# Emptiness formation probability in one-dimensional Bose liquids

Hsiu-Chung Yeh[1] and Alex Kamenev[1,2]
[1]*School of Physics and Astronomy, University of Minnesota, Minneapolis, Minnesota 55455, USA*
[2]*William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, Minnesota 55455, USA*

(Received 5 November 2019; accepted 5 February 2020; published 28 February 2020)

We study the emptiness formation probability (EFP) in interacting one-dimensional Bose liquids, which is the probability that a snapshot of its ground state reveals exactly zero particles within the interval $|x| < R$. For a weakly interacting liquid there is parametrically wide regime $n^{-1} < R < \xi$ (here $n$ is the average density and $\xi$ is the healing length) where EFP exhibits a nontrivial crossover from the Poisson to the Gaussian behavior. We employ the instanton technique [*NATO Science Series II: Mathematics, Physics and Chemistry* (2004), Vol. 221] to study quantitative details of these regime and compare it with previously reported special cases.

DOI: 10.1103/PhysRevA.101.023623

## I. INTRODUCTION

Recent precision measurements of particle number fluctuations in ultracold quantum gases [1–3] have revived interest [4–7] in large-deviation statistics in many-body systems. Emptiness formation probability (EFP) is probably the most iconic and widely studied measure of such large deviations. It plays a special role in the theory of Bethe ansatz [8] integrable models [9–12] and is a test bed for the development of nonperturbative techniques, such as the instanton calculus [13]. The EFP, $P_{\text{EFP}}(R)$, is the probability that no particles are found within the space interval $(-R, R)$ in the ground state of a one-dimensional (1D) many-body system with average density $n$:

$$P_{\text{EFP}}(R) = \prod_{i=1}^{N} \int_{|x_i| \geqslant R} dx_i \, |\Psi_g(x_1, x_2, \ldots, x_N)|^2, \tag{1}$$

where $\Psi_g(x_1, x_2, \ldots, x_N)$ is the normalized ground-state wave function of an $N$-particle system. Even in integrable models, where $\Psi_g$ is known via the Bethe ansatz, the calculation of the multiple integral over the restricted interval is still a difficult task. A similar idea was first discussed in random matrix theory (RMT) [14], where the probability that no eigenvalues fall within a certain interval of energy spectrum for different ensembles was studied [15].

For integrable systems, the problem is often formulated in terms of spin-1/2 chains, where the EFP is defined as a probability of measuring $l$ consecutive spin to be "up" in the ground state of the chain. Via a Jordan-Wigner transformation, such a formulation is equivalent to the absence of quasiparticles on $l$ consecutive sites [16]. In these cases, EFP is found to be expressed in terms of Fredholm determinants [9,16–19]. Even though EFP can be related to known mathematical constructions, how to extract its asymptotic behavior is unclear in general cases. The exact answers are known so far only in a handful of isolated points in the parameter space [16,20].

This makes EFP an attractive playground for development of approximate asymptotic techniques. Most studies have been focusing on the regime $nR \gg 1$ (see, however, Ref. [21]), where EFP is exponentially small, $P_{\text{EFP}}(R) \ll 1$. In this limit the problem may be studied within the semiclassical instanton approximation, where $\ln P_{\text{EFP}}(R)$ is associated with (twice) the classical action along a certain dynamical trajectory of the Euler-Lagrange equations [13]. Such classical problems need to be solved with the boundary conditions imposed on both "past" and "future" boundaries, which makes techniques for initial-value problems fail, either analytically or numerically. Similar structures are known in the theory of rare events in classical stochastic systems [22–25].

In this work we focus on EFP in the repulsive Lieb-Liniger (LL) model [26], of spinless bosons with the repulsive delta-potential in 1D. The ground (and excited) states of the model may be written through the Bethe ansatz [26], and its thermodynamic characteristics are known exactly in terms of the microscopic parameters [26]. In particular, one may find the sound velocity $v_s$ and thus define the healing (or correlation) length as

$$\xi = (mv_s)^{-1}, \tag{2}$$

where $m$ is a mass of bosonic particles. In the limit of impenetrable interactions (the Tonks-Girardeau limit [27]), $\xi = (\pi n)^{-1}$, the model is equivalent to free fermions. Their (squared) ground-state wave function coincides with the joint probability distribution of eigenvalues in the circular unitary random matrix ensemble [28]. The exact answer for the free-fermion EFP is thus known from RMT [29,30]:

$$-\ln P_{\text{EFP}}(R) = \frac{1}{2}\frac{R^2}{\xi^2} + \frac{1}{4}\ln(R/\xi) + O(1). \tag{3}$$

Within the instanton approach the leading term here was derived by Abanov [31] through a beautiful application of the complex-valued functions theory. The only treatment away from the Tonks-Girardeau case that we are familiar with is Ref. [32], which conjectured EFP in the limit $\xi \ll R$ (see discussion below).

Our particular focus here is on the opposite limit of weakly interacting bosons. A defining feature of this regime is that the mean distance between the particles is much shorter than the correlation length, $n^{-1} \ll \xi$. As a result, there is a wide

 

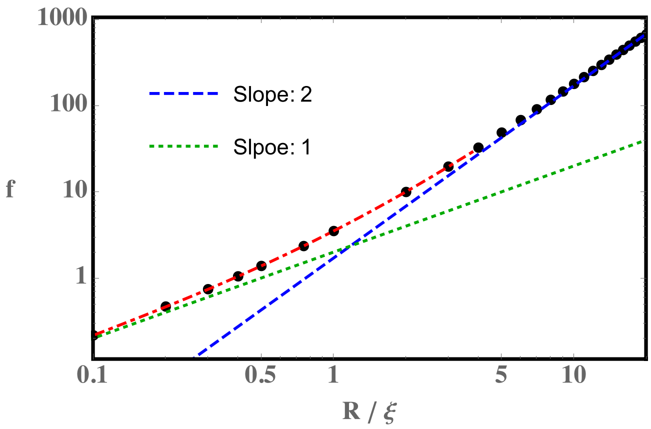


FIG. 1. Function $f(R/\xi)$, Eq. (4), for a weakly repulsive bosons on a log-log scale. The numerical results show a crossover for the exponent of EFP from linear (green dotted line) to quadratic (blue dashed line), and the red dot-dashed line is fit to the first few points.

range $n^{-1} < R < \xi$, which was not previously discussed in the literature.

Our main finding is that, through the entire range $n^{-1} \ll R$, the logarithm of EFP may be expressed as

$$-\ln P_{\rm EFP} = n\xi f\left(\frac{R}{\xi}\right), \tag{4}$$

where $f(r)$ is a universal function, as long as $n^{-1} \ll \xi$, plotted in Fig. 1. Its asymptotic limits are

$$f(r) \approx 2.01(4)r + 1.50(4)r^2 + O(r^3), \quad r \ll 1. \tag{5}$$

The leading term here is consistent with $P_{\rm EFP} \approx e^{-2Rn}$, which is the Poisson probability of finding the interval $2R$ empty of independent (i.e., noninteracting) randomly placed particles with the mean density $n$. Indeed, the limit $R \ll \xi$ is reached in the noninteracting case (i.e., $\xi \to \infty$). The latter is characterized by the uniform ground state $|\Psi_g|^2 = L^{-N}$, where $L = N/n$ is the system size. From Eq. (1), $P_{\rm EFP} = (\frac{L-2R}{L})^N \overset{N\to\infty}{\to} e^{-2Rn}$.

The other limit is

$$f(r) \approx 1.70(1)r^2 + 0.1(3)r + O(\ln r), \quad r \gg 1. \tag{6}$$

Now the leading term corresponds to the Gaussian EFP, $P_{\rm EFP} \approx \exp\{-1.7R^2 n/\xi\}$. The Gaussian large-$R$ asymptotic of the zero-temperature EFP may be argued on very general grounds [31]. The specific coefficient, found here for the weakly interacting limit, is new. It is at odds with the conjecture of Ref. [32], $-\ln P_{\rm EFP} = 4(R/\xi)^2$, which is parametrically inconsistent with our scaling, Eq. (4).

The term linear in $r$ in Eq. (6) is consistent with being zero. Indeed, in all cases with short-range interactions, where exact results are available [16,20], such a term is indeed absent. We believe that this is a generic feature of short-range interacting systems and provide a perturbative argument to that effect in Sec. III. Curiously, the Calogero-Sutherland model with the inverse-square long-range interactions exhibits a nonzero $O(r)$ term (i.e., $\approx R$ term in the large-$R$ asymptotic of $-\ln P_{\rm EFP}$) [14,33]. Our numerical accuracy is not sufficient to establish a coefficient of the $\ln R$ term in Eq. (6).

The paper is organized as follows: In Sec. II we formulate an instanton approach for calculating EFP for weakly interacting bosons. A numerical solution of the corresponding Euler-Lagrange equations, a discussion of the limiting cases, and a comparison with other works may be found in Sec. III. Appendix A presents a derivation of the hydrodynamic action in the Hamiltonian formalism and Appendix B is devoted to the free-fermion limit as a test-drive of our numerical procedure.

## II. INSTANTON CALCULUS FOR WEAKLY INTERACTING BOSONS

Here we adopt the hydrodynamic instanton approach to emptiness formation, developed by Abanov [31,34–37]. It is justified in the macroscopic emptiness regime, $n^{-1} \ll R$, where EFP is exponentially small. It is thus expected to be given by an optimal evolution trajectory in the space of the system's hydrodynamic degrees of freedom. In our case the latter are the local particle density, $\rho(x,t)$, and the local current, $j(x,t)$. The two are rigidly related by the continuity equation,

$$\partial_t \rho + \partial_x j = 0. \tag{7}$$

The classical action, which yields proper hydrodynamic equations as its extremal conditions, is given by [27]

$$S[\rho, j] = \iint dxdt \left[\frac{mj^2}{2\rho} - V(\rho)\right], \tag{8}$$

$$V(\rho) = \frac{c}{2}(\rho - n)^2 + \frac{(\partial_x \rho)^2}{8m\rho}. \tag{9}$$

The Lagrangian in Eq. (8) consists of the kinetic energy of the current along with the potential energy (equation of state) $V(\rho)$. For a weakly interacting Bose liquid the latter is quadratic in density deviations from its equilibrium value $n$, with the interaction parameter $c$. The correlation length is given by $\xi = 1/mv_s = (mnc)^{-1/2}$. It satisfies the weak-interaction criterion $n^{-1} \ll \xi$ as long as $\gamma \equiv mc/n \ll 1$. The potential energy also contains the so-called quantum pressure [27,38] term, which reflects the tendency of the condensate to maintain the uniform density throughout the system (due to the gradient terms in the underlying quantum description).

Variation of the action (8) over $\rho$ and $j$, under the continuity constraint, Eq. (7), yields a classical Euler equation of the hydrodynamic flow (with the quantum pressure contribution) [39]. Solutions of this equation do *not* lead to the formation of emptiness. The reason is that the emptiness is a large *quantum* fluctuation (similar to tunneling), which is located in a classically forbidden region of phase space. The instanton approach is based on the realization that the quantum transition amplitude is given by the path integral $\int \mathcal{D}\rho\mathcal{D}j e^{iS[\rho,j]}\delta(\partial_t\rho + \partial_x j)$, with proper boundary conditions. The integration contours over the field variables may be then deformed into the complex plane to pass through a classically forbidden stationary configuration that reaches the required emptiness. The probability of such a rare event is $P \propto |e^{iS_{\rm inst}}|^2$, where the classical action along the instanton trajectory, $S_{\rm inst}$, acquires a (positive) imaginary part.

Before proceeding with the analytical continuation to the complex plane, it is convenient to pass from a Lagrangian formalism (7) to a Hamiltonian formalism. To this end we introduce a new auxiliary field $\partial_x\theta(x,t)$ and perform the Hubbard-Stratonovich transformation for the kinetic-energy term $\sim mj^2/(2\rho)$ in $e^{iS[\rho,j]}$. This brings terms $-\rho(\partial_x\theta)^2/(2m)+j\partial_x\theta$ to the action. One may then integrate by parts the last term (assuming periodic boundary conditions in the $x$ direction) and employ the continuity relation to find

$$S[\rho,\theta]=\iint dxdt\left[\theta\partial_t\rho-\frac{\rho(\partial_x\theta)^2}{2m}-V(\rho)\right], \tag{10}$$

where we neglected the factor $\sqrt{\det[\rho]}$ from Hubbard-Stratonovich transformation since it goes beyond the accuracy of the instanton approach (see Appendix A for details). Notice that the fields $\rho$ and $\theta$ are not subject to any constraints and play the role of the canonical pair.

We are now on the position to perform the analytical continuation. Following the standard treatment of tunneling, it is achieved by the Wick rotation to imaginary time $t\to -i\tau$. The resulting equations of motions may be solved with real $\rho$ and purely imaginary $\theta$ (the integration contour in $\theta$ is deformed to pass through an imaginary saddle point). It is convenient thus to redefine $\theta\to i\theta$ such that the saddle-point solutions for both $\rho$ and $\theta$ are real functions (in imaginary time), while the new $\theta$ integration runs along the imaginary axis. The corresponding Euclidian action acquires the Hamiltonian form

$$S[\rho,\theta]=i\iint dxd\tau\,[\theta\partial_\tau\rho-\mathcal{H}(\rho,\theta)], \tag{11}$$

$$\mathcal{H}(\rho,\theta)=\frac{\rho(\partial_x\theta)^2}{2m}-\frac{c}{2}(\rho-n)^2-\frac{(\partial_x\rho)^2}{8m\rho}. \tag{12}$$

Notice that the potential $V(\rho)$ enters the effective Hamiltonian $\mathcal{H}(\rho,\theta)$ with the "wrong" sign, mirroring the inverted potential in the tunneling problem.

The equations of motion that follow from the action (11) are not the most convenient for the numerical solution. To facilitate the latter, we found it useful to perform the canonical transformation $(\rho,\theta)\to(Q,P)$ to the new pair of the conjugated fields $Q(x,\tau)=\sqrt{\rho(x,\tau)}\,e^{-\theta(x,\tau)}$ and $P(x,\tau)=\sqrt{\rho(x,\tau)}\,e^{\theta(x,\tau)}$, or conversely $\rho=PQ$ and $\theta=\frac{1}{2}\ln(P/Q)$. Substituting these into Eq. (11), one finds for the action

$$S[Q,P]=i\iint dxd\tau\,[P\partial_\tau Q-\mathcal{H}(Q,P)]+\frac{i}{2}\int dx\,PQ\ln\frac{P}{Q}\bigg|_{\tau=\tau_i}^{\tau=\tau_f}, \tag{13}$$

$$\mathcal{H}(Q,P)=-\frac{\partial_x P\partial_x Q}{2m}-\frac{c(PQ-n)^2}{2}, \tag{14}$$

where $\tau_{i(f)}$ are initial (final) times of the optimal trajectory, as discussed below.

Variables $Q$, $P$ may be considered as an analytical continuation of the real-time degrees of freedom $Q\leftrightarrow\Psi$ and $P\leftrightarrow\bar{\Psi}$. The first line of Eq. (13) is nothing but the analytical continuation of the Gross–Pitaevskii (GP) action [40], $\sim|\partial_x\Psi|^2/2m+c(|\Psi|^2-n)^2/2$. However, would we start directly from the GP action, and we would miss the boundary term, i.e., the second line in Eq. (13). This boundary term [41], $i\int dx\rho\,\theta|_{\tau=\tau_i}^{\tau=\tau_f}$, does not alter the equations of motion but contributes to the instanton action. Its contribution appears to be of paramount importance in the regime $n^{-1}<R<\xi$. To the best of our knowledge, it was first introduced in the context of classical stochastic systems by Krapivsky, Meerson, and Sasorov [24], but we discuss it here in the quantum context.

It is convenient to pass to dimensionless coordinates and fields: $x\to\xi x$, $\tau\to\tau/(nc)$, $P\to\sqrt{n}P$, $Q\to\sqrt{n}Q$. In terms of them, the Euclidean action takes the form

$$S=in\xi\left(\iint dxd\tau\left[P\partial_\tau Q+\frac{\partial_x P\partial_x Q}{2}+\frac{(PQ-1)^2}{2}\right]+\frac{1}{2}\int dx\,PQ\ln\frac{P}{Q}\bigg|_{\tau=\tau_f}^{\tau=\tau_i}\right). \tag{15}$$

The corresponding equations of motion acquire the universal parameter-free form:

$$\partial_\tau Q=\tfrac{1}{2}\partial_x^2 Q-(PQ-1)Q, \tag{16}$$

$$\partial_\tau P=-\tfrac{1}{2}\partial_x^2 P+(PQ-1)P. \tag{17}$$

These partial differential equations are known as the Ablowitz-Kaup-Newell-Segur (AKNS) system [42], which is integrable with the inverse scattering method. Remarkably, exactly these equations appear in the studies of rare events in the Kardar-Parisi-Zhang classical stochastic equation [25,43,44].

We can now specify the boundary conditions, appropriate for the emptiness formation problem. We are looking for a transition amplitude from a uniform state at a distant past, $\tau_i=-\infty$, to a state with the emptiness, i.e., zero density for $|x|<R$, at the observation time, $\tau_f=0$. This leads to the conditions: $\rho(x,-\infty)=n$ and $\rho(|x|<R,0)=0$. Outside of the interval $x\in(-R,R)$ at the observation time $\tau_f=0$, the density is not fixed and is to be integrated out in the boundary term $i\int dx\rho\,\theta|^{\tau=\tau_f}$. This fixes $\theta(|x|>R,0)=0$. In terms of the dimensionless coordinates and fields $Q$, $P$, these read as

$$PQ(x,-\infty)=1, \tag{18}$$

$$P(x,0)=\begin{cases}0, & |x|<R/\xi\\ Q(x,0), & |x|>R/\xi.\end{cases} \tag{19}$$

The zero density constraint within the emptiness interval $\rho=QP=0$, may be enforced by either $P=0$ or $Q=0$. This choice is arbitrary, since $Q$ and $P$ are interchangeable by a canonical transformation.

The program now is as follows: one needs to solve the stationary field equations (16) and (17), subject to the boundary conditions (18) and (19). The resulting instanton trajectory is to be substituted into the action (15) (including the boundary term), resulting in the instanton action $S_{\text{inst}}(R)$. The semiclassical transition amplitude is then given $e^{iS_{\text{inst}}(R)}$, resulting finally in the EFP of the form

$$-\ln P_{\text{EFP}}(R)=2\,\text{Im}\,S_{\text{inst}}(R). \tag{20}$$

One notices then that Eqs. (16) and (17) are free from any parameters, while the boundary conditions (18) and

(19) depend on the single parameter $R/\xi$. The form of the action (15) immediately implies the result, Eq. (4), where $f(R/\xi)$ is twice the value of the double integral plus the boundary term, within the large round brackets on the right-hand side of Eq. (15), evaluated along the optimal trajectory.

## III. RESULTS AND DISCUSSION

The equations of motion (16) and (17) are of the AKNS type and thus are, in principle, integrable. However, the boundary conditions (18) and (19) are *not* the initial-value problem, which could be treated with the inverse scattering approach. Although a lot is known about solutions of Eqs. (16) and (17) (see, e.g., discussion of their multisoliton configurations in Ref. [25]), we were not able to find their analytical treatment, suitable for EFP setup, formulated above. We thus resorted to a numerical approach.

We use Chernykh-Stepanov algorithm [23,45] to solve the equations of motion iteratively. The algorithm takes the advantage of the diffusive character of Eq. (16) in *forward* time and of Eq. (17) in *backward* time. The two equations are successively evolved $Q$-forward, followed by $P$-backward in time to converge to the desired solutions. The diffusive character of the equations provides stability for such iteration scheme, making the $(Q, P)$ variables advantageous over the $(\rho, \theta)$ pair. The results are still presented in terms of the more physically intuitive $(\rho, \theta)$ degrees of freedom.

At the initial backward-propagating step, we put $Q(x, \tau) = 1$ and $P(x, 0) = \theta(|x| - R)$, where $\theta(x)$ is the Heaviside step function. Then $P(x, \tau)$ is determined from backward evolution of Eq. (17) up to a large negative time $\tau = -T$. Next we update the initial condition for $Q$ from $Q(x, -T)P(x, -T) = 1$, cf. Eq. (18), and evolve Eq. (16) forward in time up to $\tau = 0$, with $P(x, \tau)$ found in the first step. This way we obtain new $Q(x, \tau)$, which we use to update the initial conditions for $P$ at $\tau = 0$, according to Eq. (19), and evolve $P$ backward in time again, etc. We then evaluate the action (15) and check that its value does not depend on the choice of the large negative initial time, $-T$.

The evolution of density and (imaginary) phase are shown in Figs. 2 and 3 for $R/\xi = 1$ and 20, respectively. The corresponding $f(R/\xi)$ is presented in Fig. 1. We numerically determine this function in the regimes $R/\xi \ll 1$ and $R/\xi \gg 1$ by curve fitting to the data points with a second-order polynomial. The best-fit coefficients are summarized in Eqs. (5) and (6), correspondingly. In this work, we are only able to determine the coefficients for the first two leading-order terms because numerical precision of data points varies from the order of $10^{-4}$ at $R/\xi = 0.1$ to the order of 1 at $R/\xi = 20$.

In $R/\xi \ll 1$ limit the system is approaching the noninteracting system. Indeed, $\xi \to \infty$ is equivalent to the limit $c \to 0$. In this case the stationary equations for $Q$ and $P$ become pure diffusion and antidiffusion, while the dynamical part of the action (15) is $\iint dx d\tau P[\partial_\tau Q - \frac{1}{2}\partial_x^2 Q]$, which is nullified on the equation of motion. The only contribution to the action is thus the boundary term

$$\frac{1}{2}\int dx PQ \ln \frac{P}{Q}\bigg|_{\tau_i=-\infty}. \tag{21}$$

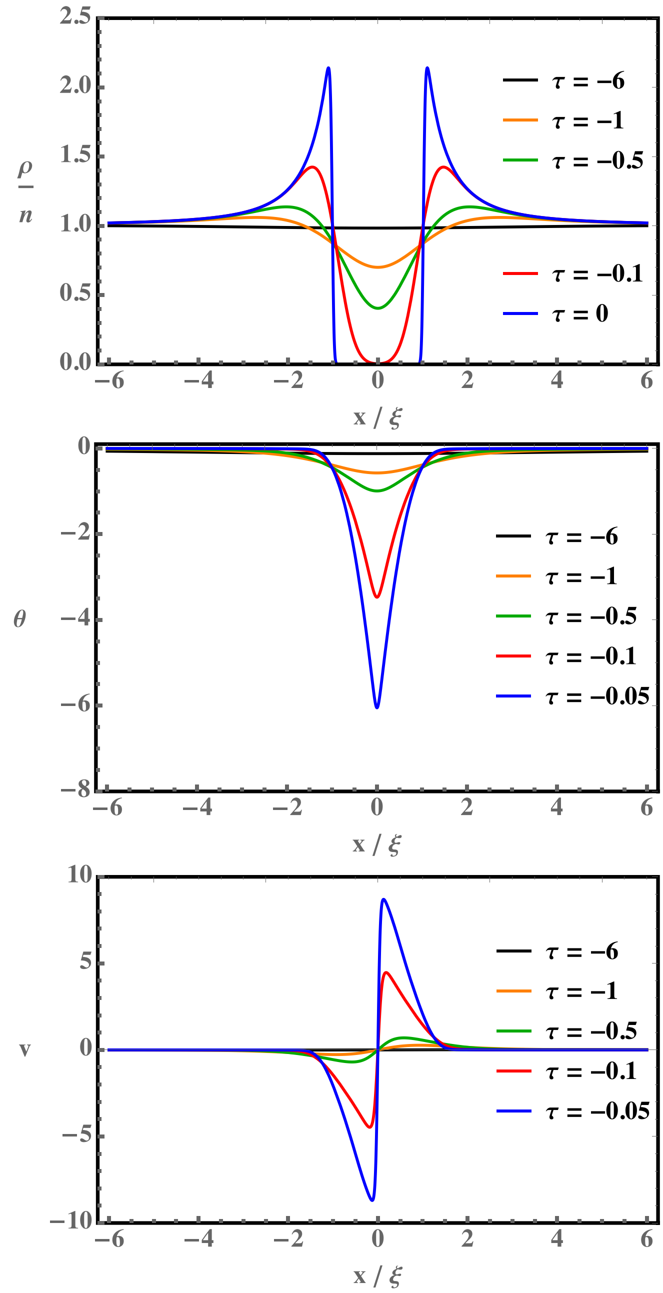


FIG. 2. Time evolution of the density $\rho(x, \tau)$, imaginary phase $\theta(x, \tau)$, and velocity $v(x, \tau)$ for weakly interacting bosons with $R/\xi = 1$. The density evolves from the uniform value $\rho = n$ at large negative $\tau$ towards the emptiness of size $2R$ at $\tau = 0$. At $\tau \to 0$ the phase tends to the negative infinity for $|x| < R$ and thus is not shown. The velocity $v$ is related to spatial gradient of phase, $v = \partial_x\theta/m$.

The final time $\tau_f = 0$ does not contribute either, in view of Eq. (19). Its numerical evaluation gives 1.005(20) in the limit $R \ll \xi$. After taking care of the factor of two in Eq. (20), this agrees with the free-boson result $-\ln P_{\rm EFP}(R) = 2nR$, as we expect.

In the opposite limit $R/\xi \gg 1$, it is useful to look at the action (11) and rescale variables in an alternative way: $x \to Rx$, $\tau \to Rm\xi\tau$, $\rho \to n\rho$, and $\theta \to (R/\xi)\theta$. The Euclidean action

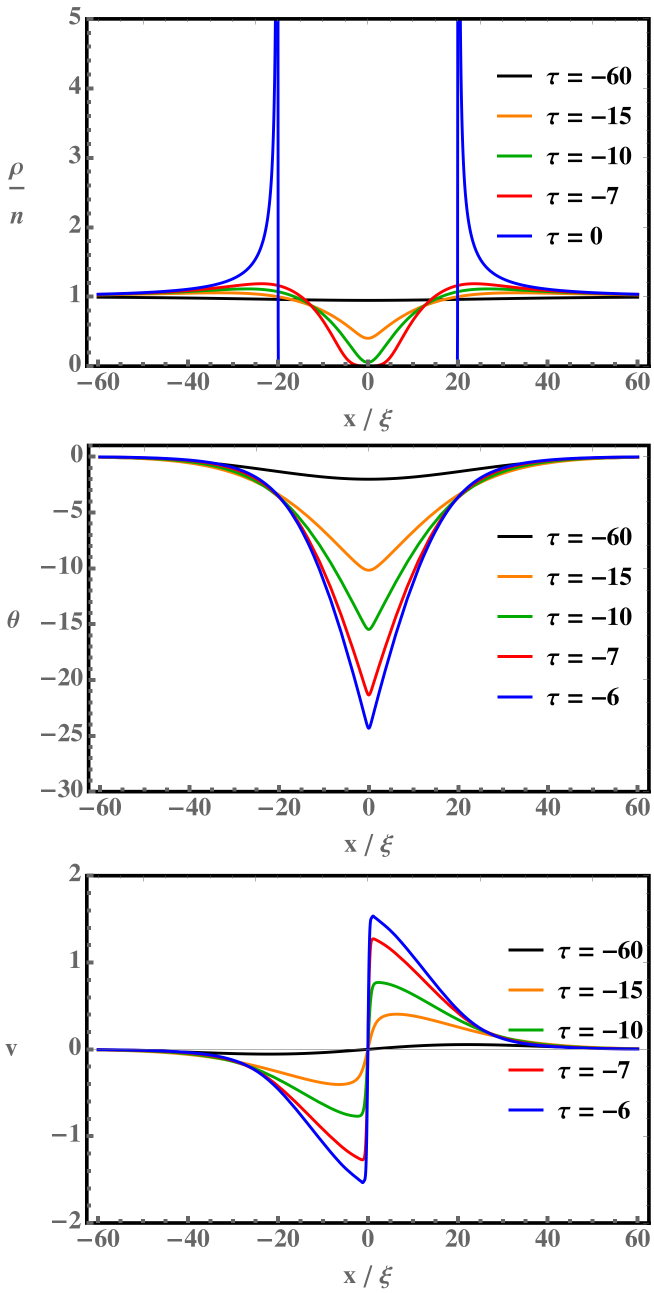


FIG. 3. Same as Fig. 2 but for $R/\xi = 20$.

takes the form

$$S = in\xi \iint dx d\tau \left[\frac{R^2}{\xi^2}\left(\theta\partial_\tau\rho - \frac{\rho(\partial_x\theta)^2}{2} + \frac{(\rho-1)^2}{2}\right) + \frac{(\partial_x\rho)^2}{8\rho}\right]. \tag{22}$$

The first line here is the leading term, $\propto(R/\xi)^2$, which is given by the hydrodynamic action without quantum pressure. It corresponds to the leading Gaussian term in EFP, Eq. (6). One notices the absence of the term linear in $R/\xi$, which is consistent with our numerical finding.

The message from Eq. (22) is that the Gaussian part of EFP in the limit $R \gg \xi$ can be found without the quantum pressure. This is in agreement with the success of such hydrodynamic theory [31] to obtain exact results vis à vis the Gaussian limit. The most notable case is the free-fermion Tonks-Girardeau limit, cf. Eq. (3). We numerically explored this known limit (see Appendix B) as a test-drive of our numerical procedureand found the coefficient 0.501(2), which should be compared with 1/2 in Eq. (3)—this provides some support to the accuracy of our results.

We conclude with a brief comparison with some previously published results on EFP. The only analytic work that we know of on EFP in a 1D interacting-boson model is a conjecture by Its, Korepin, and Waldron [32]. In the weakly interacting limit, the leading term at large $R$ is claimed to be $-\ln P_{\rm EFP} = 4(R/\xi)^2$. This is in parametric disagreement with our main result (4). There is a factor $n\xi$ missing in their conjecture and it plays an important role as a large parameter in the weak-interacting region. The only large parameter in their work is $R/\xi$, which alone is insufficient to describe the asymptotic behavior. They might have overlooked this factor in the calculation. On the other hand, calculations based on the bosonization procedure [31] are in a parametric agreement with Eq. (4). Bosonization only allows for a treatment of a small suppression of density, rather than the emptiness. If one arbitrarily takes such "small" suppression all the way to zero density, its probability is consistent with Eq. (4). There is also a number of results on EFP in antiferromagnetic spin-1/2 $XXZ$ chains with the Hamiltonian

$$H = \sum_{j=-\infty}^{\infty}\left[S_j^xS_{j+1}^x + S_j^yS_{j+1}^y + \Delta S_j^zS_{j+1}^z\right], \tag{23}$$

where $\Delta$ is the anisotropy in the $z$ direction. It is proposed in Ref. [46] that EFP in the gapless regime $-1 < \Delta \leqslant 1$ is

$$-\ln P_{\rm EFP} \approx Al^2 + B\ln l, \tag{24}$$

where $A$ and $B$ are constants depending on $\Delta$, and $l$ is the number of consecutive spin-polarized sites. An explicit expression for the coefficient $A$ was found to be

$$A = \ln\left[\frac{\Gamma^2(1/4)}{\pi\sqrt{2\pi}}\right] - \int_0^\infty \frac{dt}{t}\frac{\sinh^2(t\nu)e^{-t}}{\cosh(2t\nu)\sinh(t)}, \tag{25}$$

where parameter $\nu$ is defined through $\cos(\pi\nu) = \Delta$. The correspondence with the weakly interacting bosons may be established for $\Delta \gtrsim -1$, where the Luttinger parameter $K = 1/[2(1-\nu)] \gg 1$. Defining the correlation length (in lattice units) as $\xi = K/(\pi n)$, where the corresponding bosonic density is $n = 1/2$ in the lattice units, one finds from Eq. (25) for the leading term of EFP in the $\xi \gg 1$ limit:

$$-\ln P_{\rm EFP} = \frac{n}{2\xi}l^2, \tag{26}$$

which is in parametric agreement with our result (4).

To conclude, we have developed the instanton approach that is capable of describing a complete crossover of EFP from the Poisson to the Gaussian regime over the wide range of parameters $n^{-1} < R < \xi$ available in weakly interacting bosonic 1D systems. Such systems are now routinely realized in cold atom experiments, where EFP may be measured.

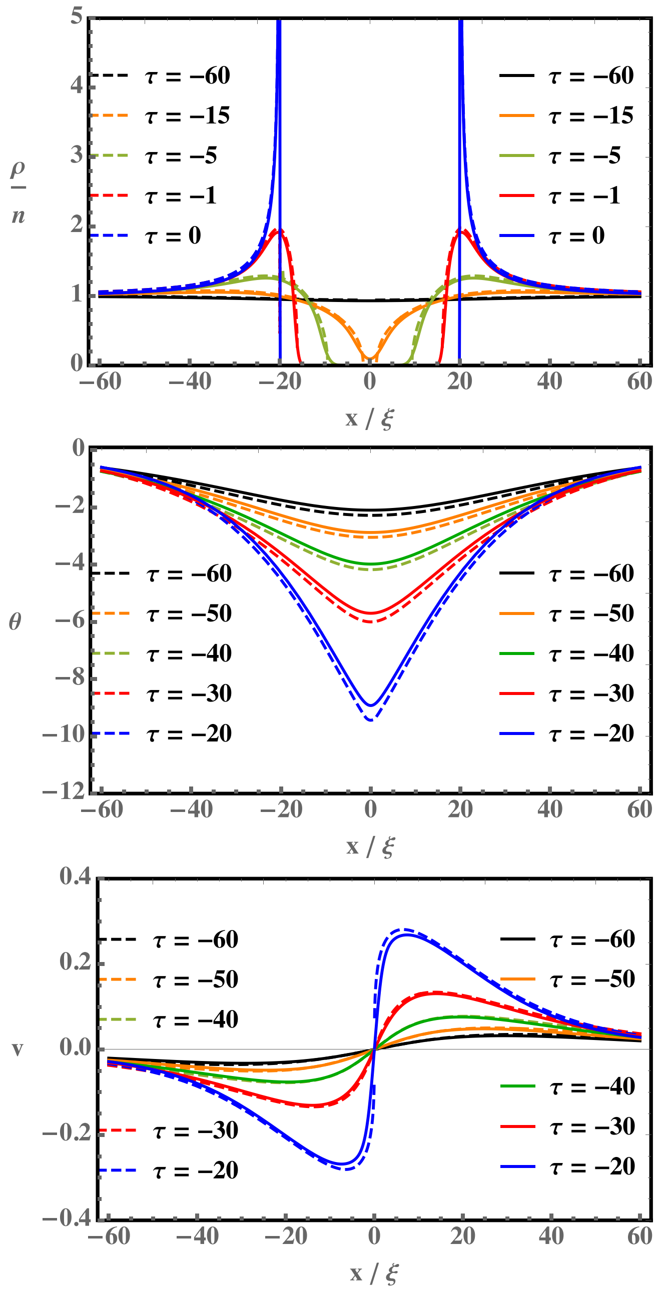


FIG. 4. The upper panel is the time evolution of the density $\rho$ for free fermions with $R/\xi = 20$. The middle and lower parts are the time evolution of phase $\theta$ and velocity $v$, respectively. The solid lines are numerical solutions of Eqs. (B3) and (B4), using the algorithm outlined in Sec. III, and dashed lines are the analytical solutions of Ref. [31].

## ACKNOWLEDGMENTS

We are indebted to A. Abanov, D. Gangardt, and B. Meerson for valuable discussions. This work was supported by NSF Grant No. DMR-1608238.

## APPENDIX A: HAMILTONIAN FORMALISM FOR HYDRODYNAMIC SYSTEM

We present details of the transition from Lagrangian formalism to Hamiltonian formalism by the Hubbard-Stratonovich transformation. The functional integral of a hydrodynamic system in Lagrangian formalism is

$$Z = \int \mathcal{D}\rho\mathcal{D}j\, e^{iS[\rho,j]}\delta(\partial_t\rho + \partial_x j), \tag{A1}$$

$$S[\rho, j] = \iint dxdt \left[\frac{mj^2}{2\rho} - V(\rho)\right], \tag{A2}$$

where the $\delta$ function $\delta(\partial_t\rho + \partial_x j)$ imposes the continuity equation on the system and $V(\rho)$ is some general potential energy (equation of state) which does not affect the following derivation. We introduce an auxiliary field $\partial_x\theta(x,t)$ by the Hubbard-Stratonovich transformation

$$Z = \int \mathcal{D}\rho\mathcal{D}j\mathcal{D}\partial_x\theta\, \sqrt{\det[\rho]}e^{iS[\rho,j,\theta]}\delta(\partial_t\rho + \partial_x j), \tag{A3}$$

$$S[\rho, j, \theta]ds = \iint dxdt\, [-\rho(\partial_x\theta)^2/(2m) + j\partial_x\theta - V(\rho)], \tag{A4}$$

where the prefactor $\sqrt{\det[\rho]}$ comes from the Gaussian functional integration and we neglect the overall constant factor in the functional integral. In principal, the prefactor $\sqrt{\det[\rho]}$ should be reformulated and absorbed into the action in the exponent. However, we can focus only on Eq. (A4) in semiclassical approximation without worrying about the contribution from the prefactor $\sqrt{\det[\rho]}$.

In the end, we first do the integration by parts on $j\partial_x\theta$ and then integrate out the field $j$. The term with $\partial_x j$ is replaced by $-\partial_t\rho$ because of the $\delta$ function $\delta(\partial_t\rho + \partial_x j)$. The action is now expressed in Hamiltonian formalism:

$$S[\rho, \theta] = \iint dxdt \left[\theta\partial_t\rho - \frac{\rho(\partial_x\theta)^2}{2m} - V(\rho)\right]. \tag{A5}$$

## APPENDIX B: FREE-FERMION LIMIT

In the free-fermion limit the hydrodynamic potential is given by

$$V(\rho) = 2\int_{\pi n}^{\pi\rho} \frac{dk}{2\pi}\frac{k^2}{2m} - \mu(\rho - n) = \frac{\pi^2(\rho^3 - 3n^2\rho + 2n^3)}{6m}, \tag{B1}$$

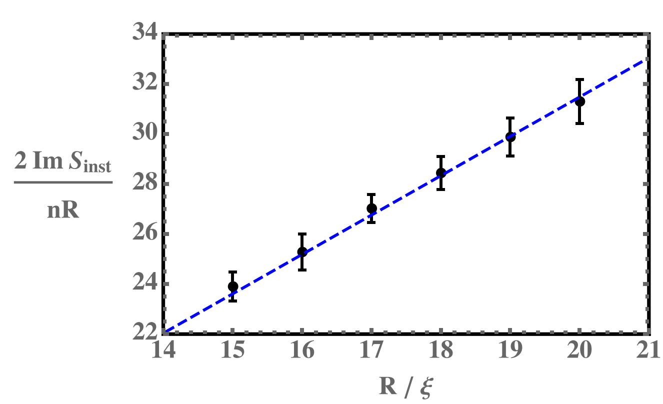


FIG. 5. $2\,\mathrm{Im}\,S_{\mathrm{inst}}/nR$ vs $R/\xi$ for free fermions. The blue dashed line is a fit to $0.501\pi R/\xi$.

where $\mu = (\pi n)^2/2m$ is the chemical potential. We substitute it in the hydrodynamic action (11) to find

$$S = i \iint dx d\tau \left[\theta \partial_\tau \rho - \frac{\rho(\partial_x \theta)^2}{2} + V(\rho) + \frac{(\partial_x \rho)^2}{8\rho}\right], \quad \text{(B2)}$$

where we kept the quantum pressure term from the weakly interacting case, since, as explained in Sec. III, it does not contribute in the large-$R$ limit anyways. We now proceed to the $Q$, $P$ variables as above and then make them dimensionless, using $\xi = 1/\pi n$ appropriate for the free fermions. The resulting equations of motion are

$$\partial_\tau Q = \tfrac{1}{2}\partial_x^2 Q - \tfrac{1}{2}(P^2Q^2 - 1)Q, \quad \text{(B3)}$$

$$\partial_\tau P = -\tfrac{1}{2}\partial_x^2 P + \tfrac{1}{2}(P^2Q^2 - 1)P, \quad \text{(B4)}$$

with the same boundary condition (18) and (19) and the modified action

$$S = in\xi \left( \iint dx d\tau \left[ P\partial_\tau Q + \frac{\partial_x P \partial_x Q}{2}, + \frac{(P^3Q^3 - 3PQ + 2)^2}{6}\right] + \frac{1}{2}\int dx\, PQ \ln \frac{P}{Q}\bigg|_{\tau=-\infty}^{\tau=0} \right). \quad \text{(B5)}$$

The instanton solution is shown in Fig. 4, where we compare it to the analytical solution (without quantum pressure) of Ref. [31]. The corresponding optimal action is shown in Fig. 5. Its best fit is given by

$$-\ln P_{\mathrm{EFP}} = 0.501(2)(R/\xi)^2 + O(\ln R/\xi), \quad \text{(B6)}$$

where we used relation $\xi = 1/\pi n$. This is in a very good agreement with exact result for the free fermions, Eq. (3) [29,30].

[1] J. Esteve, J.-B. Trebbia, T. Schumm, A. Aspect, C. I. Westbrook, and I. Bouchoule, Phys. Rev. Lett. **96**, 130403 (2006).
[2] J. Armijo, T. Jacqmin, K. V. Kheruntsyan, and I. Bouchoule, Phys. Rev. Lett. **105**, 230402 (2010).
[3] T. Jacqmin, J. Armijo, T. Berrada, K. V. Kheruntsyan, and I. Bouchoule, Phys. Rev. Lett. **106**, 230405 (2011).
[4] A. Del Campo, Phys. Rev. A **84**, 012113 (2011).
[5] M. Pons, D. Sokolovski, and A. Del Campo, Phys. Rev. A **85**, 022107 (2012).
[6] A. del Campo, New J. Phys. **18**, 015014 (2016).
[7] M. Arzamasovs and D. M. Gangardt, Phys. Rev. Lett. **122**, 120401 (2019).
[8] H. Bethe, Eur. Phys. J. A **71**, 205 (1931).
[9] V. E. Korepin, A. G. Izergin, F. H. Essler, and D. B. Uglov, Phys. Lett. A **190**, 182 (1994).
[10] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, UK, 1997), Vol. 3.
[11] J. De Gier and V. Korepin, J. Phys. A: Math. Gen. **34**, 8135 (2001).
[12] H. E. Boos, V. E. Korepin, and F. A. Smirnov, Nucl. Phys. B **658**, 417 (2003).
[13] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets* (World Scientific, Singapore, 2009).
[14] M. L. Mehta, *Random Matrices* (Elsevier, Amsterdam, 2004), Vol. 142.
[15] D. S. Dean and S. N. Majumdar, Phys. Rev. Lett. **97**, 160201 (2006).
[16] M. Shiroishi, M. Takahashi, and Y. Nishiyama, J. Phys. Soc. Jpn. **70**, 3535 (2001).
[17] N. Kitanine, J. Maillet, and V. Terras, Nucl. Phys. B **567**, 554 (2000).
[18] N. Kitanine, J. Maillet, N. Slavnov, and V. Terras, J. Phys. A: Math. Gen. **35**, L385 (2002).
[19] H. E. Boos, F. Göhmann, A. Klümper, and J. Suzuki, J. Stat. Mech.: Theory Exp. (2006) P04001.
[20] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, J. Phys. A: Math. Gen. **35**, L753 (2002).
[21] A. Bastianello, L. Piroli, and P. Calabrese, Phys. Rev. Lett. **120**, 190601 (2018).
[22] M. I. Dykman, E. Mori, J. Ross, and P. Hunt, J. Chem. Phys. **100**, 5735 (1994).
[23] V. Elgart and A. Kamenev, Phys. Rev. E **70**, 041106 (2004).
[24] P. Krapivsky, B. Meerson, and P. V. Sasorov, J. Stat. Mech.: Theory Exp. (2012) P12014.
[25] M. Janas, A. Kamenev, and B. Meerson, Phys. Rev. E **94**, 032133 (2016).
[26] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
[27] M. Kulkarni and A. G. Abanov, Phys. Rev. A **86**, 033614 (2012).
[28] F. J. Dyson, J. Math. Phys. **3**, 157 (1962).
[29] J. Des Cloizeaux and M. Mehta, J. Math. Phys. **14**, 1648 (1973).
[30] F. J. Dyson, Commun. Math. Phys. **47**, 171 (1976).
[31] A. Abanov, arXiv:cond-mat/0504307.
[32] V. Korepin, A. Its, and A. Waldron, arXiv:cond-mat/9510068.
[33] B. Sutherland, J. Math. Phys. **12**, 246 (1971).
[34] A. G. Abanov and V. E. Korepin, Nucl. Phys. B **647**, 565 (2002).
[35] A. G. Abanov and F. Franchini, Phys. Lett. A **316**, 342 (2003).
[36] F. Franchini and A. G. Abanov, J. Phys. A: Math. Gen. **38**, 5069 (2005).
[37] F. Franchini and M. Kulkarni, Nucl. Phys. B **825**, 320 (2010).
[38] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Elsevier, Oxford, 2013), Vol. 3.
[39] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Pergamon, Oxford, 1987).
[40] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[41] The straight substitution results in the boundary term $i\int dx \rho(\theta - 1/2)|_{\tau=\tau_i}^{\tau=\tau_f}$. There is, however, a global symmetry of shifting $\theta(x,t) \to \theta(x,t) + \bar{\theta}$ by a constant, associated with

$P \to Pe^{\bar{\theta}}$ and $Q \to Qe^{-\bar{\theta}}$, such that $\rho = QP$ is invariant. This allows us to bring the boundary term to the form of Eq. (13).
[42] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. **31**, 125 (1973).
[43] B. Meerson, E. Katzav, and A. Vilenkin, Phys. Rev. Lett. **116**, 070601 (2016).
[44] A. Kamenev, B. Meerson, and P. V. Sasorov, Phys. Rev. E **94**, 032108 (2016).
[45] A. I. Chernykh and M. G. Stepanov, Phys. Rev. E **64**, 026306 (2001).
[46] V. Korepin, S. Lukyanov, Y. Nishiyama, and M. Shiroishi, Phys. Lett. A **312**, 21 (2003).