Noise tolerance of Dicke states

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Starting from explicitly constructing an entanglement witness for the mixture of a Dicke state with white noise (noisy Dicke state), we provide an entanglement criterion for any multiqubit state. The criterion is necessary and sufficient for noisy W states of three, four, and five qubits. We demonstrate an entanglement criterion for any four-qubit state. The criterion is necessary and sufficient for the generalized noisy four-qubit Dicke states. We present the sufficient criterion of full separability for a noisy $2k$ qubit Dicke state with k excitations. Then we generalize the entanglement witness and criterion to the mixture of white noise and generic n-qubit Dicke state with k excitations based on strong numeric evidence. Apart from noisy Dicke states, we also consider the entanglement properties of simultaneously amplitude damped and dephased Dicke states, and depolarized multiqubit W states.

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I. INTRODUCTION

Entanglement is crucial for quantum communication and quantum computation. To determine whether a given quantum state is entangled is by no means easy. Many entanglement criteria had been developed for entanglement detection [1–6] (see [7] for a review). One of them is the positive partial transpose (PPT) criterion [1,2]. An alternative to entanglement criterion is the entanglement witness [8]. For multipartite systems, entanglement criteria [9–13] and entanglement witnesses [14–17] have also been investigated. The entanglement witness is also important in experiments [18–23]. Experimental entanglement detection relies on finitely many local measurements, rather than full quantum state tomography. The former can be put into the form of an entanglement witness. The latter becomes impractical for a large system due to the exponential increase in measurements with the number of subsystems.

Multiqubit W states generated across qubit registers can be used for various quantum information processing tasks [24] and have been experimentally prepared with photons, ions, neutral atoms, and nuclear magnetic resonance (NMR) [25–29]. An experimental W-state entanglement of 25 individually accessible atomic quantum interfaces has been generated [30]. A multipartite W state can be used for anonymous information transmission in a noisy quantum network [31]. The W-state anonymous transmission tolerates one node loss, a prominent performance compared with anonymous transmission using a Greenberger-Horne-Zeilinger (GHZ) state. Quantum key distribution and secret sharing can be implemented using W state and project measurement, teleportation of the entangled state and three-qubit dense coding were proposed using a generalized W state [32]. W states are special Dicke states [33]. A four-qubit Dicke state can be utilized for quantum network protocols such as $1 \rightarrow 3$ cloning and open-destination teleportation [34]. Dicke states can also be used in quantum metrology [35]. Experimentally, high-fidelity Dicke states have been created with photons [36,37] and trapped ions [27,38].

A prepared Dicke state may undergo different kinds of noises. The noise can be due to external additive noise, decoherence, and imperfection in preparation. In the atomic quantum interface experiment, the typical noise of preparing a multipartite W state is the additive vacuum state and double excitation state; the contribution of higher-order excitations turns out to be negligible [30]. In a quantum network, one may consider a noise model in which each qubit is subjected to the same individual decohering channel [31] (parallel or product channel later on). The decoherence of the channel can be amplitude damping, dephasing, and depolarizing. One may consider another simple noise model of global mixing the entangled state with white noise [10,39]. The external additive white noise is a locally prepared noise. Admixing locally prepared noise is different from local decoherence [40]. In the rest of the paper, we will be interested in discussing the fully separable (often abbreviated as separable below) criterion of Dicke states in a noise environment.

Recently, the entanglement criterion has been studied in the Dicke basis, with emphasis on Dicke diagonal states [41–44]. It has been shown that the PPT criterion is necessary and sufficient for the separability of a Dicke diagonal state. There are two different proofs. One demonstrates the criterion with extremal witnesses [42] and the other proves the criterion without using a witness [43]. A Dicke diagonal state is a symmetric (bosonic, more strictly) state so by construction it is either fully separable or genuinely multipartite entangled [43]. A mixed state is symmetric if its support (the space spanned by its eigenvectors of nonzero eigenvalues) is in the symmetric subspace of the Hilbert space, that is, its support is invariant under qubit permutations.

We ask whether the PPT criterion is necessary and sufficient for the entanglement of the mixture of a Dicke state with white noise or a Dicke state decohering in a noise environment

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(damped Dicke state). The answer is "no" in general, as shown later. It should be clarified that a noisy Dicke state or a damped Dicke state is no longer a Dicke diagonal state. Although the density matrices are invariant under qubit permutations, their support is not invariant under swaps. So the separability problem of a noisy Dicke state or a damped Dicke state is a rather different topic compared with that of a Dicke diagonal state.

The paper is organized as follows: We describe the preliminary details of the entanglement witness in Sec. II. Section III is devoted to the entanglement criterion for W states. We investigate the entanglement criterion for generalized four-qubit Dicke states in Sec. IV and the entanglement criterion for n-qubit Dicke states in Sec. V. We then compare the entanglement criteria obtained with the PPT criterion in Sec. VI and conclude in Sec. VII.

II. ENTANGLEMENT WITNESSES

A multipartite quantum state is called separable if it can be written as a statistical mixture of product states of its parties [45]. Otherwise, it is entangled. An entanglement witness is a Hermitian operator $\hat{W}$ that has non-negative expectation value $\mathrm{Tr}(\hat{W}\varrho_{\mathrm{sep}}) \geqslant 0$ for all separable states ϱ_{sep}, and negative expectation value $\mathrm{Tr}(\hat{W}\rho) < 0$ for at least one state ρ. We say that the entanglement of state ρ is witnessed by $\hat{W}$. A witness is called weakly optimal if there exists a separable state ϱ_{sep} such that $\mathrm{Tr}(\hat{W}\varrho_{\mathrm{sep}}) = 0$ [8]. Let a weakly optimal witness $\hat{W}=\Lambda\hat{I}-\hat{M}$, where $\hat{I}$ is the identity matrix of the system and $\hat{M}$ is a Hermitian operator; then,

$$\Lambda = \max_{\varrho_{\mathrm{sep}}} \mathrm{Tr}(\hat{M}\varrho_{\mathrm{sep}}). \tag{1}$$

A weakly optimal (hereafter, we omit "weakly optimal" for simplicity) witness leads to a necessary criterion of separability. The process for obtaining a proper witness for a given state is rather *ad hoc*. In order to make the necessary criterion more efficient, we turn to the method of matched entanglement witness [16]. Define

$$\mathcal{L}_{\min} = \min_{\hat{M}} \frac{\Lambda}{\mathrm{Tr}(\hat{M}\rho)}, \tag{2}$$

with the convention of $\Lambda > 0$ and $\mathrm{Tr}(\hat{M}\rho) > 0$. Then the state ρ is entangled when $\mathcal{L}_{\min} < 1$. An n-qubit generic state ρ is described by $4^n - 1$ parameters. So the number of parameters for describing a Hermitian operator is also $4^n - 1$. Even for moderate large number n, the minimization in (2) seems to be very difficult, if not impossible. However, when the state ρ possesses some symmetries, the number of parameters for describing it greatly reduces, and so does the number of parameters in $\hat{M}$. The numerical calculations based on (1) and (2) are useful in finding the entanglement witness of a given quantum state.

III. WITNESS FOR W STATES

A W state of n qubits is $|W_n\rangle = \frac{1}{\sqrt{n}}(|00\ldots01\rangle + |0\ldots010\rangle + \cdots + |10\ldots00\rangle)$. An entanglement criterion was proposed for the state of a usual ($n = 3$) W state mixed with white noise [46]. It is known that every entanglement criterion can be converted to a proper entanglement witness. We obtain the following witness corresponding to the entanglement criterion in [46]:

$$\begin{aligned}\hat{W} = {}& \frac{1}{d}|000\rangle\langle000| - [|001\rangle(\langle010| + \langle100|) \\ & + |010\rangle(\langle001| + \langle100|) + |100\rangle(\langle001| + \langle010|)] \\ & + d(|011\rangle + |101\rangle + |110\rangle)(\langle011| + \langle101| + \langle110|),\end{aligned} \tag{3}$$

where d is a positive parameter. Based on the structure of witness (3), we may propose a witness of an n-qubit system in the following: (The validity of it as a witness will be shown in Appendix A.)

$$\begin{aligned}\hat{W} = {}& \frac{1}{d}|0\rangle^{\otimes n}\langle0|^{\otimes n} + |0\ldots01\rangle\langle0\ldots01| \\ & + |0\ldots010\rangle\langle0\ldots010| + \cdots + |10\ldots0\rangle\langle1\ldots00| \\ & - n|W_n\rangle\langle W_n| + \frac{1}{2}dn(n-1)|D_{n,2}\rangle\langle D_{n,2}|,\end{aligned} \tag{4}$$

where $|D_{n,k}\rangle$ is a Dicke state, i.e., a state of n qubits that are invariant under the permutation of its elements,

$$|D_{n,k}\rangle = \left(C_k^n\right)^{-1/2} \sum_{\Pi} |\Pi(1^k 0^{n-k})\rangle, \tag{5}$$

and where $C_k^n = n!/[k!(n-k)!]$ is a normalization factor and k denotes the number of excitations, namely, the number of qubits on the state $|1\rangle$. The W state is a special Dicke state with $k = 1$.

Witness (4) leads to a necessary criterion of full separability for any multiqubit state.

Proposition 1. The necessary criterion of full separability for a multiqubit state ρ is

$$\sqrt{\rho_{\mathbf{0},\mathbf{0}} \sum_{\mathbf{j},\mathbf{j}'} \rho_{\mathbf{j},\mathbf{j}'}} \geqslant \frac{1}{2} \sum_{\mathbf{i}\neq\mathbf{i}'} \rho_{\mathbf{i},\mathbf{i}'}, \tag{6}$$

with $|\mathbf{0}| = 0, |\mathbf{i}| = 1, |\mathbf{i}'| = 1, |\mathbf{j}| = 2, |\mathbf{j}'| = 2$. We have defined the Hamming weight $|\mathbf{m}| = \sum_k m_k$ for a binary string $\mathbf{m} = m_1 m_2 \ldots m_n$ [or binary vector $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ for later use]. Violation of it implies entanglement.

The proof of Proposition 1 is shown in Appendix A. For a multiqubit state, where ρ completely lies in the subspace with basis $|\mathbf{m}\rangle(|\mathbf{m}| \leqslant 1)$, we have $\rho_{\mathbf{j},\mathbf{j}'} = 0$ for all $|\mathbf{j}|, |\mathbf{j}'| = 2$, and hence the left-hand side of inequality (6) is 0 and the state is entangled as far as the right-hand side of (6) is positive.

Denote the mixture of the W_n state with white noise (noisy W_n state) as

$$\rho_{W_n}(p) = pW_n + \frac{1-p}{2^n}I, \tag{7}$$

where $0 \leqslant p \leqslant 1$ and I is the $2^n \times 2^n$ identity matrix. From Proposition 1, the necessary condition of separability for the noisy W_n state follows directly:

Corollary 1. The necessary condition of full separability for state $\rho_{W_n}(p)$ is

$$p \leqslant \frac{1}{1 + 2^n\sqrt{\frac{n-1}{2n}}}. \tag{8}$$

The state $\rho_{W_3}(p)$ is fully separable when $p \leqslant 1/(1+8/\sqrt{3}) \approx 0.17797$ from either (8) or Ref. [46]. We may ask if the necessary condition (8) is also sufficient. A direct construction of the noisy W state from product states leads to the sufficient conditions for the full separability of the noisy n-qubit W state:

Corollary 2. The noisy W state $\rho_{W_n}(p)$ is fully separable if

$$p \leqslant \begin{cases} \frac{1}{1+2^n\sqrt{\frac{n-1}{2n}}} & \text{if } 2 \leqslant n \leqslant 5 \\ \frac{n}{n+(n-2)2^n} & \text{if } n \geqslant 6. \end{cases} \tag{9}$$

The proof of Corollary 2 can be found in Appendix B. Hence, the separability condition (8) is necessary and sufficient when $n \leqslant 5$. The necessary condition (8) differs from the sufficient condition (9) when $n \geqslant 6$.

Consider an amplitude damping and dephasing quantum channel $\mathcal{E}_1$, with Kraus operators A_0, A_1, A_2, with

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-r-h} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{h} \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & \sqrt{r} \\ 0 & 0 \end{pmatrix},$$

where $1-r-h = e^{-2t/T_2}$, $1-r = e^{-t/T_1}$, with T_1, T_2 being the relaxing and dephasing time, respectively [47]. Let an initial W_n state passing through a product channel $\mathcal{E}_1^n$, with each qubit being acted on by an individual channel $\mathcal{E}_1$; then we have the output state

$$\mathcal{E}_1^n(|W_n\rangle) = r(|0\rangle\langle 0|)^{\otimes n} + hW_{nd} + (1-r-h)|W_n\rangle\langle W_n|,$$

where W_{nd} is the diagonal part of the W_n state. We then have the following corollary:

Corollary 3. The simultaneously amplitude damped and dephased n-qubit W state is always entangled for any finite evolution time.

This can be shown by $\text{Tr}[\mathcal{E}_1^n(|W_n\rangle)\hat{W}] = \frac{r}{d} - (1-r-h)(n-1) < 0$ as we can set $d \to \infty$. Alternatively, we may consider the simultaneously amplitude damped and dephased W_n state as a mixed state that completely lies in the subspace with basis $|\mathbf{m}\rangle(|\mathbf{m}| \leqslant 1)$, so that it is entangled.

If we replace the channel $\mathcal{E}_1$ with a depolarizing channel $\mathcal{E}_2$ described by Kraus operator $\sqrt{1-3p}\sigma_0, \sqrt{p}\sigma_1, \sqrt{p}\sigma_2, \sqrt{p}\sigma_3$, where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices and σ_0 is the 2×2 identity matrix, with channel noise parameter $p \in [0, \frac{1}{4}]$, then the W_n state corrupted by the noise of product depolarizing channel $\mathcal{E}_2^n$ becomes $\mathcal{E}_2^n(|W_n\rangle)$. A direct application of Proposition 1 to state $\mathcal{E}_2^n(|W_n\rangle)$ leads to the following:

Corollary 4. The necessary condition for the full separability of the W_n state corrupted by the noise of the product depolarizing channel is

$$\begin{aligned} &32(8-n)p^4 - 64(2n-1)p^3 + 80(n-1)p^2 \\ &- 16(n-1)p + n - 1 \leqslant 0. \end{aligned} \tag{10}$$

We have $p \geqslant p_{th} \approx 0.111814$ for $n = 3$. The noise threshold p_{th} increases with n until $p_{th} \approx 0.121673$ for $n \to \infty$.

The prepared multipartite W state in the experiment is [30]

$$\rho_e = p_0\rho_0 + p_1\rho_1 + p_2\rho_2, \tag{11}$$

where p_0, p_1, p_2 and ρ_0, ρ_1, ρ_2 denote the population and the corresponding density matrix with zero, one, and double excitations. Here, $\rho_1 = |W_n\rangle\langle W_n|$, and the noise comes from ρ_0 and ρ_2. Applying Proposition 1, we have the necessary condition for the full separability of experimental state ρ_e:

$$\sqrt{p_0p_2\sum_{i,j}(\rho_2)_{i,j}} \geqslant \frac{n-1}{2}p_1. \tag{12}$$

Notice that the experimentally prepared W state slightly differs from our standard W state by local phases. The local phases can be removed with local unitary operations. So the state ρ_e in (11) should be treated by the corresponding local unitary operations.

IV. WITNESS FOR DICKE STATE $D_{4,2}$

For an n-qubit quantum state ρ, we consider its characteristic function (also called moments) $R_{j_1\ldots j_n} = \text{Tr}[\rho(\sigma_{j_1}\otimes\cdots\otimes\sigma_{j_n})]$, with $j_l = 0, 1, 2, 3$. Due to the symmetry of the Dicke state, most of the moments are zero. We write the operator $\hat{M}$ for constructing the witness in the following form:

$$\hat{M} = \sum_{j_1\ldots j_n=0}^{3} M_{j_1\ldots j_n}\sigma_{j_1}\otimes\cdots\otimes\sigma_{j_n}. \tag{13}$$

In many cases, it is convenient to set the coefficient $M_{j_1\ldots j_n}$ to be zero when the moment $R_{j_1\ldots j_n}$ is zero. Further, we set $M_{0\ldots 0} = 0$ without loss of generality. The moments can be measured in experiment, and hence the entanglement of a state is detected when $\sum_{\mathbf{j}\neq \mathbf{0}} M_{\mathbf{j}}R_{\mathbf{j}} > \Lambda$, where $\mathbf{j} = j_1\ldots j_n$.

A. Necessary criterion

Denote the mixture of a Dicke state with white noise as

$$\rho_{D_{n,k}} = p|D_{n,k}\rangle\langle D_{n,k}| + \frac{(1-p)}{2^n}I. \tag{14}$$

We find that the Hermitian operator $\hat{M}$ (the general method of finding $\hat{M}$ was shown in Ref. [16]) for detecting the entanglement of $\rho_{D_{4,2}}$ takes the following form (we omit the tensor product symbol $\otimes$ for simplicity):

$$\begin{aligned} \hat{M} = {} & \sigma_3\sigma_3\sigma_3\sigma_3 - (\sigma_1\sigma_1\sigma_3\sigma_3)_p - (\sigma_2\sigma_2\sigma_3\sigma_3)_p \\ & + M_1(\sigma_1\sigma_1\sigma_2\sigma_2 + \sigma_2\sigma_2\sigma_1\sigma_1) + M_2(\sigma_1\sigma_2\sigma_2\sigma_1 \\ & + \sigma_2\sigma_1\sigma_1\sigma_2) + M_3(\sigma_1\sigma_2\sigma_1\sigma_2 + \sigma_2\sigma_1\sigma_2\sigma_1) \\ & + \sigma_1\sigma_1\sigma_1\sigma_1 + \sigma_2\sigma_2\sigma_2\sigma_2, \end{aligned} \tag{15}$$

where $|M_i| \leqslant 1$, $i = 1, 2, 3$, and $M_1 + M_2 + M_3 = 1$. The subscript p represents the summation over all the permutations of Pauli operators. There are six terms in $(\sigma_1\sigma_1\sigma_3\sigma_3)_p$. Denoting the mean of the operator $\hat{M}$ on the pure product state $|\psi\rangle = |\psi_1\rangle|\psi_2\rangle|\psi_3\rangle|\psi_4\rangle$ as $\tau_M = \langle\psi|M|\psi\rangle$, where $|\psi_j\rangle$ is a pure state of the jth qubit, we have $|\psi_j\rangle\langle\psi_j| = \frac{1}{2}(\sigma_0 + x_j\sigma_1 + y_j\sigma_2 + z_j\sigma_3)$ in Bloch representation, with real vector parameters $\mathbf{r}_j = (x_j, y_j, z_j)$ and $|\mathbf{r}_j| = 1$. Then, τ_M is a homogeneous function of $x_j, y_j, z_j (j = 1, \ldots, 4)$. We need to obtain $\Lambda = \max_{|\psi\rangle}\tau_M$.

The entanglement witness for the noisy Dicke state $\rho_{D_{4,2}}$ is $\hat{W} = \Lambda\hat{I} - \hat{M}$, with $\Lambda = 1$ (proven in Appendix C) and the Hermitian operator $\hat{M}$ being shown in (15). Although the witness is originally designed for a noisy Dicke state, it leads to a necessary criterion of separability for any four-qubit quantum state.

Proposition 2. The necessary criterion of full separability for a four-qubit quantum state is

$$\sum_{|\mathbf{i}|=|\mathbf{j}|;|\mathbf{i}|=1,3} \rho_{\mathbf{i},\mathbf{j}} \geqslant \sum_{|\mathbf{i}|=|\mathbf{j}|=2;\mathbf{i}\neq\mathbf{j},\bar{\mathbf{j}}} \rho_{\mathbf{i},\mathbf{j}} + 4\max_{|\mathbf{i}|=2}(\rho_{\mathbf{i},\bar{\mathbf{i}}} + \rho_{\bar{\mathbf{i}},\mathbf{i}}), \quad (16)$$

where $\bar{\mathbf{i}} = \mathbf{1} - \mathbf{i}$ with binary string $\mathbf{1} = 1111$. Violation of the inequality implies entanglement.

To prove Proposition 2 is to show $\Lambda = 1$, which is displayed in Appendix C.

It is possible to obtain similar witnesses by changing the sign of some terms in (15), so that in such cases, some of the negative terms in the right-hand side of (16) can be replaced with their absolute values.

Instead of considering the white noise tolerance of Dicke state $D_{4,2}$, we consider the white noise tolerance of a kind of general Dicke state,

$$|GD_{4,2}\rangle = \frac{1}{\sqrt{2(1+u^2+v^2)}}[|0011\rangle + |1100\rangle + u(|0101\rangle + |1010\rangle) + v(|0110\rangle + |1001\rangle)]. \quad (17)$$

The mixed state of this general Dicke state with white noise is $\rho_{GD_{4,2}}(p) = p|GD_{4,2}\rangle\langle GD_{4,2}| + \frac{(1-p)}{2^4}I$. From (16) (and the modified formula when u or/and v are negative), we have the necessary condition of separability for $\rho_{GD_{4,2}}$:

$$p \leqslant \left(1 + 8\frac{|u| + |v| + |uv| + \max\{1, u^2, v^2\}}{1 + u^2 + v^2}\right)^{-1}. \quad (18)$$

The necessary criterion (16) can be further improved to the following:

Proposition 3. The necessary criterion of full separability for a four-qubit quantum state is

$$\sqrt{\sum_{|\mathbf{i}|=|\mathbf{j}|=1} \rho_{\mathbf{i},\mathbf{j}} \sum_{|\mathbf{i}|=|\mathbf{j}|=3} \rho_{\mathbf{i},\mathbf{j}}} \geqslant \frac{1}{2}\left[\sum_{|\mathbf{i}|=|\mathbf{j}|=2;\mathbf{i}\neq\mathbf{j},\bar{\mathbf{j}}} \rho_{\mathbf{i},\mathbf{j}} + 4\max_{|\mathbf{i}|=2}(\rho_{\mathbf{i},\bar{\mathbf{i}}} + \rho_{\bar{\mathbf{i}},\mathbf{i}})\right], \quad (19)$$

where $\bar{\mathbf{i}} = \mathbf{1} - \mathbf{i}$ with binary string $\mathbf{1} = 1111$. Violation of the inequality implies entanglement.

The proof is shown in Appendix D. If a four-qubit state ρ completely lies in the subspace with basis $|\mathbf{m}\rangle(|\mathbf{m}| \leqslant 2)$, then we have $\rho_{\mathbf{i},\mathbf{j}} = 0$ for all $|\mathbf{i}| = |\mathbf{j}| = 3$, and the state is entangled as far as the right-hand side of (19) is positive. It follows:

Corollary 5. The simultaneously amplitude damped and dephased four-qubit generalized Dicke state is always entangled for any finite evolution time.

After passing through the product simultaneously amplitude damping and dephasing channel $\mathcal{E}_1^4$, the state $|GD_{4,2}\rangle$ becomes $\mathcal{E}_1^4(|GD_{4,2}\rangle)$. It completely lies in the subspace with basis $|\mathbf{m}\rangle(|\mathbf{m}| \leqslant 2)$, and the right-hand side of (19) is always positive for state $\mathcal{E}_1^4(|GD_{4,2}\rangle)$ when $u > 0$, $v > 0$. Hence, the damped and dephased $|GD_{4,2}\rangle$ state with $u > 0, v > 0$ is entangled for any finite evolution time. We may extend the conclusion for all damped and dephased $|GD_{4,2}\rangle$ states by modifying the sign of some terms of the witness.

B. Sufficient criterion of noisy general Dicke states

The necessary criterion of separability for a four-qubit state has been shown in (16). We have the condition (18) when the states are limited to noisy general Dicke states $\rho_{GD_{4,2}}(p)$. In this section, we will illustrate that the necessary criterion (18) is also sufficient.

An inspection of the criterion (16) shows that the antidiagonal terms play quite different roles compared with the other terms. So we first consider the fully separable problems of the four-qubit states with only diagonal elements and antidiagonal elements in the computational basis. To be more concrete, we consider the following X-type unnormalized state:

$$\varrho_{1234} = \begin{pmatrix}
w^2 & & & & & & & & & & & & & & & 0 \\
 & w^2 & & & & & & & & & & & & & 0 & \\
 & & w^2 & & & & & & & & & & & 0 & & \\
 & & & w^2 & & & & & & & & & 1 & & & \\
 & & & & w^2 & & & & & & & 0 & & & & \\
 & & & & & w^2 & & & & & u^2 & & & & & \\
 & & & & & & w^2 & & & v^2 & & & & & & \\
 & & & & & & & w^2 & 0 & & & & & & & \\
 & & & & & & & 0 & w^2 & & & & & & & \\
 & & & & & & v^2 & & & w^2 & & & & & & \\
 & & & & & u^2 & & & & & w^2 & & & & & \\
 & & & & 0 & & & & & & & w^2 & & & & \\
 & & & 1 & & & & & & & & & w^2 & & & \\
 & & 0 & & & & & & & & & & & w^2 & & \\
 & 0 & & & & & & & & & & & & & w^2 & \\
0 & & & & & & & & & & & & & & & w^2
\end{pmatrix}, \quad (20)$$

where $w^2 = \max\{1, u^2, v^2\}$. The state ϱ_{1234} does not violate the necessary separable criterion (16).

In order to show that ϱ_{1234} is fully separable, we will explicitly construct it with product states. Let the product state be $|\psi\rangle = \Pi_{j=1}^4 |\psi_j\rangle$, with $|\psi_j\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi_j}|1\rangle)$. Denote $\varrho_a(\boldsymbol{\varphi}) = |\psi\rangle\langle\psi|$, where the vector $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$. Let $\mathbf{m} = (m_1, m_2, m_3, m_4)$ be a binary vector. Then the state

$$\varrho_b(\boldsymbol{\varphi}) = \frac{1}{8} \sum_{|\mathbf{m}|=0,2,4} \varrho_a(\boldsymbol{\varphi} + \mathbf{m}\pi) \tag{21}$$

is X type, and it possibly has nonzero diagonal and antidiagonal elements and all the other elements are nullified. The state $\varrho_b(\boldsymbol{\varphi})$ is complex. Then we define the real state $\varrho_c(\boldsymbol{\varphi}) = \frac{1}{2}[\varrho_b(\boldsymbol{\varphi}) + \varrho_b(-\boldsymbol{\varphi})]$. Next, let $\varphi_2 = \varphi_1 + \frac{1}{2}(\tau_u + \tau_v)$, $\varphi_3 = \varphi_1 + \frac{1}{2}(\tau_1 + \tau_v)$, $\varphi_4 = \varphi_1 + \frac{1}{2}(\tau_u + \tau_1)$, and denote $\boldsymbol{\tau} = (\tau_1, \tau_u, \tau_v)$. By integrating $\varrho_c(\boldsymbol{\varphi})$ over φ_1, we arrive at the state $\varrho_d(\boldsymbol{\tau})$, which is

$$\varrho_d(\boldsymbol{\tau}) = \varrho_{1234}/\mathrm{Tr}(\varrho_{1234}), \tag{22}$$

where we have set $\cos\tau_1 = \frac{1}{w^2}$, $\cos\tau_u = \frac{u^2}{w^2}$, $\cos\tau_v = \frac{v^2}{w^2}$. Thus, the state ϱ_{1234} is fully separable as it is a probability mixture of product states.

We may decompose the noisy general Dicke state $\rho_{GD_{4,2}}(p)$ [with p shown in (18)] into the probability mixture of the state $\widetilde{\varrho}_{12}(\eta) = \varrho'_{12}(\eta) \otimes (|01\rangle\langle 01|)_{34} + \varrho'_{12}(1/\eta) \otimes (|10\rangle\langle 10|)_{34}$, $(\eta > 0)$, its qubit permutation states, and the state ϱ_{1234}. The two-qubit state $\varrho'_{12}(\eta)$ is defined as

$$\varrho'_{12}(\eta) = \begin{pmatrix} 1 & & & \\ & 1/\eta & \pm 1 & \\ & \pm 1 & \eta & \\ & & & 1 \end{pmatrix} \tag{23}$$

in the computational basis. Define an unnormalized state $\rho'_{GD_{4,2}}(q) = 2(1 + u^2 + v^2)|GD_{4,2}\rangle\langle GD_{4,2}| + qI$, which is proportional to $\rho_{GD_{4,2}}(p)$ and $q = (1-p)(1 + u^2 + v^2)/(8p)$. The state constructed from the fully separable states is

$$\begin{aligned} \rho_{\mathrm{comp}} = {} & |u|[\widetilde{\varrho}_{14}(\eta_5) + \widetilde{\varrho}_{23}(\eta_6)] + |v|[\widetilde{\varrho}_{13}(\eta_3) + \widetilde{\varrho}_{24}(\eta_4)] \\ & + |uv|[\widetilde{\varrho}_{12}(\eta_1) + \widetilde{\varrho}_{34}(\eta_2)] + \varrho_{1234}. \end{aligned} \tag{24}$$

The state ρ_{comp} has the same off-diagonal elements as those of the state $\rho'_{GD_{4,2}}(q)$. In the case of negative u (or v, or uv), we should choose the antidiagonal elements of the two-qubit state ϱ'_{ij} to be negative. The state $\rho_{GD_{4,2}}(p)$ is fully separable if $\rho'_{GD_{4,2}}(q) \geqslant \rho_{\mathrm{comp}}$, which is reduced to the comparison of their diagonal elements, namely,

$$q \geqslant w^2 + |u| + |v| + |uv|, \tag{25}$$

$$q + 1 \geqslant w^2 + |u|\left(\eta_5^{-1} + \eta_6^{-1}\right) + |v|\left(\eta_3^{-1} + \eta_4^{-1}\right), \tag{26}$$

$$q + u^2 \geqslant w^2 + |u|(\eta_5 + \eta_6) + |uv|\left(\eta_1^{-1} + \eta_2^{-1}\right), \tag{27}$$

$$q + v^2 \geqslant w^2 + |uv|(\eta_1 + \eta_2) + |v|(\eta_3 + \eta_4). \tag{28}$$

These four inequalities can be made to be the same by adjusting the parameters $\eta_i (i = 1, \ldots, 6)$. If we set $\eta_{2i-1} = \eta_{2i}$, then all the η_i can be analytically obtained. Thus, inequality (25) is the sufficient condition of the full separability of the noisy general Dicke states. It coincides with (18). Hence we have the necessary and sufficient condition of separability (18) for the noisy general Dicke state $\rho_{GD_{4,2}}$.

V. NOISE TOLERANCE OF DICKE STATE $D_{n,k}$

It is straightforward to extend the method of obtaining the sufficient condition of separability from the four-qubit noisy Dicke state $\rho_{D_{4,2}}$ to the multiqubit noisy Dicke state $\rho_{D_{2k,k}}$. The main idea of decomposing the very noisy Dicke state $\rho_{D_{4,2}}$ into fully separable states is to find the two states. One is the two-qubit Bell diagonal state $\varrho'_{12}(\eta = 1)$ in (23). The other is the four-qubit GHZ diagonal state ϱ_{1234} in (20) with $u = v = 1$. The common point of these two states is that they are X-type states with the antidiagonal part coming from the antidiagonal part of $D_{2k,k}$, $(k = 1, 2)$, and with identity as the diagonal part.

A. Sufficient criterion for noisy $D_{2k,k}$ state

Define a $2k$-qubit X-type unnormalized state

$$\varrho_{Xk} = I + \sum_{|\mathbf{m}|=k} |\mathbf{m}\rangle\langle\overline{\mathbf{m}}|, \tag{29}$$

where $\mathbf{m} = m_1 \ldots m_{2k}$ is the binary string of length $2k$. $\overline{\mathbf{m}} = 11\ldots 1 - \mathbf{m}$ is the binary NOT of $\mathbf{m}$. The basic building material for ϱ_{Xk} is the state $\varrho_a(\varphi, \mathbf{m}) = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \bigotimes_{j=1}^{2k} |\psi_j\rangle$, with $|\psi_j\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i(m_j\pi + \varphi)})|1\rangle$. We may write the state $\varrho_a(\varphi, \mathbf{m}) = \bigotimes_{j=1}^{2k} 1/2[\sigma_0 + (-1)^{m_j} A(\varphi)]$, with operator $A(\varphi) = \cos\varphi\sigma_1 + \sin\varphi\sigma_2$. The state

$$\varrho_b(\varphi) = \frac{1}{2^{2k-1}} \sum_{|\mathbf{m}|=\mathrm{even}} \varrho_a(\varphi, \mathbf{m}) \tag{30}$$

is X type due to the fact that

$$\sum_{|\mathbf{m}|=\mathrm{even}} \bigotimes_{j=1}^{2k} [\sigma_0 + (-1)^{m_j} A(\varphi)] = 2^{2k-1}[I + A(\varphi)^{\otimes 2k}].$$

The antidiagonal element of $\varrho_b(\varphi)$ is

$$\varrho_b(\varphi)_{\mathbf{m},\overline{\mathbf{m}}} = 2^{-2k} \exp\left[-i \sum_j (-1)^{m_j} \varphi\right].$$

Hence we have

$$\varrho_{Xk} = \frac{2^{2k-1}}{\pi} \int_0^{2\pi} \varrho_b(\varphi) d\varphi. \tag{31}$$

A fully separable $2k$-qubit state generated with ϱ_{Xj} $(j = 1, \ldots, k)$ is

$$\rho_{\mathrm{sep}} = \sum_{j=1}^{k} [\varrho_{Xj} \otimes (|01\rangle\langle 01|)^{\otimes(k-j)}]_p. \tag{32}$$

The subscript p represents summation over all permutations of the subscript of the $2k$-qubit states in the bracket. The state ρ_{sep} is so constructed that it has the same off-diagonal elements as those of the state $\rho'_{D_{2k,k}} = C_k^{2k}|D_{2k,k}\rangle\langle D_{2k,k}| + qI$. The diagonal elements of ρ_{sep} are $\rho_{\mathrm{sep}\mathbf{l},\mathbf{l}} = C_{|\mathbf{l}|}^{2k} - \delta_{|\mathbf{l}|,k}$. The details are shown in Appendix E.

Thus, if $\rho'_{D_{2k,k}} \geqslant \rho_{\rm sep}$, the state $\rho'_{D_{2k,k}}$ is fully separable. We have the sufficient separable condition $q \geqslant C_k^{2k} - 2$, and it follows:

Proposition 4. The sufficient separable condition for the state $\rho_{D_{2k,k}} = p|D_{2k,k}\rangle\langle D_{2k,k}| + \frac{1-p}{2^{2k}} I$ is

$$p \leqslant \left[1 + 2^{2k}\left(1 - \frac{2}{C_k^{2k}}\right)\right]^{-1}. \tag{33}$$

B. Necessary criterion for noisy $D_{n,k}$ state

We propose the following possible witness operator $\hat{W}$ for the entanglement of any n-qubit state:

$$\hat{W} = \frac{1}{d}\sum_{|\mathbf{i}|=|\mathbf{j}|=k-1} |\mathbf{i}\rangle\langle\mathbf{j}| + d\sum_{|\mathbf{i}|=|\mathbf{j}|=k+1} |\mathbf{i}\rangle\langle\mathbf{j}| - \sum_{l=1}^{k}\frac{2l}{l+1}\sum_{|\mathbf{i}|=|\mathbf{j}|=k;|\mathbf{i}\oplus\mathbf{j}|=2l} |\mathbf{i}\rangle\langle\mathbf{j}| \tag{34}$$

for ($2k \leqslant n$). For the case of ($2k \geqslant n$), we replace k in (34) with $n-k$ to obtain a new witness. The witness (34) reduces to (4) when $k=1$, and it reduces to a mixture of witnesses in Appendix D for $\mathbf{m} = 0011, 0101, 0110$ cases when $n=4, k=2$. Numerical calculations strongly suggest that (34) is a valid entanglement witness for any n and k. We have checked the non-negativity for the mean of (34) over the product state with 1 000 000 randomly generated product states for each of $n \leqslant 20, k=2$; $n \leqslant 13, k=3$; and $n \leqslant 11, k=4$. For larger n, we have numerically verified the non-negativity with 10 000 randomly generated product states up to $n = 40, 22, 14$ for $k = 2, 3, 4$, respectively.

The following is the necessary criterion of separability for any n-qubit state:

$$\sqrt{\sum_{|\mathbf{i}|=|\mathbf{j}|=k-1}\rho_{\mathbf{i},\mathbf{j}}\sum_{|\mathbf{i}|=|\mathbf{j}|=k+1}\rho_{\mathbf{i},\mathbf{j}}} \geqslant \sum_{l=1}^{k}\frac{l}{l+1}\sum_{|\mathbf{i}|=|\mathbf{j}|=k;|\mathbf{i}\oplus\mathbf{j}|=2l}\rho_{\mathbf{i},\mathbf{j}}, \tag{35}$$

for ($2k \leqslant n$). For the case of ($2k \geqslant n$), we replace k in (35) with $n-k$.

It follows from (35) that the necessary condition of full separability for noisy Dicke state $\rho_{D_{n,k}} = p|D_{n,k}\rangle\langle D_{n,k}| + \frac{1-p}{2^n} I$ is

$$p \leqslant p_{\rm NEW} = \frac{1}{1 + 2^n\sqrt{\frac{k(n-k)}{(k+1)(n-k+1)}}}. \tag{36}$$

The necessary condition (36) coincides with proposition 4 for $n=4, k=2$.

Applying (35) to the parallel amplitude damped and dephased Dicke state $\mathcal{E}_1^n(|D_{n,k}\rangle)$ leads to the following necessary condition of full separability:

$$0 \geqslant \sum_{l=1}^{k}\frac{l}{l+1}C_k^n C_{k-l}^k C_l^{n-k}(1-r-h)^l(1-r)^{k-l},$$

for $2k \leqslant n$. A similar inequality can be obtained when $2k \geqslant n$. The inequality is always violated unless the damping factor $1-r-h = e^{-2t/T_2} = 0$, which can only be true when $t \to \infty$. Thus a Dicke state passing through a parallel amplitude damping and dephasing channel remains entangled for any finite time.

VI. A COMPARISON WITH THE POSITIVE PARTIAL TRANSPOSE CRITERION

We now compare our propositions with the results derived from the PPT criterion.

A. Noisy Dicke states

The PPT criterion for the n qubit noisy W state leads to the following result:

$$p \leqslant \frac{1}{1 + 2^n\frac{\sqrt{\lfloor\frac{n}{2}\rfloor(n-\lfloor\frac{n}{2}\rfloor)}}{n}}. \tag{37}$$

For $n=3$, we have $p \leqslant 0.20959$ as also shown in [46,48], while Proposition 1 gives $p \leqslant 0.17797$. With a simple comparison between Corollary 1 and (37) for noisy W states, we conclude that our Proposition 1 is better than the PPT criterion as the necessary criterion of separability for noisy W states with any number of qubits.

For the noisy general four-qubit Dicke states, the necessary condition of full separability obtained by the PPT criterion is simply $p \leqslant \frac{1}{9}$ for all the noisy general Dicke states $\rho_{GD_{4,2}}$. It is easy to show that the maximum of the right-hand side of (18) is $\frac{1}{9}$ when $u = v = 0$. Hence the condition (18) derived from Proposition 2 is better than the PPT criterion for the states. This is not surprising since the condition (18) is necessary and sufficient for the separability of the states. For examples, we have $p \leqslant \frac{3}{35}$ as the necessary and sufficient condition of separability for $\rho_{D_{4,2}}$, and $p \leqslant \frac{1}{13}$ as the necessary and sufficient condition of separability for the state $\Psi_{S,4}$ [14] mixed with white noise (the general Dicke state with $u = v = -\frac{1}{2}$). They are all better than the results using the PPT criterion.

Hence, Proposition 1 and Proposition 3 as necessary criteria of full separability are stronger than the PPT criterion for the noisy W_n state and the noisy general four-qubit Dicke state, respectively. Moreover, for six-qubit noisy Dicke state $\rho_{D_{6,3}}$, the PPT criterion leads to the fully separable condition $p \leqslant \frac{1}{33}$; our formula (36) leads to $p \leqslant \frac{1}{49}$.

In fact, we have the following general conjecture for the PPT criterion on noisy Dicke states:

Lemma 1. The PPT necessary criterion of separability for a noisy Dicke state $\rho = p|D_{n,k}\rangle\langle D_{n,k}| + (1-p)/2^n I$ is

$$p \leqslant p_{\rm PPT} = \frac{1}{1 + 2^n\mu_{\rm PPT}}, \tag{38}$$

where

$$\mu_{\rm PPT} = \max_{m,i,j;i\neq j}\sqrt{C_i^m C_{k-i}^{n-m} C_j^m C_{k-j}^{n-m}}/C_k^n. \tag{39}$$

The proof of Lemma 1 will be shown in Appendix F. Direct numerical calculation shows that $\mu_{\rm PPT} \leqslant \frac{1}{2}$. We have checked that it is true for $n \leqslant 500$ and all possible k, m, i, j. Thus we have a lower bound for $p_{\rm PPT}$, namely,

$$p_{\rm PPT} \geqslant p_{\rm PPT}^{lb} = \frac{1}{1 + 2^{n-1}}. \tag{40}$$

It is clear that $p_{\mathrm{NEW}} \leqslant p_{\mathrm{PPT}}^{lb} \leqslant p_{\mathrm{PPT}}$ by comparing (36) with (40), since equality in $p_{\mathrm{NEW}} \leqslant p_{\mathrm{PPT}}^{lb}$ is realized when $n=2, k=1$. For all $n \geqslant 3, k=1, \ldots, n-1$, we have $\sqrt{\frac{k(n-k)}{(k+1)(n-k+1)}} > \frac{1}{2} \geqslant \mu_{\mathrm{PPT}}$. Hence, we have $p_{\mathrm{NEW}} < p_{\mathrm{PPT}}^{lb} \leqslant p_{\mathrm{PPT}}$. Thus our new necessary criterion (36) is always tighter than the PPT criterion for a multiqubit noisy Dicke state.

B. Dicke diagonal states

It is known that the necessary and sufficient criterion for Dicke diagonal states is just the PPT criterion [42,43]. So our propositions cannot be better than the PPT criterion for Dicke diagonal states. However, we will show that Proposition 1 and Proposition 3 are still very good. If we apply Proposition 1 and Proposition 3 to a Dicke diagonal state, $\rho_{D_n} = \sum_{k=0}^{n} p_k |D_{n,k}\rangle\langle D_{n,k}|$, we have the following necessary separable conditions, respectively:

$$q_0 q_2 \geqslant q_1^2, \tag{41}$$

$$q_1 q_3 \geqslant q_2^2, \tag{42}$$

where $q_k = p_k / C_k^n$. The necessary condition (41) is exactly the non-negativity of the main submatrix of $\rho_{D_n}^{\mathrm{PT}}$ (partial transpose of ρ_{D_n}). Hence, Proposition 1 is one of the conditions derived from the PPT criterion for Dicke diagonal states.

The necessary condition (42) is the non-negativity of the main submatrix of $\rho_{D_4}^{\mathrm{PT}}$. In the special case of Dicke diagonal states, Propositions 1 and 3 as necessary criteria of full separability are comparable with the PPT criterion.

VII. CONCLUSIONS

We have demonstrated two necessary separable criteria for multiqubit states. One of the criteria can be applied to any multiqubit state; it is derived from detecting entanglement of a noisy n-qubit W state of any n. We also provide the sufficient criterion. The necessary criterion and sufficient criterion coincide with each other for the noisy W states of three, four, and five qubits. The other necessary criterion is limited to four-qubit states and it is the necessary and sufficient criterion for the full separability of the noisy generalized four-qubit Dicke states. Numeric evidence strongly suggests an entanglement criterion for a noisy Dicke state with any number of qubits and excitations. We also present the sufficient criterion of full separability for noisy Dicke states with $2k$ qubits and k excitations. Compared with the PPT criterion, our criteria are better in performance when dealing with noisy Dicke states and are comparable when dealing with Dicke diagonal states. The applications of the criteria show that initially pure Dicke states damped by independent and identical amplitude damping and dephasing channels remain entangled for any finite time of evolution. A multiqubit W state evolves to a fully separable state in an independent and identical depolarizing channel if the noise is large enough. The noise threshold approaches a constant as the qubit number of the W state tends to infinite.

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APPENDIX A: VALIDITY OF WITNESS (4) AND PROOF OF PROPOSITION 1

An entanglement witness $\hat{W}$ should meet the requirement of $\langle\psi|\hat{W}|\psi\rangle \geqslant 0$ for any product state $|\psi\rangle = \bigotimes_{j=1}^{n} |\psi_j\rangle$, where the jth-qubit state is $|\psi_j\rangle = (1+|\xi_j|^2)^{-1/2}(|0\rangle + \xi_j|1\rangle)$ with complex ξ_j. We may denote $|\psi\rangle = |\psi'\rangle|\psi_n\rangle$, where $|\psi'\rangle$ is the product of the first $(n-1)$-qubit states. Then, $\mathcal{W} = \langle\psi'|\hat{W}|\psi'\rangle$ is a 2×2 matrix with the form of

$$\mathcal{W} = \mathcal{N}\begin{bmatrix} \frac{1}{d} + d|b|^2 + c - |a|^2, & dab^* - a^* \\ da^*b - a, & d|a|^2 \end{bmatrix}, \tag{A1}$$

where $a = \sum_{j=1}^{n-1} \xi_j$, $b = \sum_{j>l;j,l\leqslant n-1} \xi_j \xi_l$, $c = \sum_{j=1}^{n-1} |\xi_j|^2$, and $\mathcal{N} = \prod_{j=1}^{n-1}(1+|\xi_j|^2)^{-1/2}$ is the normalization factor. The validity of $\hat{W}$ in (4) as a witness is equivalent to the non-negative of the matrix $\mathcal{W}$. Thus we should have $\det \mathcal{W} \geqslant 0$, which can be shown to be

$$\sum_{l,j,m} \alpha_{lm}\alpha_{lj} \geqslant 0, \tag{A2}$$

with $\alpha_{lj} = i(\xi_l^* \xi_j - \xi_l \xi_j^*)$ being real and asymmetric with respect to its subscripts. The inequality (A2) can be further written as

$$\sum_l \left(\sum_j \alpha_{lj}\right)^2 \geqslant 0. \tag{A3}$$

Inequality (A3) is always true, so that $\mathcal{W}$ is non-negative, and thus the operator $\hat{W}$ in (4) is a valid entanglement witness for any number of qubits. We can convert the valid entanglement witness to Proposition 1 in the following. A direct calculation of $\mathrm{Tr}(\rho\hat{W}) \geqslant 0$ leads to

$$\frac{1}{d}\rho_{\mathbf{0},\mathbf{0}} + d\sum_{\mathbf{j},\mathbf{j}'} \rho_{\mathbf{j},\mathbf{j}'} \geqslant \sum_{\mathbf{i}\neq\mathbf{i}'} \rho_{\mathbf{i},\mathbf{i}'}, \tag{A4}$$

with $|\mathbf{0}|=0, |\mathbf{i}|=|\mathbf{i}'|=1, |\mathbf{j}|=|\mathbf{j}'|=2$. Minimizing the left-hand side of (A4) over parameter d leads to Proposition 1. In the minimization, we should have $\sum_{\mathbf{j},\mathbf{j}'} \rho_{\mathbf{j},\mathbf{j}'} \geqslant 0$, which is guaranteed by the positivity of density matrix ρ. More explicitly, we have $\mathbf{v}\rho\mathbf{v}^\dagger \geqslant 0$ for vector $\mathbf{v} = \{v_{\mathbf{j}}\}$ with component $v_{\mathbf{j}} = 1$ if $|\mathbf{j}|=2$, and $v_{\mathbf{j}}=0$ otherwise.

APPENDIX B: PROOF OF COROLLARY 2

For the sufficient condition of separability of a noisy W state, a useful method is to make use of the zero eigenvalues of $\mathcal{W}$ and their corresponding eigenvectors. The zero eigenvalue of $\mathcal{W}$ appears when the equality is achieved in inequality (A3), namely,

$$\sum_{j=1}^{n-1} \alpha_{lj} = 0 \quad \text{for } l \in \{0, 1, \ldots, n-1\}. \tag{B1}$$

Equation (B1) reduces to $\alpha_{12} = 0$ when $n=3$. One of the solutions is $\xi_1 = \xi_2$, which leads to an unnormalized separable state,

$$\varrho_{12}=\begin{pmatrix}1/|\xi_1|^2 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & |\xi_1|^2\end{pmatrix}, \tag{B2}$$

in the computational basis of the first two qubits. The separable state ϱ_{12} is the equal probability mixture of the states $|\psi_1(\xi_1)\rangle^{\otimes 2}$ with $\xi_1=\pm|\xi_1|$ and $\xi_1=\pm i|\xi_1|$. The separable state of the three-qubit system is $\varrho_{12}\otimes|0\rangle\langle 0|_3$, where the third qubit state $|0\rangle\langle 0|_3$ comes from the eigenvector corresponding to the zero eigenvalue of matrix $\mathcal{W}$. With the permutations of qubits, we have the overall fully separable state $\varrho_{12}\otimes|0\rangle\langle 0|_3+\varrho_{13}\otimes|0\rangle\langle 0|_2+|0\rangle\langle 0|_1\otimes\varrho_{23}$. By adding some diagonal elements to it and letting $|\xi_1|=\sqrt[4]{3}$, we obtain the full separable state $\rho_{W_3}(p)$ with $p=1/(1+8/\sqrt{3})$.

For the noise tolerance of the W_n state of general n, one of the solutions to Eq. (8) is $\xi_1=\xi_2$ and $\xi_3=\cdots=\xi_{n-1}=0$. We have the separable state $\varrho_{12}\otimes|0\rangle\langle 0|_{3,4,\dots,n}$, where $|0\rangle\langle 0|_{3,4,\dots,n}=|0\rangle\langle 0|^{\otimes(n-2)}$. The last qubit state $|0\rangle\langle 0|$ is just the eigenvector corresponding to the zero eigenvalue of the matrix $\mathcal{W}$. With permutations of qubits, we have the fully separable state $\varrho_{\rm sep}=\sum_{l<j;l,j=1}^{n}\varrho_{lj}\otimes|0\rangle\langle 0|^{\otimes(n-2)}$. Rewriting $\rho_{W_n}(p)=\frac{p}{n}\rho'_{W_n}(p)$, where $\rho'_{W_n}(p)=n|W_n\rangle\langle W_n|+qI$ with $q=\frac{n(1-p)}{2^n p}$, then $\rho'_{W_n}(p)$ and $\varrho_{\rm sep}$ have the same off-diagonal elements. The state $\rho_{W_n}(p)$ is fully separable if $\rho'_{W_n}(p)-\varrho_{\rm sep}\geqslant 0$. Comparing the diagonal elements of $\rho'_{W_n}(p)$ with those of $\varrho_{\rm sep}$, we have the following inequalities for the sufficient condition of $\rho_{W_n}(p)$:

$$q\geqslant C_2^n/|\xi_1|^2,\quad 1+q\geqslant n-1,\quad q\geqslant|\xi_1|^2.$$

The solution to these inequalities is $q\geqslant\max\{\sqrt{\frac{n(n-1)}{2}},n-2\}$.

APPENDIX C: PROOF OF PROPOSITION 2

We need to show $\Lambda=\max_{|\psi\rangle}\tau_M=1$, namely, the maximal mean of operator $\hat{M}$ defined in (13) over all product states is 1. We have

$$\tau_M=\mathbf{A}\cdot\mathbf{r}_4^T\leqslant|\mathbf{A}|, \tag{C1}$$

with the vector $\mathbf{A}=\mathbf{r}_3\mathcal{M}$, where $\mathcal{M}$ is a matrix with entries

$$\mathcal{M}_{1,1}=x_1x_2+M_1y_1y_2-z_1z_2,$$
$$\mathcal{M}_{2,2}=M_1x_1x_2+y_1y_2-z_1z_2,$$
$$\mathcal{M}_{3,3}=-x_1x_2-y_1y_2+z_1z_2,$$
$$\mathcal{M}_{1,2}=M_2x_1y_2+M_3y_1x_2,$$
$$\mathcal{M}_{2,1}=M_3x_1y_2+M_2y_1x_2,$$
$$\mathcal{M}_{1,3}=\mathcal{M}_{3,1}=-x_1z_2-x_2z_1,$$
$$\mathcal{M}_{2,3}=\mathcal{M}_{3,2}=-y_1z_2-y_2z_1.$$

Notice that

$$|\mathbf{A}|^2=\mathbf{r}_3\mathcal{M}\mathcal{M}^T\mathbf{r}_3^T\leqslant|\lambda|^2_{\max}, \tag{C2}$$

where $|\lambda|_{\max}$ is the maximal absolute eigenvalue of matrix $\mathcal{M}$. We may write $\mathcal{M}=\mathcal{M}(M_1,M_2,M_3)$ to explicitly express the fact that $\mathcal{M}$ relies on the parameters M_1,M_2,M_3. Then we have $\mathcal{M}(M_1,M_2,M_3)=p_1\mathcal{M}(-1,1,1)+p_2\mathcal{M}(1,-1,1)+p_3\mathcal{M}(1,1,-1)$, where $p_i=\frac{1}{2}(1-M_i)$, $i=1,2,3$, and $\{p_i\}$ is a probability distribution. So,

$$|\lambda|_{\max}=\max_{i=1,2,3}\{|\lambda_i|_{\max}\}, \tag{C3}$$

where $|\lambda_i|_{\max}$ $(i=1,2,3)$ are the maximal absolute eigenvalues of $\mathcal{M}(-1,1,1)$, $\mathcal{M}(1,-1,1)$, and $\mathcal{M}(1,1,-1)$, respectively. However, these three matrices are mutually related (convertible) by qubit permutations, so they have the same maximal absolute eigenvalues. Thus, we have

$$|\lambda|_{\max}=|\lambda_1|_{\max}. \tag{C4}$$

The eigenvalue equation of matrix $\mathcal{M}(-1,1,1)$ is

$$(\lambda_1^2-1)(\lambda_1+\mathbf{r}_1\cdot\mathbf{r}_2)=0. \tag{C5}$$

Hence we have $|\lambda_1|_{\max}=1$.

APPENDIX D: PROOF OF PROPOSITION 3

Proof. The witness is

$$\hat{W}=\frac{1}{d}\sum_{|\mathbf{i}|=|\mathbf{j}|=1}|\mathbf{i}\rangle\langle\mathbf{j}|+d\sum_{|\mathbf{i}|=|\mathbf{j}|=3}|\mathbf{i}\rangle\langle\mathbf{j}| -\sum_{|\mathbf{i}|=|\mathbf{j}|=2;\mathbf{i}\neq\mathbf{j},\bar{\mathbf{j}}}|\mathbf{i}\rangle\langle\mathbf{j}|-4(|\mathbf{m}\rangle\langle\overline{\mathbf{m}}|+|\overline{\mathbf{m}}\rangle\langle\mathbf{m}|), \tag{D1}$$

with $\mathbf{m}=0011$ or 0101 or 0110. We should prove $\langle\psi|\hat{W}|\psi\rangle\geqslant 0$ for any product state $|\psi\rangle$. The non-negativity of the mean of witness operator $\hat{W}$ over product states is equivalent to the non-negativity of matrix $\mathcal{W}=\langle\psi'|\hat{W}|\psi'\rangle$, where ψ' is the product of the first three qubit states. We have

$$\mathcal{W}=\mathcal{N}\begin{bmatrix}\frac{1}{d}|a|^2+d|e|^2-f, & \frac{1}{d}a^*+dbe^*-h\\ \frac{1}{d}a+db^*e-h^*, & \frac{1}{d}+d|b|^2-g\end{bmatrix}, \tag{D2}$$

where $a=\sum_{j=1}^3\xi_j$, $b=\sum_{j>l}\xi_j\xi_l$, $e=\xi_1\xi_2\xi_3$, $f=\sum_{i\neq j\neq l}|\xi_i|^2\xi_j\xi_l^*$, $g=\sum_{j\neq l}\xi_j\xi_l^*$, and $h=\sum_{i\neq j,l;j>l}|\xi_i|^2(\xi_j^*+\xi_l^*)+4\xi_1^*\xi_2^*\xi_3$ in the case of $\mathbf{m}=0011$. First, we will prove the non-negativity of the matrix element $\frac{1}{d}+d|b|^2-g$ in $\mathcal{W}$. It can be rewritten as $(1,\xi_3^*)\mathcal{W}'(1,\xi_3^*)^T$, with

$$\mathcal{W}'=\begin{pmatrix}\frac{1}{d}+d|b'|^2-g' & da'b'^*-a'^*\\ da'^*b'-a' & d|a'|^2\end{pmatrix}, \tag{D3}$$

where $a'=\xi_1+\xi_2$, $b'=\xi_1\xi_2$, $g'=\xi_1\xi_2^*+\xi_1^*\xi_2$. So the matrix $\mathcal{W}'$ is a special case of the matrix $\mathcal{W}$ in (A1). Thus we have $\mathcal{W}'\geqslant 0$. Hence we have $\frac{1}{d}+d|b|^2-g\geqslant 0$. Then we will prove that the determinant of the matrix $\mathcal{W}$ in (D2) is non-negative. We can denote the determinant as $\det\mathcal{W}=\mathcal{N}^2(\frac{1}{d}\Delta_1+d\Delta_2+\Delta_3)$, with

$$\Delta_1=ah+a^*h^*-f-|a|^2g,$$
$$\Delta_2=b^*eh+be^*h^*-|e|^2g-|b|^2f,$$
$$\Delta_3=|ab-e|^2+fg-|h|^2.$$

A direct calculation shows that $\Delta_1=(\alpha_{12}+\alpha_{13}-\alpha_{23})^2$, $\Delta_2=(|\xi_3|^2\alpha_{12}+|\xi_2|^2\alpha_{13}-|\xi_1|^2\alpha_{23})^2$, and $\Delta_3=-2(\alpha_{12}+\alpha_{13}-\alpha_{23})(|\xi_3|^2\alpha_{12}+|\xi_2|^2\alpha_{13}-|\xi_1|^2\alpha_{23})$. Thus we have

$\sqrt{\Delta_1\Delta_2} = \frac{1}{2}|\Delta_3|$. Hence, $\det \mathcal{W} \geqslant 0$, so that operator (D1) is an entanglement witness.

APPENDIX E: THE DIAGONAL PART OF $\rho_{\rm sep}$ IN (32)

The reasonings for $\rho_{\mathrm{sep}\mathbf{l},\mathbf{l}} = C_{|\mathbf{l}|}^{2k} - 2\delta_{|\mathbf{l}|,k}$ are as follows: There are C_k^{2k} rows of nonzero off-diagonal elements in $\rho_{\rm sep}$; they are $|\mathbf{m}_1\rangle\langle\mathbf{m}|$ with $|\mathbf{m}| = k$ and $\mathbf{m}_1 = 0^{\otimes k}1^{\otimes k}$ for the $\mathbf{m}_1$th row. Let us pick up the elements with $|\mathbf{m}\oplus\mathbf{m}_1| = 2j$ in the row. There are $(C_j^k)^2$ such elements since by converting j bits in the $0^{\otimes k}$ part and j bits in the $1^{\otimes k}$ part of $\mathbf{m}_1$, we obtain a valid $\mathbf{m}$ with specified properties. Each element with $|\mathbf{m}\oplus\mathbf{m}_1| = 2j$ is an off-diagonal element of some ϱ_{Xj} embedded in the $\rho_{\rm sep}$ state. A ϱ_{Xj} state has C_j^{2j} off-diagonal elements. The total number of ϱ_{Xj} embedded in $\rho_{\rm sep}$ is

$$N_{Xj} = C_k^{2k}\left(C_j^k\right)^2/C_j^{2j}.$$

The diagonal part of a ϱ_{Xj} embedded in $\rho_{\rm sep}$ will contribute to $\rho_{\rm sep}$ the 2^{2j} diagonal elements $\rho_{\mathrm{sep}\mathbf{l},\mathbf{l}}$. The weight distribution of $\mathbf{l}$ is C_i^{2j} when $|\mathbf{l}| = k - j + i$. Hence, the summation of the diagonal elements $\rho_{\mathrm{sep}\mathbf{l},\mathbf{l}}$ with fixed $|\mathbf{l}|$ is

$$S_{|\mathbf{l}|} = \sum_{j=1}^{k} N_{Xj}C_{|\mathbf{l}|+j-k}^{2j}.$$

The number of diagonal elements with fixed $|\mathbf{l}|$ is $N_{|\mathbf{l}|} = C_k^{2k}$. Hence, for fixed $|\mathbf{l}|$,

$$\rho_{\mathrm{sep}\mathbf{l},\mathbf{l}} = \frac{S_{|\mathbf{l}|}}{N_{|\mathbf{l}|}} = \sum_{j=1}^{k}\left(C_j^k\right)^2 C_{|\mathbf{l}|+j-k}^{2j}/C_j^{2j}.$$

We have $\rho_{\mathrm{sep}\mathbf{l},\mathbf{l}}(|\mathbf{l}| = k) = \sum_{j=1}^{k}(C_j^k)^2 = C_k^{2k} - 1$, $\rho_{\mathrm{sep}\mathbf{l},\mathbf{l}}(|\mathbf{l}| = k+1) = \sum_{j=1}^{k}(C_j^k)^2\frac{j}{j+1} = C_{k+1}^{2k}$, and similar results for other values of $|\mathbf{l}|$.

APPENDIX F: PROOF OF LEMMA 1

We consider the negative eigenvalues of a partially transposed Dicke state. Denote the unnormalized Dicke state as

$$|\overline{D_{n,k}}\rangle = \sum_{\Pi}|\Pi(1^k0^{n-k})\rangle.$$

For the split of the n-qubit system into two parties of m and $n-m$ qubits, we may write the Dicke state as

$$|\overline{D_{n,k}}\rangle = \sum_{i=0}^{\min(m,k)}|\overline{D_{m,i}}\rangle|\overline{D_{n-m,k-i}}\rangle.$$

The transpose on the first m-qubit partite leads to matrix

$$(|\overline{D_{n,k}}\rangle\langle\overline{D_{n,k}}|)^{\rm PT} = \sum_{i=0,j=0}^{\min(m,k)}|\overline{D_{m,j}}\rangle|\overline{D_{n-m,k-i}}\rangle\langle\overline{D_{m,i}}|\langle\overline{D_{n-m,k-j}}|.$$

We then decompose the $2^n\times 2^n$ matrix into blocks. When $i\neq j$, the submatrix $V(i,j) = |\overline{D_{m,j}}\rangle|\overline{D_{n-m,k-i}}\rangle\langle\overline{D_{m,i}}|\langle\overline{D_{n-m,k-j}}| + |\overline{D_{m,i}}\rangle|\overline{D_{n-m,k-j}}\rangle\langle\overline{D_{m,j}}|\langle\overline{D_{n-m,k-i}}|$ takes the following form:

$$V(i,j) = \begin{pmatrix}\mathbf{0}_{n_1,n_1} & \mathbf{1}_{n_1,n_2}\\ \mathbf{1}_{n_2,n_1} & \mathbf{0}_{n_2,n_2}\end{pmatrix},$$

where $\mathbf{a}_{n_b,n_c}$ is a submatrix of size $n_b\times n_c$ and with each element being a. Here, $n_1 = C_j^mC_{k-i}^{n-m}$, $n_2 = C_i^mC_{k-j}^{n-m}$. It is easy to show that two of the eigenvalues of $V(i,j)$ are $\pm\sqrt{n_1n_2}$ and all the other eigenvalues are 0. Notice that for different pairs of (i,j) and (i',j'), $V(i,j)$ and $V(i',j')$ are independent block-diagonalized submatrices. So the smallest eigenvalue of the partially transposed Dicke state under partition $m|(n-m)$ is

$$\lambda_{\min} = -\max_{i,j,i\neq j}\sqrt{C_j^mC_{k-i}^{n-m}C_i^mC_{k-j}^{n-m}}/C_k^n.$$

Optimizing over all partitions and all k, we have $\mu_{\rm PPT} = |\min_{m,k}\lambda_{\min}|$, so

$$\mu_{\rm PPT} = \max_{k,m,i,j,i\neq j}\sqrt{C_j^mC_{k-i}^{n-m}C_i^mC_{k-j}^{n-m}}/C_k^n.$$

Notice that an unnormalized state $\widetilde{\rho} = \mu_{\rm PPT}I + |\overline{D_{n,k}}\rangle\langle\overline{D_{n,k}}|$ has a non-negative partial transpose, so we have a PPT noisy Dicke state $\rho = \widetilde{\rho}/\mathrm{tr}(\widetilde{\rho}) = \frac{1}{1+2^n\mu_{\rm PPT}}(|\overline{D_{n,k}}\rangle\langle\overline{D_{n,k}}| + \mu_{\rm PPT}I)$. It leads to (38).

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