Quantifying non-Gaussianity of bosonic fields via an uncertainty relation

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While Gaussian states and associated Gaussian operations are basic ingredients and convenient objects for continuous-variable quantum information, it is also realized that non-Gaussianity is an important resource for quantum information processing. The characterization and quantification of non-Gaussianity have been widely studied in the past decade, with several significant measures for non-Gaussianity introduced. In this work, by exploiting an information-theoretic refinement of the conventional Heisenberg uncertainty relation and a physical characterization of Gaussian states as minimum uncertainty states, we introduce an easily computable measure for non-Gaussianity of bosonic field states in terms of the Wigner-Yanase skew information. Fundamental properties, as well as intuitive meaning, of this measure are unveiled. The concept is illustrated by prototypical non-Gaussian states, and compared with various existent measures for non-Gaussianity. Its merit and physical significance are elucidated.

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I. INTRODUCTION

With the rapid development of continuous-variable quantum information, Gaussian states are playing an increasingly important role in both theoretical and experimental studies of quantum technology [1–7]. This is rooted in the many remarkable features of Gaussian states: (i) Gaussian states arise naturally in bosonic fields and limit laws [8,9]. (ii) Gaussian states are ubiquitous and exhibit many remarkable extremal characteristics [6,10]. (iii) Gaussian states can be neatly characterized in an elegant mathematical framework involving quadratic forms [11,12]. (iv) Gaussian states can be easily prepared and manipulated in experiments [13–15]. (v) Gaussian states encapsulate a rather wide family of states such as coherent states, squeezed coherent states, and thermal states. (vi) Gaussian states constitute a versatile resource for quantum information protocols [16–21].

However, it has also been noticed that non-Gaussian states can be exploited to improve the efficiency of certain quantum protocols [22–27], and non-Gaussianity can be regarded as a resource [28–30]. Studies on non-Gaussianity have been widely conducted in the past decade [31–38]. In this context, questions arise naturally about the characterization and quantification of the non-Gaussianity of quantum states. Several measures, which exploit the differences between the bosonic field states and associated reference Gaussian states, have been proposed to quantify non-Gaussianity [39–47]. For example, a measure for non-Gaussianity based on the Hilbert-Schmidt distance between the state itself and its reference Gaussian state is introduced in Ref. [39]. Measures for non-Gaussianity based on the relative entropy and fidelity are studied in Refs. [40–45]. Interplay between non-Gaussianity and uncertainty relations is studied in Refs. [46,47].

Non-Gaussianity is a rather rich and subtle feature of bosonic field states, and it is impossible to envision a single quantity capturing all aspects of non-Gaussianity. Different measures may yield different orderings of non-Gaussianity, which serve different purposes and may be relevant for particular tasks. Therefore, it is desirable to quantify non-Gaussianity from different angles with physical tasks and theoretical considerations in mind.

Recently, a physical characterization of Gaussian states as the minimum uncertainty states of an information-theoretic uncertainty relation was revealed [48]. This is based on a novel refinement of the conventional Heisenberg uncertainty relation involving the celebrated Wigner-Yanase skew information [49], which is a measure of quantum uncertainty and is convex in the states. Gaussian states are demonstrated to be exactly the minimum uncertainty states of this uncertainty relation (i.e., they saturate the equality in the uncertainty relation), while non-Gaussian states have larger uncertainties which strictly dominate the lower bound [48]. In sharp contrast, due to the concavity of variance, the minimum uncertainty states of the conventional uncertainty relation only consist of pure states (coherent states and squeezed coherent states) and cannot exhaust all Gaussian states in which there are many mixed ones. By virtue of the physical characterization of Gaussian states, we propose a measure for non-Gaussianity of bosonic field states and study its properties and implications.

The paper is organized as follows. In Sec. II, we review several equivalent characterizations of Gaussian states, introduce our measure for non-Gaussianity, and exhibit its

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fundamental properties. In Sec. III, we illustrate the measure through some important bosonic field states and elucidate some intuitive features of it. We make a comparative study between our measure for non-Gaussianity and other popular ones in the literature in Sec. IV. We present a brief discussion on classical versus quantum non-Gaussianity and related issues in Sec. V. Finally, we summarize the main results in Sec. VI.

II. QUANTIFYING NON-GAUSSIANITY

A single-mode bosonic field is mathematically described by the canonical commutation relation

$$[a, a^\dagger] = \mathbf{1}$$

for the annihilation operator a and the (adjoint) creation operator $a^\dagger$. Here $[X, Y] = XY - YX$ denotes the operator commutator. Gaussian states g can be mathematically characterized by the following Gaussian characteristic functions [11,12]:

$$\mathrm{tr} g D_\alpha = e^{-b|\alpha|^2 - c^*\alpha^2/2 - c\alpha^{*2}/2 + d^*\alpha - d\alpha^*},$$

where $D_\alpha = e^{\alpha a^\dagger - \alpha^* a}$ are the Weyl displacement operators, $b \geqslant 0$, $c, d \in \mathbb{C}$ are constants. Equivalently, a state is Gaussian if and only if it can be expressed as a displaced squeezed thermal state [1,3,50]

$$g = D_\alpha S_\zeta \tau_{\bar{n}} S_\zeta^\dagger D_\alpha^\dagger, \tag{1}$$

where $S_\zeta = e^{\zeta a^{\dagger 2}/2 - \zeta^* a^2/2}$ is the Stoler squeezing operator with squeezing parameter $\zeta = re^{i\phi}$, and

$$\tau_{\bar{n}} = \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1}\right)^n |n\rangle\langle n|$$

is the thermal state with average photon number $\bar{n} = \mathrm{tr}\tau_{\bar{n}} a^\dagger a$, while $|n\rangle$ are the Fock (number) states which are the eigenvectors of the number operator $N = a^\dagger a$ and constitute an orthonormal basis. We denote by $\mathcal{G}$ the set of Gaussian states.

Motivated by the consideration of minimum uncertainty, a physical characterization of Gaussian states based on an information-theoretic refinement of the conventional Heisenberg uncertainty relation was obtained recently in Ref. [48]. More precisely, it was established that the family of Gaussian states coincides exactly with the family of minimum uncertainty states of the following uncertainty relation:

$$\sqrt{U(\rho, Z_\theta)U(\rho, Z_{\theta+\pi/2})} \geqslant \tfrac{1}{2}, \quad \theta \in [0, \pi), \tag{2}$$

where

$$Z_\theta = \frac{e^{-i\theta}a + e^{i\theta}a^\dagger}{\sqrt{2}}$$

are the rotated quadrature operators, and the quantity

$$U(\rho, X) = \sqrt{V^2(\rho, X) - C^2(\rho, X)}$$

can be regarded as quantifying quantum uncertainty of the observable X in the state ρ [49]. The variance

$$V(\rho, X) = \mathrm{tr}\rho X^2 - (\mathrm{tr}\rho X)^2$$

quantifies the total uncertainty, while $C(\rho, X) = V(\rho, X) - I(\rho, X)$ may be interpreted as classical uncertainty and

$$I(\rho, X) = -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho}, X]^2$$

is the celebrated Wigner-Yanase skew information [51], which has founded many applications in quantum information theory [52–58]. It should be emphasized that, although the skew information coincides with the conventional variance for pure states, there are fundamental differences between them. For example, the skew information is convex, while the variance is concave. Due to the square root $\sqrt{\rho}$ and commutator involved in the definition of skew information, more quantum nature is captured in the skew information. In contrast, the variance is a hybrid quantity involving in general both classical and quantum uncertainties.

Uncertainty relation (2) is a specification of the following general uncertainty relation [49]:

$$\sqrt{U(\rho, X)U(\rho, Y)} \geqslant \tfrac{1}{2}|\mathrm{tr}\rho[X, Y]|$$

since $[Z_\theta, Z_{\theta+\pi/2}] = i$. We emphasize that $V(\rho, X) \geqslant U(\rho, X)$, $V(\rho, X) \geqslant I(\rho, X)$, and, for any pure state ρ,

$$U(\rho, X) = V(\rho, X) = I(\rho, X).$$

Thus the difference between uncertainty relation (2) and the conventional uncertainty relation

$$\sqrt{V(\rho, Z_\theta)V(\rho, Z_{\theta+\pi/2})} \geqslant \tfrac{1}{2}, \quad \theta \in [0, \pi), \tag{3}$$

arises only for mixed states. This feature is exactly what we need for characterizing mixed Gaussian states as minimum uncertainty states.

For Gaussian states g defined by Eq. (1) with squeezing parameter $\zeta = re^{i\phi}$ (we call ϕ the squeezing angle), by direct evaluation, we have

$$U(g, Z_\theta)U(g, Z_{\theta+\pi/2}) = \tfrac{1}{4}(1 + \sinh^2(2r)\sin^2(\phi - 2\theta)),$$

which shows that the equality in uncertainty relation (2) is achieved for the rotation parameter $\theta = \phi/2$. Consequently, any Gaussian state is the minimum uncertainty state of uncertainty relation (2) for certain θ.

Conversely, by the equality condition of the Schwarz inequality, any minimum uncertainty state ρ for uncertainty relation (2) with a specified rotation parameter θ satisfies the following equations [48]:

$$[\sqrt{\rho}, Z_\theta] = iu\{\sqrt{\rho}, Z_{\theta+\pi/2} - \mathrm{tr}\rho Z_{\theta+\pi/2}\},$$
$$[\sqrt{\rho}, Z_{\theta+\pi/2}] = -iv\{\sqrt{\rho}, Z_\theta - \mathrm{tr}\rho Z_\theta\},$$

where $u^2 = 4I^2(\rho, Z_\theta)$, $v^2 = 4I^2(\rho, Z_{\theta+\pi/2})$, and $\{X, Y\} = XY + YX$ is the anticommutator. The above relations imply that the state ρ has to be Gaussian of the form (1), with squeezing angle $\phi = 2\theta$ [48].

To summarize, let

$$\mathcal{U}_\theta = \left\{\rho : \sqrt{U(\rho, Z_\theta)U(\rho, Z_{\theta+\pi/2})} = \tfrac{1}{2}\right\}$$

be the set of minimum uncertainty states of uncertainty relation (2) with the fixed value θ, and let

$$\mathcal{G}_\theta = \{g = D_\alpha S_\zeta \tau_{\bar{n}} S_\zeta^\dagger D_\alpha^\dagger : \alpha \in \mathbb{C}, \zeta = re^{2i\theta}, r, \bar{n} \geqslant 0\}$$

be the set of Gaussian states with fixed squeezing angle 2θ; then

$$\mathcal{U}_\theta = \mathcal{G}_\theta, \quad \theta \in [0, \pi),$$

and

$$\bigcup_{\theta\in[0,\pi)} \mathcal{U}_\theta = \bigcup_{\theta\in[0,\pi)} \mathcal{G}_\theta = \mathcal{G}$$

is exactly the set $\mathcal{G}$ of all Gaussian states. In sharp contrast, the minimum uncertainty states of conventional uncertainty relation (3) only consist of pure states including coherent states and squeezed coherent states, a strict subset of Gaussian states.

Inspired by the above physical characterization of Gaussian states, we are led naturally to introduce a measure quantifying the non-Gaussianity of quantum state ρ as follows:

$$N_G(\rho) = \min_{\theta\in[0,\pi)} \sqrt{U(\rho, Z_\theta)U(\rho, Z_{\theta+\pi/2})} - \tfrac{1}{2}. \tag{4}$$

The measure $N_G(\rho)$ has the following desirable properties:

(1) $N_G(\rho) \geqslant 0$ for any state ρ, and $N_G(\rho) = 0$ if and only if the quantum state ρ is Gaussian.

(2) $N_G(\cdot)$ is invariant under displacements and phase-space rotations in the sense that

$$N_G(\rho) = N_G(U\rho U^\dagger)$$

for $U = D_\alpha$ or $e^{ita^\dagger a}$, $\alpha \in \mathbb{C}, t \in \mathbb{R}$.

(3) $N_G(\rho)$ is neither convex nor concave with respect to ρ.

Item (1) follows readily from the characterization of Gaussian states as the minimum uncertainty states [48]. Item (2) can be verified from the defining Eq. (4). To illustrate item (3), recall that convex combinations of Gaussian states are in general not Gaussian states [this implies nonconvexity of $N_G(\cdot)$], while thermal states, which are Gaussian, can be expressed as convex combinations of non-Gaussian Fock states [this implies nonconcavity of $N_G(\cdot)$].

In classical probability theory, characterization of non-Gaussianity is usually related to cumulants (moments) of higher order. In fact, one often uses kurtosis, which is defined via the fourth-order cumulant, to reflect certain aspects of non-Gaussianity of random variables. Thus, it may be useful to characterize non-Gaussianity of quantum states in terms of higher-order moments. This is exactly the approach in a recent paper [59]. However, our approach is quite different and is not just based on moment of second order due to the square root $\sqrt{\rho}$: it involves $\sqrt{\rho}$, X^2. In sharp contrast, the variance involves ρ, X^2. Consequently, in the skew information, the relative order between that of X and ρ is $\frac{2}{1/2} = 4$ (due to the appearance of X^2 and $\sqrt{\rho}$), while the relative order is 2 between that of X and ρ in the variance (due to the appearance of X^2 and ρ).

Our uncertainty relation fully characterizes all Gaussian states as saturating the inequality from the physical perspective [48], and here we are pursuing this further to quantify non-Gaussianity by the excess of the uncertainty. It is a faithful measure of non-Gaussianity.

III. EXAMPLES

We evaluate $N_G(\rho)$ for some important quantum states in order to illustrate its basic features and to gain some intuitive understanding.

A. Fock states

For the Fock states $|n\rangle$, we get from direct calculation that

$$U(|n\rangle, Z_\theta) = U\big(|n\rangle, Z_{\theta+\frac{\pi}{2}}\big) = n + \tfrac{1}{2}$$

are independent of the rotation angle θ. Therefore, the amount of non-Gaussianity is

$$N_G(|n\rangle) = \big(n + \tfrac{1}{2}\big) - \tfrac{1}{2} = n.$$

This result is pleasing and, as n increases, the non-Gaussianity increases, as expected from our intuition.

B. ON states

The ON states

$$|\text{ON}\rangle = \sqrt{1-t}|0\rangle + \sqrt{t}|n\rangle, \quad n = 1, 2, \ldots, \tag{5}$$

as superpositions of the vacuum and the Fock states, have attracted attention quite recently since they can serve as resource units for universal quantum computation [60]. By direct evaluation, we have

$$N_G(|\text{ON}\rangle) = \begin{cases} \frac{1}{2}(\sqrt{1+8t^3} - 1), & n = 1 \\ \frac{1}{2}(\sqrt{1+24t^2} - 1), & n = 2 \\ nt, & n \neq 1, 2. \end{cases}$$

We see that as n increases, the non-Gaussianity increases.

C. Schrödinger cat states

For the Schrödinger even cat states

$$|\alpha_+\rangle = \frac{1}{\sqrt{2 + 2e^{-2|\alpha|^2}}}(|\alpha\rangle + |-\alpha\rangle), \quad \alpha \in \mathbb{C},$$

we have

$$U(|\alpha_+\rangle, Z_\theta) = F - \tfrac{1}{2}(\alpha^2 e^{-2i\theta} + \alpha^{*2} e^{2i\theta}),$$
$$U\big(|\alpha_+\rangle, Z_{\theta+\frac{\pi}{2}}\big) = F + \tfrac{1}{2}(\alpha^2 e^{-2i\theta} + \alpha^{*2} e^{2i\theta}),$$

where $F = |\alpha|^2 \tanh |\alpha|^2 + 1/2$ is independent of the rotation angle θ. The minimum of the product of quantum uncertainties

$$\min_{\theta\in[0,\pi)} U(|\alpha_+\rangle, Z_\theta)U(|\alpha_+\rangle, Z_{\theta+\pi/2}) = F^2 - \tfrac{1}{4} \max_{\theta\in[0,\pi)} (\alpha^2 e^{-2i\theta} + \alpha^{*2} e^{2i\theta})^2$$

is achieved when $\theta = \arg(\alpha)$, which implies that the non-Gaussianity of $|\alpha_+\rangle$ is

$$N_G(|\alpha_+\rangle) = \sqrt{\big(|\alpha|^2 \tanh |\alpha|^2 + \tfrac{1}{2}\big)^2 - |\alpha|^4} - \tfrac{1}{2}. \tag{6}$$

Similarly, for the odd Schrödinger cat states

$$|\alpha_-\rangle = \frac{1}{\sqrt{2 - 2e^{-2|\alpha|^2}}}(|\alpha\rangle - |-\alpha\rangle), \quad \alpha \in \mathbb{C},$$

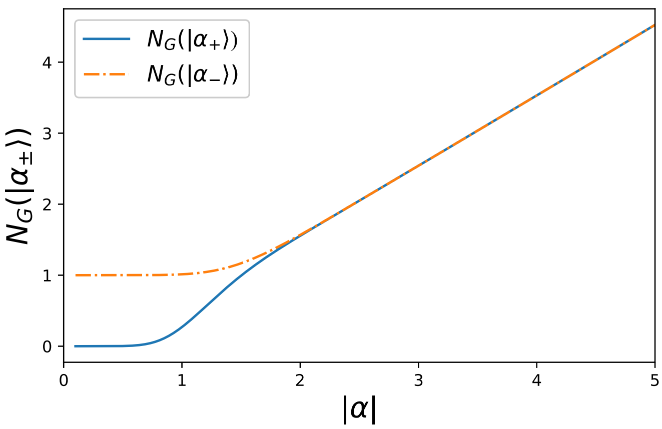

FIG. 1. Non-Gaussianity of even and odd Schrödinger cat states.

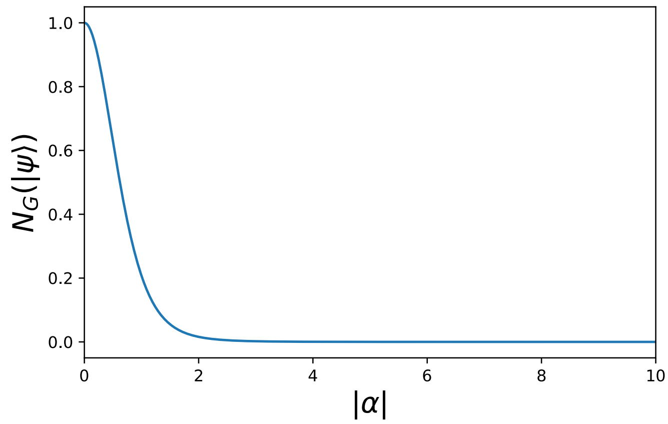

FIG. 2. Non-Gaussianity of photon-added coherent states.

we have

$$N_G(|\alpha_-\rangle) = \sqrt{\left(|\alpha|^2 \coth |\alpha|^2 + \tfrac{1}{2}\right)^2 - |\alpha|^4} - \tfrac{1}{2}. \qquad (7)$$

In Fig. 1, we plot non-Gaussianity for even and odd Schrödinger cat states with respect to $|\alpha|$. As can be seen clearly, when $|\alpha|$ is small, non-Gaussianity of odd Schrödinger cat states (which approach the Fock state $|1\rangle$ when $|\alpha| \to 0$) is larger than that of the even Schrödinger cat states (which approach the vacuum state $|0\rangle$ when $|\alpha| \to 0$), though the difference is very small for large $|\alpha|$.

D. Photon-subtracted squeezed coherent states

Although the squeezed coherent states $S(r)|\alpha\rangle$ are Gaussian states, the photon-subtracted states [61],

$$|\psi_{\rm ps}\rangle = \frac{1}{\sqrt{s+t}} a S(r)|\alpha\rangle,$$

are non-Gaussian. Here $s = |\alpha \cosh r - \alpha^* \sinh r|^2$, $t = \sinh^2 r$ and $r > 0$ is the squeezing strength. Indeed, after direct calculation, we obtain

$$N_G(|\psi_{\rm ps}\rangle) = \frac{1}{2}\frac{\sqrt{(s+3t)(s^2+3t^2)}}{(s+t)^{3/2}} - \frac{1}{2},$$

which is always positive because $r > 0$. Interestingly, the photon-subtracted squeezed vacuum states $S(r)|0\rangle$ actually lead to squeezed Fock states $aS(r)|0\rangle = S(r)|1\rangle$ with constant non-Gaussianity 1, independent of the squeezing parameter r.

E. Photon-added coherent states

For the photon-added coherent states

$$|\psi\rangle = \frac{1}{\sqrt{1+|\alpha|^2}} a^\dagger |\alpha\rangle,$$

the minimum of $U(|\psi\rangle, Z_\theta)U(|\psi\rangle, Z_{\theta+\pi/2})$ is achieved when $\theta = \arg(\alpha)$ and

$$N_G(|\psi\rangle) = \frac{1}{2}\sqrt{\frac{(|\alpha|^2+3)(|\alpha|^4+3)}{(|\alpha|^2+1)^3}} - \frac{1}{2}.$$

We plot in Fig. 2 the non-Gaussianity $N_G(|\psi\rangle)$ with respect to $|\alpha|$. As can be seen, when $|\alpha|$ increases, the non-Gaussianity actually decreases. This is intuitive since the effect of adding a photon to a coherent state with a large amplitude tends to be negligible.

F. Fock-diagonal states

For the Fock-diagonal states

$$\rho_{\rm F} = \sum_{n=0}^{\infty} p_n |n\rangle\langle n|,$$

we have

$$N_{\rm G}(\rho_{\rm F}) = \sqrt{\left(\bar{n} + \frac{1}{2}\right)^2 - \left(\sum_{n=0}^{\infty} \sqrt{p_n p_{n+1}}(n+1)\right)^2} - \frac{1}{2} \qquad (8)$$

with $\bar{n} = {\rm tr}\,\rho_{\rm F} a^\dagger a = \sum_{n=0}^{\infty} n p_n$ being the average photon numbers. The class of Fock-diagonal states is rather broad. Let us consider some special cases.

Case 1. For the states

$$\rho_\lambda = \lambda \tau_{\bar{n}} + (1-\lambda)|0\rangle\langle 0|, \quad \lambda \in [0,1],$$

which are mixtures of vacuum state $|0\rangle$ and the thermal states

$$\tau_{\bar{n}} = \frac{1}{\bar{n}+1}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n |n\rangle\langle n|,$$

the amount of non-Gaussianity $N_G(\rho_\lambda)$ is given by Eq. (8) with

$$p_0 = \frac{\lambda}{\bar{n}+1} + 1 - \lambda,$$

$$p_n = \frac{\lambda}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n, \quad n = 1, 2, \ldots.$$

We depict the non-Gaussianity $N_G(\rho_\lambda)$ versus the mixing parameter λ for different average photon numbers $\bar{n}$ of the thermal states in Fig. 3. Both thermal states and vacuum state are Gaussian, yet their mixtures are non-Gaussian. We emphasize that the set of Gaussian states is not convex. Moreover, the non-Gaussianity increases with increasing mean thermal photon number, which is as expected for any reasonable and consistent measure for non-Gaussianity. Otherwise, suppose on the contrary that it is decreasing with increasing mean

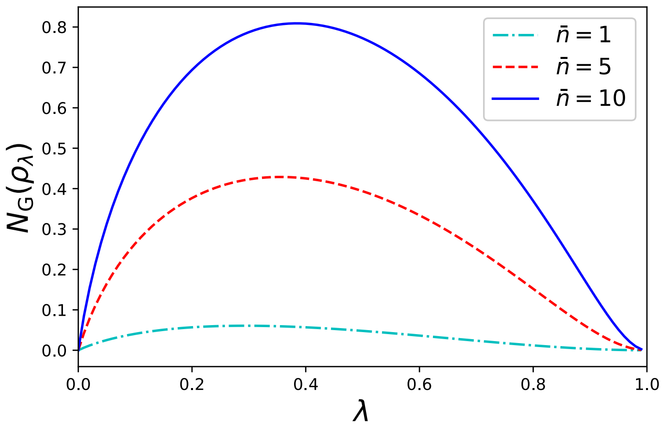

FIG. 3. Non-Gaussianity of $\rho_\lambda = \lambda\tau_{\bar{n}} + (1-\lambda)|0\rangle\langle 0|$ versus λ for different $\bar{n}$.

thermal photon number; then since for $\bar{n} = 0$, ρ_λ is just the vacuum state, and the mixture is trivially the vacuum, which is Gaussian with $N_G(\rho_\lambda) = 0$. Now as $\bar{n}$ increases, if $N_G(\rho_\lambda)$ decreases, then this quantity is identically zero for any mixture of thermal states and vacuum, and thus is useless.

Case 2. For comparison, we calculate the mixtures of thermal states with the single-photon state; that is,

$$\sigma_\lambda = \lambda\tau_{\bar{n}} + (1-\lambda)|1\rangle\langle 1|.$$

The amount of non-Gaussianity $N_G(\sigma_\lambda)$ is given by Eq. (8) with

$$p_1 = \frac{\lambda\bar{n}}{(\bar{n}+1)^2} + 1 - \lambda,$$

$$p_n = \frac{\lambda}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n, \quad n \neq 1.$$

In Fig. 4, we plot the non-Gaussianity $N_G(\sigma_\lambda)$ versus the degree of mixture λ for different $\bar{n}$. As has been calculated before, the non-Gaussianity for the single-photon state is $N_G(|1\rangle\langle 1|) = 1$. Thus, it is interesting to observe that after mixing with a thermal state which is Gaussian, non-Gaussianity may become larger.

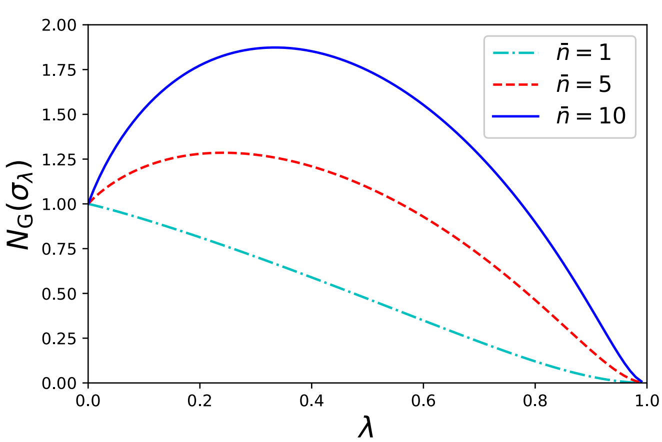

FIG. 4. Non-Gaussianity of $\sigma_\lambda = \lambda\tau_{\bar{n}} + (1-\lambda)|1\rangle\langle 1|$ versus λ for different $\bar{n}$.

Case 3. The truncated thermal states $\tau_{\hat{0}}$,

$$\tau_{\hat{0}} = \frac{1}{\bar{n}}\sum_{n=1}^{\infty}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n |n\rangle\langle n|,$$

can be obtained from the thermal states by removing the vacuum component $|0\rangle\langle 0|$. Actually, the elimination of the vacuum component from a quantum state can lead to a state which is described by a negative Wigner function, thus de-Gaussifying the quantum state [3]. The non-Gaussianity of $\tau_{\hat{0}}$ can be calculated from Eq. (8) as

$$N_G(\tau_{\hat{0}}) = \frac{1}{2}\sqrt{5 + \frac{4}{\bar{n}+1}} - \frac{1}{2},$$

which is a decreasing function of the average photon number $\bar{n}$. This means that the effect on the non-Gaussianity of removing vacuum is more apparent for thermal states with small average photon numbers.

IV. COMPARISON

In this section, we make a comparative study of our measure for non-Gaussianity with several popular measures in the literature. We elucidate certain advantages and the convenience of our measure. First, we review briefly four measures for non-Gaussianity, which are based on (1) Hilbert-Schmidt distance, (2) fidelity, (3) relative entropy, and (4) Wehrl entropy, respectively.

A. Measure based on Hilbert-Schmidt distance

In terms of the Hilbert-Schmidt distance between the quantum state ρ and its reference Gaussian state ρ_g (the Gaussian state with the same mean and variance as ρ), Genoni *et al.* proposed the following measure for non-Gaussianity [39]:

$$N_H(\rho) = \frac{1}{2}\frac{\mathrm{tr}(\rho - \rho_g)^2}{\mathrm{tr}\rho^2}.$$

This measure has some nice properties and can be straightforwardly calculated. For the Fock states $|n\rangle$, analytic expressions are given as

$$N_H(|n\rangle\langle n|) = \frac{n+1}{2n+1} - \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n,$$

which seem rather complicated; e.g., one cannot read off the monotonicity with respect to n readily, though it can be shown that as n increases, $N_H(|n\rangle\langle n|)$ increases, with the limiting value $1/2$ for $n \to \infty$. In contrast, our measure for non-Gaussianity, Eq. (4), yields a neat expression

$$N_G(|n\rangle\langle n|) = n.$$

The monotonic relation between non-Gaussianity and the photon number n is quite apparent and cannot be simpler. Also, our measure is boundless; it tends to infinity as n increases.

B. Measure based on fidelity

The following measure for non-Gaussianity

$$N_F(\rho) = 1 - F(\rho, \rho_g)$$

TABLE I. Comparison between various measures for non-Gaussianity.

	Fock states $\|n\rangle\langle n\|$	Fock-diagonal states $\sum_{n=0}^{\infty} p_n\|n\rangle\langle n\|$
$N_{\rm H}$	$\frac{n+1}{2n+1}-\frac{1}{n+1}\left(\frac{n}{n+1}\right)^n$	$\frac{1}{2}\left(1+\frac{\sum_{n=0}^{\infty}(b_n^2-2b_np_n)}{\sum_{n=0}^{\infty}p_n^2}\right)$
$N_{\rm F}$	$1-\sqrt{\frac{n^n}{(n+1)^{n+1}}}$	$1-\sum_{n=0}^{\infty}\sqrt{b_np_n}$
$N_{\rm R}$	$(n+1)\ln(n+1)-n\ln n$	$(\bar{n}+1)\ln(\bar{n}+1)-\bar{n}\ln\bar{n}+\sum_{n=0}^{\infty}p_n\ln p_n$
$N_{\rm W}$	$\ln(n+1)-n-\ln n!+n\psi(n+1)$	$1+\ln(\bar{n}+1)-\frac{1}{\pi}\int\bar{p}(\alpha)\ln\bar{p}(\alpha)d^2\alpha$
$N_{\rm G}$	n	$\sqrt{\left(\bar{n}+\frac{1}{2}\right)^2-\left(\sum_{n=1}^{\infty}n\sqrt{p_{n-1}p_n}\right)^2}-\frac{1}{2}$

based on the fidelity (equivalently, the Bures distance) was introduced by Ghiu *et al.* [44]. Here $F(\rho,\tau)=\mathrm{tr}((\sqrt{\rho}\tau\sqrt{\rho})^{1/2})$ is the fidelity between ρ and τ. For the Fock states $|n\rangle$, and more generally, for the Fock-diagonal states $\rho_{\rm F}$, we have

$$N_{\rm F}(|n\rangle)=1-\sqrt{\frac{n^n}{(n+1)^{n+1}}},$$

$$N_{\rm F}(\rho_{\rm F})=1-\sum_{n=0}^{\infty}\sqrt{b_np_n},\quad b_n=\frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n,$$

both of which are rather involved.

C. Measure based on relative entropy

An informational measure for non-Gaussianity was initiated in Ref. [39], and further pursued in Refs. [40–42]. This is defined as the relative entropy between a quantum state and its associated reference Gaussian state as follows:

$$N_{\rm R}(\rho)=S(\rho|\rho_{\rm g})=S(\rho_{\rm g})-S(\rho),$$

where $\rho_{\rm g}$ is the associated reference Gaussian state of ρ, i.e., the unique Gaussian state with the same mean and variance as that of ρ, and $S(\rho)=-\mathrm{tr}\rho\ln\rho$ is the von Neumann entropy. It turns out that $N_{\rm R}(\rho)=\inf_g S(\rho|g)$ where the inf is over all Gaussian states g. This measure enjoys a lot of nice properties [40]. It is known that, for the Fock states $|n\rangle$,

$$N_{\rm R}(|n\rangle)=(n+1)\ln(n+1)-n\ln n,$$

while for the Fock-diagonal states $\rho_{\rm F}=\sum_{n=0}^{\infty}p_n|n\rangle\langle n|$, we have

$$N_{\rm R}(\rho_{\rm F})=(\bar{n}+1)\ln(\bar{n}+1)-\bar{n}\ln\bar{n}+\sum_{n=0}^{\infty}p_n\ln p_n.$$

D. Measure based on Wehrl entropy

By virtue of the Wehrl entropy of the Husimi function $\langle\alpha|\rho|\alpha\rangle$ with $|\alpha\rangle$ being the coherent states, Ivan *et al.* proposed the following measure for non-Gaussianity [42]:

$$N_{\rm W}(\rho)=S_{\rm W}(\rho_{\rm g})-S_{\rm W}(\rho),$$

where $\rho_{\rm g}$ is the associated reference Gaussian state of ρ, and

$$S_{\rm W}(\rho)=-\frac{1}{\pi}\int\langle\alpha|\rho|\alpha\rangle\ln\langle\alpha|\rho|\alpha\rangle d^2\alpha$$

is the Wehrl entropy.

For the Fock states $|n\rangle$, it can be evaluated that

$$N_{\rm W}(|n\rangle\langle n|)=\ln(n+1)-n-\ln n!+n\psi(n+1),$$

where

$$\psi(n+1)=\sum_{k=1}^{n}\frac{1}{k}-\gamma$$

is the digamma function and $\gamma\approx 0.5772$ is the Euler constant. Moreover, for the Fock-diagonal states $\rho_{\rm F}=\sum_{n=0}^{\infty}p_n|n\rangle\langle n|$, we have

$$N_{\rm W}(\rho_{\rm F})=1+\ln(\bar{n}+1)-\frac{1}{\pi}\int\bar{p}(\alpha)\ln\bar{p}(\alpha)d^2\alpha,$$

where $\bar{p}(\alpha)=\sum_{n=0}^{\infty}p_n\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}$. The above expressions are rather heavy.

We summarize the above comparison in Table I, in which $\bar{n}=\sum_{n=0}^{\infty}np_n$, and

$$b_n=\frac{1}{1+\bar{n}}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n,\quad \bar{p}(\alpha)=\sum_{n=0}^{\infty}p_n\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}.$$

We see the simplicity and intuitive significance of $N_{\rm G}(\cdot)$. Our measure for the Fock states is particularly simple and intuitive, in sharp contrast to the complicated expressions in other measures.

V. CLASSICAL VERSUS QUANTUM NON-GAUSSIANITY

Recall that $\mathcal{G}$ denotes the set of Gaussian states, whose complement $\mathcal{G}^c$ is the set of non-Gaussian states. We have introduced a measure, i.e., $N_{\rm G}(\cdot)$, to quantify the degree of non-Gaussianity of states in $\mathcal{G}^c$. Taking into account quantumness, it is natural to further divide the set of non-Gaussian states $\mathcal{G}^c$ into classical non-Gaussian states $\mathcal{G}^c_{\rm cl}$ and quantum non-Gaussian states $\mathcal{G}^c_{\rm qu}$. A non-Gaussian state is called classical non-Gaussian if it can be expressed as a probabilistic mixture of Gaussian states. Otherwise, it is called quantum non-Gaussian (or genuine non-Gaussian). Consequently, the set of quantum states $\mathcal{S}$ can be divided into

$$\mathcal{S}=\mathcal{G}\bigcup\mathcal{G}^c,$$

while

$$\mathcal{G}^c=\mathcal{G}^c_{\rm cl}\bigcup\mathcal{G}^c_{\rm qu}.$$

Moreover, the union $\mathcal{G}\bigcup\mathcal{G}^c_{\rm cl}$ of Gaussian states $\mathcal{G}$ and classical non-Gaussian states $\mathcal{G}^c_{\rm cl}$ constitutes a convex set.

The issue of a clear discrimination between classical and quantum non-Gaussianity is an extremely important, subtle, and open issue, which lies beyond the scope of our present work. It is desirable to characterize and quantify quantum non-Gaussianity from various perspectives. Some remarkable progress was made in Refs. [28–30,62–68]. In particular, photon number probabilities were employed to detect quantum non-Gaussianity from both theoretical and experimental perspectives in Refs. [62,63]. Quantum non-Gaussianity was quantified via the negativity of Wigner functions in Ref. [64]. Detection of quantum non-Gaussianity via s-parametrized quasiprobability functions was studied in Ref. [65]. Quantum non-Gaussian depth of single-photon states was addressed in Ref. [66]. The interplay between quantum non-Gaussianity and nonclassicality was investigated in Ref. [67]. A faithful hierarchy of n-photon quantum non-Gaussianity was obtained in Ref. [68]. Quantifying quantum Gaussianity via uncertainty relation is worth further investigation.

By the way, we emphasize that although non-Gaussianity and nonclassicality are related concepts, they are fundamentally different. A Gaussian state may be classical (such as the coherent states) or nonclassical (such as squeezed coherent states). A non-Gaussian state may be classical (such as mixtures of two coherent states) or nonclassical (such as the Schrödinger cat states). Moreover, the set of classical states is convex, while the set of Gaussian states is not convex. Both non-Gaussianity and nonclassicality are important resources in quantum information tasks.

VI. DISCUSSION

We have introduced a measure for non-Gaussianity of single-mode bosonic field states which involves only simple optimization and thus is computable. The measure is based on a physical characterization of Gaussian states as minimum uncertainty states of an informational uncertainty relation involving the Wigner-Yanase skew information. We have further evaluated the measure for several important quantum states to illustrate its significance and intuitive meaning. Comparisons with several popular measures for non-Gaussianity are made, which elucidate some simplicity and convenience of our measure. The results may shed certain light on the quantitative aspect of non-Gaussianity as a resource for quantum information tasks.

In quantum metrology, the resolution precision can be assessed by quantum Fisher information in view of the celebrated Cramér-Rao inequality. It has been established that there are classical non-Gaussian states (mixtures of coherent states) which provide larger resolution than coherent states with the same mean number of photons [69]. This provides an example of a quantum information task in which classical non-Gaussianity is helpful. Our quantity is based on the Wigner-Yanase skew information $I(\rho, X)$, which is also a version of quantum Fisher information and is closely related to the commonly used quantum Fisher information $F(\rho, X)$ based on a symmetric logarithmic derivative through the inequality chain $I(\rho, X) \leqslant F(\rho, X) \leqslant 2I(\rho, X)$ [52]. Because of this connection, our criteria for non-Gaussianity may play a role in certain tasks of quantum metrology. This is an important issue worth further investigation.

For simplicity, we have only treated single-mode bosonic fields. From both experimental and theoretical perspectives, it will be desirable to extend the results to multimode cases, which seems nontrivial since correlations between different modes may emerge and cause complication. Furthermore, it will be interesting to study Gaussian and non-Gaussian characters of quantum processes from the perspective of uncertainty.

Recently, the nonclassicality of bosonic field states was also quantified in terms of Wigner-Yanase skew information [70,71]. In view of the present quantification of non-Gaussianity involving the Wigner-Yanase skew information, the interplay between non-Gaussianity and nonclassicality calls for further exploration from this informational perspective.

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