Lévy flights for light in ordered lasers

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Lévy flights for light have been demonstrated in disordered systems with and without optical gain and remained unobserved in ordered ones. In the present article, we investigate, numerically and experimentally, Lévy flights for light in ordered systems due to an ordered (conventional) laser. The statistical analysis was performed on the intensity fluctuations of the output spectra upon repeated identical experimental realizations. We found out that the optical gain and the mirror reflectivity are critical parameters governing the fluctuation statistics. We identified Lévy regimes for gain around the laser threshold, and Gaussian-Lévy-Gaussian crossovers were unveiled when increasing the gain from below to above threshold. The experimental results were corroborated by Monte Carlo simulations, and the fluctuations were associated to a Langevin noise source that takes into account the randomness of the spontaneous emission, which seeds the laser emission and can cause large fluctuations of the output spectra from shot to shot under identical experimental realizations.

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I. INTRODUCTION

Lévy-stable is an interdisciplinary study that has been used to describe problems such as geology, finance, and animal foraging [1]. The central limit theorem states that the sum of random variables with finite variance results in Gaussian probability distributions. Instead, the Lévy-stable is explained by the generalized central limit theorem, where the averages may or not be finite, but the variance diverges. The probability to observe a variable x in the limit $x \to \infty$ is proportional to $x^{-(1+\alpha)}$; Gaussian distributions for $\alpha \geqslant 2$, and Lévy distributions for $0 < \alpha < 2$.

In the photonic context, Lévy flights were reported for light propagating in disordered materials without [2] and with gain [3]. In the former regime, Lévy flights were observed for the transmitted light intensity through a specially designed material where the step-length distribution could be chosen by mixes of high-refractive-index scattering particles (TiO_2) and glass microspheres with tailored size distribution [2]. When introducing gain in the disordered materials, the multiple scattering of light provides feedback, and laser action can occur without a resonant cavity, which is called random laser [4]. Lévy distributions have been reported for the output intensity of random lasers: under identical experimental realizations the output spectra change stochastically from one realization to another, and the corresponding probability distributions are Gaussian or Lévy [5].

Despite the large quantity of articles concerned with the demonstration and understanding of Lévy distributions for the output intensity fluctuations of random lasers [5–8], to the best of our knowledge there is not any report in ordered systems. The aim of the present article is to investigate the Lévy distributions for the output intensity fluctuations of ordered (conventional) lasers using as prototype a Q-switched Nd:YAG. The experimental results are supported by Monte Carlo numerical simulations.

II. EXPERIMENT

The experiments were performed using a Q-switched Nd:YAG laser operating at 1064 nm with pulse duration of ~6 ns. The trivalent neodymium ions (Nd^{3+}) in the YAG crystal were optically excited by flashlamps having an applied voltage fixed at 1550 V with a frequency of 20 Hz. The optical gain was varied by the delay between the Q switch (QS) and the flashlamps, which means control of the initial density of ions in the upper level of the laser transition (N_0). The QS frequency was 10 Hz. Each time the QS was activated, the system was able to lase, and we name it a shot. Keeping the QS delay fixed, each shot is considered an identical realization of the experiment under fairly identical conditions, since the time between the laser shots (10 ms) is much larger than the spontaneous relaxation of the Nd^{3+}, which is on the order of 100 μs. The output spectra were acquired using a spectrometer equipped with a CCD and triggered with the QS. The CCD exposure time was 70 ms, which is smaller than the time between optical pulses, ensuring single-shot spectrum acquisition. The spectral range was from 350 to 1150 nm with a resolution of 2.0 nm. For each QS delay, 1000 spectra were acquired to perform the statistical analysis.

The laser characterization was performed by decreasing the QS delay, i.e., increasing N_0 from below the threshold (only spontaneous emission) to well above the threshold

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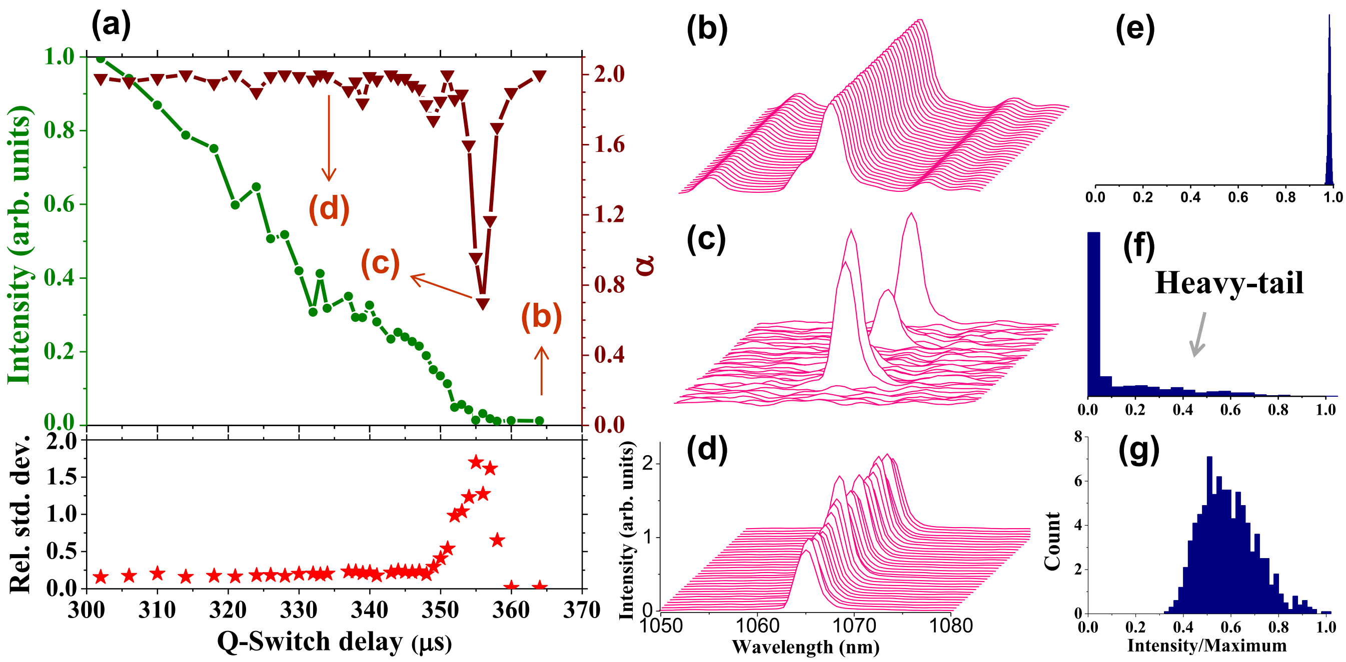

FIG. 1. Characterization of the output of a Nd:YAG laser over 1000 identical realizations for each gain value, which was controlled by the delay between the flashlamp and the Q switch: (a) average output intensity, Lévy exponent (α), and relative standard deviation as a function of the Q-switch delay; (b)–(d) 30 output spectra for the Q-switch delay values indicated in Fig. 1(a); (e)–(g) corresponding unnormalized probability distributions (histogram) for the output intensity fluctuations considering the 1000 spectra.

(laser regime). Figure 1(a) shows the average output intensity and the corresponding relative standard deviation (RSD) as a function of the QS delay. The first 30 spectra for each value of the QS delay marked in Fig. 1(a) are shown in Figs. 1(b)–1(d). For a Q-switch delay of 364 μs, the gain is below the laser threshold, and one has only spontaneous emission. The spectral intensity fluctuations from shot to shot [Fig. 1(b)] are mild and due to flashlamp intensity fluctuations, which are very small [less than 1%, bottom of Fig. 1(a)]. The corresponding probability distribution for the intensity fluctuations is Gaussian [Fig. 1(e)]. Increasing the gain by decreasing the QS delay to 356 μs, the gain reaches the threshold for laser emission and the spectral intensity fluctuation from shot to shot is very strong [Fig. 1(c)] with maximum RSD [Fig. 1(a)]. Noticeably, the corresponding probability distribution is non-Gaussian with a heavy tail, characteristics of Lévy distributions [Fig. 1(f)]. By increasing the gain further, the intensity fluctuations are smoothed [Fig. 1(d) for QS delay of 336 μs], the RSD decreases [Fig. 1(a)], and the shape of the probability distribution changes, recovering the Gaussian profile [Fig. 1(g)]. To analyze the Gaussian-Lévy-Gaussian crossovers in detail, the parameter α, determined by the curve-fitting procedure described in Ref. [9], is plotted as a function of the QS delay in Fig. 1(a). The Gaussian-Lévy-Gaussian transitions are evident as α changes from 2.0 to 0.7 at the onset of laser emission and from 0.7 back to 2.0 at larger gain.

III. MONTE CARLO SIMULATIONS

The transition from the fluorescence to the laser regime can be described by a simple coupled rate equation system for the population inversion and the energy at the laser transition, which is a deterministic system, and the fluctuations are not explained. On the other hand, in Monte Carlo simulations the fluctuations arise naturally. Based on previous outstanding publications on random lasers [6–8], we considered a one-dimensional system (1D), and the steps were as follows: (1) The active ions were in the upper level of the laser transition with an initial population density N_0 (gain). (2) The density of ions which spontaneously relaxes in a time interval dt is $N\tau dt$, where N is the density of excited ions in time t and τ is the lifetime of the energy level. We multiplied it by $\beta = 0.1$ to consider that not all of the excited ions contribute to photons at the cavity axis and to include nonradiative relaxations of the upper energy level of the laser transition. To mimic the linewidth of a real transition, the frequencies of the photons were due to a weighed random Cauchy distribution centered at the resonance frequency with FWHM of 100 (arbitrary units). (3) The spontaneously created photons whose direction coincides with the cavity axis have initial direction with 50/50 percent of probability to right/left. Once created, the photons propagate to right or left until the boundary of the 1D medium. (4) At each step of the photons in the amplifying region, they can create other photons by stimulated emission with a rate $\gamma N n_\omega$. (5) At the boundaries, there is probability δ which varies from 0.0 to 1.0 with the photon return to the gain medium, which can be understood as a measurement of the reflectivity of mirrors. Here $\delta = 1.0$ corresponds to a closed cavity, while for $\delta = 0.0$ there is a completely open system without feedback for laser action. Each time photons escape from the gain medium, the number of photons in each frequency is counted. Each photon evolves independently, and it is essential in the simulations to observe the return from the

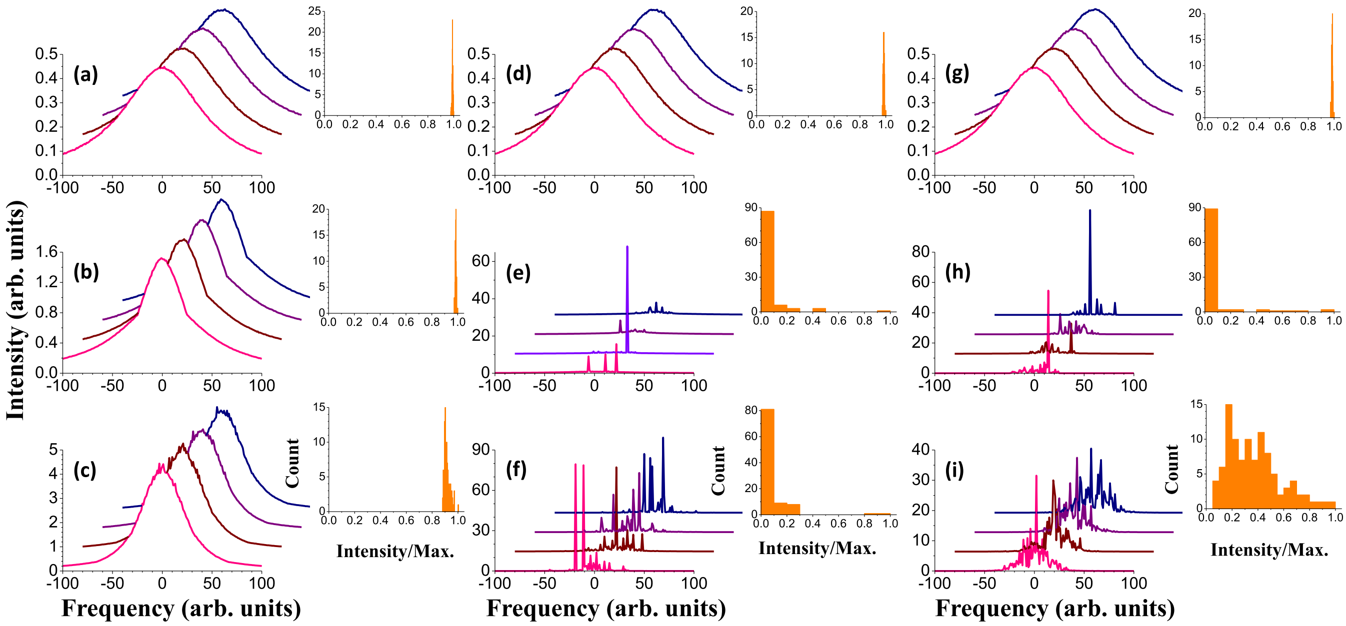

FIG. 2. Output spectra of an ordered laser generated by Monte Carlo simulations. The rows correspond to an initial population N_0 equaling 10^6, 2.5×10^6, and 5.5×10^6, respectively, and the columns correspond to mirror reflectivity (δ) of 0.0, 0.1, and 0.3, respectively. The corresponding unnormalized probability distributions (histogram) for the intensity fluctuations at the central frequency considering the 100 spectra for each pair of N_0 and δ are displayed in the insets.

Lévy to the Gaussian regime. For each pair of values of N_0 and δ, 100 spectra were simulated.

Some of the simulated spectra and the corresponding probability distributions (insets) are shown in Fig. 2. The first, second, and third column corresponding to δ equals 0.0, 0.1, and 0.3, respectively, while the rows corresponding to N_0 equal 10^6, 2.5×10^6, and 5.5×10^6. Increasing the gain for $\delta = 0.0$ [Figs. 2(a)–2(c)], we observe only a spectral narrowing and an increase in the output intensity that is characteristic of amplification of the spontaneous emission during the photon propagation in the gain medium [10]. Noticeably, the corresponding probability distributions are Gaussian [insets of Figs. 2(a)–2(c)]. On the other hand, for $\delta = 0.1$ the transition from the fluorescent to the laser regime is evident [Figs. 2(d) and 2(e)] by the appearance of narrow peaks with giant intensities [Fig. 2(e)] and from the crossover in the statistical regimes from Gaussian to a heavy tail (Lévy) [insets of Figs. 2(d) and 2(e)]. Finally, for $\delta = 0.3$ we observe the first transition from a smooth to a spiky spectrum with the change in the probability distributions from Gaussian [Fig. 2(g)] to Lévy [Fig. 2(h)]. At gain well above the threshold, there was a tendency of spike suppression with a lot of narrow peaks superimposed to the background of the spontaneous emission and the Gaussian probability distribution [Fig. 2(i)], in excellent agreement with the experimental results of Fig. 1. The spikes are not observed in the experimental results [Figs. 1(b)–1(d)] due to the low resolution of the spectrometer. However, the large-intensity lasing peaks and very small intensities (nonlasing) under the same experimental conditions observed in Fig. 1(c) are reproduced [Fig. 2(e)]. In the nonlasing spectra, the energy is distributed among the modes due to intrinsic competition for the available gain and/or dumped out as nonradiative relaxations and/or as photons whose directions are different from the cavity axis. These two last effects are accounted for by the β parameter we considered in the spontaneous emission [11].

The influence of the mirror reflectivity on the output intensity fluctuation of the laser was investigated numerically and is resumed in Fig. 3, which shows the parameter α as a function of the gain (N_0) for different values of δ. The transition from spontaneous emission to the laser regime is evident by the

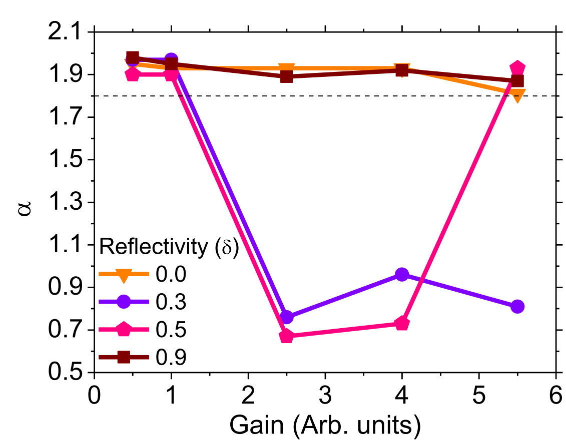

FIG. 3. Characterization of fluctuation statistics ($\alpha \geqslant 2.0$ – Gaussian, and $\alpha < 2.0$ – Lévy) for the output intensity of ordered lasers simulated by the Monte Carlo method. The α parameter is given as a function of the gain (N_0) for different values of the mirror reflectivity (δ). Due to the limited number of simulations for each pair of N_0 and δ, we considered the dashed line a guide to decide between Gaussian ($\alpha \geqslant 1.8$) or Lévy ($\alpha < 1.8$) distributions.

crossover from the Gaussian ($\alpha \geqslant 2.0$) to the Lévy regimes ($\alpha < 2.0$). (Due to the limited number of realizations, we considered Lévy regimes $\alpha < 1.8$, the dashed line in Fig. 3.) This first transition and the return to the Gaussian regime at large gain depend on δ. For $\delta = 0$ the distribution is always Gaussian due to the absence of feedback for laser action. For small values of $\delta \neq 0$, the return is slow. On the other hand, for $\delta = 0.5$ there is a fast return from the Lévy to the Gaussian regime, and for $\delta = 0.9$ the large fluctuations are suppressed, which agrees with recent results on specially designed laser cavities [12].

IV. DISCUSSION

At this point, an analogy with the behavior reported for random lasers is unveiled [8]: the mirror reflectivity plays a role analog to the scattering strength in random lasers. In random lasers, the scattering strength has a large influence on the observation of Lévy distributions when increasing the gain from below to well above the threshold. For low scattering strengths one has the transition from Gaussian to Lévy distributions at the threshold and a slow return to the Gaussian regime at larger gain. Upon increasing the scattering strength, there is a fast return to the Gaussian regime, and at much larger scattering strength there is no Gaussian-Lévy crossover when crossing the threshold, i.e., the probability distributions are Gaussian independently of the amount of optical gain. The emergence of the large fluctuations close to the random laser threshold is associated to the multimode nature of the random laser and the depletion of the gain by amplification of optical modes. The modes compete for the available gain, and only the "lucky" photons [13] are amplified, giving rise to peaks with large intensities in the output spectra under the Lévy regime. The randomness of scatterer positions in random lasers is not critical for the emergence of Lévy distributions [14]. The return from a Lévy to Gaussian regime at large gain and the absence of Gaussian-Lévy crossover at large scattering strengths are due to the spatial overlapping and temporal coincidence of a large number of optical modes which compete for the available gain, depleting it and suppressing strong amplifications.

Summing up, in the ordered laser (laser with cavity) of the present work, one has that for gain just above the threshold, just a few optical modes are amplified, and the rare large peak intensities are due to "lucky" photons [13] with random frequency, which take the gain and dominate the emitted spectrum. In each realization of the experiment under fairly identical conditions, the frequency of the "lucky" photon is a random variable weighted by the spontaneous emission curve [Fig. 2(e)]. Increasing the gain from below to above the threshold occurs during the first statistical distribution transition from Gaussian to Lévy. By increasing the gain further, there is a large spatial overlapping and temporal coincidence among the optical modes, which depletes the gain and leads to self-averaged Gaussian distributions. For δ closest to 1.0, the fraction of spontaneously emitted photons which are fed back to the gain volume is large, implying a large spatial overlapping and temporal coincidence of the optical modes even for gain in the vicinity of the threshold. Then the probability distributions are Gaussian in all regimes, i.e., fluorescent and lasing ($\alpha \approx 2.0$ for $\delta = 0.9$ in Fig. 3). In this context the intensity fluctuations can be understood by a *Langevin noise source* due to the randomness of the spontaneous emission, which seeds the laser emission and can cause large fluctuations of the output spectra from shot to shot under identical experimental realizations. Similar to random lasers, the emergence of the large fluctuations close to the laser threshold is associated with the multimode nature of the resonator and depletion of the gain by amplification of optical modes. The observation of Lévy or Gaussian statistics strongly depends on the spatial overlap and temporal coincidence of the optical modes, which can be controlled by the amount of gain and mirror reflectivity. Under this picture, large fluctuations are expected to disappear around the threshold of single longitudinal mode cavities due to the absence of mode interaction via competition for the available gain.

In connection with random lasers, the mirrors of the resonator can be thought of as scatterers, since each time photons arrive on them there are probabilities for scape of the resonator, or they can be backreflected, with direct influence on the dwell time of photons inside the resonator, and consequently, the spatial and temporal superposition of modes which compete for the available gain. The mirror reflectivity and the gain are critical parameters in determining the output fluctuation statistics, similar to the scattering strength and gain in random lasers, where photons can perform Lévy flights [8,14]. Due to these ordered and random laser similarities, we denominated the dynamics of the photons inside the ordered system of the present work as Lévy flights. It is worth mentioning that the term optical rogue waves has been used to describe noisy processes with long-tailed probability distributions under different physical underlying mechanisms involving soliton propagation, supercontinuum generation, and modulation instability, whose analogies with oceanic rogue waves are due to the nonlinear Schrödinger equation [15,16]. Notwithstanding the ocean rogue waves counterpart, the term rogue waves has also been invoked to describe strong fluctuations in lasers associated to, e.g., nonlinear transformations of random Gaussian fluctuations of the excitation source into long-tailed probability distributions [16,17]. In both scenarios, the analyses are performed in terms of the temporal evolution of the involved beams. On the other hand, the present investigation is done by means of repeated experimental realizations under the same initial conditions. Recently, intensity fluctuations in ordered lasers were deeply investigated in Refs. [10,18], aiming at the demonstration of replica symmetry breaking. The strong fluctuations were associated to the nonlinear interaction among the optical modes by the third-order nonlinear susceptibility of the gain medium. Here, the output fluctuations were described without taking into account the phases of the modes, i.e., only intensity feedback was considered.

V. CONCLUSION

In summary, the output intensity fluctuations of ordered lasers were investigated aiming at the probability distributions. The emergence of large fluctuations close to the laser threshold was associated to the multimode interaction via competition for the available gain. Lévy distributions were unveiled when the fraction of emitted photons fed back to the gain medium was small, and for gain around the threshold due to the "lucky" photons with random frequency, which take the gain and dominate the emitted spectrum. Below the threshold the distributions were Gaussian due to the absence of gain. Well above the threshold and for large mirror reflectivities (independent of the gain), the distributions were Gaussian due to the superposition of a large number of modes, which depletes the gain and suppresses strong amplifications. In an analogy with random lasers, in which the output fluctuations depend on the scattering strength and gain, and light is known to perform Lévy flights, we denominated the dynamics of the photons inside the ordered system of the present work as Lévy flights. The present results show that the Lévy distributions are not exclusive of disordered systems and can enlightening the discussion of optical mode competition for the available gain with nonlinear interaction among them and frustration induced by disorder in the photonic-magnetic analogy [19]. Finally, the present results open several possibilities for the connection between lasers and the statistical physics of complex systems, and signal that studies until now restricted for disordered (random) lasers can be performed in closed-cavity ones, like the recently demonstrated turbulence hierarchy in random lasers [20].

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