Quantifying the resource content of quantum channels: An operational approach

Lu Li,[1,*] Kaifeng Bu,[1,2,†] and Zi-Wen Liu[3,4,‡]
[1]*School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, People's Republic of China*
[2]*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*
[3]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*
[4]*Center for Theoretical Physics, Research Laboratory of Electronics, and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

(Received 18 December 2018; accepted 23 January 2020; published 26 February 2020)

We propose a general method to operationally quantify the "resourcefulness" of quantum channels via channel discrimination, an important information processing task. A main result is that the maximum success probability of distinguishing a given channel from the set of free channels by free probe states is exactly characterized by the resource generating power, i.e., the maximum amount of resource produced by the action of the channel, given by the trace distance to the set of free states. We apply this framework to the resource theory of quantum coherence, as an informative example. The general results can also be easily applied to other resource theories such as entanglement, magic states, and asymmetry.

DOI: 10.1103/PhysRevA.101.022335

I. INTRODUCTION

Understanding and utilizing various forms of quantum resources represents a main theme of quantum information science. To this end, a powerful framework known as the quantum resource theory is being actively developed in recent years to systematically study the quantification and manipulation of quantum resources (see [1] for a recent review). In fact, the resource features of certain quantum effects, in particular quantum entanglement, have already been carefully studied earlier [2–4], but a key observation underlying the recent interests in the resource theory framework is that the theories of different kinds of resource properties (stemming from different physical constraints) can share a largely common structure and a wide range of general approaches and results [5–12]. Indeed, this idea has been successfully applied to the study of various other key quantum resources, such as coherence [13–15], superposition [16], magic states [17,18], thermal nonequilibrium [19,20], asymmetry [21,22], etc.

The well-established schemes of resource theory (at a nonabstract level; see, e.g., [23,24] for abstract, category-theoretic formulations that do not rely on the explicit mathematical structures of the object space) mostly handle in particular static resources encoded in quantum states (density operators). However, certain quantum processes or channels can represent dynamical quantum resources which play natural and fundamental roles in broad scenarios. The systematic study of channel resource theories is blueprinted recently by [25], but we are still at an early stage of developing the complete theory.

The quantification of resource is a central topic of all kinds of resource theories. In particular, one is interested in the operational interpretation of certain resource measures, i.e., how they correspond to the value of the resource in achieving some operational task. In state resource theories, general operational resource measures can be induced by several tasks, e.g., resource interconversion [5,6,12] and resource erasure [9]. However, for quantum channels, we only know that the smooth log-robustness characterizes the randomness cost of the task of one-shot resource erasure [25] at the general level. (Note that the quantification of channel resources have been previously considered in various specific contexts, such as entanglement [26], coherence [27,28], non-Gaussianity [29], and magic [30]).

In this work, we suggest a simple and general scheme to quantify the resourcefulness of quantum channels based on quantum channel discrimination, a fundamental problem in quantum information [31–33]. (Note that channel discrimination is already known to play key roles in the characterization of state resources [10,11,34–36].) The core question here is how well one can distinguish a quantum channel from another by optimizing over input probe states and output measurements. We find that the maximum success probability of distinguishing the given channel from the set of free operations by all free probe states is exactly characterized by the maximum amount of resource that can be generated by the channel, i.e., the resource generating power, as measured by the trace-norm distance of resource. This resource generating power satisfies several desirable properties, such as faithfulness, convexity, submultiplicity, and monotonicity. Besides, the advantage of using a resource state as the probe state, compared with free probe states, is upper bounded by the trace-norm measure of resource. As a prominent example, we analyze in depth the widely studied resource theory of coherence, the structure of which allows for further results. Our study leads to several new understandings of the coherence theory. Our approaches

*lilu93@zju.edu.cn
†kfbu@fas.harvard.edu
‡zliu1@perimeterinstitute.ca; zwliu@mit.edu

apply to many other important resource theories, such as entanglement theory, as we shall also briefly demonstrate.

II. MAIN RESULTS

Given a finite dimensional Hilbert space $\mathcal{H}$, let $\mathcal{D}(\mathcal{H})$ denote the set of all quantum states on $\mathcal{H}$. Assume the set of free states $\mathcal{F}$ to be a nonempty, convex, and closed subset of $\mathcal{D}(\mathcal{H})$. Let $\mathfrak{F}$ be the set of free quantum channels, or completely positive and trace preserving (CPTP) maps. Channels in $\mathfrak{F}$ must map all free states to free states.

Define the resource generating and increasing power $(\Omega/\widetilde{\Omega})$ of channel $\mathcal{N}:\mathcal{D}(\mathcal{H})\longrightarrow\mathcal{D}(\mathcal{H})$ as follows. Given some resource monotone of states ω and the set of free states $\mathcal{F}$,

$$\Omega(\mathcal{N}) := \max_{\rho\in\mathcal{F}} \omega(\mathcal{N}(\rho)), \tag{1}$$

$$\widetilde{\Omega}(\mathcal{N}) := \max_{\rho\in\mathcal{D}(\mathcal{H})} [\omega(\mathcal{N}(\rho)) - \omega(\rho)]. \tag{2}$$

Note that the complete versions of resource generating and increasing power can also be defined which, in addition, optimize over any ancilla space (see [25] for extended discussions).

A representative type of resource monotone is the distance to $\mathcal{F}$. More explicitly, given some distance measure D, one can define resource measure ω_D for any quantum state ρ as follows:

$$\omega_D(\rho) := \min_{\sigma\in\mathcal{F}} D(\rho,\sigma). \tag{3}$$

The resource generating and increasing power given by ω_D is denoted $\Omega_D/\widetilde{\Omega}_D$. It can be shown that they are actually equivalent for contractive distance metrics (see the proof in Appendix A).

Proposition 1. If the distance measure D satisfies the triangle inequality and the data processing inequality (i.e., nonincreasing under CPTP maps), then we have

$$\Omega_D(\mathcal{N}) = \widetilde{\Omega}_D(\mathcal{N}). \tag{4}$$

Of particular importance to this work is the trace distance $\frac{1}{2}\|\rho-\sigma\|_1 := \frac{1}{2}\mathrm{Tr}|\rho-\sigma|$, which we denote by subscript “1”.

Here, we aim at establishing connections between the resource generating power of a channel and its nonfree feature in the task of channel discrimination. Given two channels $\mathcal{N}$ and $\mathcal{M}$, and the same probe state ρ going through the channels $\mathcal{N}$, $\mathcal{M}$, respectively, then the success probability of distinguishing $\mathcal{N}$ and $\mathcal{M}$ by the probe state ρ is the success probability of distinguishing $\mathcal{N}(\rho)$ and $\mathcal{M}(\rho)$ as follows:

$$p_{\rm succ}(\mathcal{N},\mathcal{M},\rho) = \max_{\{\Pi,\mathbb{I}-\Pi\}} \left\{\tfrac{1}{2}\,\mathrm{Tr}\,[\mathcal{N}(\rho)\Pi] + \tfrac{1}{2}\,\mathrm{Tr}\,[\mathcal{M}(\rho)(\mathbb{I}-\Pi)]\right\}, \tag{5}$$

where the maximization is taken over all POVM $\{\Pi,\mathbb{I}-\Pi\}$. By the Holevo-Helstrom theorem [37], $p_{\rm succ}(\mathcal{N},\mathcal{M},\rho) = \frac{1}{2}+\frac{1}{4}\|\mathcal{N}(\rho)-\mathcal{M}(\rho)\|_1$.

The success probability of distinguishing $\mathcal{N}$ from the set of channels $\mathfrak{F}$ by the probe state ρ is defined as

$$p_{\rm succ}(\mathcal{N},\mathfrak{F},\rho) := \min_{\mathcal{M}\in\mathfrak{F}} p_{\rm succ}(\mathcal{N},\mathcal{M},\rho), \tag{6}$$

and the maximum success probability of distinguishing $\mathcal{N}$ from $\mathfrak{F}$ by using any free state or any quantum state (denoted by Q) as the probe state are, respectively, given by

$$p_{\rm succ}(\mathcal{N},\mathfrak{F},\mathcal{F}) := \max_{\rho\in\mathcal{F}} p_{\rm succ}(\mathcal{N},\mathfrak{F},\rho), \tag{7}$$

$$p_{\rm succ}(\mathcal{N},\mathfrak{F},Q) := \max_{\rho\in\mathcal{D}(\mathcal{H})} p_{\rm succ}(\mathcal{N},\mathfrak{F},\rho). \tag{8}$$

The following result provides an exact characterization of the success probability $p_{\rm succ}(\mathcal{N},\mathfrak{F},\mathcal{F})$.

Theorem 1. Given a quantum channel $\mathcal{N}$ and the set of free channels $\mathfrak{F}$. The maximum success probability of discriminating $\mathcal{N}$ from $\mathfrak{F}$ by the set of free states $\mathcal{F}$ is only directly related to the resource increasing power given by trace distance (which equals the generating power due to Proposition 1) of $\mathcal{N}$ as follows:

$$p_{\rm succ}(\mathcal{N},\mathfrak{F},\mathcal{F}) = \tfrac{1}{2}+\tfrac{1}{2}\widetilde{\Omega}_1(\mathcal{N}) = \tfrac{1}{2}+\tfrac{1}{2}\Omega_1(\mathcal{N}). \tag{9}$$

The proof of this theorem is provided in Appendix A. We now show that $\Omega_1(\mathcal{N})$ satisfies the basic conditions for resource quantifiers of quantum channels, e.g., normalized, and monotone under left and right compositions with free channels [25]. More specifically see the following.

Proposition 2. The trace-norm resource generating power $\Omega_1(\mathcal{N})$ satisfies the following properties:

(i) $\Omega_1(\mathcal{N})\geqslant 0$, and $\Omega_1(\mathcal{N})=0$ if $\mathcal{N}\in\mathfrak{F}$. Moreover, if $\mathfrak{F}$ includes all CPTP maps which maps all free states to free states (resource nongenerating maps), then $\Omega_1(\mathcal{N})=0$ iff $\mathcal{N}\in\mathfrak{F}$.

(ii) For any $\mathcal{M}_1,\mathcal{M}_2\in\mathfrak{F}$, we have

$$\Omega_1(\mathcal{M}_1\circ\mathcal{N}\circ\mathcal{M}_2)\leqslant\Omega_1(\mathcal{N}). \tag{10}$$

(iii) Given a set of quantum channels $\{\mathcal{N}_i,p_i\}_i$ with $\sum_i p_i=1$,

$$\Omega_1\left(\sum_i p_i\mathcal{N}_i\right)\leqslant\sum_i p_i\Omega_1(\mathcal{N}_i). \tag{11}$$

Moreover, if the free states on $\mathcal{H}_A\otimes\mathcal{H}_B$ is defined as a convex combination of the tensor product of free states on $\mathcal{H}_A$ and $\mathcal{H}_B$, i.e., $\mathcal{F}_{AB}=\mathrm{Conv}\{\mathcal{F}_A\otimes\mathcal{F}_B\}$, then resource generating power $\Omega_1(\mathcal{N})$ also satisfies the following properties.

(iv) Given two channels $\mathcal{N}_1$ and $\mathcal{N}_2$, it holds that

$$\Omega_1(\mathcal{N}_1\otimes\mathcal{N}_2)\geqslant\max\{\Omega_1(\mathcal{N}_1),\Omega_1(\mathcal{N}_2)\}. \tag{12}$$

(v) Given two channels $\mathcal{N}_1$ and $\mathcal{N}_2$, it holds that

$$\Omega_1(\mathcal{N}_1\otimes\mathcal{N}_2)\leqslant\Omega_1(\mathcal{N}_1)+\Omega_1(\mathcal{N}_2). \tag{13}$$

In fact, each of the above properties holds under weaker assumptions. The proof for more general distance measures is provided in Appendix B. Due to property (i), Theorem 1 also indicates that resource nongenerating channels are effectively indistinguishable from each other by free probe states. Due to property (iv), it is easy to define a regularized version of $\Omega_1(\mathcal{N})$ by $\Omega_1^\infty(\mathcal{N})=\lim_{n\to\infty}\frac{1}{n}\Omega_1(\mathcal{N}^{\otimes n})$, which is invariant under tensoring, i.e., $\Omega_1^\infty(\mathcal{N}^{\otimes 2})=\Omega_1^\infty(\mathcal{N})$. However, this is not the focus of this work.

Since $\mathcal{F}\subset\mathcal{D}(\mathcal{H})$, we have $p_{\rm succ}(\mathcal{N},\mathfrak{F},Q)\geqslant p_{\rm succ}(\mathcal{N},\mathfrak{F},\mathcal{F})$. If the probe state ρ is not a free state, then the resource in ρ may help improve the success probability of discriminating the given channel $\mathcal{N}$ from the set of free channels. Here we provide an upper bound on the advantage of using a resource probe state.

Theorem 2. Given a quantum channel $\mathcal{N}$, a quantum state ρ, and the set of free channels $\mathfrak{F}$. The advantage provided by the state ρ compared with all free states to distinguish any given channel $\mathcal{N}$ from $\mathfrak{F}$ is upper bounded by the trace-norm distance of resource:

$$p_{\text{succ}}(\mathcal{N},\mathfrak{F},\rho)-p_{\text{succ}}(\mathcal{N},\mathfrak{F},\mathcal{F})\leqslant\tfrac{1}{2}\omega_1(\rho). \tag{14}$$

The proof is presented in Appendix C. A direct corollary is the following bound on the success probability of discriminating $\mathcal{N}$ from free channels by any probe state ρ.

Corollary 1. Given a quantum channel $\mathcal{N}$, a quantum state ρ, and the set of free channels $\mathfrak{F}$, the success probability $p_{\text{succ}}(\mathcal{N},\mathfrak{F},\rho)$ is upper bounded by

$$p_{\text{succ}}(\mathcal{N},\mathfrak{F},\rho)\leqslant\tfrac{1}{2}+\tfrac{1}{2}\Omega_1(\mathcal{N})+\tfrac{1}{2}\omega_1(\rho). \tag{15}$$

III. EXAMPLE

As an application of the above general framework, we now focus on quantum coherence, a prominent quantum feature emerging from the superposition principle of quantum mechanics. Coherence represents a key quantum resource which has a variety of applications in quantum information science, including quantum metrology [38], thermodynamics [39,40], and biology [41,42]. In recent years, the resource theory of coherence has drawn a lot of attention, where the manipulation and characterization of coherence in quantum states are thoroughly investigated (see [15,43] for a review). Now we extend the study to quantum channels following the idea in the last section, that is, to characterize the coherence value of a channel by its distinguishability from the typical sets of coherence-free channels.

Given a fixed basis $\{|i\rangle\}_{i=0}^{d-1}$ for a d-dimensional system, any quantum state which is diagonal in the reference basis is called an incoherent state and is a free state in the resource theory of coherence. The set of incoherent states is denoted by $\mathcal{I}$. Let Δ denote the fully dephasing channel in the given basis, which is defined as $\Delta(\rho)=\sum_i\langle i|\rho|i\rangle|i\rangle\langle i|$. Δ is a prominent example of the resource destroying map [7].

There are several individually motivated choices of free operations in the resource theory of coherence. The following four, which collectively emerge from the relations with Δ and can be broadly generalized via the theory of resource destroying map [7], are considered most important: (1) maximally incoherent operations (MIO) [44], the maximum possible set of coherence-free operations that contains all quantum operations $\mathcal{M}$ that maps incoherent states to incoherent states, i.e., $\mathcal{M}(\mathcal{I})\subset\mathcal{I}$; (2) incoherent operations (IO) [13], containing $\mathcal{M}$ that admit a set of Kraus operators $\{K_i\}$ such that $\mathcal{M}(\cdot)=\sum_i K_i(\cdot)K_i^\dagger$ and $K_i\mathcal{I}K_i^\dagger\subset\mathcal{I}$ for any i; (3) dephasing-covariant operations (DIO) [7,44], containing $\mathcal{M}$ such that $[\Delta,\mathcal{M}]=0$; (4) strictly incoherent operations (SIO) [44,45], containing all $\mathcal{M}$ admitting a set of Kraus operators $\{K_i\}$ such that $\Delta(K_i\rho K_i^\dagger)=K_i\Delta(\rho)K_i^\dagger$ for any i and any quantum state ρ.

Several operational motived coherence measures have been introduced and here we consider the coherence measure defined by l_1-norm distance [13], trace-norm distance [46], and robustness [47],

$$C_{l_1}(\rho):=\min_{\sigma\in\mathcal{I}}\|\rho-\sigma\|_{l_1}, \tag{16}$$

$$C_1(\rho):=\tfrac{1}{2}\min_{\sigma\in\mathcal{I}}\|\rho-\sigma\|_1, \tag{17}$$

$$C_R(\rho)=\min\{t\geqslant 0:\rho+t\sigma\in\mathcal{I},\sigma\in\mathcal{D}(\mathcal{H})\}. \tag{18}$$

In fact, in single-qubit system $\mathbb{C}^2$, the trace norm of coherence C_1 is equal to l_1 norm of coherence C_{l_1} [46,48] and the robustness of coherence C_R [47] up to a scalar 2.

In the resource theory of coherence, certain coherence generating power can also be used to characterize the cost of simulating the given channel by incoherent operations [49,50] and the capacity of a channel to generate maximally coherent states [27]. Besides, the ability of a quantum channel to detect nonclassicality has also been introduced to quantify the resource of channels in terms of trace distance [28] and relative entropy [28,51].

First, it follows from Theorem 1 that the success probability of distinguishing $\mathcal{N}$ from the set of free operations $\mathfrak{I}$, where $\mathfrak{I}$ can be any of $\{SIO, IO, DIO, MIO\}$, is universally determined by the trace-norm coherence generating power.

Proposition 3. Given a quantum channel $\mathcal{N}$ and the set of coherence-free operations $\mathfrak{I}\in\{SIO, IO, DIO, MIO\}$, the maximum success probability of distinguishing $\mathcal{N}$ from $\mathfrak{I}$ by incoherent states is

$$p_{\text{succ}}(\mathcal{N},\mathfrak{I},\mathcal{I})=\tfrac{1}{2}+\tfrac{1}{2}\widetilde{\mathcal{C}}_1(\mathcal{N})=\tfrac{1}{2}+\tfrac{1}{2}\mathcal{C}_1(\mathcal{N}). \tag{19}$$

Again, the result indicates that channels in MIO are mutually indistinguishable by incoherent states since $C_1(\mathcal{N})=0$. Therefore, the task of discriminating a channel from coherence-free ones gives an operational interpretation for the coherence generating power. Compared with [34,35], which only consider the effect of coherence in the probe states in channel discrimination, the results here reveal the roles of coherence in quantum channels in this task.

Since the trace norm of coherence $C_1\leqslant 1-1/d$ [52,53], the success probability $p_{\text{succ}}(\mathcal{N},\mathfrak{I},\mathcal{I})\leqslant 1-1/(2d)$. For example, for the Hadamard gate H on single-qubit system $\mathbb{C}^2$, we have $p_{\text{succ}}(H,\mathfrak{I},\mathcal{I})=3/4$, which follows from the fact that $\mathcal{C}_1(H)=1/2$ (see Appendix A for the calculation of $\mathcal{C}_1$ in the single-qubit system). Due to the equivalence between trace-norm distance and robustness of coherence, it may be expected that this theorem can be experimentally testified in a future work, as the robustness of coherence can be measured in experiment [54,55].

Obviously, $p_{\text{succ}}(\mathcal{N},\mathfrak{I},Q)\geqslant p_{\text{succ}}(\mathcal{N},\mathfrak{I},\mathcal{I})$ for any quantum channel. There exists some quantum channel $\mathcal{N}$ such that the inequality is strict, which shows that the resource of probe states is useful for distinguishing the given channel from the set of free operations.

Proposition 4. For $\mathfrak{I}\in\{SIO, IO\}$, there exists some quantum channel $\mathcal{N}$ such that

$$p_{\text{succ}}(\mathcal{N},\mathfrak{I},Q)>p_{\text{succ}}(\mathcal{N},\mathfrak{I},\mathcal{I}). \tag{20}$$

The proof is presented in Appendix E. The above result shows that the resource feature in probe states is useful for improving the success probability of distinguishing the given channel from the set of free operations $\mathfrak{I} \in \{SIO, IO\}$. However, whether the similar result holds for MIO or DIO is unknown.

By applying Theorem 2 to the resource theory of coherence, we obtain the following upper bound on the success probability when we choose a coherent state as the probe state.

Proposition 5. Given a set of free operations $\mathfrak{I} \in \{SIO, IO, DIO, MIO\}$ and a probe state ρ. For any quantum channel $\mathcal{N}$, we have

$$p_{\text{succ}}(\mathcal{N}, \mathfrak{I}, \rho) - p_{\text{succ}}(\mathcal{N}, \mathfrak{I}, \mathcal{I}) \leqslant \tfrac{1}{2} C_1(\rho). \tag{21}$$

If we restrict the measurement in the channel discrimination to be an incoherent POVM, i.e., diagonal in the given basis $\{|i\rangle\}_i$, then the success probability to distinguish the given two channels by a probe state ρ is

$$p^I_{\text{succ}}(\mathcal{N}, \mathcal{M}, \rho) = \max_{\substack{\{\Pi, \mathbb{I} - \Pi\} \\ \text{diagonal}}} \left\{ \tfrac{1}{2} \operatorname{Tr}[\mathcal{N}(\rho)\Pi] + \tfrac{1}{2} \operatorname{Tr}[\mathcal{M}(\rho)(\mathbb{I} - \Pi)] \right\}. \tag{22}$$

In this case, the success probability of distinguishing the given channel $\mathcal{N}$ from the set of free operations $\mathfrak{I} \in \{SIO, IO, DIO, MIO\}$ is equal to the probability of random guessing.

Theorem 3. Given a quantum channel $\mathcal{N}$ and the set of free operations $\mathfrak{I} \in \{SIO, IO, DIO, MIO\}$, the success probability by incoherent POVM is

$$p^I_{\text{succ}}(\mathcal{N}, \mathfrak{I}, \rho) = \tfrac{1}{2}, \tag{23}$$

for any $\rho \in \mathcal{D}(\mathcal{H})$.

The proof is provided in Appendix E. This indicates that the restriction of incoherent POVM will eliminate the advantage provided by the coherence of the state and channel components of our channel discrimination task. Note that Ref. [28] considers a slightly different scenario (for example, the order of taking minimization over channels and maximization over states is different, which implies the quantity in Ref. [28] could be larger than the quantity we define here, and the set of free operations there consists of detection-incoherent operations, which is different from the sets we consider), where, in contrast, it is possible to detect coherence even by free measurements.

The general results Theorems 1 and 2 can also be applied to other resource theories, such as entanglement, magic states, and so on. For instance, in the resource theory of bipartite entanglement, the free states are separable states, and the free operations are typically chosen to be local operations and classical communication (LOCC), or separable operations (SEP)—the maximal set of entanglement nongenerating operations. Then we have the following.

Proposition 6. Given the set of free operations $\mathfrak{I} \in \{\text{LOCC, SEP}\}$ and a probe state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. For any quantum channel $\mathcal{N}$, we have

$$p_{\text{succ}}(\mathcal{N}, \mathfrak{I}, \rho) - p_{\text{succ}}(\mathcal{N}, \mathfrak{I}, \mathcal{I}) \leqslant \tfrac{1}{2} E_1(\rho_{AB}), \tag{24}$$

where $E_1(\rho_{AB}) := \min_{\sigma \in \text{Sep}(A:B)} \|\rho_{AB} - \sigma\|_1$ and $\text{Sep}(A:B)$ denotes the set of separable states on $\mathcal{H}_A \otimes \mathcal{H}_B$.

As for the free measurement case, in general, we can also define the free measurement $\{\Pi, \mathbb{I} - \Pi\}$, where Π and $\mathbb{I} - \Pi$ are proportional to some free states. If a resource theory has resource destroying channel λ and $\lambda^\dagger$ is a resource destroying channel as well, then Theorem 3 is still true (see Appendix E). However, whether Theorem 3 can be applied to other convex resource theories is unknown.

IV. CONCLUSION

This work considers the fundamental task of channel discrimination from a resource theory perspective, which leads to an intuitive and general framework of operationally quantifying the resource value of quantum channels by how efficiently they can be distinguished from the resource-free ones. The key observation is that the maximum success probability of distinguishing a channel from the set of free operations by all free states is characterized by the trace-norm resource generating power of the channel. As the resource generating power satisfies the properties like positivity, convexity, submultiplicity, and the monotonicity under free operations, it establishes an operational framework of quantifying resource in quantum channels. We demonstrate the power of this framework in the resource theory of quantum coherence. In addition to the de-generalized results, we also show that restricting to incoherent POVMs in this task will eliminate any advantage over random guessing. Our results shed new light on the operational resource theory of quantum channels and in particular the resource theory of coherence. We hope that the framework will lead to more interesting results for a variety of resource theories and information processing tasks.

Several problems are left for future work. First, what we studied here is essentially the worst-case success probability of discriminating from free channels and its universal correspondences in a general resource theory setting. In specific cases, it could also be interesting to analyze the average-case success probability, where we take certain averages instead of minimizing over the set of free channels. Furthermore, the resource measures involving optimizations are generally hard to evaluate for large system size, but in certain cases they may be efficiently computable by, e.g., semidefinite programs. We leave the evaluation of relevant resource measures for future studies.

Note added. Recently, we became aware of a recent work by Liu and Yuan [56], which establishes general connections between the resource generating and increasing power and channel distillation and dilution tasks.

ACKNOWLEDGMENTS

This research was supported by the Templeton Religion Trust Grant TRT 0159 and ARO Grant W911NF1910302. L.L. and K.B. acknowledge Arthur Jaffe for his support and help. K.B. also acknowledges the support of Academic Awards for Outstanding Doctoral Candidates from Zhejiang University. Z.-W.L. is supported by AFOSR, ARO, and Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

APPENDIX A: CONNECTIONS BETWEEN CHANNEL DISCRIMINATION AND RESOURCE GENERATING/INCREASING POWER

Given a distance measure $D : \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \to \mathbb{R}_+$, we consider the following conditions.

(1) Positivity: $D(\rho,\sigma) \geqslant 0$, $D(\rho,\sigma) = 0$ iff $\rho = \sigma$.

(2) Pseudojoint convexity: $D(\sum_i p_i\rho_i, \sum_i p_i\sigma_i) \leqslant \max_i D(\rho_i,\sigma_i)$ with $\sum_i p_i = 1$.

(2′) Joint convexity: $D(\sum_i p_i\rho_i, \sum_i p_i\sigma_i) \leqslant \sum_i p_i D(\rho_i,\sigma_i)$ with $\sum_i p_i = 1$.

(3) Data processing inequality: $D(\mathcal{N}(\rho), \mathcal{N}(\sigma)) \leqslant D(\rho,\sigma)$ for any CPTP map $\mathcal{N}$.

(4) Triangle inequality: $D(\rho,\sigma) \leqslant D(\rho,\tau) + D(\tau,\sigma)$ for any $\tau \in \mathcal{D}(\mathcal{H})$.

Here, we assume the distance measure always satisfies the condition (1), i.e., positivity.

Lemma 1. For any given distance measure D and quantum channel $\mathcal{N}$, it holds that

$$\Omega_D(\mathcal{N}) = \max_{\rho\in\mathcal{F}} \min_{\mathcal{M}\in\mathfrak{F}} D(\mathcal{N}(\rho), \mathcal{M}(\rho)). \tag{A1}$$

Proof. First, we have

$$\begin{aligned}
&\max_{\rho\in\mathcal{F}} \min_{\mathcal{M}\in\mathfrak{F}} D(\mathcal{N}(\rho), \mathcal{M}(\rho))\\
&\quad\geqslant \max_{\rho\in\mathcal{F}} \min_{\sigma\in\mathcal{F}} D(\mathcal{N}(\rho), \sigma)\\
&\quad= \max_{\rho\in\mathcal{F}} \omega_D(\mathcal{N}(\rho))\\
&\quad= \Omega_D(\mathcal{N}),
\end{aligned}$$

where the inequality comes from the fact that $\mathcal{M}(\rho) \in \mathcal{F}$ for any $\rho \in \mathcal{F}$.

Besides, for any $\rho \in \mathcal{F}$, we can define the quantum channel $\mathcal{N}_\rho$ as $\mathcal{N}_\rho(\tau) = \sigma^\star_{\mathcal{N}(\rho)}$ for any quantum state $\tau \in \mathcal{D}(\mathcal{H})$ with $\sigma^\star_{\mathcal{N}(\rho)} \in \mathcal{F}$ and $\omega_D(\mathcal{N}(\rho)) = D(\mathcal{N}(\rho), \sigma^\star_{\mathcal{N}(\rho)})$. It is easy to verify that $\mathcal{N}_\rho$ is a free operation, i.e., $\mathcal{N}_\rho \in \mathfrak{F}$. Thus,

$$\begin{aligned}
&\max_{\rho\in\mathcal{F}} \min_{\mathcal{M}\in\mathfrak{F}} D(\mathcal{N}(\rho), \mathcal{M}(\rho))\\
&\quad\leqslant \max_{\rho\in\mathcal{F}} D(\mathcal{N}(\rho), \mathcal{N}_\rho(\rho))\\
&\quad= \max_{\rho\in\mathcal{F}} D(\mathcal{N}(\rho), \sigma^\star_{\mathcal{N}(\rho)})\\
&\quad= \max_{\rho\in\mathcal{F}} \omega_D(\mathcal{N}(\rho))\\
&\quad= \Omega_D(\mathcal{N}),
\end{aligned}$$

where the inequality comes from the fact that $\mathcal{N}_\rho \in \mathfrak{F}$ and $\mathcal{N}_\rho$ maps any quantum state to the free state $\sigma^\star_{\mathcal{N}(\rho)}$. ■

Lemma 2. If the distance measure D satisfies the triangle inequality and the data processing inequality (i.e., nonincreasing under CPTP maps), then we have

$$\Omega_D(\mathcal{N}) = \widetilde{\Omega}_D(\mathcal{N}). \tag{A2}$$

Proof. It is obvious that $\Omega_D \leqslant \widetilde{\Omega}_D$, thus we only need to prove $\widetilde{\Omega}_D \leqslant \Omega_D$.

For any quantum state $\rho \in \mathcal{D}(\mathcal{H})$, we have

$$\begin{aligned}
&\omega_D(\mathcal{N}(\rho)) - \omega_D(\rho)\\
&\quad= \min_{\sigma\in\mathcal{F}} D(\mathcal{N}(\rho), \sigma) - \min_{\tau\in\mathcal{F}} D(\rho, \tau)\\
&\quad= \max_{\tau\in\mathcal{F}}[\min_{\sigma\in\mathcal{F}}(D(\mathcal{N}(\rho), \sigma) - D(\rho, \tau))]\\
&\quad\leqslant \max_{\tau\in\mathcal{F}} \min_{\sigma\in\mathcal{F}}[D(\mathcal{N}(\rho), \sigma) - D(\mathcal{N}(\rho), \mathcal{N}(\tau))]\\
&\quad\leqslant \max_{\tau\in\mathcal{F}} \min_{\sigma\in\mathcal{F}} D(\mathcal{N}(\tau), \sigma)\\
&\quad= \max_{\tau\in\mathcal{F}} \omega_D(\mathcal{N}(\tau))\\
&\quad= \Omega_D(\mathcal{N}),
\end{aligned}$$

where the first inequality comes from the data processing inequality and the second inequality comes from the triangle inequality of D. Therefore, we have $\widetilde{\Omega}_D(\mathcal{N}) \leqslant \Omega_D(\mathcal{N})$. ■

Proof of Theorem 1. It is easy to verify that the trace norm satisfies the data processing inequality and the triangle inequality. Thus, according to Lemmas 1 and 2, we have

$$\widetilde{\Omega}_1(\mathcal{N}) = \Omega_1(\mathcal{N}) = \tfrac{1}{2} \max_{\rho\in\mathcal{F}} \min_{\mathcal{M}\in\mathfrak{F}} \|\mathcal{N}(\rho) - \mathcal{M}(\rho)\|_1.$$

Besides, the success probability $p_{\text{succ}}(\mathcal{N}, \mathfrak{F}, \mathcal{F})$ can be expressed as

$$\begin{aligned}
p_{\text{succ}}(\mathcal{N}, \mathfrak{F}, \mathcal{F}) &= \tfrac{1}{2} + \tfrac{1}{4} \max_{\rho\in\mathcal{F}} \min_{\mathcal{M}\in\mathfrak{F}} \|\mathcal{N}(\rho) - \mathcal{M}(\rho)\|_1\\
&= \tfrac{1}{2} + \tfrac{1}{2}\widetilde{\Omega}_1(\mathcal{N})\\
&= \tfrac{1}{2} + \tfrac{1}{2}\Omega_1(\mathcal{N}).
\end{aligned}$$

■

Corollary 2. If we take the distance measure D to be max-relative entropy $D_{\max}$ or fidelity D_F, then we have

$$\widetilde{\Omega}_D(\mathcal{N}) = \Omega_D(\mathcal{N}) = \max_{\rho\in\mathcal{F}} \min_{\mathcal{M}\in\mathfrak{F}} D(\mathcal{N}(\rho), \mathcal{M}(\rho)), \tag{A3}$$

where $D_F(\rho,\sigma) = \sqrt{1 - F^2(\rho,\sigma)}$ with $F(\rho,\sigma) = \text{Tr}\,[|\sqrt{\rho}\sqrt{\sigma}|]$.

Proof. It has been proved that $D_{\max}$ satisfies the data processing inequality [57] and the triangle inequality follows directly from the definitions. Besides, it has been proved that D_F satisfies the data processing inequality [58] and the triangle inequality [59,60]. ■

Now, let us consider the example of coherence. In the single-qubit system, it has been proved that the trace norm of coherence C_1 is equivalent to the l_1 norm of coherence C_{l_1} [46,48] and the analytic form of coherence generating power for unitary operations has been obtained in [49]. Therefore, we have the following corollary.

Corollary 3. Given a single-qubit unitary $U = [U_{ij}]_{i,j=1,2}$, the coherence generating power by the trace norm is

$$\mathcal{C}_1(U) = \max_{i=1,2} |U_{i1}U_{i2}|, \tag{A4}$$

especially, for the Hadamard gate H, $\mathcal{C}_1(H) = 1/2$.

APPENDIX B: PROPERTIES OF $\Omega_D(\mathcal{N})$

Now, let us investigate the properties of $\Omega_D(\mathcal{N})$ for any distance measure D. We assume that the free states on $\mathcal{H}_A \otimes \mathcal{H}_B$ are defined as the convex combination of the tensor product of free states on $\mathcal{H}_A$ and $\mathcal{H}_B$, i.e., $\mathcal{F}_{AB} = \text{Conv}\,\{\mathcal{F}_A \otimes \mathcal{F}_B\}$.

Lemma 3. Given any distance measure D, $\Omega_D(\cdot)$ has the following properties.

(i) $\Omega_D(\mathcal{N}) \geqslant 0$, and $\Omega_D(\mathcal{N}) = 0$ if $\mathcal{N} \in \mathfrak{F}$. Moreover, if $\mathfrak{F}$ includes all CPTP maps which maps all free states to free states, then $\Omega_D(\mathcal{N}) = 0$ iff $\mathcal{N} \in \mathfrak{F}$.

(ii) If the distance measure D satisfies the data processing inequality, for any $\mathcal{M}_1, \mathcal{M}_2 \in \mathfrak{F}$,

$$\Omega_D(\mathcal{M}_1 \circ \mathcal{N} \circ \mathcal{M}_2) \leqslant \Omega_D(\mathcal{N}). \tag{B1}$$

(iii) If the distance measure D satisfies joint convexity, given a set of quantum channels $\{\mathcal{N}_i, p_i\}_i$ with $\sum_i p_i = 1$,

$$\Omega_D\left(\sum_i p_i \mathcal{N}_i\right) \leqslant \sum_i p_i \Omega_D(\mathcal{N}_i). \tag{B2}$$

(iv) If the distance measure D satisfies the pseudojoint convexity and data processing inequality, given two channels $\mathcal{N}_1$ and $\mathcal{N}_2$, it holds that

$$\Omega_D(\mathcal{N}_1 \otimes \mathcal{N}_2) \geqslant \max\{\Omega_D(\mathcal{N}_1), \Omega_D(\mathcal{N}_2)\}. \tag{B3}$$

(v) If the distance measure D satisfies the pseudojoint convexity, data processing inequality, and triangle inequality, given two channels $\mathcal{N}_1$ and $\mathcal{N}_2$, it holds that

$$\Omega_D(\mathcal{N}_1 \otimes \mathcal{N}_2) \leqslant \Omega_D(\mathcal{N}_1) + \Omega_D(\mathcal{N}_2). \tag{B4}$$

Proof. (i) This comes directly from the definition.

(ii) For any $\mathcal{M} \in \mathfrak{F}$,

$$\begin{aligned}
&\Omega_D(\mathcal{M} \circ \mathcal{N}) \\
&= \max_{\rho \in \mathcal{F}} \min_{\mathcal{M} \in \mathfrak{F}} D(\mathcal{M} \circ \mathcal{N}(\rho), \mathcal{M}(\rho)) \\
&\leqslant \max_{\rho \in \mathcal{F}} \min_{\mathcal{M} \in \mathfrak{F}} D(\mathcal{M} \circ \mathcal{N}(\rho), \mathcal{M} \circ \mathcal{M}(\rho)) \\
&\leqslant \max_{\rho \in \mathcal{F}} \min_{\mathcal{M} \in \mathfrak{F}} D(\mathcal{N}(\rho), \mathcal{M}(\rho)) \\
&= \Omega_D(\mathcal{N}),
\end{aligned}$$

where the first inequality comes from the fact that $\mathcal{M} \circ \mathcal{M} \in \mathfrak{F}$ for any $\mathcal{M} \in \mathfrak{F}$ and the second inequality comes from the data processing inequality.

Besides,

$$\Omega_D(\mathcal{N} \circ \mathcal{M}) = \max_{\rho \in \mathcal{F}} \omega_D(\mathcal{N}(\mathcal{M}(\rho))) \leqslant \max_{\rho \in \mathcal{F}} \omega_D(\mathcal{N}(\rho)),$$

where the inequality comes from the fact $\mathcal{M}(\mathcal{F}) \subset \mathcal{F}$. ■

(iii) Since D is jointly convex, then the corresponding resource monotone ω_D is convex, i.e., $\omega_D(\sum_i p_i \rho_i) \leqslant \sum_i p_i \omega_D(\rho_i)$. Thus,

$$\begin{aligned}
&\Omega_D\left(\sum_i p_i \mathcal{N}_i\right) \\
&= \max_{\rho \in \mathcal{F}} \omega_D\left(\sum_i p_i \mathcal{N}_i(\rho)\right) \\
&\leqslant \max_{\rho \in \mathcal{F}} \sum_i p_i \omega_D(\mathcal{N}_i(\rho)) \\
&\leqslant \sum_i p_i \max_{\rho \in \mathcal{F}} \omega_D(\mathcal{N}_i(\rho)) \\
&= \sum_i p_i \Omega_D(\mathcal{N}_i).
\end{aligned}$$

(iv) We only need to prove that $\max\{\omega_D(\rho_1), \omega_D(\rho_2)\} \leqslant \omega_D(\rho_1 \otimes \rho_2)$.

First,

$$\min_{\tau_{12} \in \mathcal{F}_{12}} D(\rho_1 \otimes \rho_2, \tau_{12}) \geqslant \min_{\tau_1 \in \mathcal{F}_1} D(\rho_1, \tau_1), \tag{B5}$$

where $\tau_1 = \mathrm{Tr}_2[\tau_{12}]$ and the inequality comes from the data processing inequality. Hence, we have $\omega_D(\rho_1 \otimes \rho_2) \geqslant \omega_D(\rho_1)$. Similarly, we have $\omega_D(\rho_1 \otimes \rho_2) \geqslant \omega_D(\rho_2)$.

(v) We only need to prove that $\omega_D(\rho_1 \otimes \rho_2) \leqslant \omega_D(\rho_1) + \omega_D(\rho_2)$. Due to the data processing inequality, we have

$$D(\rho, \sigma) = D(\rho \otimes \tau, \sigma \otimes \tau), \tag{B6}$$

because both partial trace and tensoring with a quantum state are CPTP maps.

Therefore, we have

$$\begin{aligned}
&\min_{\tau_{12} \in \mathcal{I}} D(\rho_1 \otimes \rho_2, \tau_{12}) \\
&\leqslant D(\rho_1 \otimes \rho_2, \tau_1 \otimes \tau_2) \\
&\leqslant D(\rho_1 \otimes \rho_2, \tau_1 \otimes \rho_2) + D(\tau_1 \otimes \rho_2, \tau_1 \otimes \tau_2) \\
&= D(\rho_1, \tau_1) + D(\rho_2, \tau_2) \\
&= \omega_D(\rho_1) + \omega_D(\rho_2),
\end{aligned}$$

where the free states τ_1 and τ_2 are chosen to satisfy the conditions $\omega_D(\rho_1) = D(\rho_1, \tau_1)$ and $\omega_D(\rho_2) = D(\rho_2, \tau_2)$.

Proof of Proposition 2. Since the trace norm satisfies the joint convexity, data processing inequality and triangle inequality, then Proposition 2 comes directly from Lemma 3. ■

APPENDIX C: UPPER BOUND FOR $p_{\mathrm{succ}}(\mathcal{N}, \mathfrak{F}, \rho)$

Proof of Theorem 2. Since

$$\tfrac{1}{2} \min_{\mathcal{M} \in \mathfrak{F}} \|\mathcal{N}(\rho) - \mathcal{M}(\rho)\|_1 \leqslant \omega_1(\mathcal{N}(\rho)),$$

then by Theorem 2 and the definition of $p_{\mathrm{succ}}(\mathcal{N}, \mathfrak{F}, \rho)$, we have

$$\begin{aligned}
&p_{\mathrm{succ}}(\mathcal{N}, \mathfrak{F}, \rho) - p_{\mathrm{succ}}(\mathcal{N}, \mathfrak{F}, \mathcal{F}) \\
&\quad = \tfrac{1}{4} \min_{\mathcal{M} \in \mathfrak{F}} \|\mathcal{N}(\rho) - \mathcal{M}(\rho)\|_1 - \tfrac{1}{2} \widetilde{\Omega}_1(\mathcal{N}) \\
&\quad \leqslant \tfrac{1}{2}(\omega_1(\mathcal{N}(\rho)) - \widetilde{\Omega}_1(\mathcal{N})) \\
&\quad \leqslant \tfrac{1}{2} \omega_1(\rho),
\end{aligned}$$

where the second inequality comes from the fact that

$$\omega_1(\mathcal{N}(\rho)) - \omega_1(\rho) \leqslant \widetilde{\Omega}_1(\mathcal{N}),$$

for any $\rho \in \mathcal{D}(\mathcal{H})$. Thus, we complete the proof. ■

APPENDIX D: IMPROVEMENT FROM COHERENT STATES IN CHANNEL DISCRIMINATION

Proof of Proposition 4. It has been shown that there exists some quantum channel $\mathcal{N}_* \in MIO$ but not IO, i.e., there exists some quantum state ρ such that $\mathcal{N}_*(\rho) \neq \mathcal{M}(\rho)$ for any $\mathcal{M} \in IO$ [61], which implies that

$$\max_{\mathcal{M} \in IO} \|\mathcal{N}_*(\rho) - \mathcal{M}(\rho)\|_1 > 0.$$

Thus, we have $p_{\text{succ}}(\mathcal{N}_*, IO, Q) > 1/2$. However, due to Proposition 6, we have

$$\begin{aligned} p_{\text{succ}}(\mathcal{N}_*, SIO, \mathcal{I}) &= p_{\text{succ}}(\mathcal{N}_*, IO, \mathcal{I}) \\ &= p_{\text{succ}}(\mathcal{N}_*, MIO, \mathcal{I}) = 1/2, \end{aligned}$$

as $\mathcal{N}_* \in MIO$. Thus, we have

$$p_{\text{succ}}(\mathcal{N}_*, IO, Q) > p_{\text{succ}}(\mathcal{N}_*, IO, \mathcal{I}).$$

Besides, since $SIO \subset IO$, then $p_{\text{succ}}(\mathcal{N}_*, SIO, Q) \geqslant p_{\text{succ}}(\mathcal{N}_*, IO, Q)$. Therefore,

$$p_{\text{succ}}(\mathcal{N}_*, SIO, Q) > p_{\text{succ}}(\mathcal{N}_*, SIO, \mathcal{I}).$$ ■

APPENDIX E: DISCRIMINATION WITH INCOHERENT MEASUREMENT

Proof of Theorem 3. It is easy to see that

$$\begin{aligned} &\max_{\substack{\{\Pi, \mathbb{I}-\Pi\} \\ \Pi \text{diagonal}}} \left\{\tfrac{1}{2}\operatorname{Tr}[\mathcal{N}(\rho)\Pi] + \tfrac{1}{2}\operatorname{Tr}[\mathcal{M}(\rho)(\mathbb{I}-\Pi)]\right\} \\ &= \max_{\substack{\{\Pi, \mathbb{I}-\Pi\} \\ \Pi \text{diagonal}}} \left\{\tfrac{1}{2}\operatorname{Tr}[\mathcal{N}(\rho)\Delta(\Pi)] + \tfrac{1}{2}\operatorname{Tr}[\mathcal{M}(\rho)(\mathbb{I}-\Delta(\Pi))]\right\} \\ &= \max_{\substack{\{\Pi, \mathbb{I}-\Pi\} \\ \Pi \text{diagonal}}} \left\{\tfrac{1}{2}\operatorname{Tr}[\Delta^\dagger \circ \mathcal{N}(\rho)\Pi] + \tfrac{1}{2}\operatorname{Tr}[\Delta^\dagger \circ \mathcal{M}(\rho)(\mathbb{I}-\Pi)]\right\} \\ &\leqslant \max_{\{\Pi, \mathbb{I}-\Pi\}} \left\{\tfrac{1}{2}\operatorname{Tr}[\Delta^\dagger \circ \mathcal{N}(\rho)\Pi] + \tfrac{1}{2}\operatorname{Tr}[\Delta^\dagger \circ \mathcal{M}(\rho)(\mathbb{I}-\Pi)]\right\} \\ &= \tfrac{1}{2} + \tfrac{1}{4}\|\Delta^\dagger \circ \mathcal{N}(\rho) - \Delta^\dagger \circ \mathcal{M}(\rho)\|_1. \end{aligned}$$

Besides, $\Delta^\dagger$ satisfies the conditions that $\Delta^\dagger(\mathcal{D}(\mathcal{H})) \subset \mathcal{I}$ and $\Delta^\dagger(\rho) = \rho$ for any $\rho \in \mathcal{I}$, which implies that

$$\tfrac{1}{2}\min_{\mathcal{M}\in\mathfrak{J}} \|\Delta^\dagger \circ \mathcal{N}(\rho) - \Delta^\dagger \circ \mathcal{M}(\rho)\|_1 = C_1(\Delta^\dagger \circ \mathcal{N}(\rho)) = 0.$$ ■

Note that, in any other resource theory with resource destroying channel λ, we can also define the free measurement $\{\Pi, \mathbb{I}-\Pi\}$, where Π and $\mathbb{I}-\Pi$ are proportional to some free states. Then it is easy to see that the above proof still works for the free measurement case if λ satisfies the condition that $\lambda^\dagger(\mathcal{D}(\mathcal{H})) \subset \mathcal{F}$ and $\lambda^\dagger(\rho) = \rho$ for any $\rho \in \mathcal{F}$, i.e., $\lambda^\dagger$ is a resource destroying map [7].

[1] E. Chitambar and G. Gour, Quantum resource theories, Rev. Mod. Phys. **91**, 025001 (2019).
[2] M. B. Plenio and S. Virmani, An introduction to entanglement measures, Quantum Inf. Comput. **7**, 1 (2007).
[3] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2010).
[5] M. Horodecki and J. Oppenheim, (quantumness in the context of) resource theories, Int. J. Mod. Phys. B **27**, 1345019 (2013).
[6] F. G. S. L. Brandão and G. Gour, Reversible Framework for Quantum Resource Theories, Phys. Rev. Lett. **115**, 070503 (2015).
[7] Z.-W. Liu, X. Hu, and S. Lloyd, Resource Destroying Maps, Phys. Rev. Lett. **118**, 060502 (2017).
[8] B. Regula, Convex geometry of quantum resource quantification, J. Phys. A **51**, 045303 (2018).
[9] A. Anshu, M.-H. Hsieh, and R. Jain, Quantifying Resources in General Resource Theory with Catalysts, Phys. Rev. Lett. **121**, 190504 (2018).
[10] R. Takagi, B. Regula, K. Bu, Z.-W. Liu, and G. Adesso, Operational Advantage of Quantum Resources in Subchannel Discrimination, Phys. Rev. Lett. **122**, 140402 (2019).
[11] P. Skrzypczyk and N. Linden, Robustness of Measurement, Discrimination Games, and Accessible Information, Phys. Rev. Lett. **122**, 140403 (2019).
[12] Z.-W. Liu, K. Bu, and R. Takagi, One-Shot Operational Quantum Resource Theory, Phys. Rev. Lett. **123**, 020401 (2019).
[13] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying Coherence, Phys. Rev. Lett. **113**, 140401 (2014).
[14] A. Winter and D. Yang, Operational Resource Theory of Coherence, Phys. Rev. Lett. **116**, 120404 (2016).
[15] A. Streltsov, G. Adesso, and M. B. Plenio, Colloquium: Coherence, Rev. Mod. Phys. **89**, 041003 (2017).
[16] T. Theurer, N. Killoran, D. Egloff, and M. B. Plenio, Resource Theory of Superposition, Phys. Rev. Lett. **119**, 230401 (2017).
[17] V. Veitch, S. A. H. Mousavian, D. Gottesman, and J. Emerson, The resource theory of stabilizer quantum computation, New J. Phys. **16**, 013009 (2014).
[18] M. Howard and E. Campbell, Application of a Resource Theory for Magic States to Fault-Tolerant

Quantum Computing, Phys. Rev. Lett. **118**, 090501 (2017).

[19] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Resource Theory of Quantum States Out of Thermal Equilibrium, Phys. Rev. Lett. **111**, 250404 (2013).

[20] M. Horodecki and J. Oppenheim, Fundamental limitations for quantum and nanoscale thermodynamics, Nat. Commun. **4**, 2059 (2013).

[21] G. Gour, I. Marvian, and R. W. Spekkens, Measuring the quality of a quantum reference frame: The relative entropy of frameness, Phys. Rev. A **80**, 012307 (2009).

[22] I. Marvian and R. W. Spekkens, Extending Noether's theorem by quantifying the asymmetry of quantum states, Nat. Commun. **5**, 3821 (2014).

[23] B. Coecke, T. Fritz, and R. W. Spekkens, A mathematical theory of resources, Inf. Comput. **250**, 59 (2016).

[24] T. Fritz, Resource convertibility and ordered commutative monoids, Math. Struct. Comput. Sci. **27**, 850 (2017).

[25] Z.-W. Liu and A. Winter, Resource theories of quantum channels and the universal role of resource erasure, arXiv:1904.04201.

[26] C. H. Bennett, A. W. Harrow, D. W. Leung, and J. A. Smolin, On the capacities of bipartite Hamiltonians and unitary gates, IEEE Trans. Inf. Theory **49**, 1895 (2003).

[27] K. Ben Dana, M. García Díaz, M. Mejatty, and A. Winter, Resource theory of coherence: Beyond states, Phys. Rev. A **95**, 062327 (2017).

[28] T. Theurer, D. Egloff, L. Zhang, and M. B. Plenio, Quantifying the Coherence of Operations, Phys. Rev. Lett. **122**, 190405 (2019).

[29] Q. Zhuang, P. W. Shor, and J. H. Shapiro, Resource theory of non-Gaussian operations, Phys. Rev. A **97**, 052317 (2018).

[30] X. Wang, M. M. Wilde, and Y. Su, Quantifying the magic of quantum channels, New J. Phys. **21**, 103002 (2019).

[31] A. Acín, Statistical Distinguishability Between Unitary Operations, Phys. Rev. Lett. **87**, 177901 (2001).

[32] G. Wang and M. Ying, Unambiguous discrimination among quantum operations, Phys. Rev. A **73**, 042301 (2006).

[33] S. Pirandola, R. Laurenza, C. Lupo, and J. L. Pereira, Fundamental limits to quantum channel discrimination, npj Quantum Inf. **5**, 50 (2019).

[34] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, Robustness of Coherence: An Operational and Observable Measure of Quantum Coherence, Phys. Rev. Lett. **116**, 150502 (2016).

[35] K. Bu, U. Singh, S.-M. Fei, A. K. Pati, and J. Wu, Maximum Relative Entropy of Coherence: An Operational Coherence Measure, Phys. Rev. Lett. **119**, 150405 (2017).

[36] J. Bae, D. Chruściński, and M. Piani, More Entanglement Implies Higher Performance in Channel Discrimination Tasks, Phys. Rev. Lett. **122**, 140404 (2019).

[37] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).

[38] V. Giovannetti, S. Lloyd, and L. Maccone, Advances in quantum metrology, Nat. Photonics **5**, 222 (2011).

[39] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Quantum Coherence, Time-Translation Symmetry, and Thermodynamics, Phys. Rev. X **5**, 021001 (2015).

[40] M. Lostaglio, D. Jennings, and T. Rudolph, Description of quantum coherence in thermodynamic processes requires constraints beyond free energy, Nat. Commun. **6**, 6383 (2015).

[41] M. B. Plenio and S. F. Huelga, Dephasing-assisted transport: quantum networks and biomolecules, New J. Phys. **10**, 113019 (2008).

[42] F. Levi and F. Mintert, A quantitative theory of coherent delocalization, New J. Phys. **16**, 033007 (2014).

[43] M.-L. Hu, X. Hu, J. Wang, Y. Peng, Y.-R. Zhang, and H. Fan, Quantum coherence and geometric quantum discord, Phys. Rep. **762-764**, 1 (2018).

[44] E. Chitambar and G. Gour, Comparison of incoherent operations and measures of coherence, Phys. Rev. A **94**, 052336 (2016).

[45] E. Chitambar and G. Gour, Critical Examination of Incoherent Operations and a Physically Consistent Resource Theory of Quantum Coherence, Phys. Rev. Lett. **117**, 030401 (2016).

[46] L.-H. Shao, Z. Xi, H. Fan, and Y. Li, Fidelity and trace-norm distances for quantifying coherence, Phys. Rev. A **91**, 042120 (2015).

[47] M. Piani, M. Cianciaruso, T. R. Bromley, C. Napoli, N. Johnston, and G. Adesso, Robustness of asymmetry and coherence of quantum states, Phys. Rev. A **93**, 042107 (2016).

[48] S. Rana, P. Parashar, and M. Lewenstein, Trace-distance measure of coherence, Phys. Rev. A **93**, 012110 (2016).

[49] K. Bu, A. Kumar, L. Zhang, and J. Wu, Cohering power of quantum operations, Phys. Lett. A **381**, 1670 (2017).

[50] M. G. Díaz, K. Fang, X. Wang, M. Rosati, M. Skotiniotis, J. Calsamiglia, and A. Winter, Using and reusing coherence to realize quantum processes, Quantum **2**, 100 (2018).

[51] X. Yuan, Relative entropies of quantum channels with applications in resource theory, Phys. Rev. A **99**, 032317 (2019).

[52] J. Chen, S. Grogan, N. Johnston, C.-K. Li, and S. Plosker, Quantifying the coherence of pure quantum states, Phys. Rev. A **94**, 042313 (2016).

[53] X.-D. Yu, D.-J. Zhang, G. F. Xu, and D. M. Tong, Alternative framework for quantifying coherence, Phys. Rev. A **94**, 060302(R) (2016).

[54] Y.-T. Wang, J.-S. Tang, Z.-Y. Wei, S. Yu, Z.-J. Ke, X.-Y. Xu, C.-F. Li, and G.-C. Guo, Directly Measuring the Degree of Quantum Coherence using Interference Fringes, Phys. Rev. Lett. **118**, 020403 (2017).

[55] W. Zheng, Z. Ma, H. Wang, S.-M. Fei, and X. Peng, Experimental Demonstration of Observability and Operability of Robustness of Coherence, Phys. Rev. Lett. **120**, 230504 (2018).

[56] Y. Liu and X. Yuan, Operational resource theory of quantum channels, arXiv:1904.02680.

[57] N. Datta, Min- and max-relative entropies and a new entanglement monotone, IEEE Trans. Inf. Theory **55**, 2816 (2009).

[58] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, Noncommuting Mixed States Cannot be Broadcast, Phys. Rev. Lett. **76**, 2818 (1996).

[59] A. Gilchrist, N. K. Langford, and M. A. Nielsen, Distance measures to compare real and ideal quantum processes, Phys. Rev. A **71**, 062310 (2005).

[60] A. Uhlmann, Noncommuting mixed states cannot be broadcast, Rep. Math. Phys. **9**, 273 (1976).

[61] K. Bu and C. Xiong, A note on cohering power and de-cohering power, Quantum Inf. Comput. **13**, 1206 (2017).