Measurement-device-independent verification of channel steering

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Extending the concept of steerability for quantum states, channel steerability is an ability to remotely control the given channel from a coherently extended party. Verification of channel steering can be understood as certifying coherence of the channel in a one-sided device-independent manner with respect to a bystander. Here we propose a method to verify channel steering in a measurement-device-independent way. To do this, we first obtain Choi matrices from given channels and use the canonical method of measurement-device-independent verification of quantum steering. As a consequence, exploiting channel-state duality which interconverts steerability of channels and that of states, channel steering is verified. We further analyze the effect of imperfect preparation of entangled states used in the verification protocol, and find that the threshold of the undesired noise that we can tolerate is bounded from below by steering robustness.

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I. INTRODUCTION

Quantum steering is a nonclassical phenomenon in that local measurements on one side can induce ensembles of local quantum states on the other side, which cannot be explained by any classical correlations or local hidden states. This phenomenon has attracted great attention since it was implied in Einstein-Podolsky-Rosen's seminal paper [1] and mentioned by Schrödinger [2] as a "paradox." Recently, steering has been put into mathematical rigor [3,4], and a number of theoretical developments [5–21] and experimental realizations [19–30] have followed.

The concept of steering of quantum states can be extended to quantum channels [31]. Channel steering is a nonclassical phenomenon in that local measurements on a bystander's side can induce an instrument of reduced channels on the other side, which cannot be explained by any classical correlations or reduced subchannels. An operational meaning of the steering is verification of nonseparability when the steered party is trusted while the steering party is untrusted [4]. Thus an analogous operational meaning can be obtained for the channel steering; the channel steering is verification of coherent extension of the channel, when input and output parties of the reduced channels are trusted but a bystander is untrusted. Coherent extension of the channel means that there is information leakage to a bystander more than classical randomness [31]. Therefore, verification of the channel steering can be regarded as identifying whether the bystander has access to quantum information being transmitted through the original channel.

The analogy between the concepts of state steering and channel steering enables channel-state duality which preserves steerability [31]. In Ref. [31], it was proved that a given quantum channel is steerable if and only if the corresponding quantum state obtained by channel-state duality is steerable. This result links the state steering and channel steering; the two concepts are not independent topics; rather, they are interconvertible. One can see the analogy between the state steering and channel steering in Fig. 1.

Meanwhile, in the definition of the steering, whether the parties are trusted or not is important. We say that a party is trusted when it is reliable and it has perfect control over its experiment. In the case of the state steering, this implies that one can reconstruct the local state of the steered party, while for the channel steering this implies that one can reconstruct the reduced channel. In practical situations, however, these assumptions are hard to achieve because reliability of experimenters, accuracy of the measurement apparatus, and protecting quantum processing from external noise are imperfect. The steering verification protocol is free from such prerequisites with respect to only one party (the steering party in the state steering or the bystander in the channel steering), so that it is called one-sided device independent (1s-DI). Recently, to alleviate the prerequisites, alternative verification schemes that remove trust on the parties in entanglement verification [32] and steering verification [33] were proposed. Although the alternative schemes still require trust on generating devices or measurement devices of external parties, this does not require perfect control over the experiments of the parties. We can thus say that these are alleviated scenarios that are called measurement-device independent (MDI). In entanglement verification and steering verification, canonical techniques to convert the entanglement criterion and steering criterion from their original scenarios to the MDI scenarios have been well established [34–36]. The MDI method was exploited in quantum key distribution tasks to overcome detector-side channel attacks and obtain better secure distances [37,38].

In this paper, motivated by previous developments, we propose an MDI verification scheme of channel steering. To do this, we first convert steerability of the channels to that of the states with the aid of the channel-state duality. Subsequently, we transform the steering witness in the 1s-DI

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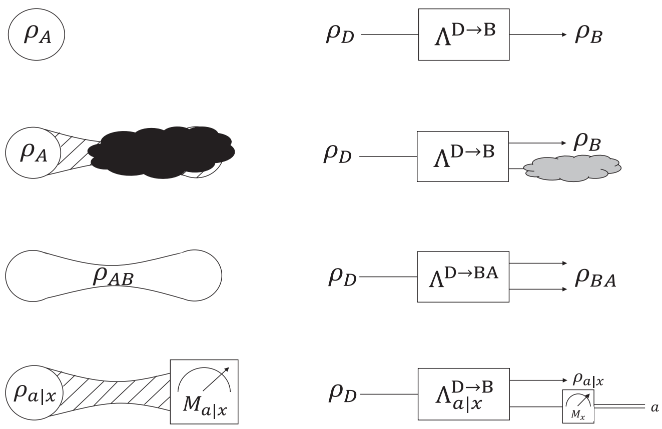

FIG. 1. Comparison between state steering and channel steering. Left: Given a local state ρ_A, an extended state ρ_{AB} is such that one can retrieve the original local state ρ_A by discarding system B. If a set of measurements $\{M_{a|x}\}_{a,x}$ is performed on system B, system A collapses to an assemblage $\{\rho_{a|x}\}_{a,x}$. By checking whether measurements on system B really steered system A, we determine steerability of the state. Right: Channel steering can be defined in an analogous way. Given a channel from system D to system B, $\Lambda^{D\to B}$, one can come up with an extended channel with one input D and two output systems B and A such that the original channel $\Lambda^{D\to B}$ is retrieved by discarding an added output system A, a bystander. If a set of measurements $\{M_{a|x}\}_{a,x}$ is performed on a bystander's system, the channel reduces to an instrument from D to B, $\{\Lambda^{D\to B}_{a|x}\}_{a,x}$. By checking whether measurements on a bystander's system really steered a channel from D to B, we determine steerability of the channel.

scenario to the steering criterion in the MDI scenario. We then verify steerability of the state obtained by channel-state duality, using an MDI steering criterion. This leads to verification of the channel steering in an MDI manner. Furthermore, we show that the channel-state duality of steerability is still preserved as long as we use a bipartite state that is pure and of the full Schmidt rank although it is different from the originally intended one. This means that misalignment of bipartite pure states for channel-state duality does not change the protocol. We analyze the effect of noise in generating bipartite pure states, and find that even if the preparation is corrupted by noise we are still able to authenticate steerability of the channel, provided that the portion of the noise is below some threshold which is bounded by the steering robustness introduced in Ref. [12]. This will allow us to inspect information leakage from the given channel to the bystander without the assumption of perfect control over the experiment.

The rest of this paper is organized as follows. In Sec. II, we review state steering and MDI verification. In Sec. III, the concept of channel steering is explained. In order to combine the aforementioned concepts, we introduce channel-state duality, and show that channel-state duality of steerability is still preserved using any bipartite pure state with the full Schmidt rank in Sec. IV. In Sec. V, we propose a scheme to verify channel steering in an MDI way. We consider the effect of imperfect state preparation, and conclude that MDI verification of the channel steering is possible up to some amount of noise. Further, we find that the allowed portion of noise is given by steering robustness. We conclude the paper in Sec. VI.

II. QUANTUM STATE STEERING

A positive semidefinite operator ρ with unit trace is called a quantum state or a state. A quantum substate or a substate is defined as a positive semidefinite operator ρ_a with its trace less than or equal to unity [12]. An ensemble $\mathcal{E}$ of a state ρ is a set of substates $\{\rho_a\}_a$ such that the sum of all elements is the state ρ, $\sum_a \rho_a = \rho$. A collection of ensembles $\{\mathcal{E}_x\}_x = \{\rho_{a|x}\}_{a,x}$ shall be called a state assemblage or simply an assemblage. A state assemblage $\{\rho_{a|x}\}_{a,x}$ is called unsteerable if every substate $\rho_{a|x}$ arises from classical processing of some ensemble $\{\rho_\lambda\}_\lambda$ as

$$\rho_{a|x} = \sum_\lambda p(a|x,\lambda)\rho_\lambda, \tag{1}$$

with some probability distribution $p(a|x,\lambda)$. Normalized substates $\{\rho_\lambda/\mathrm{Tr}[\rho_\lambda]\}_\lambda$ are conventionally referred to as hidden states. If an assemblage is not unsteerable, it is called steerable.

One can construct similar concepts for observables. A positive-operator-valued measurement (POVM) is a set of positive semidefinite operators $\{M_a\}_a$ such that the sum of all elements equals an identity, $\sum_a M_a = I$. A collection of POVMs, $\{M_{a|x}\}_{a,x}$, is called a measurement assemblage. A measurement assemblage $\{M_{a|x}\}_{a,x}$ is called jointly measurable or compatible if every POVM element $M_{a|x}$ arises from classical processing of some POVM $\{M_\lambda\}_\lambda$, usually named the grand POVM, as

$$M_{a|x} = \sum_\lambda p(a|x,\lambda)\, M_\lambda, \tag{2}$$

with some probability distribution $p(a|x,\lambda)$. If a measurement assemblage is not compatible, it is called incompatible.

It is straightforward to see a one-to-one correspondence between state assemblages and measurement assemblages. One can obtain a state assemblage from an extended state by performing measurements in a measurement assemblage on a partial state as

$$\rho_{a|x} = \mathrm{Tr}_B[\rho_{AB}(I \otimes M_{a|x})]. \tag{3}$$

Conversely, one can construct a measurement assemblage from a state assemblage as

$$M_{a|x} := (\tilde{\rho})^{-\frac{1}{2}}\tilde{\rho}_{a|x}(\tilde{\rho})^{-\frac{1}{2}}, \tag{4}$$

where $\rho = \sum_a \rho_{a|x}$, and the tilde denotes an operator projected on range(ρ). It is proved that they have steerability correspondence, i.e., a state assemblage is unsteerable if and only if a measurement assemblage in Eq. (4) is compatible [13].

A convex combination of two unsteerable assemblages yields an unsteerable assemblage [39], which means that the set of unsteerable assemblages is convex. Therefore, for every steerable assemblage $\{\rho_{a|x}\}_{a,x}$, we can draw a hyperplane which separates $\{\rho_{a|x}\}_{a,x}$ and the set of unsteerable assemblages. Such a separation is realized by a set of positive semidefinite operators $\{F_{a|x}\}_{a,x}$, called the steering witness, such that

$$\sum_{a,x} \mathrm{Tr}[F_{a|x}\rho_{a|x}] > \sup_{\{\sigma_{a|x}^{US}\}} \sum_{a,x} \mathrm{Tr}\big[F_{a|x}\sigma_{a|x}^{US}\big], \qquad (5)$$

where the supremum is taken over the set of unsteerable assemblages. The right-hand side of Eq. (5) is usually called the steering bound and is conventionally denoted by α.

In an experiment, to obtain knowledge of a state assemblage, one needs to have a perfect measurement apparatus because the state tomography via imperfect measurements can yield an incorrect state assemblage different from the real one. Therefore, not to mislead the determination of steerability, we require high reliability of the experimenter. In the case of steering, the steering party is untrusted while the steered party is trusted, thus only the steering party is free from the reliability requirement. This property is called one-sided device independence of steering verification.

Recently, based on the semiquantum nonlocal games [32], another scheme of steering verification was introduced [33], where an experimenter who has a state assemblage is questioned in quantum states and answers in real numbers. According to the experimenter's responses, one can determine whether the state assemblage is steerable or not. This does not require reliability of the experimenter (i.e., steered party), and it is thus called a measurement-device-independent scenario. The MDI steering verification was experimentally demonstrated [35] and the canonical way to convert the 1s-DI steering witness to that in the MDI scenario was proposed [36] in an analogous way with the entanglement witness [34]. We will revisit this topic in Sec. V.

III. QUANTUM CHANNEL STEERING

A completely positive and trace-preserving linear map $\Lambda[\,\cdot\,]$ is called a quantum channel. Every quantum channel $\Lambda[\,\cdot\,]$ can be written as $\sum_i K_i[\,\cdot\,]K_i^\dagger$ with $\sum_i K_i^\dagger K_i = I$, where the operators $\{K_i\}_i$ are called Kraus operators [40]. A quantum subchannel is defined as a completely positive and trace nonincreasing linear map $\Lambda_a[\,\cdot\,]$ [12]. Every quantum subchannel $\Lambda_a[\,\cdot\,]$ can be written as $\sum_i K_i[\,\cdot\,]K_i^\dagger$ with $\sum_i K_i^\dagger K_i \leqslant I$, where the set of operators $\{K_i\}_i$ is a subset of a set of Kraus operators. An instrument of a quantum channel Λ is a set of quantum subchannels $\{\Lambda_a\}_a$ such that the sum of every element is the quantum channel, $\sum_a \Lambda_a = \Lambda$. A collection of instruments $\{\Lambda_{a|x}\}_{a,x}$ is called a channel assemblage. A channel assemblage is said to be unsteerable if every subchannel $\Lambda_{a|x}$ arises from classical processing of some instrument $\{\Lambda_\lambda\}_\lambda$ as

$$\Lambda_{a|x} = \sum_\lambda p(a|x,\lambda)\Lambda_\lambda, \qquad (6)$$

with some probability distribution $p(a|x,\lambda)$. If a channel assemblage is not unsteerable, it is called steerable.

For a quantum channel from a system D to B, $\Lambda^{D\to B}$, we can come up with a channel extension with one sender D but two receivers B and A, $\Lambda^{D\to BA}$, which satisfies $\Lambda^{D\to B} = \mathrm{Tr}_A \circ \Lambda^{D\to BA}$. If an extended quantum channel $\Lambda^{D\to BA}$ can be expressed as a sum of decompositions into quantum state and quantum subchannel,

$$\Lambda^{D\to BA} = \sum_\lambda \Lambda_\lambda^{D\to B} \otimes \rho_\lambda^A, \qquad (7)$$

we call it an incoherent extension. If a channel extension is not incoherent, it is called coherent.

There is a correspondence between an instrument and a POVM; for every POVM, we can obtain an instrument by performing measurements on one side of an extended channel, and for every instrument we can find a POVM which induces the given instrument from an extended channel. Along this correspondence, one can consider every channel assemblage $\{\Lambda_{a|x}^{D\to B}\}_{a,x}$ as an induced one from an extended channel being performed by POVM $\{M_{a|x}^A\}_{a,x}$:

$$\Lambda_{a|x}^{D\to B}[\,\cdot\,] = \mathrm{Tr}_A[(I\otimes M_{a|x})\,\Lambda^{D\to BA}[\,\cdot\,]]. \qquad (8)$$

One can then define channel steerability of an extended channel $\Lambda^{D\to BA}$ as steerability of an instrument of the reduced subchannels $\{\Lambda_{a|x}^{D\to B}\}_{a,x}$ obtained by Eq. (8). As a consequence, the following two statements are equivalent: (i) channel assemblage is steerable, and (ii) the bystander (say, Alice) has the ability to remotely control the channel $\Lambda^{D\to B}$ by performing measurements $\{M_{a|x}\}_{a,x}$ on her side. Furthermore, it was proved that verification of the channel steering is equivalent to corroborating that the channel extension is coherent when Alice is not trusted [31]. This is an analogous argument with the state steering in that steering verification is equivalent to corroborating that the shared state is entangled when Alice is not trusted [4].

IV. CHANNEL-STATE DUALITY

The two aforementioned concepts of steering can be linked via channel-state duality, or Choi-Jamiołkowski isomorphism [41,42]. For a quantum channel $\Lambda^{D\to B} : \mathcal{B}(\mathcal{H}_D) \to \mathcal{B}(\mathcal{H}_B)$, we can construct an extended quantum state $\rho_{CB} \in \mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_B)$, where $\mathcal{H}_C$ is a Hilbert space isomorphic to $\mathcal{H}_D$, by

$$\rho^{CB} = (I^C \otimes \Lambda^{D\to B})[\,|\Phi\rangle\langle\Phi|\,]. \qquad (9)$$

Here, $|\Phi\rangle \in \mathcal{H}_C \otimes \mathcal{H}_D$ is the maximally entangled state

$$|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle, \qquad (10)$$

where $\{|i\rangle\}_i$ is an orthonormal basis of Hilbert space $\mathcal{H}_D$ (and thus of $\mathcal{H}_C$). In this paper, symbol $|\Phi\rangle$ will be exclusively used to denote the maximally entangled state. The mapping from a quantum channel to a quantum state in Eq. (9) is bijection if we restrict the range of the mapping to quantum states where the reduced state after tracing out system B is the maximally mixed state. A resultant quantum state in Eq. (9) is called a Choi matrix of the channel $\Lambda^{D\to B}$.

Channel-state duality links the interesting properties of channels and states. For example, (i) a channel is unital if and only if the reduced state after tracing out system C of the dual state is the maximally mixed state, (ii) it is a measure-and-prepare channel if and only if its dual state is separable [43], and (iii) it is an entanglement binding [44] if and only if the dual state is bound entangled [45]. The correspondence between the properties of channels and states is also shown to hold for steerability [31]. A channel extension $\Lambda^{D\to BA}$ of the channel $\Lambda^{D\to B}$ is unsteerable if and only if its Choi matrix is unsteerable by any local measurements on A. One can reexpress this correspondence between channel assemblages and state assemblages; a channel assemblage $\{\Lambda_{a|x}\}_{a.x}$ is unsteerable if and only if the state assemblage that consists of its Choi matrices is unsteerable.

In this paper, we find that steerability correspondence can be obtained not only by the maximally entangled state but also by any bipartite pure state $|\psi\rangle$ with the full Schmidt rank. From now on, for simplicity, we will denote the state assemblage obtained from Choi-Jamiołkowski isomorphism of a channel assemblage using $|\psi\rangle$, $\{(I^C \otimes \Lambda_{a|x}^{D\to B})[\,|\psi\rangle\langle\psi|\,]\}_{a,x}$, as $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$ and call it a *Choi assemblage obtained by* $|\psi\rangle$.

Theorem 1. Let $|\psi\rangle \in \mathcal{H}_C \otimes \mathcal{H}_D$ be a pure state with the full Schmidt rank. Then a channel assemblage $\{\Lambda_{a|x}^{D\to B}\}_{a,x}$ is unsteerable if and only if the Choi assemblage obtained by $|\psi\rangle$, $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$, is unsteerable.

Proof. To prove the sufficiency, let us assume that the given channel assemblage $\{\Lambda_{a|x}^{D\to B}\}_{a,x}$ is unsteerable. Then we can write $\Lambda_{a|x}^{D\to B} = \sum_\lambda p(a|x,\lambda)\,\Lambda_\lambda$, and the Choi assemblage obtained by $|\psi\rangle$ reads $\{\sum_\lambda p(a|x,\lambda)\,(I\otimes\Lambda_\lambda)[\,|\psi\rangle\langle\psi|\,]\}_{a,x}$, where $\{\,(\,I\,\otimes\,\Lambda_\lambda\,)[\,|\psi\rangle\langle\psi|\,]\}_\lambda$ is an ensemble. This is exactly the form of Eq. (1), which proves the sufficiency.

To prove the necessity, let us assume that a Choi assemblage of the given channel assemblage $\{\Lambda_{a|x}^{D\to B}\}_{a,x}$ obtained by $|\psi\rangle$ is unsteerable. Then there exists some conditional probability distribution $p(a|x,\lambda)$ and a set of quantum states $\{\sigma_\lambda^{CB}\}_\lambda$ which satisfy

$$\big(I^C \otimes \Lambda_{a|x}^{D\to B}\big)[\,|\psi\rangle\langle\psi|\,] = \sum_\lambda p(a|x,\lambda)\sigma_\lambda^{CB}. \tag{11}$$

The polar decomposition of σ_λ^{CB} can be written as

$$\sigma_\lambda^{CB} = \sum_i p_{i,\lambda}|\psi_{i,\lambda}\rangle\langle\psi_{i,\lambda}|, \tag{12}$$

with some pure states $|\psi_{i,\lambda}\rangle \in \mathcal{H}_C \otimes \mathcal{H}_B$ and probability distributions $p_{i,\lambda}$. Using the property of the maximally entangled state, the pure states can be expressed as

$$|\psi\rangle = (I\otimes K)|\Phi\rangle = (K^T\otimes I)|\Phi\rangle, \tag{13}$$

$$|\psi_{i,\lambda}\rangle = (I\otimes K_{i,\lambda})|\Phi\rangle = (K_{i,\lambda}^T\otimes I)|\Phi\rangle, \tag{14}$$

with some operators K and $K_{i,\lambda}$ in $B(\mathcal{H}_D)$ [and thus in $B(\mathcal{H}_C)$]. Here, K is a full-rank operator due to the full Schmidt-rank assumption of $|\psi\rangle$. This guarantees that K is invertible. From Eqs. (11) and (13), summing over a and tracing out system B yields $(K^\dagger K)^T/d = \sum_\lambda \sigma_\lambda^C$. Meanwhile, from Eqs. (12) and (14), tracing out system B gives $\sigma_\lambda^C = \sum_i p_{i,\lambda}(K_{i,\lambda}^\dagger K_{i,\lambda})^T/d$. Combining the two results, we obtain an equality of $(K^\dagger K)^T = \sum_{i,\lambda} p_{i,\lambda}(K_{i,\lambda}^\dagger K_{i,\lambda})^T$ which guarantees that $\{\sqrt{p_{i,\lambda}}K_{i,\lambda}K^{-1}\}_{i,\lambda}$ is a set of Kraus operators. The quantum states σ_λ^{CB} can then be written as

$$\sigma_\lambda^{CB} = \big(I^C\otimes\Lambda_\lambda^{D\to B}\big)[|\psi\rangle\langle\psi|], \tag{15}$$

where $\Lambda_\lambda^{D\to B}[\,\cdot\,] = \sum_i p_{i,\lambda}K_{i,\lambda}K^{-1}\,[\,\cdot\,]\,(K^{-1})^\dagger K_{i,\lambda}^\dagger$ is a subchannel and $\{\Lambda_\lambda\}_\lambda$ is an instrument. Furthermore, it is straightforward to show that a mapping

$$A \mapsto K^{-1}\,A\,(K^{-1})^\dagger \tag{16}$$

is bijective. The bijectivity of the map (16) and the channel-state duality together with Eqs. (11) and (15) complete the necessity proof. ■

We note that steerability correspondence proved in Ref. [31] can be derived as a special case of Theorem 1 that is obtained by setting K to be the identity.

V. MDI VERIFICATION OF CHANNEL STEERING

In order to determine steerability of a channel assemblage $\{\Lambda_{a|x}^{D\to B}\}_{a,x}$ as in the form of Eq. (6), one needs to perform channel tomography to ascertain whether it can be obtained from classical processing of some instrument. However, channel tomography requires perfect control over input and output systems. For example, experimenters must be reliable, and their generating device for input states and measurement device for tomography must be accurate enough. Consequently, for a deficient apparatus or untrustworthy experimenters, one cannot be assured of results of steerability determination. Therefore, to alleviate the demands, MDI verification of the channel steering is needed, thus in this section we will propose how to verify channel steering in an MDI manner.

For the first step, we will verify steerability of the Choi assemblage obtained by $|\psi\rangle$ instead of that of the channel assemblage, employing the channel-state duality of steering. Suppose a channel assemblage $\{\Lambda_{a|x}^{D\to B}\}_{a,x}$ of which one intends to find out steerability. One can obtain the corresponding Choi assemblage obtained by $|\psi\rangle$, $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$, by preparing bipartite pure states with the full Schmidt rank $|\psi\rangle\langle\psi| \in B(\mathcal{H}_C\otimes\mathcal{H}_D)$ and transmitting part D through subchannels $\Lambda_{a|x}^{D\to B}$ while preserving part C. In this process, one who supervises the experiment actively participates in the verification protocol by generating quantum states, and we shall call him or her the *referee*. To verify steerability of the Choi assemblage obtained by $|\psi\rangle$ in an MDI way, we will adopt the canonical method of an MDI verification protocol proposed in Refs. [34,36]. Let $\{F_{a|x}\}_{a,x} \subset B(\mathcal{H}_C\otimes\mathcal{H}_B)$ be a steering witness for the Choi assemblage $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$ such that

$$\sum_{a,x}\mathrm{Tr}\big[F_{a|x}\,\rho_{a|x}^{|\psi\rangle}\big] > \sup_{\{\sigma_{a|x}^{US}\}}\sum_{a,x}\mathrm{Tr}\big[F_{a|x}\,\sigma_{a|x}^{US}\big] := \alpha, \tag{17}$$

where the supremum is taken over all unsteerable assemblages $\{\sigma_{a|x}^{US}\}_{a,x} \subset B(\mathcal{H}_C\otimes\mathcal{H}_B)$. The existence of such a witness is guaranteed by the convexity of unsteerable assemblages [39] and the hyperplane separation theorem. Based on Refs. [34,36], we can decompose a steering witness into

$$F_{a|x} - \frac{\alpha}{|\mathcal{X}|}I = \sum_{z,y}\beta_{1,1,a}^{z,y,x}\,\tau_z\otimes\omega_y, \tag{18}$$

for any a and x, where $\mathcal{X}$ is an index set of x, and $\{\tau_z\}_z \subset B(\mathcal{H}_C)$ and $\{\omega_y\}_y \subset B(\mathcal{H}_B)$ are tomographically complete sets of states. Here, the decomposition of the steering witness does not need to be unique. To accomplish the MDI verification of steerability, the referee prepares quantum states $\tau_z^T \in B(\mathcal{H}_{C'})$ and $\omega_y^T \in B(\mathcal{H}_{B'})$, where $\mathcal{H}_{X'}$ is a Hilbert space isomorphic to $\mathcal{H}_X$ for $X \in \{C, B\}$, and provides them to parties C and B,

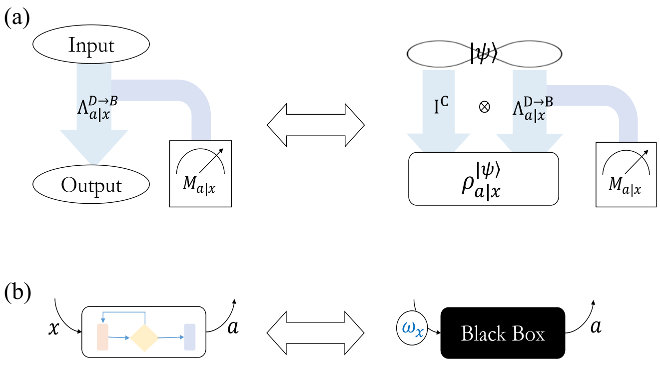

FIG. 2. Schematic of an MDI verification protocol of channel steering. (a) Due to Theorem 1, steerability of the channel assemblage (left) is equivalent to steerability of the Choi assemblage obtained by $|\psi\rangle$ (right). Thus we first obtain Choi assemblage using bipartite pure states $|\psi\rangle$ with the full Schmidt rank, and verify steerability of the resultant Choi assemblage $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$. (b) Left: 1s-DI verification of state steering can be accomplished by requesting and receiving classical information from the experimenter when the whole process can be trusted. It is depicted as a white box which takes and yields classical information. Right: If we encode the question in quantum states, MDI verification of state steering is possible without any trust in the process. It is depicted as a black box because we do not need to concern ourselves with what is taking place inside. Therefore, by verifying the obtained Choi assemblage in the MDI protocol, MDI verification of the channel steering is accomplished.

say Charlie and Bob, respectively. In response to the input, Charlie (Bob) reports one of the values, {0, 1}. It will become clear that zero corresponds to measurement failure and 1 corresponds to measurement success. Within the dichotomic options, they try to maximize the value called the score:

$$I(\{\rho_{a|x}^{|\psi\rangle}\}_{a,x},\beta)=\sum_{a,x,y,z}\beta_{1,1,a}^{z,y,x}P(1,1,a|\tau_z^T,\omega_y^T,x),\qquad(19)$$

where $P(1,1,a|\tau_z^T,\omega_y^T,x)$ is a probability for which both Charlie and Bob report 1 when the bystander's input is x and output is a. The meaning of the score can be understood as a sum of the payoff that Charlie and Bob obtain for each round according to their input and output, where the payoff is set to be $\beta_{1,1,a}^{z,y,x}$ if their outputs are both 1 (measurement success) and zero if one of their outputs is zero (measurement failure). Charlie and Bob will determine their optimal choice based on their measurement outcome performed on the given state τ_z^T, ω_y^T, and their part of the output substate $\rho_{a|x}^{|\psi\rangle}$. Thus the score can be written as

$$\begin{aligned}I(\{\rho_{a|x}^{|\psi\rangle}\}_{a,x},\beta)\\ =\sum_{a,x,y,z}\beta_{1,1,a}^{z,y,x}\,\mathrm{Tr}[(Q^{C'C}\otimes P^{BB'})(\tau_z^T\otimes\rho_{a|x}^{|\psi\rangle}\otimes\omega_y^T)].\end{aligned}\qquad(20)$$

With this expression, we can show that the score is positive for the steerable assemblage $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$ while zero for any unsteerable assemblage $\{\sigma_{a|x}^{US}\}_{a,x}$. This scheme is a straightforward extension from Refs. [34,36] to tripartite states, and a sketch of the protocol is provided in Fig. 2. In this figure, we used white-box and black-box pictures to focus on the "trust" in the process while simplifying other details.

Theorem 2. For any Choi assemblage obtained by $|\psi\rangle$, we can construct the corresponding steering criterion without an assumption on measurement devices.

Proof. Let us first prove that the score is nonpositive (thus zero) for unsteerable assemblages. For any unsteerable assemblage $\{\sigma_{a|x}^{US}\}_{a,x}$, one can write $\sigma_{a|x}^{US}=\sum_\lambda p(a|x,\lambda)\sigma_\lambda^{CB}$ for some probability distribution $p(a|x,\lambda)$ and ensemble $\{\sigma_\lambda^{CB}\}_\lambda$, as in Eq. (1). Therefore, for any choice of joint POVM elements $Q^{C'C}$, $P^{BB'}$, the score (20) reads

$$\begin{aligned}&I(\{\sigma_{a|x}^{US}\}_{a,x},\beta)\\ &=\sum_{a,x,y,z,\lambda}\beta_{1,1,a}^{z,y,x}p(a|x,\lambda)\\ &\quad\times\mathrm{Tr}\big[(Q^{C'C}\otimes P^{BB'})\big(\tau_z^T\otimes\sigma_\lambda^{CB}\otimes\omega_y^T\big)\big]\\ &=\sum_{a,x,y,z,\lambda}\beta_{1,1,a}^{z,y,x}\;p(a|x,\lambda)\;\mathrm{Tr}\big[\big(R_\lambda^{C'B'}\big)\big(\tau_z^{C'}\otimes\omega_y^{B'}\big)^T\big],\end{aligned}\qquad(21)$$

where $R_\lambda^{C'B'}$ is a reduced POVM element, defined by $R_\lambda^{C'B'}=\mathrm{Tr}_{CB}[\,(Q^{C'C}\otimes P^{BB'})(I\otimes\sigma_\lambda^{CB}\otimes I)\,]$. Since a reduced POVM element is a positive semidefinite operator, unless it is the zero operator, one can convert the reduced POVM element to the substate $\rho_{R,\lambda}:=(R_\lambda^{C'B'})^T/N$ so that $\{\rho_{R,\lambda}\}_\lambda$ forms an ensemble, where N is a normalizing constant $N=\mathrm{Tr}[Q^{C'C}\otimes P^{BB'}]$. Substituting $R_\lambda^{C'B'}$ by $\rho_{R,\lambda}$, the score reads

$$\begin{aligned}&I(\{\sigma_{a|x}^{US}\}_{a,x},\beta)\\ &=\mathrm{Tr}\left[\sum_{a,x,\lambda}p(a|x,\lambda)\,\big(R_\lambda^{C'B'}\big)^T\sum_{z,y}\beta_{1,1,a}^{z,y,x}\,\big(\tau_z^{C'}\otimes\omega_y^{B'}\big)\right]\\ &=N\sum_{a,x}\mathrm{Tr}\left[\sum_\lambda p(a|x,\lambda)\;\rho_{R,\lambda}\left(F_{a|x}-\frac{\alpha}{|\mathcal{X}|}I\right)\right]\\ &=N\sum_{a,x}\mathrm{Tr}\left[\rho_{a|x}^{US}\left(F_{a|x}-\frac{\alpha}{|\mathcal{X}|}I\right)\right]\leqslant 0,\end{aligned}\qquad(22)$$

where $\rho_{a|x}^{US}=\sum_\lambda p(a|x,\lambda)\rho_{R,\lambda}$, thus the last inequality in Eq. (22) holds by the definition of the steering witness. This shows that any choice of POVM elements gives a nonpositive score, thus the optimal choice of Charlie and Bob to maximize the score is to report zero as output. Consequently, the score is zero for any unsteerable assemblage.

Meanwhile, for the Choi assemblage $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$, Charlie and Bob can obtain a positive score by projecting the system $C'C$ and BB' into the maximally entangled state $|\Phi\rangle\langle\Phi|$:

$$\begin{aligned}&I(\{\rho_{a|x}^{|\psi\rangle}\}_{a,x},\beta)\\ &=\sum_{a,x,y,z}\beta_{1,1,a}^{z,y,x}\,\mathrm{Tr}\big[\,(|\Phi\rangle\langle\Phi|\otimes|\Phi\rangle\langle\Phi|)\big(\tau_z^T\otimes\rho_{a|x}^{|\psi\rangle}\otimes\omega_y^T\big)\,\big]\\ &=\sum_{a,x}\mathrm{Tr}\left[\sum_{z,y}\beta_{1,1,a}^{z,y,x}\,(\tau_z\otimes\omega_y)\,\rho_{a|x}^{|\psi\rangle}\right]/\,d_Cd_B\\ &=\sum_{a,x}\mathrm{Tr}\left[\left(F_{a|x}-\frac{\alpha}{|\mathcal{X}|}I\right)\rho_{a|x}^{|\psi\rangle}\right]/\,d_Cd_B>0,\end{aligned}\qquad(23)$$

where we used the property of the maximally entangled state, $\mathrm{Tr}[\,|\Phi\rangle\langle\Phi|(A\otimes B)\,] = \mathrm{Tr}[A^T B]\,/\,d_A$, for the second equality. The last inequality is derived by the definition of the steering witness. Therefore we can always obtain a positive score for the Choi assemblage $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$, which completes the proof that the score (20) is a steering criterion in the MDI scenario.

Here, in order to avoid misleading, we point out that MDI verification of the channel steering is *not* free from trust or perfect control over the experiment. Although an MDI scenario is independent from measurement devices, it still requires generation of quantum states such as bipartite pure states with the full Schmidt rank $|\psi\rangle\langle\psi|$ to obtain the Choi assemblage from the channel $\Lambda^{D\to B}$, and sets of tomographically complete quantum states $\{\tau_z\}_z$ and $\{\omega_y\}_y$. Thus we still need to trust or control the generating devices. Nevertheless, we can say that the MDI verification is better than the original scenario for the following reasons.

First, in the 1s-DI scenario, it is usually not easy to test the assumptions such as reliability of experimenters or accuracy of the measurement apparatus. In contrast, in the MDI scenario, the assumption of the generation of quantum states is open to a test from an external party. Any ombudsman can bring their own measurement apparatus and test generated quantum states via state tomography. In this sense, we can say that the MDI verification is more reliable.

Second, in the MDI scenario, measurement inefficiency does not affect steerability verification, along a similar line with the MDI entanglement verification [34]. When losses occur to one of the parties, they lose quantum correlations and thus their assemblage is described by an unsteerable one that contributes zero to the score. Therefore, the effect is that the probability part, $\mathrm{Tr}[(Q^{C'C}\otimes P^{BB'})(\tau_z^T\otimes\rho_{a|x}^{CB}\otimes\omega_y^T)]$, is multiplied by measurement efficiencies of Charlie and Bob. Since a positive factor does not change sign, unless one of the measurement efficiencies of Charlie and Bob is zero, the score is nonpositive for any unsteerable assemblage and positive for the Choi assemblage $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$. This guarantees loss tolerance of the MDI steering verification with respect to Charlie and Bob.

Third, in the MDI scenario, imperfect generation of quantum states can be analyzed and its effect on the verification protocol can be quantified. For imperfect preparation of quantum questions, when the type of the questions and form of the score are set in a two-qubit system, the effect of misaligned quantum states to the score is exactly quantified in Ref. [35] and generalized for inefficient measurements in Ref. [46]. Regardless of the type of the quantum channel through which quantum questions are transmitted, there is no chance for passing the steering verification using an unsteerable assemblage because the effect of quantum channels can be absorbed in the optimal POVM elements [35]. Furthermore, for imperfect generation of pure states with the full Schmidt rank which are used for the channel-state duality, although we cannot obtain perfect Choi assemblage obtained by $|\psi\rangle$, we can still verify the channel steering in an MDI way as follows.

We note that misalignment in state preparation with generating different bipartite pure states of the full Schmidt rank does not change the protocol because the channel-state duality holds for any bipartite pure state with the full Schmidt rank. We also consider the case that we fail to generate a perfect full Schmidt-rank pure state. The generated state is then a classical mixture of the full Schmidt-rank pure state and some undesired noise that is unsteerable as

$$\rho_w^{CD} = w|\psi\rangle\langle\psi| + (1-w)\sigma_{\mathrm{noise}}, \tag{24}$$

where σ_{noise} denotes unsteerable noise and $0\leqslant w\leqslant 1$. A typical example is a colored noise, $\sigma_{\mathrm{noise}} = \sum_i p_i|ii\rangle\langle ii|$, generated by decoherence of the bipartite pure state $|\psi\rangle\langle\psi| = \sum_{i,j}\sqrt{p_ip_j}|ii\rangle\langle jj|$. We observe that if all off-diagonal parts of the pure state $|\psi\rangle\langle\psi|$ disappear it results in the colored noise. If we use the state in Eq. (24) for the channel-state duality, we obtain

$$\begin{aligned}&\big(I^C\otimes\Lambda_{a|x}^{D\to B}\big)\big(\rho_w^{CD}\big)\\&\quad= w\,\rho_{a|x}^{|\psi\rangle} + (1-w)\,\big(I\otimes\Lambda_{a|x}^{D\to B}\big)[\,\sigma_{\mathrm{noise}}\,],\end{aligned} \tag{25}$$

where $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$ is the Choi assemblage obtained by $|\psi\rangle$ from the given channel assemblage $\{\Lambda_{a|x}^{D\to B}\}_{a,x}$. One can consider Eq. (25) as a convex combination of two state assemblages $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$ and $\{(I\otimes\Lambda_{a|x}^{D\to B})[\,\sigma_{\mathrm{colored}}\,]\}_{a,x}$. According to the theorems in Ref. [39], local operations do not increase steerability of a state assemblage, thus $\{(I\otimes\Lambda_{a|x}^{D\to B})[\,\sigma_{\mathrm{noise}}\,]\}_{a,x}$ remains unsteerable. Meanwhile, from the definition of steering measures such as steerable weight [47] or steering robustness [12], the steering measure is zero if and only if the assemblage is unsteerable. Furthermore, forenamed steering measures are convex monotones [39], which means that, for any real number $0\leqslant r\leqslant 1$ and two assemblages $\{\rho_{a|x}\}_{a,x}$ and $\{\sigma_{a|x}\}_{a,x}$,

$$S[\,r\rho_{a|x} + (1-r)\,\sigma_{a|x}] \leqslant r\,S(\rho_{a|x}) + (1-r)\,S(\sigma_{a|x}), \tag{26}$$

where S is the steering measure. As a consequence, a nonzero steering measure of Eq. (25) implies a nonzero steering measure of the Choi assemblage obtained by $|\psi\rangle$, $\{\rho_{a|x}^{|\psi\rangle}\}_{a,x}$. Therefore, by verifying steerability of Eq. (25), we end up with verifying channel assemblage $\{\Lambda_{a|x}^{D\to B}\}_{a,x}$.

We need to find out how much unsteerable noise can be tolerated. In other words, this is the least amount of noise required to obliterate steerability of the assemblage. This lower bound is determined by *steering robustness of the assemblage* $R(\{\rho_{a|x}\})$ [12]:

$$\begin{gathered}R(\{\rho_{a|x}\}) = \inf_{\{\sigma_{a|x}\}} t \quad \text{such that}\\ \frac{\rho_{a|x} + t\,\sigma_{a|x}}{1+t} \text{ is an unsteerable assemblage,}\end{gathered} \tag{27}$$

where the infimum is taken over the set of all assemblages. If we restrict $\{\sigma_{a|x}\}$ to a set of unsteerable assemblages, the obtained value, say $R^{US}(\{\rho_{a|x}\})$, will be larger than or equal to $R(\{\rho_{a|x}\})$, and we determine the amount of allowed noise by $\frac{R^{US}(\{\rho_{a|x}^{|\psi\rangle}\})}{1+R^{US}(\{\rho_{a|x}^{|\psi\rangle}\})}$. Therefore, the threshold of the noise in generating the pure state $|\psi\rangle$ to verify steerability of the channel assemblage is lower bounded by $\frac{R(\{\rho_{a|x}^{|\psi\rangle}\})}{1+R(\{\rho_{a|x}^{|\psi\rangle}\})}$. This tells us that, even if any bipartite pure state with the full Schmidt rank can be used to obtain steerability correspondence, in a practical situation, we should use states of which their Choi assemblage has high steering robustness to endure the undesired noise.

We finally pose several questions. First, how much would be a gap between R and R^{US}? Second, how can we find the pure state $|\psi\rangle$ which gives the largest steering robustness (and R^{US}) for a given channel assemblage $\{\Lambda_{a|x}^{D\to B}\}_{a,x}$? (A mixed state cannot give the largest value due to the linearity of the quantum channel and convexity of steering robustness.) Third, does a Choi assemblage preserve the order of the steering measure for pure states? In other words, for a given channel assemblage $\{\Lambda_{a|x}^{D\to B}\}_{a,x}$ and for pure states $|\psi\rangle$ and $|\phi\rangle$ such that $S(|\psi\rangle) \leqslant S(|\phi\rangle)$, where $S(|X\rangle)$ is a steering measure of pure state $|X\rangle$, would it be true that $S(\{\rho_{a|x}^{|\psi\rangle}\}) \leqslant S(\{\rho_{a|x}^{|\phi\rangle}\})$? These deserve further investigations.

VI. CONCLUSION

We have proposed a way to verify the channel steering in an MDI manner. We first converted a channel assemblage to a state assemblage via Choi-Jamiołkowski isomorphism for a bipartite pure state with the full Schmidt rank, and then determined steerability of the state assemblage. We applied the canonical method of MDI verification of state steering to the Choi assemblage obtained by $|\psi\rangle$, and showed that the steering criterion, called the score, is nonpositive for any unsteerable state assemblages while it is positive for the Choi assemblage. As a consequence of steerability correspondence we proved that this verifies steerability of the given channel assemblage.

We further analyzed the situation of imperfect preparation of a pure state with the full Schmidt rank to obtain the Choi assemblage. A typical case would be that the pure state suffers from decoherence and thus some portion of it is converted into colored noise, just like a Werner state. We showed that not only for colored noise, but also for any type of unsteerable noise, can we verify channel steering in an MDI manner for a portion of undesired noise bounded from below by steering robustness of the state assemblage.

Our paper leads to several open questions.

(1) How much is a gap between R and R^{US}?

(2) How can we find the pure state which gives the maximal steering robustness for a given channel assemblage?

(3) Does a Choi assemblage preserve the order of the steering measure for pure states?

We leave these problems for future research.

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