) or we have a free particle moving on the half line ($\alpha = 0$).

As $\nu \to 0^+$, so $\alpha \to (-1/4)^+$, the two zeros of β_γ merge into a single fixed point

$$\gamma_{\rm UV} = \gamma_{\rm IR} \equiv \gamma_{\rm BKT} = 0. \tag{20}$$

In Ref. [26], it was shown that this is equivalent to a BKT-like phase transition. Now, the solution of (17) is (15). If $c \neq 0$, we are outside the BKT fixed point (20), and there is again a massive RG flow. We can only take the limit $R \to 0$ if $c < 0$, and we find that this flow is associated with a bound state with energy

$$\kappa = 2e^{\frac{1}{c}-\mathscr{C}_E}/L_0, \quad \nu = 0. \tag{21}$$

Again, the bound state is independent both of R, so it persists in the limit $R \to 0$.

III. SUPERSYMMETRY IN THE CONTINUOUS-SCALING PHASE

The discussion in the previous section can be summarized as follows:

[1]See, however, the unpublished version of Ref. [29] by one of the present authors.

[2]Technically, we could also have, separately, the situation where $L < R$, but this is inconsistent with the condition that R is smaller than any scale of the theory; in particular, we cannot take the limit $R \to 0$.

[3]Alternatively, the bound state can be seen to appear because for $R \to 0$ the regularized potential has a delta function at the origin, which is known to support a bound state, depending on the coupling.

In the strongly attractive range ($\nu = i\tilde{\nu}$). Only a discrete subgroup of scaling symmetry remains.

In the strongly repulsive range ($\nu \geqslant 1$). Continuous scaling symmetry exists. Moreover, normalizability of the states imposes $B_{\nu,k} = 0$, thus eliminating the anomalous scaling L that appears in the renormalization process.

In the weak-medium range ($0 \leqslant \nu < 1$). Continuous scaling symmetry exists on the two fixed points (18) of the RG flow. But the theory is free to flow from the UV (or BKT) fixed point where $L = 0$, and develops a finite anomalous scale. The flow might go toward the IR fixed point where $L = \infty$ and scaling symmetry is restored; or, if $\varepsilon = -1$, the flow might develop a massive limit associated with a bound state. In any case, scaling symmetry is broken outside the fixed points. Worse, in the massive flows, the bound state is definitely paradoxical if the potential is repulsive (or if $\alpha = 0$).

Here we come to the point of this paper, which is to show that the entire continuous-scaling phase, consisting of the weak-medium plus the strongly repulsive ranges, is unified by a SUSY of the Hamiltonian (1), which is destroyed by theories that flow away from the UV fixed point.[4] SUSY fixes $L = 0$ for every $\nu \geqslant 0$, thus restoring scaling symmetry. In particular, the energies of the bound states (19) and (21) become infinite when $L = 0$, and they are excised from the spectrum.

A. SUSY quantum mechanics

A nonrelativistic quantum system is said to be supersymmetric if its Hamiltonian H_+ can be factored as [34]

$$H_+ = \frac{\hbar^2}{2m} Q^\dagger Q, \tag{22}$$

where the operator Q and its conjugate $Q^\dagger$ are given by

$$Q = \frac{d}{dx} + \frac{\sqrt{2m}}{\hbar} W(x), \quad Q^\dagger = -\frac{d}{dx} + \frac{\sqrt{2m}}{\hbar} W(x) \tag{23}$$

for a function $W(x)$, called the superpotential, which determines the Schrödinger potential as

$$\frac{2m}{\hbar^2} V_+(x) = \left[\frac{\sqrt{2m}}{\hbar} W(x)\right]^2 - \frac{\sqrt{2m}}{\hbar}\frac{d}{dx} W(x). \tag{24}$$

Given such a Hamiltonian H_+, there is a "partner" Hamiltonian H_- with the inverse factorization, i.e.,

$$H_- = \frac{\hbar^2}{2m} Q Q^\dagger, \tag{25}$$

whose corresponding potential has a flipped sign:

$$\frac{2m}{\hbar^2} V_-(x) = \left[\frac{\sqrt{2m}}{\hbar} W(x)\right]^2 + \frac{\sqrt{2m}}{\hbar}\frac{d}{dx} W(x). \tag{26}$$

SUSY stems from the fact that the matrix operators

$$\mathscr{H} = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix}, \quad \mathscr{Q} = \begin{bmatrix} 0 & 0 \\ Q & 0 \end{bmatrix}, \quad \mathscr{Q}^\dagger = \begin{bmatrix} 0 & Q^\dagger \\ 0 & 0 \end{bmatrix}, \tag{27}$$

form the closed superalgebra $\mathfrak{sl}(1|1)$:

$$\{\mathscr{Q}, \mathscr{Q}^\dagger\} = \mathscr{H}, \quad [\mathscr{H}, \mathscr{Q}] = [\mathscr{H}, \mathscr{Q}^\dagger] = 0, \quad \{\mathscr{Q}, \mathscr{Q}\} = \{\mathscr{Q}^\dagger, \mathscr{Q}^\dagger\} = 0. \tag{28}$$

These operators act on the space of "superstates" generated by

$$|\psi_k^+\rangle = \begin{bmatrix} \psi_k^+ \\ 0 \end{bmatrix}, \quad |\psi_k^-\rangle = \begin{bmatrix} 0 \\ \psi_k^- \end{bmatrix}, \tag{29}$$

where $\psi_k^\pm(x)$ are eigenstates of $H_\pm$:

$$\frac{2m}{\hbar} H_\pm \psi_k^\pm = k^2 \psi_k^\pm, \quad A = 1, 2. \tag{30}$$

The ψ_k^+ sector is said to be bosonic, and the ψ_k^- to be fermionic, and we follow this nomenclature. The factorization (22) relates the partner's spectra $\{E^\pm\}$ and their eigenvectors $\psi_k^\pm$. The vacuum state, defined as $|0\rangle = |\psi_0^+\rangle + |\psi_0^-\rangle$, is annihilated by the charge $\mathscr{Q}$, viz. $\mathscr{Q}|0\rangle = 0$. When the vacuum energy is zero, this implies that ψ_0^+ is a solution of $Q\psi_0^+ = 0$:

$$\psi_0^+(x) \sim \exp\left[-\tfrac{\sqrt{2m}}{\hbar}\textstyle\int dx\, W(x)\right], \quad H_+\psi_0^+ = 0. \tag{31}$$

If this function is square integrable, then the vacuum lies completely in the bosonic sector, $|0\rangle = |\psi_0^+\rangle$, and the discrete energy spectra are related as $E_n^- = E_{n+1}^+$, with $n = 0, 1, 2, 3, \cdots$. If, however, ψ_0^+ is *not* square integrable, then SUSY is said to be spontaneously broken and the energy spectra are completely degenerated, with the partner wave functions related by

$$\psi_k^-(x) = \frac{1}{k} Q\psi_k^+(x), \quad \psi_k^+(x) = \frac{1}{k} Q^\dagger \psi_k^-(x). \tag{32}$$

This will be our case of interest. Note that the construction of SUSY partners is completely algebraic, and insensitive to whether one of the Hamiltonians should, eventually, not be self-adjoint.

B. SUSY of the inverse square potential

The gist of this paper is that quantum mechanics with the inverse square potential is SUSY quantum mechanics, with the superpotential

$$\frac{\sqrt{2m}}{\hbar} W_\nu(x) = -\frac{\nu + \frac{1}{2}}{x}, \tag{33}$$

associated with the operators Q_ν and $Q_\nu^\dagger$. Indeed, the SUSY partners given by (24) and (26) are

$$\frac{2m}{\hbar^2} V_+(x) = \frac{\nu^2 - \frac{1}{4}}{x^2}, \quad \frac{2m}{\hbar^2} V_-(x) = \frac{(\nu+1)^2 - \frac{1}{4}}{x^2}. \tag{34}$$

Both are inverse square potentials, hence we say that (1) is shape invariant under SUSY, whose effect is to change the coupling as

$$\nu \mapsto \tilde{\nu} = \nu + 1. \tag{35}$$

Thus the partner of the BKT phase transition potential, which has $\alpha_+ = -\frac{1}{4}$, is the potential with $\alpha_- = \frac{3}{4}$, which is the lowest value of the coupling in the strongly repulsive range (see Table I). In general, *the entire weak-medium interval* $\alpha \in [-\frac{1}{4}, \frac{3}{4})$ *has been mapped by SUSY to the strongly repulsive interval* $\alpha \in [\frac{3}{4}, \frac{15}{4})$.

[4] Or the self-adjoint extensions with $L > 0$.

Looking at the eigenstates, first we see that the vacuum, $\psi^+_{\nu,0}$, is such that

$$Q_\nu \psi^+_{\nu,0} = 0, \quad \text{hence} \quad \psi^+_{\nu,0} = A_{\nu,0}\, x^{\frac{1}{2}+\nu}. \tag{36}$$

This function is not square integrable on $[0,\infty)$. Therefore, SUSY is spontaneously broken, the partner spectra are completely degenerated, and every eigenvector of H_+ is related to an eigenvector of H_-. From Eq. (32) we can find the partners to the wave functions $\psi^+_{\nu,k}(x)$ given in (5). Making use of a recurrence formula for the Bessel functions,[5] we find

$$\begin{aligned}\psi^-_{\nu,k}(x) &= \frac{1}{k}\left(\frac{d}{dx} - \frac{\nu+\frac{1}{2}}{x}\right)\psi^+_{\nu,k}(x)\\ &= \sqrt{x}[-A_{\nu,k}J_{\nu+1}(kx) - B_{\nu,k}N_{\nu+1}(kx)]\\ &\equiv \psi^+_{\nu+1,k}(x).\end{aligned}$$

Indeed, the map (35) appears. Most importantly, the integration constants are related by

$$A_{\nu+1,k} = -A_{\nu,k}, \quad B_{\nu+1,k} = -B_{\nu,k}. \tag{37}$$

We emphasize that the fact that SUSY is spontaneously broken is crucial; if this were not the case, the relation (37) would not hold.

Equation (37) has a remarkable consequence. Suppose we start with a theory in the weak-medium range, with

$$-\tfrac{1}{4} \leqslant \alpha_+ = \nu^2 - \tfrac{1}{4} < \tfrac{3}{4}. \tag{38}$$

We renormalize the theory, and the constants $A_{\nu,k}$ and $B_{\nu,k}$ are related by the anomalous scale L as in Eq. (13). The (fermionic) partner model has the coupling

$$\alpha_- = (\nu+1)^2 - \tfrac{1}{4} > \tfrac{3}{4}, \tag{39}$$

which is in the strongly repulsive range, where normalizability fixes $B_{\nu+1,k} = 0$, cf. Eq. (7). But then Eq. (37) forces us to make $B_{\nu,k} = 0$, hence $L = 0$, thus *selecting the UV point for the model in the medium-weak range*.

Hamiltonians in the weak-medium range have one single bound state with energy $E \sim -1/L^2$ given by (19) or (21). By fixing $L = 0$, thus $E = -\infty$, these states are excised from the spectrum. This solves the paradox of bound states in repulsive or free potentials with $\alpha \in [0, \frac{3}{4})$. This is also consistent with the fact that there could be no bound states even in the attractive weak-medium range, $\alpha \in [-\frac{1}{4}, 0)$, because SUSY is a symmetry between the spectra of these models and the spectra of strongly repulsive models with $\alpha \in [\frac{3}{4}, 2)$.

In the strongly attractive range, $\nu = i\tilde{\nu}$ becomes an imaginary number and the superpotential (33) also becomes imaginary. One could argue that the important thing would be for V_+ to be real, but

$$\frac{2m}{\hbar^2}V_+(x) = -\frac{\left(\tilde{\nu}^2 + \frac{1}{4}\right)}{x^2}, \quad \frac{2m}{\hbar^2}V_-(x) = \frac{\frac{3}{4} - \tilde{\nu}^2 + 2i\tilde{\nu}}{x^2},$$

so the partner potential $V_-(x)$ is complex and we have non-Hermitian quantum mechanics [37,38]. In this sense, SUSY is a property of the inverse square potential only in the range where scaling symmetry has not been discretely broken.

Since the inverse square potential is shape invariant, we can repeat the procedure but now taking $V_-(x)$ as a *bosonic* model, whose fermionic partner will have the Bessel index $\nu + 2$. Going on like this generates a chain of models with indices $\nu + n$, all having the same spectrum and constants related as

$$\begin{aligned}&B_{\nu+n,\,k} = 0,\\ &A_{\nu+n+1,\,k} = -A_{\nu+n,\,k} = \cdots = (-1)^n A_{\nu,k}\,; \quad n \in \mathbb{N}.\end{aligned} \tag{40}$$

Every model in this chain but the first one lies in the regime of strongly repulsive couplings.

For the purpose of illustration, let us consider explicitly the case $\nu = 1$, $\alpha = \frac{3}{4}$. This is the smaller value of α and ν inside the strongly repulsive regime. The general renormalized energy eigenstate is (5), with the integration constants related by the anomalous scale according to (13), that is,

$$\psi_{1,k}(x) = A_{1,k}\sqrt{x}\big[J_1(kx) - \varepsilon\pi\left(\tfrac{1}{2}kL\right)^2 N_1(kx)\big]. \tag{41}$$

The function $\sqrt{kx}N_1(kx)$ is not square integrable at $x = 0$ and so it must be absent, hence we must set $L = 0$ to make $\psi_{1,k}$ a good wave function. But let us carry on with $L \neq 0$ for a while. Starting from $\psi_{1,k}$, we can obtain two different SUSY partners, depending on whether we take it to be the in the bosonic or in the fermionic sector. In the latter case, we have $\psi_{1,k} = \psi^-_{0,k}$, and we can recover the partner function $\psi^+_{0,k}$ by applying $Q_0^\dagger$ according to (32). The result is

$$\begin{aligned}\psi^+_{0,k} &= \frac{1}{k}Q_0^\dagger\psi^-_{0,k}\\ &= \frac{1}{k}\left[-\frac{d}{dx} - \frac{0+1/2}{x}\right]\psi_{1,k}(x)\\ &= A_{1,k}\sqrt{x}\left[-J_0(kx) + \varepsilon\pi\left(\frac{1}{2}kL\right)^2 N_0(kx)\right].\end{aligned} \tag{42}$$

This is the general solution of the potential with $\nu = 0$, and $\alpha = -\frac{1}{4}$, the threshold of the strongly attractive range. The function $\sqrt{x}N_0(kx)$ vanishes at $x = 0$, and looking only at (42) there is no reason to set $L = 0$. But now recall that if we do not set $L = 0$, then $\psi^-_{0,k}$ is not well-defined. Keeping track of L before setting it to zero also reveals a subtlety of the $\nu = 0$ solution. Comparing the relative constant in the last line of (42) with formula (16), one finds a nontrivial relation between the anomalous scales:

$$\begin{aligned}-\frac{1}{\pi}\varepsilon\left(\frac{1}{2}kL\right)^{-2} &= -\frac{2}{\pi}\ln\left(e^{-\frac{1}{c}+\mathscr{C}_E}kL_0\right)\\ &= -\frac{1}{\pi}\left[-\frac{2}{c} + 2\mathscr{C}_E + \ln\left(k^2L_0^2\right)\right].\end{aligned} \tag{43}$$

Matching powers, we see that fixing $L = 0$ corresponds to fixing the dimensionless constant $c = 0$ in Eqs. (14)–(16), not $L_0 = 0$. Of course, if $c = 0$ the coupling (15) is fixed to lie on the BKT fixed point (20), and the length scale L_0 disappears. Recall that the only way for the theory to flow away from the BKT fixed point where $c = 0$ is in the direction $c < 0$.

Next, we can see what happens if (41) is taken to be the bosonic sector of a SUSY partner, i.e., if $\psi_{1,k} = \psi^+_{1,k}$. Then

[5] If $\mathscr{C}_\nu(z)$ is a solution of the Bessel equation with index ν, then $\mathscr{C}_{\nu\pm1} = \mp\mathscr{C}'_\nu(z) + (\nu/z)\mathscr{C}_\nu(z)$ is a solution of the Bessel equation with index $\nu \pm 1$; see Ref. [36] Sec. 10.6(i).

(32) gives its fermionic partner $\psi^-_{1,k}$ by applying Q_1, viz.

$$\begin{aligned}\psi^-_{1,k} &= \frac{1}{k} Q_1 \psi^+_{1,k} \\ &= \frac{1}{k}\left[\frac{d}{dx} - \frac{1+1/2}{x}\right]\psi_{1,k}(x) \\ &= A_{1,k}\sqrt{x}\left[-J_2(kx) + \varepsilon\pi\left(\frac{1}{2}kL\right)^2 N_2(kx)\right].\end{aligned} \tag{44}$$

This is the general solution further inside the strongly repulsive range, with $\nu = 2$, $\alpha = \frac{15}{4}$. Here there is no ambiguity, the function $\sqrt{x}N_2(kx)$ also diverges at $x = 0$, and we must set $L = 0$ to regularize both $\psi^\pm_{1,k}$ at the same time.

IV. EXAMPLES AND GENERALIZATIONS

Any $V(x)$ which has the *asymptotic* form of an inverse square potential near $x = 0$ must be subject to the same renormalization process described in Sec. II, irrespective of its form at large x. As a consequence, the anomalous scale L may appear in these theories. As we give the three examples below, we would like to call attention to a generalization of the method described in Sec. III. If $V(x)$ is supersymmetric, i.e., if it has the form (24) and if SUSY is spontaneously broken, we can use the same procedure as above to fix the anomalous scale, even if (unlike the inverse square potential) $V(x)$ is not shape invariant under SUSY.

A. The radial motion of a free particle

The most basic appearance of the inverse square potential is in the radial Schrödinger equation for a free particle:

$$\frac{2m}{\hbar^2}V(x) = \frac{\ell(\ell+1)}{x^2}, \quad \ell \in \mathbb{N}, \quad x > 0. \tag{45}$$

For every $\ell \geqslant 1$, the number $\alpha_\ell = \ell(\ell+1)$ lies in the strongly repulsive regime (4a), but the s wave, with $\ell = 0$, lies in the weak-medium range, $\alpha_0 = 0$. Of course, there is no physical reason for the existence of a bound state associated with a scale L—this is simply a free particle. In fact, there is no renormalization needed at all since there are no interactions; the singularity at $x = 0$ is just a problem of the spherical coordinate system.

Nevertheless, this gives an interesting illustration of how the use of SUSY solves paradoxes introduced by a (here forceful) renormalization. The superpotential associated with (45) is $(\sqrt{2m}/\hbar)W_\ell(x) = -(\ell+1)/x$ and the wave function for $\ell = 0$ with $B_{0,k} = 0$ is

$$\psi_{0,k}(x) = A_{0,k}\sqrt{x}J_{1/2}(kx). \tag{46}$$

Following the steps of Sec. III, this function generates the solution for every ℓ by applying multiple operators Q_ℓ constructed from the chain of partner models with degenerate spectra:

$$\psi_{\ell,k}(x) = k^{-\ell}Q_\ell Q_{\ell-1}\cdots Q_1\psi_{0,k}(x) \sim \sqrt{x}J_{\ell+1/2}(kx). \tag{47}$$

In this particular case, the action of the Q operators is equivalent to a recurrence relation between spherical Bessel functions, see, e.g., [19].

B. The generalized Calogero-Moser-Sutherland potential

A nontrivial example is the (shifted) Calogero-Moser-Sutherland potential[6] [39,40]

$$\frac{2m}{\hbar^2}V_+(x) = \frac{\nu^2 - 1/4}{\sinh^2(x/a)} + (\nu + 1/2)^2, \quad x > 0, \tag{48}$$

with $0 < \nu < 1$ (we exclude $\nu = 0$ for simplicity). The solution of (3) is given by hypergeometric functions,

$$\begin{aligned}\psi^+_{\nu,k}(x) = a^{\frac{1}{2}}&\left[\sinh\left(\frac{1}{a}x\right)\right]^{\nu+\frac{1}{2}}\left[\cosh\left(\frac{1}{a}x\right)\right]^{\frac{3}{2}} \\ &\times\left\{\frac{(a/2)^\nu}{\nu\Gamma(\nu)}k^\nu A_{\nu,k}\,F\left[\omega,\bar\omega;1+\nu;-\sinh^2\left(\frac{1}{a}x\right)\right]\right. \\ &-\frac{B_{\nu,k}}{k^\nu\left[\sinh\left(\frac{1}{a}x\right)\right]^{2\nu}}\frac{\Gamma(\nu)}{\pi(a/2)^\nu} \\ &\left.\times F\left[\omega-\nu,\bar\omega-\nu,1-\nu;-\sinh^2\left(\frac{1}{a}x\right)\right]\right\},\end{aligned} \tag{49}$$

where $\omega \equiv 1 - \frac{\nu}{2} - \frac{i}{2}\sqrt{2mEa^2/\hbar^2 - (\nu+1/2)^2}$. The integration constants where chosen such that the asymptotic forms of (49) and of (5) coincide near the singularity $x = 0$.

Here renormalization is indeed necessary, and, as a result, the ratio $B_{\nu,k}/A_{\nu,k}$ is given by Eq. (13). The model is described by the superpotential

$$\frac{\sqrt{2m}}{\hbar}W(x) = -\frac{1}{a}\left(\nu + \frac{1}{2}\right)\coth(x/a). \tag{50}$$

The solution of the zero mode, $\psi^+_{\nu,0}(x) \sim [\sinh(x/a)]^{\nu+1/2}$, obtained from solving $Q\psi^+_{\nu,0}(x) = 0$, is not normalizable. Therefore the spectrum of (48) is the same as that of its superpartner:

$$\begin{aligned}\frac{2m}{\hbar^2}V_-(x) &= \left(\frac{\sqrt{2m}}{\hbar}W(x)\right)^2 + \frac{\sqrt{2m}}{\hbar}\frac{dW(x)}{dx} \\ &= \frac{(\nu+1)^2 - 1/4}{\sinh^2(x/a)} + (\nu+1/2)^2.\end{aligned} \tag{51}$$

Near the origin, we find the inverse square potential with a strongly repulsive coupling:

$$(2m/\hbar^2)V_-(x) \approx \left[(\nu+1)^2 - \tfrac{1}{4}\right]/x^2.$$

There is no renormalization in this model, and since $\psi^-_{\nu,k} = (1/k)Q\psi^+_{\nu,k}(x)$, we must fix $L = 0$, i.e., $B_{\nu,k} = 0$ in (49).

C. The Calogero-Moser-Sutherland potential

With this last example, we will show that, in potentials which are asymptotically inverse square, SUSY restricts the energy spectrum more than the normalizability condition alone. It fixes $L = 0$ even when there are square-integrable solutions at the IR fixed point where $L \to \infty$.

[6]This is an hyperbolic generalization of the Calogero-Moser-Sutherland trigonometric potential [8], which is the subject of the next example, cf. Ref. [41].

The Calogero-Moser-Sutherland potential [8] and its superpotential are

$$\frac{\sqrt{2m}}{\hbar}W(x) = -\frac{\pi}{a}\frac{(\nu+1/2)}{\sin(\pi x/a)}, \quad 0 < x < a, \tag{52}$$

$$\frac{2m}{\hbar^2}V_+(x) = \frac{\pi^2}{a^2}\frac{(\nu+1/2)}{\sin^2(\pi x/a)}\left[\nu + \frac{1}{2} - \cos\left(\frac{\pi}{a}x\right)\right], \tag{53}$$

with $0 < \nu < 1$. The potential is an infinite well, with inverse square behavior at the boundaries. Near $x = 0$, it goes as $(2m/\hbar^2)V_+(x) \approx (\nu^2 - 1/4)/x^2$ which has a medium-weak coupling. Near $x = a$, it goes as

$$(2m/\hbar^2)V_+(x) \approx \frac{(\nu+1)^2 - \frac{1}{4}}{(a-x)^2},$$

with a strongly repulsive coupling.

The general solution for the stationary wave function is ($k = \sqrt{2mE/\hbar^2}$):

$$\begin{aligned}\psi^+_{\nu,k}(x) = &\left[\sin\left(\frac{\pi}{2a}x\right)\right]^{\nu+\frac{1}{2}}\left[\cos\left(\frac{\pi}{2a}x\right)\right]^{\nu+\frac{3}{2}} \\ &\times\left[C^{\rm UV}_{\nu,k}F\left[1+\nu-\frac{ka}{\pi}, 1+\nu+\frac{ka}{\pi}; 1+\nu; \sin^2\left(\frac{\pi x}{2a}\right)\right]\right. \\ &\left.+\frac{C^{\rm IR}_{\nu,k}}{\left[\sin\left(\frac{\pi x}{2a}\right)\right]^{2\nu}}F\left[1-\frac{ka}{\pi}, 1+\frac{ka}{\pi}; 1-\nu; \sin^2\left(\frac{\pi x}{2a}\right)\right]\right].\end{aligned} \tag{54}$$

Near $x = 0$, as expected, $\psi^+_{\nu,k}(x) \approx f_{\rm UV}(x) + f_{\rm IR}(x)$, where $f_{\rm UV}(x) \propto C^{\rm UV}_{\nu,k}\, x^{\nu+\frac{1}{2}}$ and $f_{\rm IR}(x) \propto C^{\rm IR}_{\nu,k}\, x^{-\nu+\frac{1}{2}}$. Both solutions as square integrable at the origin, and the renormalization procedure fixes the ratio $C^{\rm IR}_{\nu,k}/C^{\rm UV}_{\nu,k} \propto L^{2\nu}$. On the other hand, *neither* of the solutions is square integrable in $x = a$, unless one of the hypergeometrics is a polynomial—which gives a discrete condition on k —and the other solution is discarded by fixing the respective $C = 0$, i.e., by choosing one of the fixed points. The discrete spectra on the UV point ($L = 0$) and on the IR point ($L = \infty$) are

$$k^{\rm UV}_{\nu,n} = \frac{\pi}{a}(n+1+\nu), \qquad C^{\rm IR}_{\nu,k} = 0, \tag{55}$$

$$k^{\rm IR}_{\nu,n} = \frac{\pi}{a}(n+1), \qquad C^{\rm UV}_{\nu,k} = 0, \tag{56}$$

where $n \in \mathbb{N}$.

We therefore have two classes of normalized, renormalized eigenfunctions, each class with a different spectrum. Renormalization (or the self-adjoint extension) around $x = 0$ tells us that the two spectra cannot coincide, since they correspond to two different fixed points, but it does not give any clear criterion for the preference of one over the other. It is SUSY that chooses the spectrum unambiguously.

Indeed, the partner potential of (53) is

$$\frac{2m}{\hbar^2}V_-(x) = \frac{\pi^2}{a^2}\frac{(\nu+1/2)}{\sin^2(\pi x/a)}\left[\nu + \frac{1}{2} + \cos\left(\frac{\pi}{a}x\right)\right]. \tag{57}$$

Near the origin, $V_-(x) \sim [(\nu+1)^2 - 1/4]/x^2$, with a strongly repulsive coupling, while near $x = a$ it goes as $(\nu^2 - 1/4)/(a-x)^2$. In fact, the partner potentials (53) and (57) are the same, they are simply reflected about $x = a/2$. By itself, solving the Schrödinger equation for $V_-(x)$ leads to the same ambiguity for the spectrum, but the ambiguity is resolved after we relate the partner wave functions. For the UV case, we have

$$\psi^{-{\rm UV}}_{\nu,n}(x) = \frac{1}{k_n}Q\,\psi^{+{\rm UV}}_{\nu,n}(x) \sim C^{\rm UV}_{\nu,n}x^{\nu+\frac{3}{2}}[1 + O(x^2)]. \tag{58}$$

It is possible to show that $\psi^{-{\rm UV}}_{\nu,n}$ is normalized if $\psi^{+{\rm UV}}_{\nu,n}$ is normalized, and both generate the same spectrum (55). Meanwhile, the fermionic partner of the IR solution,

$$\psi^{-{\rm IR}}_{\nu,n} \propto Q\,\psi^{+{\rm IR}}_{\nu,n}(x) \sim C^{\rm IR}_{\nu,n}\,x^{-\nu-\frac{1}{2}}[1 + O(x^2)], \tag{59}$$

is clearly not square integrable at the origin. Therefore the IR spectrum is not consistent with SUSY, which selects the UV fixed point.

V. DISCUSSION

Our main result is the proof that there is a SUSY in the parameter space of inverse square potentials with coupling α. It relates the weak-medium ($-\frac{1}{4} \leqslant \alpha < \frac{3}{4}$) and the strongly repulsive ($\alpha \geqslant \frac{3}{4}$) ranges of the potential. As a consequence of this relation, SUSY removes the anomalous scale that appears in the quantization of the weak-medium range, by setting $L = 0$ and forbidding the renormalized theory to leave the UV fixed point of its RG flow. Meanwhile, in the strongly attractive range, there is no SUSY description, since the would-be partner-potentials of (1) for $\alpha < -\frac{1}{4}$ become complex functions and the Hamiltonians are non-Hermitian.

Hence *dynamical SUSY is a property of the the potential (1) only before and at the BKT-like phase transition; it disappears completely at the discrete-scaling phase*. We consider the existence of a dynamical SUSY, by itself, to be a noteworthy fact about the renormalized Hamiltonian (1).

The inverse square potential with a weak-medium coupling appears in many different contexts, and our argument for fixing $L = 0$ results in rather nontrivial consequences. The first example is the radial scattering of a charged, nonrelativistic, spinless particle by a thin solenoid (the Aharonov-Bohm effect). One of the present authors has shown that in this scenario up to two phase shifts have to be renormalized [29], introducing up to two anomalous quantum scales. The results of the present paper, however, strongly suggest that these scales should be set to zero, thus recovering the usual formula for the cross-section after SUSY is imposed.

The Schrödinger equation (3) also describes fluctuations of asymptotically AdS domain walls. In the gauge–gravity correspondence, the spectrum of (1) is related to the mass spectrum of particles living the AdS boundary and controlled by the bulk geometry which can be singular [13–15]. In this context, the ambiguity of the renormalized solutions in the weak-medium coupling range, here codified in the scale L, are related to the necessity of assigning holographic boundary conditions at a singularity (not at the AdS boundary), which is an unphysical situation. In a recent paper [16], three of the present authors have shown that the SUSY transformation for the fluctuations corresponds to a symmetry of the bulk $d+1$-dimensional domain wall relating large and small scales. With the appropriate translation, the same arguments presented here can be used to fix $L = 0$, then fix the ambiguities in the boundary conditions discussed in Refs. [13–15].

We would like to finally make a comment about the similarity between the inverse-square potential and the two-dimensional Dirac-delta potential.[7] The latter is another famous case of how the RG equations and dimensional transmutation can appear in nonrelativistic quantum mechanics [32,42,43]. In two dimensions, a Dirac-delta potential $V(\mathbf{x}) = \lambda\delta^2(\mathbf{x})$ has the same scaling dimension as the kinetic term, and the Hamiltonian is classically scale invariant. When $\lambda > 0$ and the potential is repulsive, this symmetry is preserved upon quantization. But for an attractive potential, with $\lambda < 0$, the existence of a bound state leads to spontaneous breaking of scaling symmetry and introduces an anomalous logarithmic correction of the scattering cross-section. These effects, which Jackiw has called "doubtlessly the most elementary manifestation of quantum mechanical symmetry breaking," [43] are very similar to the ones in the inverse-square potential after the BKT-like phase transition. Such similarity is made even stronger after SUSY fixes scaling invariance of the whole region with $\alpha \geqslant -\frac{1}{4}$, providing a clear-cut division into only *two* (rather than three) different regions of parameter space—where scaling symmetry is either necessarily broken or it is not broken at all.

[7] We thank an anonymous referee for bringing this point to our attention.

APPENDIX: RENORMALIZED SOLUTIONS FOR $\alpha \leqslant -1/4$

In the strongly repulsive regime, the Bessel index becomes imaginary, $\nu = i\tilde{\nu}$ with $\alpha = -(\tilde{\nu} + \frac{1}{4})$. The renormalized zero-energy solution is

$$\psi_{\tilde{\nu};0}(x) = C_{\tilde{\nu}}(0)\sqrt{x}\sin[\tilde{\nu}\ln(x/\tilde{L}) + \delta], \tag{A1}$$

with a length scale $\tilde{L}$ and a dimensionless integration constant δ which cannot be fixed by a boundary condition. By the same steps as before, one finds the running coupling [25]:

$$\gamma(R) = \tilde{\nu}\cot[\tilde{\nu}\ln(R/\tilde{L}) + \delta]. \tag{A2}$$

The shallow bound states referred to in the main text can be found by looking at solutions for $E < 0$ which are regular at $x = \infty$; we find

$$\psi_{\tilde{\nu}}(x) \propto K_{i\tilde{\nu}}(\kappa x) \sim \sin[\tilde{\nu}\ln(\kappa x/2) + \theta_{\tilde{\nu}}],$$
$$\kappa = \sqrt{-2mE/\hbar^2}, \quad \theta_{\tilde{\nu}} = \frac{1}{2i}\ln[\Gamma(1 - i\tilde{\nu})/\Gamma(1 + i\tilde{\nu})],$$

where we used $\kappa x \ll 1$. Continuity then results in $\kappa_n = \kappa_0 e^{-\frac{\pi}{\tilde{\nu}}n}$, $\kappa_0 \equiv (2/\tilde{L})e^{(\delta-\theta_{\tilde{\nu}})/\tilde{\nu}}$, $n \in \mathbb{Z}$. This is the discrete energy spectrum $E_n \sim -\kappa_n^2$.

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