QED calculations of the nuclear recoil effect on the bound-electron *g* factor

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A fully relativistic approach is applied to the evaluation of the nuclear recoil effect on the bound-electron g factor in hydrogenlike ions to first order in the electron-to-nucleus mass ratio m/M and to all orders in αZ. The calculations are performed in the range $1 \leqslant Z \leqslant 20$ for g factors of the $1s$, $2s$, $2p_{1/2}$, and $2p_{3/2}$ states. The αZ dependence of the nontrivial QED recoil contribution as a function of Z is studied.

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I. INTRODUCTION

In recent decades, considerable progress in theoretical and experimental investigations of the bound-electron g factor in few-electron ions has been achieved (for a review, see, e.g., Refs. [1,2] and references therein). For instance, high-precision measurements of the g factor in hydrogenlike ions accompanied by elaborate quantum electrodynamics (QED) calculations lead to the most accurate determination of the electron mass [3–8]. On the other hand, a comparison of experimental data and theoretical predictions provides a stringent test of the magnetic sector of bound-state QED. The g-factor investigations in lithiumlike [9–13] and boronlike [14] ions create possibilities to study the many-electron QED effects on the Zeeman splitting. There are also proposals on how to employ these studies for an independent determination of the fine-structure constant [15–17]. Moreover, the measurements of the g factor of ions with nonzero nuclear spin will make possible the precise determination of nuclear magnetic moments [18–21].

The measurement of the isotope shift of the ground-state g factor in Li-like calcium [11] has triggered special interest in the relativistic calculations of the g-factor contribution due to the nuclear recoil effect. The fully relativistic description of this effect on the atomic g factor requires the development of QED approaches which are beyond the usual Furry picture formalism [22], i.e., beyond the external-field approximation which treats the nucleus merely as a source of the classical electromagnetic field. A fully relativistic evaluation of the recoil contribution to the g factor of the $1s$ state was performed in Ref. [23] using the QED formalism developed in Ref. [24]. In Ref. [25] the effective four-component operators to treat the nuclear recoil effect on the atomic g factor within the lowest-order relativistic (Breit) approximation were derived. With the help of these operators, precise theoretical predictions for the nuclear recoil contribution to the bound-electron g factor in lithiumlike ions were obtained [25,26]. The possibility of probing the fully relativistic QED recoil contribution on a few-percent level in a specific difference of the g factors of heavy H- and Li-like ions was discussed in Ref. [27]. Finally, the nuclear recoil contribution to the bound-electron g factor in B-like ions was considered in Refs. [28–32].

The present study is devoted to the high-precision QED evaluation of the nuclear recoil effect on the bound-electron g factor of the $1s$, $2s$, $2p_{1/2}$, and $2p_{3/2}$ states in H-like ions in the range $Z = 1$–20. For the s states, previous calculations of the QED recoil contribution to the g factor were extended in order to cover all the ions within the range specified. For particular ions which were considered previously [23,25], the accuracy of the theoretical predictions has improved. For the $2p_{1/2}$ state, to date, this term has been evaluated for $Z \geqslant 20$ only [30]. The QED recoil contribution to the g factor of the $2p_{3/2}$ state has not been considered previously. The αZ dependence of all the obtained values is studied and the leading orders in αZ are extracted. Such investigations may be useful for a comparison of the numerical all-order and analytical αZ-expansion approaches to the nuclear recoil effect on the g factor (see, e.g., the corresponding analysis for binding energies [33,34]). The nuclear recoil effect on the g factor of few-electron ions comprises the one-electron contribution evaluated in the present work and the many-electron contributions which can be calculated within the Breit approximation employing the corresponding effective operators [25]. These calculations are in demand in view of the presently implemented ARTEMIS experiment [35,36] at GSI in Darmstadt and ALPHATRAP experiment at the Max-Planck-Institut für Kernphysik (MPIK) in Heidelberg [14,37]. These experiments are expected to attain an accuracy of 10^{-9}–10^{-10} and better for the g factors of low- and high-Z few-electron ions [2]. Therefore, the proper treatment of the nuclear recoil effect, which contributes to the bound-electron g factor on the level of 10^{-8}–10^{-5}, is an urgent task.

Relativistic units ($\hbar = 1$ and $c = 1$) and Heaviside charge units ($e^2 = 4\pi\alpha$, where $e < 0$) are employed throughout the paper.

II. THEORETICAL METHODS

The fully relativistic theory of the nuclear recoil effect on the bound-electron g factor to first order in the electron-to-nucleus mass ratio m/M and to all orders in αZ (α is the fine-structure constant and Z is the nuclear charge number) was formulated in Ref. [24]. Let us briefly review the basic results obtained therein for a hydrogenlike ion. The ion with a spinless nucleus is assumed to be placed in the homogeneous magnetic field $\boldsymbol{\mathcal{H}}$ described by the classical vector potential of the form $\mathbf{A}_{\rm cl}(\mathbf{r}) = [\boldsymbol{\mathcal{H}} \times \mathbf{r}]/2$. Within the zeroth-order approximation, the electron obeys the Dirac equation with the spherically symmetric binding potential of the pointlike nucleus $V(r) = -\alpha Z/r$,

$$h^{\rm D}|n\rangle \equiv (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m + V)|n\rangle = \varepsilon_n|n\rangle, \quad (1)$$

where $\boldsymbol{\alpha}$ and β are the Dirac matrices and $\mathbf{p}$ is the momentum operator. For simplicity, we direct the z axis along the magnetic field $\boldsymbol{\mathcal{H}} = \mathcal{H}\mathbf{e}_z$. Then the contribution to the Dirac Hamiltonian due to the coupling with $\boldsymbol{\mathcal{H}}$ reads $-e\boldsymbol{\alpha}\cdot\mathbf{A}_{\rm cl}(\mathbf{r}) = \mu_0\mathcal{H}m[\mathbf{r}\times\boldsymbol{\alpha}]_z$, where $\mu_0 = |e|/2m$ is the Bohr magneton. According to Ref. [24], the nuclear recoil contribution to the g factor of the state $|a\rangle$ with the Dirac energy ε_a and the angular momentum projection m_a is conveniently represented by the sum of two terms $\Delta g = \Delta g_{\rm L} + \Delta g_{\rm H}$, where

$$\Delta g_{\rm L} = \frac{1}{m_a}\frac{m}{M}(\langle\delta a|[\mathbf{p}^2 - 2\mathbf{p}\cdot\mathbf{D}(0)]|a\rangle - \langle a|\{[\mathbf{r}\times\mathbf{p}]_z - [\mathbf{r}\times\mathbf{D}(0)]_z\}|a\rangle), \quad (2)$$

$$\Delta g_{\rm H} = \frac{1}{m_a}\frac{m}{M}\frac{i}{2\pi}\int_{-\infty}^{\infty}d\omega\{\langle\delta a|B^k_-(\omega)G(\omega+\varepsilon_a)B^k_+(\omega)|a\rangle + \langle a|B^k_-(\omega)G(\omega+\varepsilon_a)B^k_+(\omega)|\delta a\rangle$$
$$+ \langle a|B^k_-(\omega)G(\omega+\varepsilon_a)([\mathbf{r}\times\boldsymbol{\alpha}]_z - \langle a|[\mathbf{r}\times\boldsymbol{\alpha}]_z|a\rangle)G(\omega+\varepsilon_a)B^k_+(\omega)|a\rangle\}. \quad (3)$$

Here $|\delta a\rangle = \sum_n^{\varepsilon_n\neq\varepsilon_a}|n\rangle\langle n|[\mathbf{r}\times\boldsymbol{\alpha}]_z|a\rangle(\varepsilon_a-\varepsilon_n)^{-1}$ is the wave-function correction due to the external magnetic field, $G(\omega) = \sum_n|n\rangle\langle n|[\omega-\varepsilon_n(1-i0)]^{-1}$ is the Dirac-Coulomb Green's function, $B^k_\pm(\omega) = D^k(\omega) \pm [p^k, V]/(\omega+i0)$, $[A,B] = AB - BA$, $D^k(\omega) = -4\pi\alpha Z\alpha^l D^{lk}(\omega)$, and

$$D^{lk}(\omega,\mathbf{r}) = -\frac{1}{4\pi}\left[\frac{\exp(i\sqrt{\omega^2+i0}\,r)}{r}\delta_{lk} + \nabla^l\nabla^k\frac{\exp(i\sqrt{\omega^2+i0}\,r)-1}{\omega^2 r}\right] \quad (4)$$

is the transverse part of the photon propagator in the Coulomb gauge with the branch of the square root fixed by the condition $\mathrm{Im}(\sqrt{\omega^2+i0}) > 0$. The summation over the repeated indices is implied. The zero-energy-transfer limit $\omega\to 0$ of the vector $D^k(\omega)$ appearing in Eq. (2) has the form

$$\mathbf{D}(0) = \frac{\alpha Z}{2r}\left[\boldsymbol{\alpha} + \frac{(\boldsymbol{\alpha}\cdot\mathbf{r})}{r^2}\mathbf{r}\right]. \quad (5)$$

Therefore, the vector product $[\mathbf{r}\times\mathbf{D}(0)]_z$ in Eq. (2) can be also written as $\alpha Z[\mathbf{r}\times\boldsymbol{\alpha}]_z/2r$.

The low-order contribution $\Delta g_{\rm L}$ can be derived from the relativistic Breit equation. The operators $\mathbf{p}^2$ and $[\mathbf{r}\times\mathbf{p}]_z \equiv l_z$ (l_z is the orbital angular momentum) in Eq. (2) correspond to the nonrelativistic limit whereas the terms with the vector $\mathbf{D}(0)$ provide the lowest-order relativistic correction. In the meantime, the derivation of the higher-order part $\Delta g_{\rm H}$ requires application of bound-state QED beyond the Breit approximation. For this reason, in the following we will refer to this part as the QED one. We should note that the formalism developed in Ref. [24] can be easily adopted to treat the nuclear recoil effect on the bound-electron g factors of ions with one electron over the closed shells. To this end, the representation in which the closed shells are regarded as belonging to the vacuum is to be employed (see, e.g., Refs. [25,38]).

In the case of the pointlike nucleus which is considered in the present study, the calculations of the low-order part $\Delta g_{\rm L}$ can be performed analytically for an arbitrary state of the hydrogenlike ion. The operators $\mathbf{p}^2$ and $\mathbf{p}\cdot\mathbf{D}(0)$ in Eq. (2) are invariant under rotation. Therefore, only the component of $|\delta a\rangle$ possessing the same angular quantum numbers as the unperturbed wave function $|a\rangle$ contributes. This component can be obtained by employing the generalized virial relations for the Dirac equation [39], which result in

$$|\delta a\rangle_{\kappa m_a} = \begin{pmatrix} X(r)\Omega_{\kappa m_a}(\hat{\mathbf{r}}) \\ iY(r)\Omega_{-\kappa m_a}(\hat{\mathbf{r}})\end{pmatrix}, \quad (6)$$

where

$$X(r) = \frac{\kappa m_a}{j(j+1)}\left\{\left[\frac{2\kappa(m+\varepsilon_a)-m}{2m^2}r + \frac{\kappa\alpha Z}{m^2}\right]f(r) + \frac{\kappa-2\kappa^2}{2m^2}g(r)\right\}, \quad (7)$$

$$Y(r) = \frac{\kappa m_a}{j(j+1)}\left\{\left[\frac{2\kappa(m-\varepsilon_a)+m}{2m^2}r - \frac{\kappa\alpha Z}{m^2}\right]g(r) + \frac{\kappa+2\kappa^2}{2m^2}f(r)\right\}. \quad (8)$$

Here κ is the Dirac angular quantum number of the state $|a\rangle$, $j = |\kappa| - 1/2$ is the total angular momentum, and g and f are the large and small radial components of the unperturbed wave function

$$|a\rangle = \begin{pmatrix} g(r)\Omega_{\kappa m_a}(\hat{\mathbf{r}}) \\ if(r)\Omega_{-\kappa m_a}(\hat{\mathbf{r}})\end{pmatrix}. \quad (9)$$

Applying the formulas presented in Ref. [39], one can obtain the expression for the low-order part of the nuclear recoil contribution to the bound-electron g factor [24] in the point-nucleus case,

$$\Delta g_{\rm L} = -\frac{m}{M}\frac{2\kappa^2\varepsilon_a^2 + \kappa m\varepsilon_a - m^2}{2m^2 j(j+1)}. \quad (10)$$

For the $n=1$ and $n=2$ states, Eq. (10) leads to

$$\Delta g_{\rm L}^{1s} = \frac{m}{M}\frac{2}{3}(1-\gamma_1)(1+2\gamma_1), \tag{11}$$

$$\Delta g_{\rm L}^{2s} = \frac{m}{M}\frac{1}{3}[2-\sqrt{2(1+\gamma_1)}][1+\sqrt{2(1+\gamma_1)}], \tag{12}$$

$$\Delta g_{\rm L}^{2p_{1/2}} = \frac{m}{M}\frac{1}{3}[2+\sqrt{2(1+\gamma_1)}][1-\sqrt{2(1+\gamma_1)}], \tag{13}$$

$$\Delta g_{\rm L}^{2p_{3/2}} = \frac{m}{M}\frac{2}{15}(1-\gamma_2)(1+2\gamma_2), \tag{14}$$

where $\gamma_1=\sqrt{1-(\alpha Z)^2}$ and $\gamma_2=\sqrt{4-(\alpha Z)^2}$. The leading orders in αZ read

$$\Delta g_{\rm L}^{1s} = \frac{m}{M}\left[(\alpha Z)^2 - \frac{1}{12}(\alpha Z)^4 + \cdots\right], \tag{15}$$

$$\Delta g_{\rm L}^{2s} = \frac{m}{M}\left[\frac{1}{4}(\alpha Z)^2 + \frac{11}{192}(\alpha Z)^4 + \cdots\right], \tag{16}$$

$$\Delta g_{\rm L}^{2p_{1/2}} = \frac{m}{M}\left[-\frac{4}{3} + \frac{5}{12}(\alpha Z)^2 + \cdots\right], \tag{17}$$

$$\Delta g_{\rm L}^{2p_{3/2}} = \frac{m}{M}\left[-\frac{2}{3} + \frac{7}{30}(\alpha Z)^2 + \cdots\right]. \tag{18}$$

It can be seen that for the s states ($\kappa=-1$) the nonrelativistic contribution to $\Delta g_{\rm L}$ vanishes and the αZ expansion starts with the term of pure relativistic [$\sim(\alpha Z)^2$] origin. For the p states ($\kappa=1$ or $\kappa=-2$), there is a nonzero nonrelativistic limit of the nuclear recoil effect on the bound-electron g factor.

The higher-order part $\Delta g_{\rm H}$ is evaluated numerically. It is naturally divided into three contributions depending on the number of **D** vectors. The term without **D** is referred to as the Coulomb contribution, while the terms including one and two **D** vectors are termed the one-transverse-photon (tr1) and two-transverse-photon (tr2) contributions, respectively. The ω integration for the simplest Coulomb contribution can be carried out analytically by employing Cauchy's residue theorem

$$\begin{aligned}\Delta g_{\rm H}^{\rm Coul} = \frac{1}{m_a}\frac{m}{M}\Bigg\{&\sum_{n<0}\frac{\langle\delta a|[p^k,V]|n\rangle\langle n|[p^k,V]|a\rangle + \langle a|[p^k,V]|n\rangle\langle n|[p^k,V]|\delta a\rangle}{(\varepsilon_a-\varepsilon_n)^2}\\ &+2\sum_{n<0}\frac{\langle a|[p^k,V]|n\rangle\langle n|([\mathbf{r}\times\boldsymbol{\alpha}]_z - \langle a|[\mathbf{r}\times\boldsymbol{\alpha}]_z|a\rangle)|n\rangle\langle n|[p^k,V]|a\rangle}{(\varepsilon_a-\varepsilon_n)^3}\\ &+\sum_{n_1<0}\sum_{n_2}^{\varepsilon_{n_2}\neq\varepsilon_{n_1}}\frac{\langle a|[p^k,V]|n_1\rangle\langle n_1|[\mathbf{r}\times\boldsymbol{\alpha}]_z|n_2\rangle\langle n_2|[p^k,V]|a\rangle}{(\varepsilon_a-\varepsilon_{n_1})^2(\varepsilon_{n_1}-\varepsilon_{n_2})} + \sum_{n_2<0}\sum_{n_1}^{\varepsilon_{n_1}\neq\varepsilon_{n_2}}\frac{\langle a|[p^k,V]|n_1\rangle\langle n_1|[\mathbf{r}\times\boldsymbol{\alpha}]_z|n_2\rangle\langle n_2|[p^k,V]|a\rangle}{(\varepsilon_a-\varepsilon_{n_2})^2(\varepsilon_{n_2}-\varepsilon_{n_1})}\Bigg\},\end{aligned} \tag{19}$$

where the notation $n<0$ implies that the corresponding summation runs over the negative-energy part of the spectrum only, $\varepsilon_n \leqslant -mc^2$. The ω integration for the $\Delta g_{\rm H}^{\rm tr1}$ and $\Delta g_{\rm H}^{\rm tr2}$ terms is performed numerically using Wick's rotation. An example of the integration contour employed in the present calculations is shown in Fig. 1. The branch cuts of the photon propagator (4), the poles of the Green's function $G(\omega+\varepsilon_a)$, and the pole $1/(\omega+i0)$ of the vector $B^k(\omega)$ are depicted as well. The contour is chosen to avoid the singularities near $\omega=0$ and go around the poles of the bound states with $\varepsilon_n<\varepsilon_a$. This is done since particular care is required at low values of the integration variable ω. As it is for the low-order part $\Delta g_{\rm L}$, the expression sandwiched between $|a\rangle$ and $|\delta a\rangle$ in Eq. (3) conserves the angular quantum numbers. For this reason, Eqs. (6)–(8) can also be employed to calculate the corresponding contribution to the higher-order part. Finally, the summation over the intermediate electron states is carried out using the finite basis sets constructed from B splines [40,41].

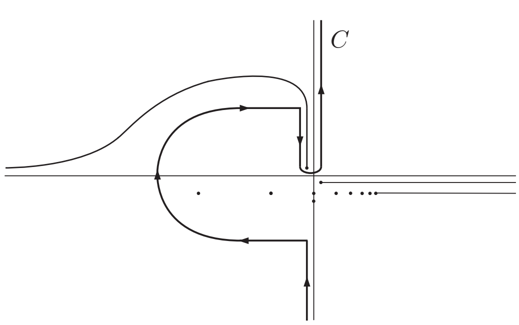

FIG. 1. Poles and branch cuts of the integrand for the part with $|\delta a\rangle$ of the one-transverse-photon contribution and the integration contour C used for the evaluation of this correction.

Concluding this section, we note that the finite nuclear size correction to the g factor can be taken into account by replacing $V=-\alpha Z/r$ in Eq. (1) with the potential of the extended nucleus. In the case of the nuclear recoil effect, this replacement allows one to take into account the nuclear size correction only partially. The similar situation takes place in the case of the nuclear recoil effect on binding energies [42]. The uncertainty due to this approximate treatment of the nuclear size correction to the recoil effect was discussed, e.g., in Refs. [26,43].

III. RESULTS AND DISCUSSION

In this section we present our results for the nontrivial QED part of the nuclear recoil effect on the bound-electron g factor of the $1s$, $2s$, $2p_{1/2}$, and $2p_{3/2}$ states in hydrogenlike ions with

TABLE I. Higher-order (QED) nuclear recoil contribution to the g factor of the $1s$ state. The results are expressed in terms of the function $P^{(5|1)}(\alpha Z)$ defined by Eq. (20). The individual terms of $P^{(5|1)}(\alpha Z) = P^{(5|1)}_{\rm Coul}(\alpha Z) + P^{(5|1)}_{\rm tr1}(\alpha Z) + P^{(5|1)}_{\rm tr2}(\alpha Z)$ are shown.

Z	$P^{(5\|1)}_{\rm Coul}(\alpha Z)$	$P^{(5\|1)}_{\rm tr1}(\alpha Z)$	$P^{(5\|1)}_{\rm tr2}(\alpha Z)$	$P^{(5\|1)}(\alpha Z)$
1	−1.114 14	100.701 20	−80.820 02	18.767 04
2	−1.097 54	53.527 79	−36.986 89	15.443 37
3	−1.081 83	37.449 50	−22.808 37	13.559 30
4	−1.066 93	29.245 93	−15.919 60	12.259 40
5	−1.052 77	24.230 28	−11.900 49	11.277 02
6	−1.039 31	20.827 13	−9.293 87	10.493 96
7	−1.026 49	18.355 87	−7.481 93	9.847 44
8	−1.014 29	16.473 49	−6.159 02	9.300 18
9	−1.002 67	14.988 00	−5.157 11	8.828 21
10	−0.991 61	13.783 31	−4.376 46	8.415 24
11	−0.981 06	12.785 01	−3.754 25	8.049 70
12	−0.971 02	11.943 11	−3.249 02	7.723 07
13	−0.961 45	11.222 77	−2.832 40	7.428 92
14	−0.952 35	10.598 92	−2.484 31	7.162 26
15	−0.943 68	10.053 04	−2.190 20	6.919 16
16	−0.935 44	9.571 16	−1.939 26	6.696 45
17	−0.927 62	9.142 49	−1.723 32	6.491 56
18	−0.920 19	8.758 63	−1.536 08	6.302 36
19	−0.913 14	8.412 86	−1.372 63	6.127 09
20	−0.906 47	8.099 79	−1.229 07	5.964 25

TABLE II. Higher-order (QED) nuclear recoil contribution to the g factor of the $2s$ state. The results are expressed in terms of the function $P^{(5|2)}(\alpha Z)$ defined by Eq. (20). The individual terms of $P^{(5|2)}(\alpha Z) = P^{(5|2)}_{\rm Coul}(\alpha Z) + P^{(5|2)}_{\rm tr1}(\alpha Z) + P^{(5|2)}_{\rm tr2}(\alpha Z)$ are shown.

Z	$P^{(5\|2)}_{\rm Coul}(\alpha Z)$	$P^{(5\|2)}_{\rm tr1}(\alpha Z)$	$P^{(5\|2)}_{\rm tr2}(\alpha Z)$	$P^{(5\|2)}(\alpha Z)$
1	−1.114 17	100.968 78(1)	−80.657 09	19.197 53(1)
2	−1.097 64	53.796 72	−36.823 55	15.875 53
3	−1.082 07	37.720 11	−22.644 33	13.993 71
4	−1.067 36	29.518 50	−15.754 62	12.696 53
5	−1.053 44	24.505 08	−11.734 36	11.717 28
6	−1.040 28	21.104 40	−9.126 41	10.937 72
7	−1.027 81	18.635 83	−7.312 96	10.295 05
8	−1.016 02	16.756 36	−5.988 40	9.751 95
9	−1.004 85	15.273 98	−4.984 70	9.284 43
10	−0.994 28	14.072 59	−4.202 12	8.876 19
11	−0.984 29	13.077 78	−3.577 84	8.515 65
12	−0.974 85	12.239 56	−3.070 44	8.194 27
13	−0.965 94	11.523 08	−2.651 52	7.905 62
14	−0.957 54	10.903 28	−2.301 03	7.644 71
15	−0.949 63	10.361 62	−2.004 41	7.407 58
16	−0.942 19	9.884 13	−1.750 86	7.191 09
17	−0.935 22	9.460 05	−1.532 20	6.992 64
18	−0.928 69	9.080 95	−1.342 14	6.810 12
19	−0.922 60	8.740 12	−1.175 77	6.641 74
20	−0.916 93	8.432 17	−1.029 21	6.486 03

$Z = 1$–20 evaluated for pointlike nuclei. For further consideration, it is useful to introduce the dimensionless functions $P^{(k|n)}(\alpha Z)$ defined as

$$\Delta g_{\rm H} = \frac{m}{M}\frac{(\alpha Z)^k}{n^3} P^{(k|n)}(\alpha Z), \tag{20}$$

where n is the principal quantum number and arbitrary integer k can be chosen for the convenient representation of the results.

The higher-order nuclear recoil contributions to the g factors of the $1s$ and $2s$ states are presented in Tables I and II, respectively. The results are shown in terms of the function $P^{(5|1)}(\alpha Z)$ for the $1s$ state and $P^{(5|2)}(\alpha Z)$ for the $2s$ state. For particular ions, this contribution was considered earlier in Refs. [11,23,25]. Our present results are in agreement with the previous ones but are calculated to a higher accuracy. The uncertainties are estimated by studying the convergence of the ω integration in Eq. (3) as well as by increasing the size of the basis employed. When the uncertainty is not specified, all the digits presented are assumed to be correct.

In Ref. [23], the behavior of the higher-order contribution $\Delta g_{\rm H}$ for the $1s$ state as a function of αZ when Z tends to zero was studied. It was found that the total result exhibits the $(\alpha Z)^5$ behavior, whereas the one-transverse-photon and two-transverse-photon terms taken separately behave as $(\alpha Z)^4$. Moreover, the individual contributions to $\Delta g^{\rm tr1}_{\rm H}$, namely, the parts with and without $|\delta a\rangle$, include even the lower power of αZ and manifest the $(\alpha Z)^3$ behavior. In the present work we study the QED recoil contribution to the g factors of the $1s$ and $2s$ states for small Z. It turns out that the higher-order part of the nuclear recoil effect $\Delta g_{\rm H}$ is rather similar for the g factors of both s states. This fact is clearly demonstrated in Figs. 2 and 3, where the Coulomb, one-transverse-photon, and two-transverse-photon contributions as well as the total values of the $\Delta g_{\rm H}$ correction are plotted for the $1s$ and $2s$ states in terms of the functions $P^{(4|1)}(\alpha Z)$ and $P^{(4|2)}(\alpha Z)$, respectively. One can see that for both states these functions for the $\Delta g^{\rm tr1}_{\rm H}$ and $\Delta g^{\rm tr2}_{\rm H}$ terms possess nonzero limits at $\alpha Z \to 0$ which cancel each other in the sum. The appearance of the curves is almost the same. We have performed the calculations of the higher-order contribution $\Delta g_{\rm H}$ for a series of Z including fractional values and, using the least-squares analysis, we fitted the results obtained to the form

$$P^{(5|1)}_{1s}(\alpha Z) = A^{51}_{1s}\ln(\alpha Z) + A^{50}_{1s} + \alpha Z(\cdots), \tag{21}$$

$$P^{(5|2)}_{2s}(\alpha Z) = A^{51}_{2s}\ln(\alpha Z) + A^{50}_{2s} + \alpha Z(\cdots), \tag{22}$$

where the function P is defined by Eq. (20). By analyzing the dependence of the results on the number of varying parameters in the fit and the number of fitting points, we have found that for the $1s$ state $A^{51}_{1s} = -5.1(2)$ and $A^{50}_{1s} = -6.6(5)$ and for the $2s$ state $A^{51}_{2s} = -5.1(2)$ and $A^{50}_{2s} = -6.2(5)$. The coefficients obtained for the $1s$ state are in agreement with those of Ref. [23] but have higher accuracy.

Since the coefficients of the logarithmic terms for the $1s$ and $2s$ states in Eqs. (21) and (22) are the same, at least

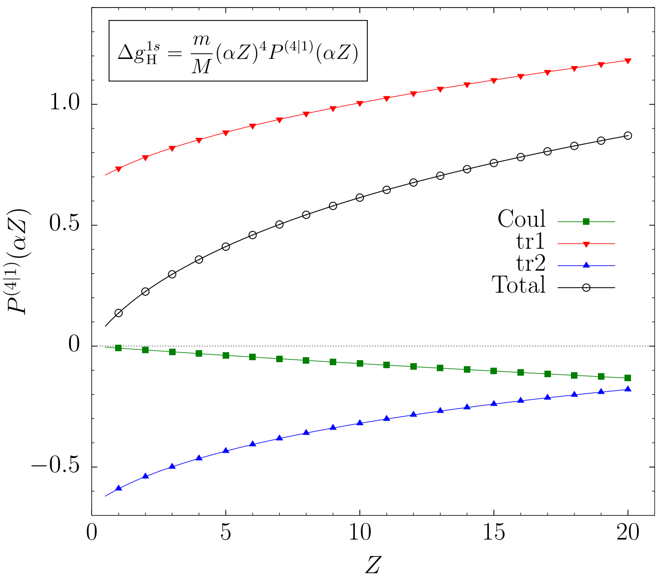

FIG. 2. Coulomb, one-transverse-photon, and two-transverse-photon contributions to the higher-order nuclear recoil effect on the g factor of the $1s$ state. The results are presented in terms of the function $P^{(4|1)}(\alpha Z)$ defined by Eq. (20). Note that $P^{(4|1)}(x) = xP^{(5|1)}(x)$.

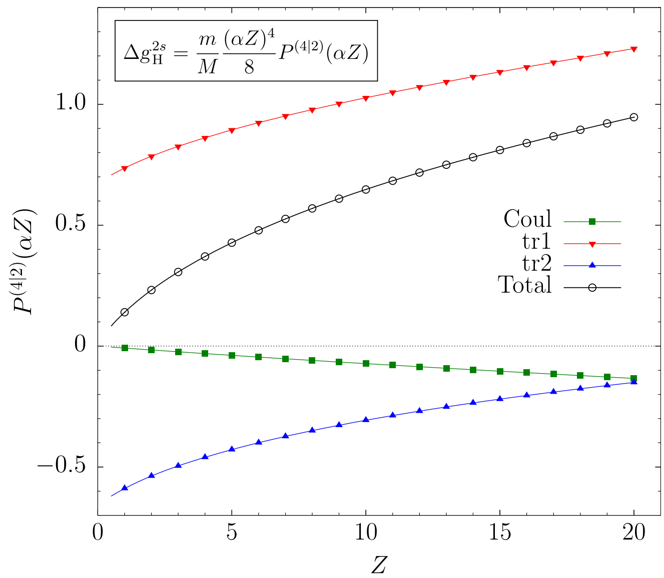

FIG. 3. Coulomb, one-transverse-photon, and two-transverse-photon contributions to the higher-order nuclear recoil effect on the g factor of the $2s$ state. The results are presented in terms of the function $P^{(4|2)}(\alpha Z)$ defined by Eq. (20). Note that $P^{(4|2)}(x) = xP^{(5|2)}(x)$.

within the numerical uncertainty of the present fit, it is also useful to consider the weighted difference $\eta \equiv 8\Delta g_{\rm H}^{2s} - \Delta g_{\rm H}^{1s}$ [we recall that, compared to the $1s$ state, for the $2s$ state the additional factor $1/8$ is separated in the definition of the function $P(\alpha Z)$]. In Fig. 4 the difference η is plotted together with the individual contributions to it in terms of the function $Q^{(5)}(\alpha Z)$ defined according to

$$\eta = \frac{m}{M}(\alpha Z)^5 Q^{(5)}(\alpha Z), \tag{23}$$

$$Q^{(5)}(\alpha Z) = P_{2s}^{(5|2)}(\alpha Z) - P_{1s}^{(5|1)}(\alpha Z). \tag{24}$$

The plots in Fig. 4 clearly show that the logarithmic terms indeed cancel each other in this difference. Moreover, the terms of order $(\alpha Z)^4$ vanish in the one-transverse-photon and two-transverse-photon contributions to the difference η. Finally, the leading terms of order $(\alpha Z)^5$ in the Coulomb parts of $\Delta g_{\rm H}^{1s}$ and $\Delta g_{\rm H}^{2s}$ also cancel each other. Therefore, the limit of $Q^{(5)}(\alpha Z)$ at $\alpha Z \to 0$ is finite and it is related to the coefficients A_{1s}^{50} and A_{2s}^{50} in Eqs. (21) and (22) as

$$Q^{(5)}(0) = A_{2s}^{50} - A_{1s}^{50}. \tag{25}$$

The limit of the function $Q^{(5)}(\alpha Z)$ at $\alpha Z \to 0$ can be determined by a least-squares fitting. We obtain $Q^{(5)}(0) = 0.43$ for the total value of the weighted difference η and $Q_{\rm Coul}^{(5)}(0) \equiv 0$, $Q_{\rm tr1}^{(5)}(0) = 0.27$, and $Q_{\rm tr2}^{(5)}(0) = 0.16$ for the Coulomb, one-transverse-photon, and two-transverse-photon contributions, respectively.

The QED recoil contributions to the g factors of the $2p_{1/2}$ and $2p_{3/2}$ states are given in Tables III and IV, respectively. For illustrative purposes, the results obtained are also plotted in Figs. 5 and 6. We note that for the p states the $\Delta g_{\rm H}$ contribution possesses the $(\alpha Z)^3$ behavior in contrast to the $(\alpha Z)^5$ behavior found for the s states. This fact is apparently related to the existence of the nonzero nonrelativistic limit for $\Delta g_{\rm L}^{2p_j}$ in Eqs. (17) and (18) whereas the low-order contributions $\Delta g_{\rm L}^{1s}$ and $\Delta g_{\rm L}^{2s}$ in Eqs. (15) and (16) are of a pure relativistic origin. For these reasons, the results in Tables III and IV and in Figs. 5 and 6 are expressed in terms of the function $P^{(3|2)}(\alpha Z)$. From these data, one can conclude that for small Z the higher-order part of the nuclear recoil effect for the $2p_{1/2}$

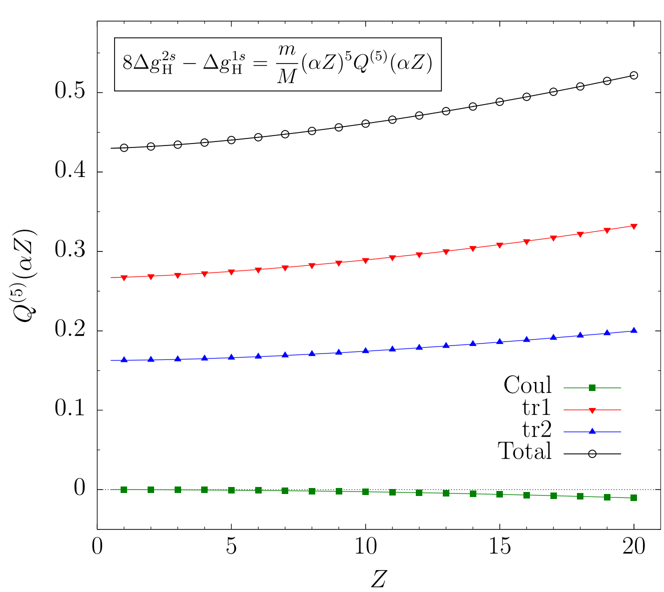

FIG. 4. Coulomb, one-transverse-photon, and two-transverse-photon terms of the weighted difference of the higher-order nuclear recoil contributions to the g factors of the $2s$ and $1s$ states, $8\Delta g_{\rm H}^{2s} - \Delta g_{\rm H}^{1s}$. The results are presented in terms of the function $Q^{(5)}(\alpha Z)$ defined by Eq. (23).

TABLE III. Higher-order (QED) nuclear recoil contribution to the g factor of the $2p_{1/2}$ state. The results are presented in terms of the function $P^{(3|2)}(\alpha Z)$ defined by Eq. (20). The individual terms of $P^{(3|2)}(\alpha Z) = P^{(3|2)}_{\rm Coul}(\alpha Z) + P^{(3|2)}_{\rm tr1}(\alpha Z) + P^{(3|2)}_{\rm tr2}(\alpha Z)$ are shown.

Z	$P^{(3\|2)}_{\rm Coul}(\alpha Z)$	$P^{(3\|2)}_{\rm tr1}(\alpha Z)$	$P^{(3\|2)}_{\rm tr2}(\alpha Z)$	$P^{(3\|2)}(\alpha Z)$
1	-1.78×10^{-9}	0.421 036	0.003 339	0.424 375
2	-2.76×10^{-8}	0.424 393	0.006 814	0.431 206
3	-1.36×10^{-7}	0.427 798	0.010 393	0.438 191
4	-4.20×10^{-7}	0.431 243	0.014 061	0.445 303
5	-1.00×10^{-6}	0.434 721	0.017 806	0.452 526
6	-2.04×10^{-6}	0.438 227	0.021 619	0.459 844
7	-3.70×10^{-6}	0.441 759	0.025 493	0.467 249
8	-6.20×10^{-6}	0.445 314	0.029 422	0.474 730
9	-9.76×10^{-6}	0.448 889	0.033 401	0.482 280
10	-1.46×10^{-5}	0.452 483	0.037 425	0.489 894
11	-2.11×10^{-5}	0.456 094	0.041 491	0.497 564
12	-2.95×10^{-5}	0.459 722	0.045 595	0.505 287
13	-4.00×10^{-5}	0.463 364	0.049 734	0.513 057
14	-5.31×10^{-5}	0.467 021	0.053 904	0.520 872
15	-6.91×10^{-5}	0.470 692	0.058 104	0.528 727
16	-8.84×10^{-5}	0.474 376	0.062 331	0.536 619
17	−0.000 111	0.478 074	0.066 583	0.544 546
18	−0.000 139	0.481 786	0.070 857	0.552 505
19	−0.000 170	0.485 511	0.075 153	0.560 494
20	−0.000 207	0.489 251	0.079 468	0.568 511

TABLE IV. Higher-order (QED) nuclear recoil contribution to the g factor of the $2p_{3/2}$ state. The results are presented in terms of the function $P^{(3|2)}(\alpha Z)$ defined by Eq. (20). The individual terms of $P^{(3|2)}(\alpha Z) = P^{(3|2)}_{\rm Coul}(\alpha Z) + P^{(3|2)}_{\rm tr1}(\alpha Z) + P^{(3|2)}_{\rm tr2}(\alpha Z)$ are shown.

Z	$P^{(3\|2)}_{\rm Coul}(\alpha Z)$	$P^{(3\|2)}_{\rm tr1}(\alpha Z)$	$P^{(3\|2)}_{\rm tr2}(\alpha Z)$	$P^{(3\|2)}(\alpha Z)$
1	-1.70×10^{-10}	0.211 964	0.000 179	0.212 143
2	-2.57×10^{-9}	0.215 070	0.000 379	0.215 449
3	-1.24×10^{-8}	0.218 186	0.000 594	0.218 781
4	-3.75×10^{-8}	0.221 312	0.000 820	0.222 132
5	-8.78×10^{-8}	0.224 446	0.001 053	0.225 499
6	-1.75×10^{-7}	0.227 588	0.001 291	0.228 879
7	-3.13×10^{-7}	0.230 737	0.001 532	0.232 268
8	-5.15×10^{-7}	0.233 892	0.001 773	0.235 665
9	-7.97×10^{-7}	0.237 054	0.002 014	0.239 067
10	-1.18×10^{-6}	0.240 223	0.002 251	0.242 472
11	-1.67×10^{-6}	0.243 398	0.002 483	0.245 879
12	-2.29×10^{-6}	0.246 579	0.002 710	0.249 286
13	-3.07×10^{-6}	0.249 767	0.002 928	0.252 692
14	-4.01×10^{-6}	0.252 962	0.003 137	0.256 094
15	-5.14×10^{-6}	0.256 163	0.003 335	0.259 493
16	-6.47×10^{-6}	0.259 371	0.003 521	0.262 885
17	-8.03×10^{-6}	0.262 587	0.003 692	0.266 271
18	-9.84×10^{-6}	0.265 809	0.003 849	0.269 649
19	-1.19×10^{-5}	0.269 039	0.003 990	0.273 017
20	-1.43×10^{-5}	0.272 277	0.004 112	0.276 375

and $2p_{3/2}$ states is determined mainly by the one-transverse-photon contribution. The two-transverse-photon contribution is of the next order in αZ, while the Coulomb contribution is almost negligible.

Evaluating the limits of the QED recoil contributions to the g factors of the $2p_{1/2}$ and $2p_{3/2}$ states at $\alpha Z \to 0$, we obtain

$$P^{(3|2)}_{2p_{1/2}}(0) = 0.417\,74(5), \quad P^{(3|2)}_{2p_{3/2}}(0) = 0.208\,87(3). \qquad (26)$$

Based on Eqs. (17), (18), and (26), we note that the ratio of the QED recoil contributions to the g factor of the p states coincides with the analogous ratio for the low-order parts in the $\alpha Z \to 0$ limit,

$$\lim_{\alpha Z \to 0} \frac{\Delta g_{\rm L}^{2p_{1/2}}}{\Delta g_{\rm L}^{2p_{3/2}}} = \lim_{\alpha Z \to 0} \frac{\Delta g_{\rm H}^{2p_{1/2}}}{\Delta g_{\rm H}^{2p_{3/2}}} = 2. \qquad (27)$$

In the recent experiment [14], the ground-state g factor of $^{40}{\rm Ar}^{13+}$ was measured to an accuracy of 10^{-9}. The higher-order QED term evaluated in this paper amounts to $\Delta g_{\rm H}[^{40}{\rm Ar}^{13+}] = 2.1 \times 10^{-9}$. This contribution, which is two times larger than the to-date experimental uncertainty, has to be taken into account, provided the many-electron QED and recoil corrections are evaluated to the required accuracy [14].

In addition, the theoretical value of the isotope shift in the atomic g factor is determined mainly by the nuclear recoil and nuclear size effects. The measurement of the isotope difference of the bound-electron g factor in lithiumlike calcium [11] and the corresponding theoretical calculation [25] being in good agreement with each other pave the way for QED tests beyond the Furry picture in the strong-coupling regime. In this regard, high-precision measurements of the isotope shift of the bound-electron g factor in boronlike ions are highly anticipated since the isotope dependence of the Zeeman effect can be evaluated to a very high accuracy, exceeding significantly the accuracy of the g-factor calculations.

IV. CONCLUSION

To summarize, in this paper we have evaluated the nuclear recoil effect of first order in m/M on the bound-electron g factor of the $n = 1$ and $n = 2$ states in H-like ions in the range $Z = 1$–20. The calculations are performed to all orders in αZ within the fully relativistic approach. A numerical analysis of the behavior of the nuclear recoil contributions as functions of Z was conducted. As the result, accurate theoretical predictions of the first-order (in m/M) nuclear recoil effect on the bound-electron g factor in hydrogenlike ions were obtained. The calculated values can be also used for the g factor of few-electron ions. However, in the latter case the total nuclear recoil contribution comprises additionally the many-electron part which can be evaluated within the Breit approximation employing the effective relativistic operators derived in Ref. [25]. The study of the nuclear recoil effect performed in the present work is in demand in connection with the forthcoming experiments at the HITRAP/FAIR in Darmstadt and at the MPIK in Heidelberg.

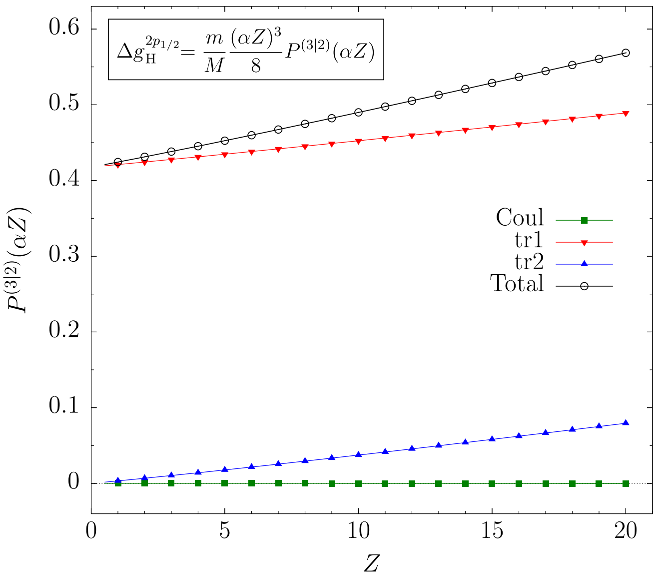

FIG. 5. Coulomb, one-transverse-photon, and two-transverse-photon contributions to the higher-order nuclear recoil effect on the $2p_{1/2}$ g factor. The results are presented in terms of the function $P^{(3|2)}(\alpha Z)$ defined by Eq. (20).

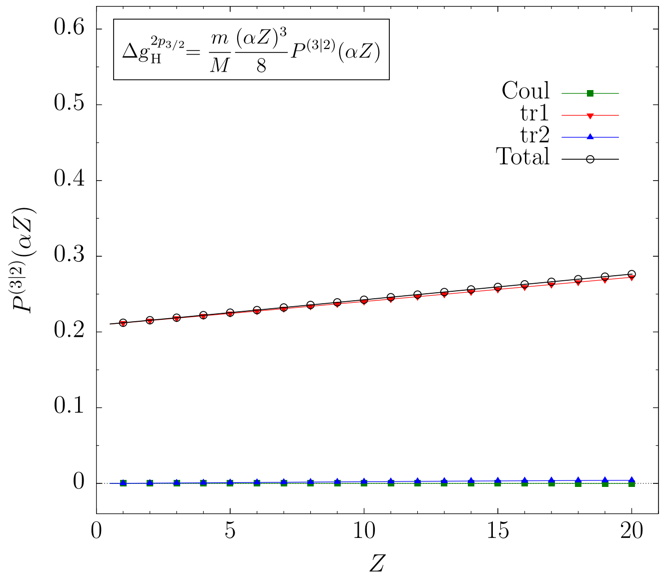

FIG. 6. Coulomb, one-transverse-photon, and two-transverse-photon contributions to the higher-order nuclear recoil effect on the $2p_{3/2}$ g factor. The results are presented in terms of the function $P^{(3|2)}(\alpha Z)$ defined by Eq. (20).

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