Aharonov-Bohm effect for bound states from the interaction of the magnetic quadrupole moment of a neutral particle with axial fields

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The behavior of a neutral particle with magnetic quadrupole moment that interacts with axial magnetic and electric fields is analyzed. From the interaction of the magnetic quadrupole moment with the axial magnetic field, a spectrum of energy analogous to a Coulomb potential in two dimensions is obtained. Furthermore, the presence of an axial electric field is also considered. From the interaction of the magnetic quadrupole moment with this electric field, an analog of the Aharonov-Bohm effect is obtained. Finally, the Aharonov-Bohm effect for bound states is analyzed when this neutral particle system is subject to the two-dimensional harmonic oscillator.

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I. INTRODUCTION

The interaction of an electron with a uniform magnetic field gives rise to a discrete spectrum of energy, where this system has become the simplest model for studying the quantum Hall effect [1]. This interaction has also been taken into account in condensed matter systems with the purpose of studying the magnetization, for instance, in quantum rings [2–4] and quantum dots [5,6]. In the current literature, this discrete spectrum of energy is known as Landau levels [7]. Furthermore, this interaction has inspired works with neutral particles. For instance, by considering the interaction between a neutral particle with an induced electric dipole moment and a field configuration determined by a uniform magnetic field and a nonuniform electric field, an analog of the Landau quantization has been dealt with in Refs. [8–11]. Moreover, an analog of the doubly anharmonic oscillator has been obtained in Ref. [12]. Another perspective of this neutral particle system has been given in Ref. [13], where a geometric quantum phase is obtained. The arising of a geometric quantum phase in the wave function of the neutral particle corresponds to an Aharonov-Bohm-type effect [14,15].

Furthermore, the interaction of quantum particles with nonuniform magnetic fields has also drawn attention in recent decades. An interesting example is the He-McKellar-Wilkens effect [16,17]. It is the arising of a geometric quantum phase from the interaction of the permanent electric dipole moment of a neutral particle with a radial magnetic field proportional to the inverse of the radial distance. Later, by considering a radial magnetic field proportional to the radial distance, it is shown in Refs. [18–20] that analogs of the Landau levels can be obtained from the interaction of the permanent electric dipole moment of the neutral particle with this nonuniform magnetic field. Another point of view of working with nonuniform magnetic fields has been given in Ref. [21]. There, a radial magnetic field produced by a nonuniform magnetization parallel to the symmetry axis of a long cylindrical wire is considered. Then, a geometric quantum phase is obtained from the interaction of the electric dipole moment of a neutral particle with this nonuniform magnetic field. This idea of having a magnetic field produced by a nonuniform magnetization has also been explored in Ref. [22]. In this case, it is considered an axial magnetic field. Then, bound-state solutions to the Schrödinger-Pauli equation are obtained from the interaction between a neutral particle with permanent magnetic dipole moment and a field configuration given by a nonuniform electric field and the axial magnetic field. Another neutral particle system, whose interaction with a nonuniform magnetic field has been explored, is given by a neutral particle with electric quadrupole moment [23]. In this neutral particle system, a geometric quantum phase has been obtained from the interaction of the electric quadrupole moment with a nonuniform axial magnetic field produced by an electric current density.

Hence, neutral particle systems in the presence of nonuniform magnetic fields have drawn attention to the possibility of achieving analogs of the Landau levels or analogs of the Aharonov-Bohm effect [14,15]. In this work, we go further in the studies of neutral particle systems in the presence of a nonuniform magnetic field in search of analogs of the Aharonov-Bohm effect [15]. We consider a system of a neutral particle that possesses a magnetic quadrupole moment. This particular neutral particle system has brought attention in the context of quantum mechanics due to the work of Chen [23], where it is shown that the interaction of the magnetic quadrupole moment with an induced electric field can yield the appearance of a geometric quantum phase. Since then, quantum effects associated with the interaction of the magnetic quadrupole moment of a neutral particle with external fields has been investigated in the literature [24–32]. Therefore, in the present work, we show that the interaction of the magnetic quadrupole moment of the neutral particle with a nonuniform axial magnetic field can give rise to a spectrum of energy analogous to the Coulomb potential (in two dimensions). In the following, we also consider the

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presence of an induced electric field, and thus we analyze an analog of the Aharonov-Bohm effect for bound states [15]. Finally, we analyze the Aharonov-Bohm effect for bound states by including the two-dimensional harmonic oscillator.

The structure of this paper is as follows: In Sec. II, we introduce the Schrödinger equation that describes the interaction of the magnetic quadrupole moment of a neutral particle with external fields. Then, we search for bound-state solutions to the Schrödinger equation when the magnetic quadrupole moment interacts with a nonuniform axial magnetic field. In Sec. III, we consider the presence of an induced electric field and the nonuniform axial magnetic field. Then, we analyze the Aharonov-Bohm effect for bound states. In Sec. IV, we include the two-dimensional harmonic oscillator. We search for bound-state solutions to the Schrödinger equation and discuss the Aharonov-Bohm effect for bound states. In Sec. V, we present our conclusions.

II. INTERACTION WITH A NONUNIFORM MAGNETIC FIELD

Great interest in chemical physics [33–50] and quantum physics [24–28,30] on spinless particles (atoms or molecules) with a magnetic quadrupole moment has been reported in the literature. From Refs. [51,52], the potential energy of atoms and molecules with a magnetic quadrupole moment in their rest frame is $U_m = -\sum_{i,j} M_{ij}\,\partial_i B_j$, where $\vec{B}$ is the magnetic field and M_{ij} is the magnetic quadrupole moment tensor. Besides, the tensor M_{ij} is a symmetric and traceless. Recently, the quantum description of a spinless particle with a magnetic quadrupole moment when it moves with velocity $v \ll c$ (c is the velocity of light) has been discussed [30–32]. The time-independent Schrödinger equation that describes the interaction of the magnetic quadrupole moment of the spinless particle with magnetic and electric fields is [30–32]

$$\mathcal{E}\psi = \frac{1}{2m}\left[\hat{p} - \frac{1}{c^2}(\vec{M}\times\vec{E})\right]^2\psi - \vec{M}\cdot\vec{B}\,\psi. \tag{1}$$

Observe that the components of the vector $\vec{M}$ are determined by $M_i = \sum_j M_{ij}\,\partial_j$, and the fields $\vec{B}$ and $\vec{E}$ are the electric and magnetic fields in the laboratory frame, respectively.

In this section, we analyze a scalar potential proportional to the inverse of the radial distance that stems from the interaction of the magnetic quadrupole moment of a spinless particle with a nonuniform magnetic field. Let us consider a medium with a current density $\vec{J} = -\frac{B_0}{r}\,\hat{\varphi}$, where $B_0 > 0$ is a constant and r is the radial coordinate [23]. By dealing with the cylindrical symmetry, this current density produces the magnetic field

$$\vec{B} = B_0 \ln\frac{r}{r_0}\hat{z}, \tag{2}$$

where r_0 is a constant. As pointed out in Ref. [23], since the current density $\vec{J}$ decreases for a large r, we have that any effect from collisions can be neglected; hence, we simplify the system to a single-particle problem.

Henceforth, we work with the units $\hbar = 1$ and $c = 1$. Let us assume that the non-null components of the tensor M_{ij} are given by

$$M_{rz} = M_{zr} = M, \tag{3}$$

where M is a constant ($M > 0$). In this way, we have that the last term of the right-hand side of Eq. (1) yields

$$V_{\text{eff}}(r) = -\vec{M}\cdot\vec{B} = -\frac{M\,B_0}{r}. \tag{4}$$

Thereby, the interaction of the magnetic quadrupole moment given in Eq. (3) with the nonuniform magnetic field (2) gives rise to a scalar potential proportional to the inverse of the radial distance. Since $B_0 > 0$ and $M > 0$, we have that the effective scalar potential (4) plays the role of an attractive scalar potential. Furthermore, the time-independent Schrödinger equation (1) becomes (with $\hbar = 1$ and $c = 1$)

$$\mathcal{E}\psi = -\frac{1}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\right]\psi - \frac{M\,B_0}{r}\,\psi. \tag{5}$$

Let us write $\psi(r,\varphi,z) = Z(z)\Phi(\varphi)f(r)$; then, by substituting ψ into Eq. (5) we obtain that $Z(z) = e^{ikz}$ and $\Phi(\varphi) = e^{il\varphi}$, where $-\infty < k < \infty$ and $l = 0, \pm1, \pm2, \ldots$. Besides, for the function $f(r)$, we have

$$f'' + \frac{1}{r}f' - \frac{l^2}{r^2}f + \frac{MB_0}{r}\,f + \zeta^2 f = 0, \tag{6}$$

with $\zeta^2 = 2m\mathcal{E} - k^2$. Note that the asymptotic behavior of the radial equation when $r \to \infty$ is determined by $f'' \approx -\zeta^2 f$. Therefore, in search of bound-state solutions, let us consider $\zeta^2 = -\tau^2$ and rewrite Eq. (6) in the form

$$f'' + \frac{1}{r}f' - \frac{l^2}{r^2}f + \frac{MB_0}{r}\,f - \tau^2 f = 0. \tag{7}$$

By defining $x = 2\tau\,r$, the radial equation (7) becomes

$$f'' + \frac{1}{x}f' - \frac{l^2}{x^2}f + \frac{\delta}{x}f - \frac{1}{4}f = 0, \tag{8}$$

where we have defined the parameter $\delta = \frac{MB_0}{2\tau}$. Thereby, by imposing that $f(x) \to 0$ when $x \to \infty$ and $x \to 0$, the solution to Eq. (8) is given by

$$f(x) = e^{-\frac{x}{2}}x^{|l|}\,F(x), \tag{9}$$

where $F(x)$ is an unknown function. By substituting Eq. (9) into Eq. (8), we obtain the following equation for the function $F(x)$:

$$xF'' + [2|l| + 1 - x]F' + \left[\delta - |l| - \tfrac{1}{2}\right]F = 0. \tag{10}$$

Hence, Eq. (10) corresponds to the Kummer equation or the confluent hypergeometric equation [53], and thus $F(x) = {}_1F_1\left(|l| + \frac{1}{2} - \delta,\, 2|l| + 1,\, x\right)$ is the confluent hypergeometric function. Its behavior for large values of its argument is given by [53]

$$\underset{1}{{}_1F}\,(a,\, b; x) \approx \frac{\Gamma(b)}{\Gamma(a)}\,e^x\,x^{a-b}[1 + O(|x|^{-1})]; \tag{11}$$

therefore, it diverges when $x \to \infty$. With the aim of having $f(x) \to 0$ when $x \to \infty$, we must impose that $a = -n$ ($n = 0, 1, 2, 3, \ldots$), i.e., $|l| + \frac{1}{2} - \delta = -n$. With this condition, the confluent hypergeometric function becomes well behaved when $x \to \infty$. In this way, with $(-\tau^2) = 2m\mathcal{E} - k^2$ and from the condition $|l| + \frac{1}{2} - \delta = -n$, we obtain

$$\mathcal{E}_{n,\,l} = -\frac{1}{8m}\frac{M^2 B_0^2}{[n + |l| + 1/2]^2} + \frac{k^2}{2m}. \tag{12}$$

Hence, the interaction of the magnetic quadrupole momentum (3) with the magnetic field (2) yields a discrete spectrum of energy analogous to the Coulomb potential. This occurs due to the presence of the effective scalar potential (4) that stems from this interaction and plays the role of an attractive Coulomb-type potential. Observe that this discrete set of energy levels is achieved for large values of the radial distance r. This occurs due to the effects of the electric current density $\vec{J}$ that disturbs the system. Since $\vec{J}$ vanishes for large values of r, the bound states can only be achieved in this particular case when the value of r is large enough in such a way that we can neglect $\vec{J}$. This agrees with Ref. [23], where it is shown that a geometric quantum phase is obtained through a path that possesses a large contour.

It is worth noting that this behavior of achieving the discrete set of energy levels for large values of the radial distance r has already been reported in the literature. Reference [54] pointed out an analogous behavior with respect to an electron that interacts with a uniform magnetic field in the presence of an antidot potential. In the Tan-Inkson model [54], the antidot potential is proportional to r^{-2}; then it is shown that the Landau levels [7] are disturbed by the antidot potential for small values of the radial distance. Therefore, the (unperturbed) Landau levels are obtained for large values of the radial distance r.

III. AHARONOV-BOHM EFFECT FOR BOUND STATES

Recently, a linear electric field parallel to a uniform magnetic field was considered in Ref. [55]. There, it was shown that the presence of this axial electric field modifies the degeneracy of the Landau levels. In this section, we bring another perspective. We consider an electric field parallel to the axial magnetic field (2). By contrast, the electric field is produced by a time-dependent magnetic field. Then, we show that an analogous effect of the Aharonov-Bohm effect for bound states [15,30] can occur in the magnetic quadrupole system analyzed in the previous section. Let us discuss the case where the components of the magnetic quadrupole moment are given by

$$\begin{aligned} M_{rz} &= M_{zr} = M, \\ M_{rr} &= M_{\varphi\varphi} = M, \\ M_{zz} &= -2M, \end{aligned} \tag{13}$$

where M is a constant ($M > 0$). This magnetic quadrupole moment of the neutral particle interacts with the axial magnetic field (2) and also with the electric field [23]:

$$\vec{E} = E_0 \ln\frac{r}{r_0}\,\hat{z}. \tag{14}$$

This electric field is produced by a time-dependent magnetic field $\vec{B} = \frac{E_0\,t}{r}\,\hat{\varphi}$, where E_0 is a constant ($E_0 > 0$). As shown in Ref. [23], the interaction of the induced electric field (14) with the magnetic quadrupole moment given in Eq. (13) yields the geometric quantum phase:

$$\phi_1 = \oint \vec{A}_{\text{eff}} \cdot d\vec{r} = \oint (\vec{M} \times \vec{E}) \cdot d\vec{r} = -2\pi\, M\, E_0. \tag{15}$$

Discussions about the topological nature of the geometric quantum phase ϕ_1 can be found in Refs. [16,17,23,56–63]. In this way, we can write the effective vector potential given in wave equation (1) in the form

$$\vec{A}_{\text{eff}} = \vec{M} \times \vec{E} = -\frac{\phi_1}{2\pi\,\rho}\,\hat{\varphi}. \tag{16}$$

Before analyzing the time-independent Schrödinger equation (1), it is worth noting that the time-dependent magnetic field does not interact with the magnetic quadrupole moment (13); therefore, only the magnetic field (2) does interact with the magnetic quadruple moment (13). Thereby, the time-independent Schrödinger equation (1) becomes (with $\hbar = 1$ and $c = 1$)

$$\begin{aligned} \mathcal{E}\psi = &-\frac{1}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\right]\psi \\ &-\frac{i}{m}\frac{\phi_1}{2\pi\,r^2}\frac{\partial\psi}{\partial\varphi} + \frac{1}{2m}\left(\frac{\phi_1}{2\pi\,r}\right)^2\psi - \frac{M\,B_0}{r}\,\psi. \end{aligned} \tag{17}$$

By following the steps from Eq. (5) to Eq. (12), we must replace l with $l + \frac{\phi_1}{2\pi}$ from Eq. (6) to Eq. (10), and thus we obtain the energy levels:

$$\mathcal{E}_{n,\,l} = -\frac{1}{8m}\frac{M^2 B_0^2}{\left[n + \left|l + \frac{\phi_1}{2\pi}\right| + 1/2\right]^2} + \frac{k^2}{2m}. \tag{18}$$

Observe that the energy levels (18) depend on the geometric quantum phase ϕ_1. As we have seen in Eq. (15), this geometric quantum phase arises from the interaction of the magnetic quadrupole moment of the neutral particle (13) with the axial electric field (14). This dependence of the energy levels on the geometric quantum phases is an analog of the Aharonov-Bohm effect for bound states [15,30]. As discussed in the previous section, the spectrum of energy (18) is valid for large values of r since the electric current density $\vec{J}$ disturbs the system, but it can be neglected for large values of r.

IV. INTERACTION WITH THE TWO-DIMENSIONAL HARMONIC OSCILLATOR

Let us consider the magnetic field (2), the electric field (14), and the magnetic quadrupole moment (13). Thus, let us include the two-dimensional harmonic oscillator potential $V(r) = \frac{1}{2}\,m\omega\,r^2$. By following the steps from Eq. (5) to Eq. (6), the radial equation becomes (with $\hbar = 1$ and $c = 1$)

$$f'' + \frac{1}{r}f' - \frac{\left(l + \frac{\phi_1}{2\pi}\right)^2}{r^2}f + \frac{M\,B_0}{r}f - m^2\omega^2\,r^2\,f + \zeta^2\,f = 0. \tag{19}$$

Next, let us perform the change of variables $y = \sqrt{m\omega}\,r$; thus, we have in Eq. (19):

$$f'' + \frac{1}{y}R' - \frac{\left(l + \frac{\phi_1}{2\pi}\right)^2}{y^2}f + \frac{\alpha}{y}f - y^2\,f + \frac{\zeta^2}{m\omega}f = 0, \tag{20}$$

where we have defined the following parameter:

$$\alpha = \frac{M B_0}{\sqrt{m\omega}}. \tag{21}$$

Let us also impose that $f(y) \to 0$ when $y \to 0$ and $y \to \infty$. Thereby, the solution to Eq. (21) can be written in terms of an unknown function $H(y)$ as

$$f(y) = e^{-\frac{y^2}{2}}\, y^{|l+\frac{\phi_1}{2\pi}|}\, H(y). \tag{22}$$

By substituting Eq. (22) into Eq. (20), we have that the function $H(y)$ is the solution to the biconfluent Heun equation [32,64]:

$$H'' + \left[\frac{2\left|l + \frac{\phi_1}{2\pi}\right| + 1}{y} - 2y\right]H' + \left[\nu + \frac{\alpha}{y}\right]H = 0, \tag{23}$$

where $\nu = \frac{\zeta^2}{m\omega} - 2 - 2|l + \frac{\phi_1}{2\pi}|$. Thereby, $H(y) = H(2|l + \frac{\phi_1}{2\pi}|,\, 0,\, \frac{\zeta^2}{m\omega},\, 2\alpha,\, -y)$ is the biconfluent Heun function.

Henceforth, we focus on the search for polynomial solutions to Eq. (23). For this purpose, let us write the solution to Eq. (23) as power-series expansion around the origin: $H(y) = \sum_{j=0}^{\infty} a_j\, y^j$ [7,65,66]. Then, by substituting it into Eq. (23), we obtain the relation

$$a_1 = -\frac{M B_0}{\sqrt{m\omega}(2|l + \frac{\phi_1}{2\pi}| + 1)}\, a_0, \tag{24}$$

and also the recurrence relation

$$\begin{aligned} a_{j+2} = &-\frac{\alpha}{(j+2)\left(j+2+2\left|l+\frac{\phi_1}{2\pi}\right|\right)}\, a_{j+1} \\ &-\frac{(\nu - 2j)}{(j+2)\left(j+2+2\left|l+\frac{\phi_1}{2\pi}\right|\right)}\, a_j. \end{aligned} \tag{25}$$

From the recurrence relation (25), we have that a polynomial of degree n is built when we impose that

$$a_{n+1} = 0, \tag{26}$$

and

$$\nu = 2n, \tag{27}$$

where $n = 1, 2, 3, 4, \ldots$. Besides, n corresponds to the quantum number associated with the radial modes. In search of bound-state solutions, let us construct a polynomial of first degree to the function $H(y)$. For $n = 1$, we have that the condition (26) yields $a_{n+1} = a_2 = 0$. By using Eqs. (24) and (25) we can calculate the coefficient a_2 and thus obtain the relation

$$\omega_{1,\,l} = \frac{M^2 B_0^2}{2m\left(2\left|l + \frac{\phi_1}{2\pi}\right| + 1\right)}. \tag{28}$$

Relation (28) yields the allowed values of the angular frequency of the two-dimensional harmonic oscillator that permit us to obtain a polynomial of first degree to $H(y)$. The label $\omega_{n,\,l}$ used in Eq. (28) means that each radial mode n yields a different set of allowed values for the angular frequency. Therefore, not all values of the angular frequency are allowed for a polynomial of first degree, just those determined by relation (28).

Next, by taking $n = 1$ in condition (27), we obtain

$$\mathcal{E}_{1,\,l,\,k} = \omega_{1,\,l}\left[\left|l + \frac{\phi_1}{2\pi}\right| + 2\right] + \frac{k^2}{2m}. \tag{29}$$

Then, by using Eq. (28), we have

$$\mathcal{E}_{1,\,l,\,k} = \frac{M^2 B_0^2}{2m\left(2\left|l + \frac{\phi_1}{2\pi}\right| + 1\right)}\left[\left|l + \frac{\phi_1}{2\pi}\right| + 2\right] + \frac{k^2}{2m}. \tag{30}$$

Hence, when the magnetic quadrupole moment (13) interacts with the magnetic field (2) and the electric field (14) in the presence of the two-dimensional harmonic oscillator, we obtain the allowed energies (30) associated with the radial mode $n = 1$. These allowed energies are achieved by searching for a polynomial of first degree to the function $H(y)$. For other values of the radial mode n, other expressions for the allowed energies can be obtained. Furthermore, we can see that the allowed energies (30) depend on the geometric quantum phase ϕ_1, which gives rise to an analog of the Aharonov-Bohm effect for bound states [15,30]. We also need to observe that the allowed energies (30) are achieved for large values of the radial distance r because of the electric current density $\vec{J}$ that disturbs the system.

V. CONCLUSIONS

We have obtained bound-state solutions to the Schrödinger equation that describe the interaction of the magnetic quadrupole moment of a neutral particle with axial magnetic and electric fields. We have started by analyzing the interaction of the magnetic quadrupole momentum (3) with an axial magnetic field (2), where this axial magnetic field is produced by a nonuniform electric current density. We have seen that a spectrum of energy analogous to the Coulomb potential in two dimensions can be achieved.

We have gone further by considering the presence of an axial electric field and a different magnetic quadrupole moment tensor. This axial electric field is produced by a time-dependent magnetic field. First of all, we have seen that this magnetic quadrupole moment does not interact with the time-dependent magnetic field. On the other hand, there was the interaction of the magnetic quadrupole moment with the induced electric field. From this interaction with the axial electric field, we have observed the appearance of a geometric quantum phase. Therefore, by solving the Schrödinger equation for the interaction of the magnetic quadrupole moment of the neutral particle (13) with the axial electric field (14) and the axial magnetic field (2), we have also obtained a spectrum of energy analogous to the Coulomb potential in two dimensions. Besides, these energy levels depend on the geometric quantum phase, which corresponds to an analog of the Aharonov-Bohm effect for bound states [15,30].

Finally, we have analyzed the interaction of the magnetic quadrupole moment (13) with the magnetic field (2) and the electric field (14) subject to the harmonic oscillator potential. Then, by searching for polynomial solutions to the biconfluent Heun equation, we have obtained the allowed energies associated with the radial mode $n = 1$. We have seen that these allowed energies depend on the geometric quantum phase;

hence, there is an analog effect of the Aharonov-Bohm effect for bound states [15,30].

We need to observe that in all cases discussed in this work, the bound states can only be achieved for large values of the radial distance r. Since the axial magnetic field (2) is produced by an electric current density proportional to r^{-1}, then this electric current density can disturb the system. Therefore, the bound states can only be achieved for large values of the radial distance r, where the electric current density can be neglected.

It is worth noting the possibility of examining quantum effects on the interaction of the magnetic quadrupole moment of a neutral particle with these axial magnetic and electric fields in an elastic medium that possesses disclinations or dislocations [67,68]. In recent decades, it has been shown that the topology of disclinations and dislocations can modify the electronic properties of the elastic medium and raises discussion about the Aharonov-Bohm effect [69–82].

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