Editors' Suggestion

Massive phonons and gravitational dynamics in a photon-fluid model

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We theoretically investigate the excitation dynamics in a photon fluid with both local and nonlocal interactions. We show that the interplay between locality and an infinite-range nonlocality gives rise to a gapped Bogoliubov spectrum of elementary excitations which, at lower momenta, correspond to massive particles (phonons) with a relativistic energy-momentum relation. In this regime and in the presence of an inhomogeneous flow the density fluctuations are governed by the massive Klein-Gordon equation on the acoustic metric and thus propagate as massive scalar fields on a curved spacetime. We finally demonstrate that in the nonrelativistic limit the phonon modes behave as self-gravitating quantum particles with an effective Schrödinger-Newton dynamics, although with a finite-range gravitational interaction and a nonzero cosmological constant. Our photon fluid represents a viable alternative to Bose-Einstein condensate models for "emergent gravity" scenarios and offers a promising setting for analog simulations of semiclassical gravity and quantum gravity phenomenology.

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I. INTRODUCTION

Analog gravity models provide a powerful testbed for several aspects of classical and quantum field theories in curved spacetime [1–3]. The general idea is that under appropriate conditions the elementary excitations in condensed-matter systems evolve as fields on a curved spacetime induced by the medium. The paradigmatic example is provided by sound waves in an inhomogeneous flowing fluid [4,5]. In spite of the fact that the background fluid is nonrelativistic, the elementary excitations of the flow (phonons) experience a curved spacetime: their evolution is governed by the Klein-Gordon equation for a massless particle in a curved background, the geometry of which is specified by a Lorentzian metric tensor (acoustic metric). As a result, the phonon dynamics exhibits an effective Lorentz invariance with the local speed of sound playing the role of the speed of light. The coefficients of the acoustic metric depend on the fluid density, which determines also the sound speed, and the flow velocity. Hence, by tailoring the properties of the flow it is possible to simulate gravitational spacetimes and related phenomena, such as, e.g., Hawking radiation, super-radiance, and cosmological particle production.

Analog-gravity scenarios have been proposed and realized in a variety of physical systems, including Bose-Einstein condensates (BECs) [6,7], surface waves [8], supefluid ^{3}He [9] and Fermi liquids [10], dielectrics [11,12], moving- and nonlinear-optical media [13–15], and exciton-polariton condensates [16,17]. Signatures of the Hawking process have been reported in different setups [18–24] and in a recent experiment the observation of super-radiance has been achieved [25].

As an alternative to the above systems, photon fluids have recently attracted considerable attention. Photon fluids belong to the family of the so-called quantum fluids of light [26], together with exciton-polariton and photon BECs [27]. While the last two are driven-dissipative systems based on nonlinear optical cavities, photon fluids simply rely on the nonlinear propagation of light. A laser beam propagating through a self-defocusing medium can be described in terms of a weakly interacting Bose gas, where the repulsive photon-photon interaction arises from a third-order nonlinearity [28,29] and the propagation coordinate acts as an effective time variable. Recent experiments in these systems provided evidence of collective many-photon phenomena, such as superfluidity and its breakdown [30,31] and nonequilibrium precondensation of classical waves [32]. In analogy with BECs, the collective excitations of the mean flow (i.e., small ripples of the transverse optical field) propagate according to the Bogoliubov dispersion relation [33], as recently demonstrated in thermo-optical [34] and Kerr media [35]. As a consequence, for the longer wavelengths a Lorentz invariant phononic regime takes place where soundlike waves propagate with a constant speed determined by the photon-fluid density. The latter is proportional to the optical intensity while the background flow velocity is controlled via the gradient of the phase profile. All these features make these systems particularly suitable for the realization of analog gravity experiments [36–44].

All the above systems are generally characterized by a gapless dispersion relation at small momenta [45] typical of *massless* collective excitations. Therefore most of the theoretical research in this area, and all ongoing experiments, have naturally focused on the simulation of massless fields propagating through a curved spacetime. A notable exception is the work by Visser and Weinfurtner [46], who first proposed a method to give rise to a spectrum of massive relativistic particles in a BEC system. The model describes a two-component BEC with an additional Raman coupling which deforms the spectrum of normal-mode excitations. Interestingly, in appropriate conditions one of the two phonon modes remains massless while the second acquires effective mass. Subsequent investigations by Girelli *et al.* introduced a modified BEC Hamiltonian with a U(1) symmetry-breaking term [47]. This modification provides a mass to the excitations and gives rise to a kind of analog gravitational dynamics. Remarkably, the gravitational potential is sourced by (a function of) the density distribution of the excitations which thus play the role of the matter in this system. These are important extensions with respect to usual analog models, since such massive excitation fields could enable simulations of quantum gravity phenomenology (e.g., Lorentz-violating dispersion relations [48]) and emergent gravity scenarios [49].

In the following we consider a photon-fluid model for light propagating in a defocusing medium with both local and nonlocal optical nonlinearities. In contrast to the purely local case, the first-order excitations satisfy a massive version of the Bogoliubov dispersion relation in a Bose gas, with the nonlocal term being the mass-generating mechanism. For the longer wavelengths, the spectrum approximates that of a massive particle with a relativistic energy-momentum relation and, in the presence of inhomogeneous flows, the density fluctuations are described by the massive Klein-Gordon equation on the acoustic metric, thus closely mimicking the propagation of massive scalar fields on a curved spacetime. Even more importantly, in the nonrelativistic limit the phonons behave as massive self-gravitating quantum particles: their dynamics obeys the Schrödinger equation in a gravitational potential the source of which depends on the phononic mass density distribution via a modified Poisson equation. Unlike the Newtonian theory, we find that the range of the gravitational interaction is finite and a cosmological constant is also present. In spite of these significant differences with respect to standard gravity, such photon fluid nonetheless remains an interesting workbench for analog simulations of semiclassical gravity scenarios. Most analog-gravity models indeed are dealing with massless excitations that in Newtonian theory cannot act as sources of a gravitational field. This system is thus one of the very few in which a form of semiclassical gravitational dynamics can be shown to emerge.

The paper is organized as follows. In Sec. II, we introduce the modified nonlinear Schrödinger equation (NSE) with local and nonlocal nonlinearities and the related photon-fluid model. We then derive the Bogoliubov–de Gennes equations governing the dynamics of the first-order fluctuations of the optical field and the corresponding dispersion relation for a generic nonlocal function. In Sec. III we focus on a thermo-optical nonlocal nonlinearity, showing that in the defocusing case the photon fluid is stable and allows for the propagation of massive phonons, while it undergoes a Jeans instability and supports tachyonic excitations in the focusing case. The rest of the paper is devoted to analyze the fully stable defocusing regime. In Sec. IV, we address the problem of inhomogeneous flows and derive a massive Klein-Gordon equation on the acoustic metric that will provide the basis for the subsequent discussion on the emergent gravitational scenario. In Sec. V, we introduce the Newtonian limit of the acoustic metric which allows us to identify the gravitational potential with inhomogeneities in the photon-fluid density. We then derive the nonrelativistic phonon dynamics from the Klein-Gordon equation for the optical field excitations. Finally, we show that a (modified) Poisson equation for the potential is encoded in the backreaction equation describing the first corrections to the mean-field dynamics induced by the fluctuations. The conclusions are presented in Sec. VI.

II. PHOTON-FLUID MODEL AND ELEMENTARY EXCITATIONS

The propagation of a monochromatic optical beam oscillating at angular frequency ω in a two-dimensional (2D) nonlinear medium can be described within the paraxial approximation in terms of the NSE [50]

$$\partial_z E = \frac{i}{2k}\nabla^2 E - i\frac{k}{n_0}E\,\Delta n(|E|^2, \mathbf{r}, z) \tag{1}$$

where E is the slowly varying envelope of the optical field, z is the propagation coordinate, $k = 2\pi n_0/\lambda$ is the wave number, λ is the vacuum wavelength, and n_0 is the linear refractive index. The Laplacian term $\nabla^2 E$ defined with respect to the transverse coordinates $\mathbf{r} = (x, y)$ accounts for diffraction and Δn is the nonlinear optical response of the medium. For a local (Kerr) defocusing nonlinearity, $\Delta n = n_2|E|^2$ with $n_2 > 0$, Eq. (1) is formally identical to the 2D Gross-Pitaevskii equation for a dilute boson gas with repulsive contact interactions, where the optical field E corresponds to the complex order parameter and the intensity-dependent refractive index Δn provides the interaction potential. The dynamics takes place in the transverse plane (x, y) of the laser beam so that the propagation coordinate z plays the role of an effective time variable $t = (n_0/c)z$, where c is the speed of light in vacuum. We remark that the analogy between photon fluids and condensates is here limited to the level of the mean-field evolution equations: as such, the system is purely classical and the optical field would correspond to the ground-state wave function of a BEC at zero temperature.

We consider an optical medium with both local and nonlocal third-order nonlinearities $\Delta n(|E|^2, \mathbf{r}, z) = n_2|E|^2 + \hat{\mathrm{n}}_{\mathrm{nl}}|E|^2$, where $\hat{\mathrm{n}}_{\mathrm{nl}}$ is the convolution operator

$$\hat{\mathrm{n}}_{\mathrm{nl}} \equiv \gamma\,(R*) = \gamma\,\theta \int d\mathbf{r}'dz' R(\mathbf{r}-\mathbf{r}', z-z') \tag{2}$$

where * denotes the convolution operation, γ is a coefficient that depends on the specific nonlocal process and $R(\boldsymbol{r}, z)$ is the medium response function. In the following we take $n_2 > 0$, since local repulsive interactions are required to observe a dynamically stable photon fluid on which sound waves can propagate, while $\theta = 1\,(-1)$ corresponds to a defocusing (focusing) nonlocal term, respectively.

The optical nonlocality originates from the fact that the nonlinear change in refractive index at any given position depends both on the local and on the surrounding field intensity through the convolution kernel R. Similar nonlinear responses arise in semiconductor materials with both Kerr and thermo-optical nonlinearities [51], in nematic liquid crystals with competing orientational and thermal effects [52], and in BECs with simultaneous local and long-range (e.g., dipolar) interactions [53,54].

The corresponding hydrodynamic formulation of the NSE is obtained by means of the Madelung transform $E = \rho^{1/2}e^{i\phi}$:

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{3}$$

$$\partial_t \psi + \frac{1}{2}v^2 = -\frac{c^2}{n_0^3}n_2\rho - \frac{c^2}{n_0^3}\hat{\mathrm{n}}_{\mathrm{nl}}\rho + \frac{c^2}{2k^2n_0^2}\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}} \tag{4}$$

where the optical intensity ρ corresponds to the fluid density and $\mathbf{v} = \frac{c}{kn_0}\nabla\phi \equiv \nabla\psi$ is the flow velocity. On the right-hand side of (4), the first term provides the local repulsive interactions related to the positive bulk pressure $P = \frac{c^2 n_2}{2n_0^3}\rho^2$. The second one gives rise to a nonlocal interaction potential, while the last term, directly related to diffraction, is the analog of the Bohm quantum potential the gradient of which corresponds to the so-called quantum pressure.

We finally observe how a z-dependent response function would actually lead to a "noncausal" photon fluid, as the nonlocal interactions would depend on both directions of $z = (c/n_0)t$. Such noncausality is actually fictitious since it originates from the mapping of the spatial z direction into a time coordinate. However, it suggests that a safe interpretation of the propagation coordinate in terms of a time variable would require a z-independent response kernel.

Bogoliubov–de Gennes equations and excitation spectrum

The first-order complex fluctuations $\varepsilon(\mathbf{r}, t)$ of the optical field can be described in terms of Bogoliubov excitations on top of the photon fluid. Linearizing Eq. (1) around a background solution, $E = E_0(1 + \varepsilon + \ldots)$ with $E_0 = \rho_0 e^{i\phi_0}$, we obtain the nonlocal Bogoliubov–de Gennes equations

$$\left(\partial_T - i\frac{c}{2kn_0}\partial_S\right)\varepsilon = -i\frac{\omega}{n_0}(n_2 + \hat{n}_{\rm nl})\rho_0(\varepsilon + \varepsilon^*), \quad (5)$$

$$\left(\partial_T + i\frac{c}{2kn_0}\partial_S\right)\varepsilon^* = i\frac{\omega}{n_0}(n_2 + \hat{n}_{\rm nl})\rho_0(\varepsilon + \varepsilon^*) \quad (6)$$

in which we have defined the usual comoving derivative $\partial_T = \partial_t + \mathbf{v_0}\cdot\nabla$, with $\mathbf{v_0} = \frac{c}{kn_0}\nabla\phi_0$, and the spatial differential operator $\partial_S = \frac{1}{\rho_0}\nabla\cdot(\rho_0\nabla)$. In the spatially homogeneous case where both the background density ρ_0 and the velocity $\mathbf{v_0}$ do not depend on the transverse coordinates, the plane-wave solutions of Eqs. (5) and (6) satisfy the dispersion relation

$$\Omega^2 = c_s^2 K^2\left[1 + \theta\frac{\gamma}{n_2}\tilde{R}(K, n_0\Omega/c) + \frac{\xi^2}{4\pi^2}K^2\right] \quad (7)$$

where K is the wave number of the mode, $\Omega = \Omega' - \mathbf{K}\cdot\mathbf{v_0}$ is its angular frequency in the locally comoving background frame, and $\tilde{R}$ is the three-dimensional Fourier transform of the response function $R(\mathbf{r}, z)$. We remark that here the angular frequency Ω' actually corresponds to the longitudinal wave number K_z expressed in temporal units via $\Omega' = K_z c/n_0$ while $\mathbf{K} = (K_x, K_y)$ is the transverse wave vector.

In analogy to purely local BECs and photon fluids [36], we defined in Eq. (7) the sound speed $c_s^2 \equiv \frac{dP(\rho_0)}{d\rho} = \frac{c^2 n_2}{n_0^3}\rho_0$ and the healing length $\xi = \lambda/2\sqrt{n_0 n_2 \rho_0}$ as the characteristic length separating the linear (phononic) and quadratic (single-particle) regime of the dispersion relation for $\gamma = 0$.

The length ξ determines the critical wave number $K_c = 2\pi/\xi$ associated to the breakdown of Lorentz invariance, generally expected to occur in quantum gravity phenomenology at the Planck scale. Low-energy modes with $K \ll K_c$ propagate indeed at the invariant universal speed c_s, while at higher momenta $K \gg K_c$ the terms arising from the quantum pressure become dominant and the group velocity of excitations increases with K.

Within the paraxial approximation and in the presence of nonlocal processes with negligible longitudinal dependence, the main contribution to the nonlocality comes from the K dependence of $\tilde{R}$ and we can thus safely assume $\tilde{R}(K, n_0\Omega/c) \simeq \tilde{R}(K, 0)$. This is indeed the case of thermo-optical nonlinearities dominated by the transverse diffusion of heat [34,55] that we will discuss in the next sections. From now on we shall ignore the z dependence of R.

We finally observe that on the basis of Eq. (7) wave propagation for focusing nonlinearities $\theta = -1$ is allowed only for wave numbers K such that $\Omega^2 > 0$, i.e., $\tilde{R}(K) < \frac{n_2}{\gamma}(1 + K^2/K_c^2)$. Negative values of Ω^2 correspond to exponentially growing modes characteristic of linearly unstable flows. In the defocusing case $\theta = 1$, the system is neutrally stable to perturbations of all wave numbers, hence supporting traveling waves. While the plane-wave solution is always modulationally stable, instabilities and wave-breaking phenomena [56] are expected in the presence of inhomogeneous beams and/or discontinuous response kernels. In spite of this fact, stable operation in nonlocal photon fluids has been experimentally demonstrated even in the presence of background inhomogeneities [30,42].

III. THERMO-OPTICAL NONLOCALITY AND MASSIVE EXCITATIONS

The functional form of $\tilde{R}(K)$ depends on the specific nonlocal process under consideration. A case of particular interest is provided by light propagation in thermo-optical media, where the change of refractive index $\hat{n}_{\rm nl}\rho = n_{\rm th}$ arises from the temperature increase due to the residual laser absorption. The heat diffuses through the material and eventually across the boundaries of the medium. As a result, the shape of the response function will strongly depend also on the transverse boundary conditions [57]. This might open interesting perspectives in experiments since it could be possible to tailor the nonlocal response of the medium, e.g., by acting on the geometry of the sample, in order to modify the dispersion (7) and in turn the physical properties of the collective excitations [30].

In the limit of an infinite medium in the two transverse dimensions $n_{\rm th}$ is coupled to the optical intensity through the stationary heat equation [58,59]

$$-\nabla^2 n_{\rm th} = \frac{\alpha|\beta|}{\kappa}\rho \quad (8)$$

where κ is the thermal conductivity of the material, α is its linear absorption coefficient, and $\beta = |\partial n_{\rm th}/\partial T|$ is the change in the refractive index with respect to the temperature. The heat equation (8) dictates that the corresponding range of the nonlocal interactions between photons is infinite (infinite-range nonlocality) [59]. The nature of the nonlinearity (focusing or defocusing, leading to attractive or repulsive interactions) depends on the sign of β. Here we take the absolute value $|\beta|$ since the sign is already considered in (2) by the coefficient θ.

Fourier transforming the expression $n_{\rm th} = \gamma\theta R(\mathbf{r}) * \rho$ and Eq. (8) one can readily verify that $\tilde{n}_{\rm th} = \gamma\theta\tilde{R}(K)\tilde{\rho} = \frac{\alpha|\beta|}{\kappa K^2}\tilde{\rho}$ and thus $\tilde{R}(K) \propto 1/K^2$. This implies that the convolution integral $\hat{n}_{\rm nl}\rho$ is, up to a constant, the solution of Eq. (8) with $R(\mathbf{r})$ being the Green's function of the 2D Laplacian operator. Hence, the following relation holds: $\gamma\theta\nabla^2 R(\mathbf{r}) = -(\alpha|\beta|/\kappa)\delta(\mathbf{r}-\mathbf{r}')$.

A. Defocusing nonlocal nonlinearity: Massive phonons

Using the above $\tilde{R}(K) = \frac{\alpha|\beta|}{\kappa\gamma K^2}$ in Eq. (7) and considering a defocusing nonlinearity $\theta = 1$ we find

$$\Omega^2 = \Omega_0^2 + c_s^2 K^2\left(1 + \frac{\xi^2}{4\pi^2}K^2\right) \tag{9}$$

where $\Omega_0 = c\sqrt{\frac{\alpha|\beta|}{\kappa n_0^3}\rho_0}$ has indeed the dimensions of a frequency. Hence, we can thus identify $\hbar\Omega_0$ with the rest energy of a particle and write $\hbar\Omega_0 = mc_s^2$, where m is the rest mass and c_s plays the role of the light speed. Defining the excitation momentum $p = \hbar K$ and the critical momentum $p_c = \hbar K_c = h/\xi$, Eq. (9) can be rewritten in the form

$$\mathcal{E}^2 = m^2c_s^4 + c_s^2p^2\left(1 + \frac{p^2}{p_c^2}\right). \tag{10}$$

The properties of the above dispersion strongly depend on the thermo-optical coefficients of the material used to produce the photon fluid. Here we are interested in investigating the regime in which $p_c \gg mc_s$. In this case, Eq. (10) is a generalization of the Bogoliubov dispersion relation describing *massive* collective excitations with high-energy Lorentz-violating corrections. Similar modified dispersion laws with extra momentum-dependent terms appear in several phenomenological approaches to quantum gravity, where p_c is typically associated to the Planck momentum [60].

The dispersion curve (10) interpolates between three different regimes depending on the fluctuation momentum.

When $p \gg p_c$, the quartic term dominates and Eq. (10) approximates the free-particle behavior $\mathcal{E} \approx c_s p^2/p_c$. Using the above definitions of p_c and c_s and the photon momentum $p_\gamma = \hbar n_0 k$ we get $\mathcal{E} \approx cp^2/(2p_\gamma)$ (or equivalently $\mathcal{E} \approx p^2/2m_\gamma$ introducing an effective photon mass $m_\gamma = p_\gamma/c$). Therefore in analogy to BEC analog models, the excitation energy tends to the energy of the individual particles forming the background fluid, i.e., in our case the photons.

In the intermediate regime, $mc_s \lesssim p \ll p_c$, we obtain the "relativistic" dispersion relation for a massive particle $\mathcal{E} \approx \sqrt{p^2c_s^2 + m^2c_s^4}$, with the speed of sound playing the role of the speed of light. These are collective excitations exactly like usual phonons in local quantum fluids, but possessing a finite rest mass.

At lower momenta, $p \ll mc_s$, the phonon modes enter the nonrelativistic regime: the energy-momentum relation reduces to $\mathcal{E} \approx p^2/2m + mc_s^2$, where the first term is the kinetic energy of a particle of mass m and the second is its constant rest mass energy.

We notice that the rest frequency Ω_0 depends only on the strength of the thermo-optical (nonlocal) nonlinearity. The latter is thus responsible for the generation of the gap in the dispersion relation and hence for the onset of the excitation mass. On the other hand, the "invariant" limit speed c_s is determined solely on the local defocusing effect.

The rest frequency Ω_0 can also be expressed in terms of the sound speed as $\Omega_0 = c_s\sqrt{\frac{\alpha|\beta|}{\kappa n_2}} = c_s/\lambda\!\!\!^{-}_C$. The characteristic length scale $\lambda\!\!\!^{-}_C$, given by the square root of the ratio between the local and nonlocal coefficients, corresponds to the acoustic analog of the reduced Compton wavelength of the particle $\lambda\!\!\!^{-}_C = \hbar/(mc_s)$. The inverse of this length defines the above nonrelativistic limit through $K \ll \lambda\!\!\!^{-}{}_C^{-1}$ and, as we shall see in Sec. V, it provides also some of the fundamental characteristic scales of the emergent gravitational force.

We conclude the section briefly discussing a more realistic model of the thermo-optical nonlinearity [30,56,61] given by

$$-\nabla^2 n_{\rm th} + \frac{n_{\rm th}}{\sigma^2} = \frac{\alpha|\beta|}{\kappa}\rho \tag{11}$$

in which the effect of the distant boundaries has been included in the distributed loss term $-\Delta n_{\rm th}/\sigma^2$, where σ is the length scale of the nonlocal interaction.

Equation (11) allows us to continuously describe the transition from an infinite-range to a finite-range thermo-optical nonlocality and has provided a theoretical framework for the phenomenological Lorentzian response adopted in previous experiments (see, e.g., [34,40]). The Fourier-transformed response associated to (11) $\gamma\tilde{R} = \frac{\alpha|\beta|}{\kappa}\frac{\sigma^2}{1+\sigma^2K^2}$ has indeed a Lorentzian shape, where $2/\sigma$ is its full width at half maximum. The dispersion (10) is thus modified as

$$\mathcal{E}^2 = m^2c_s^4\frac{p^2}{p_{\rm nl}^2 + p^2} + c_s^2p^2\left(1 + \frac{p^2}{p_c^2}\right) \tag{12}$$

where we introduced the nonlocal momentum $p_{\rm nl} = \hbar/\sigma$. The above response kernel reduces to the ideal form of the infinite space model $\gamma\tilde{R}(K) = (\frac{\alpha|\beta|}{\kappa})/K^2$ in the limit of $\sigma K \gg 1$. Such a regime can be reasonably reproduced by means of suitable background optical beams comprising wave vectors only of $K \gg 1/\sigma$. This procedure has been implemented in a lead-doped glass experiment [55]. Before being launched into the nonlinear medium, the laser beam has been passed through a phase mask generating a ring-shaped beam with zero intensity at $K = 0$ and large-enough transverse wave vectors. Using this technique, the authors demonstrated a nonlocal thermo-optical nonlinearity with $\sigma K \approx 20$. In this case $p \gg p_{\rm nl}$, and Eq. (12) well approximates the massive Bogoliubov dispersion (10). However, for finite $p_{\rm nl}$ the two relations will eventually differ at arbitrarily low momenta, as the gap in (12) arises only in the singular limit of an infinite-range nonlocality, $p_{\rm nl} = 0$ $(\sigma \to \infty)$.

B. Focusing nonlocal nonlinearity: Jeans instability

For $\theta = -1$ the hydrodynamic equations Eqs. (3) and (4) together with Eq. (11) describe a $(2+1)$-dimensional quantum fluid with local repulsive and finite-range attractive interactions. In the ideal case of an infinite medium the model reproduces the nonlinear evolution of a self-gravitating BEC [62], where the nonlocal change of refractive index $n_{\rm th}$, the solution of Eq. (8), mimics a Newtonian potential generated by the fluid mass density. In the absence of local interactions, Eq. (1)-(8) are indeed formally equivalent to the Schrödinger-Newton equation in two spatial dimensions [55,63], originally proposed by Diosi [64] and Penrose [65] as a model for quantum wave-function collapse (see also [66] for further discussion).

Concerning the dynamics of elementary excitations, in the more general case the dispersion relation reads

$$\mathcal{E}^2 = -m^2 c_s^4 \frac{p^2}{p_{\rm nl}^2 + p^2} + c_s^2 p^2\left(1 + \frac{p^2}{p_c^2}\right). \tag{13}$$

The negative sign in front of the rest energy term originating from the attractive photon interactions gives rise to two fundamentally different behaviors at high and low momenta. The critical wave number $K = K_J$ separating these two regimes is implicitly defined by the condition at which the wave frequency (energy) vanishes:

$$\frac{\sigma^2}{\lambda\!\!\!^{-}_C{}^2}\frac{1}{1+\sigma^2 K^2} - \left(1 + \frac{K^2}{K_c^2}\right) = 0. \tag{14}$$

For high wave numbers $K > K_J$, the local repulsive interactions and the quantum pressure are sufficiently strong to counterbalance the nonlocal attractive forces and the waves are freely oscillating. In the opposite case, we have growing excitation modes revealing the linear instability of the system (see also [67,68] for a discussion in self-gravitating BECs).

In the hydrodynamic (Thomas-Fermi) approximation ($K \ll K_c$) and for $(\sigma K) \gg 1$, Eq. (14) simply reduces to the ordinary Jeans instability condition, where K_J corresponds to the Compton wave number of the particle $K_J = \lambda\!\!\!^{-}_C{}^{-1}$. In astrophysics such instability is thought to be responsible for the collapse of interstellar gas clouds eventually leading to star formation.

In the stable regime, $K > \lambda\!\!\!^{-}_C{}^{-1}$, and for $K \ll K_c$ Eq. (13) yields the tachyonic dispersion relation $\mathcal{E} \approx \sqrt{p^2c_s^2 - m^2c_s^4}$, with real energy and momentum and imaginary rest mass. Excitations of any wave number in fact propagate at supersonic group velocities, with the invariant c_s being now a lower limit for propagation speeds. From now on we will focus on the fully stable case of a photon fluid with defocusing nonlocal nonlinearity.

IV. INHOMOGENEOUS FLOWS AND MASSIVE KLEIN-GORDON EQUATION

For purely local interactions $\gamma = 0$, a formal equivalence can be established between phonons propagating on top of the photon fluid and the evolution of scalar fields in curved spacetime [36]. The equation of motion is typically derived by linearizing Eqs. (3) and (4) around a background state since phonons, i.e., the acoustic elementary excitations, are defined as the *first-order fluctuations* of the quantities describing the mean fluid flow: $\rho = \rho_0 + \epsilon\rho_1 + O(\epsilon^2)$, $\psi = \psi_0 + \epsilon\psi_1 + O(\epsilon^2)$. When the terms arising from quantum pressure are negligible, the phonon dynamics is fully described by a single second-order equation for the linearized velocity potential, which has the form of the Klein-Gordon equation for a massless scalar field

$$\Box\psi_1 \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu\, \psi_1) \tag{15}$$

propagating in a $(2+1)$-dimensional curved spacetime the geometry of which is described by the acoustic metric $g_{\mu\nu}$, with inverse $g^{\mu\nu}$ and determinant g:

$$g_{\mu\nu} = \left(\frac{\rho_0}{c_s}\right)^2 \begin{pmatrix} -\left(c_s^2 - v_0^2\right) & -\mathbf{v_0^T} \\ -\mathbf{v_0} & \mathbf{I} \end{pmatrix} \tag{16}$$

where $\mathbf{I}$ is the two-dimensional identity matrix.

The above scenario is deeply modified in the presence of both local and nonlocal nonlinearities. In this context, it is convenient to derive the acoustic metric directly from the nonlocal Bogoliubov–de Gennes equations (5) and (6).

To this end, we apply the operator $(\partial_T + i\frac{c}{2kn_0}\partial_S)(\frac{1}{\rho_0})$ to Eq. (5) and we obtain

$$\begin{aligned}\left(\partial_T + i\frac{c}{2kn_0}\partial_S\right)\frac{1}{\rho_0}\left(\partial_T - i\frac{c}{2kn_0}\partial_S\right)\varepsilon \\ = \frac{c_s^2}{\rho_0}\partial_S\varepsilon - i\gamma\left(\frac{\omega}{n_0}\partial_T + i\frac{c^2}{2n_0^3}\partial_S\right)\frac{1}{\rho_0}R*[\rho_0(\varepsilon+\varepsilon^*)].\end{aligned} \tag{17}$$

As in the local case, we remind the reader that the gravitational analogy holds in the phononic regime in which the dispersion relation takes the relativistic form with a limit propagation speed, i.e., for wave numbers $K \ll K_c$. The corresponding equation for the excitation field can thus be obtained by ignoring the higher-order spatial derivatives in Eq. (17), which indeed are responsible for the Lorentz-breaking K^4 terms in the dispersion relation. A close inspection of Eq. (17) suggests that such approximation corresponds to take the diffractionless limit $k \to \infty$ [44] or, equivalently, neglect the quantum pressure terms arising from the linearized hydrodynamic equations [36]. In this limit Eqs. (3) and (4) indeed reduce to the Navier-Stokes equations for a barotropic, irrotational, and inviscid fluid, in which the Lorentz symmetry associated to phonon dynamics is not explicitly broken [5].

Under this approximation and using the fact that the background density ρ_0 satisfies the continuity equation (3) with $\mathbf{v} = \mathbf{v_0}$, Eq. (17) can be rewritten as

$$\begin{aligned}\Box\varepsilon &= -i\gamma\frac{\omega}{n_0}(\partial_T + \nabla\cdot\mathbf{v_0})R*[\rho_0(\varepsilon+\varepsilon^*)] \\ &\quad + \gamma\frac{c^2}{2n_0^3}\nabla\cdot(\nabla - \nabla\ln\rho_0)R*[\rho_0(\varepsilon+\varepsilon^*)]\end{aligned} \tag{18}$$

where

$$\Box \equiv (\partial_T + \nabla\cdot\mathbf{v_0})\partial_T - \nabla\cdot\left(c_s^2\nabla\right) \tag{19}$$

is precisely the d'Alambertian operator associated with the acoustic metric $g_{\mu\nu}$.

In the purely local case $\gamma = 0$, we recover the usual Klein-Gordon equation for a massless particle on curved spacetime, here described by the complex field ε. For a spatially homogeneous background, with constant density ρ_0 and constant flow velocity $\mathbf{v_0} = \frac{c}{kn_0}\nabla\phi_0$, the operator $\partial_T = \partial_t + \mathbf{v_0}\cdot\nabla$ commutes with the convolution operation and using Eqs. (5) and (6) we find that the wave equation (18) becomes independent of ε^*:

$$\left(\partial_{TT}^2 - c_s^2\nabla^2\right)\varepsilon = c_s^2\frac{\gamma}{n_2}R(r)*\nabla^2\varepsilon. \tag{20}$$

It is immediate to verify that the Fourier transform of Eq. (20) leads to the dispersion law (7).

The complex fluctuations ε can be easily linked to the real density and phase perturbations though the relations $\rho_1 = \rho_0(\varepsilon + \varepsilon^*)$ and $\phi_1 = (i/2)(\varepsilon^* - \varepsilon)$. By means of these expressions and using the relation between the optical phase and velocity potential of the flow, $\psi_1 = (c/kn_0)\phi_1$, one can split Eq. (18) into the following system of wave equations:

$$\Box\psi_1 = -\gamma\frac{c^2}{n_0^3}(\partial_T + \nabla\cdot\mathbf{v_0})R * \rho_1, \tag{21}$$

$$\Box\left(\frac{\rho_1}{\rho_0}\right) = \gamma\frac{c^2}{n_0^3}\nabla\cdot(\nabla - \nabla\ln\rho_0)R * \rho_1. \tag{22}$$

For local fluids $\gamma = 0$ Eq. (21) reduces to the massless Klein-Gordon equation (15) for the velocity-potential perturbations ψ_1 and an equation of the same form is satisfied also by the relative density fluctuations ρ_1/ρ_0.

In the ideal case of infinite-range thermo-optical nonlinearity, $\sigma \to \infty$, the response function satisfies $\gamma\nabla^2 R(\mathbf{r}) = -(\alpha|\beta|/\kappa)\delta(\mathbf{r} - \mathbf{r}')$. Using this result and considering a nearly homogeneous background density [69], Eq. (22) takes the form of the massive Klein-Gordon equation in curved spacetime:

$$\Box\rho_1 + \Omega_0^2\rho_1 = 0. \tag{23}$$

In the more realistic case of finite-range thermo-optical nonlinearities, Eq. (23) remains basically valid for perturbations with wave numbers $1/\sigma \ll K \ll K_c$.

V. EMERGENT GRAVITATIONAL DYNAMICS

In the previous sections we have seen that phonon excitations in our system behave as massive particles with a relativistic energy-momentum relation. In the presence of inhomogeneous flows we also derived a massive Klein-Gordon equation on the acoustic metric for the density fluctuations that thus reproduce the evolution of massive scalar fields on a curved spacetime. The spacetime curvature which mimics the gravitational field arises from the inhomogeneity of the background, the dynamics of which, however, is governed by a nonrelativistic nonlinear equation [see Eq. (1) or, equivalently, Eqs. (3) and (4)]. As such the analogy works only at the *kinematical* level: the fluctuations propagate in a given background solution associated to a specific spacetime configuration. All effects due to gravitational backreaction are neglected, i.e., the spacetime geometry is not modified by the perturbations propagating on it. While under certain conditions it is possible to extend the analogy and include in a geometric framework even the evolution of the background [70–72], there is no possibility in general to describe the *dynamics* of the acoustic metric in terms of something similar to Einstein's equations. The situation changes if one considers relativistic models, and interesting progresses in this direction have been made, e.g., in the framework of relativistic BECs [73].

Nevertheless, as mentioned before a kind of gravitational dynamics may emerge even in nonrelativistic BECs upon suitable modifications of the standard equations to break the U(1) symmetry associated with the conservation of particle number [47]. In such a modified model, the massive excitations feel a Newtonian gravitational potential the source of which is related to the excitation density.

In the following we show that a similar scenario arises also in our nonlocal photon fluid.

A. Newtonian limit

In Sec. IV we treated the general case of an inhomogeneous background, i.e., of an arbitrary curved spacetime simulating a generic gravitational field. Since the background dynamics is nonrelativistic we expect to find at most a kind of Newtonian gravity, as previously shown in other nonrelativistic frameworks [47]. We thus focus on a nearly homogeneous background corresponding to a weak gravitational field because, in analogy to the weak-field approximation of general relativity (GR), it is in this limit that a Newtonian-like gravity is expected to emerge.

In GR the weakness of the gravitational field allows for the decomposition of the metric into a flat Minkowski spacetime, $\eta_{\mu\nu}$, plus a small perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$ where $h_{\mu\nu}$ represents the weak deviations from flatness $g_{\mu\nu}$, i.e., the gravitational field. In this regime the Newtonian potential is related to the metric though the equation $g_{00} = \eta_{00} + h_{00} \simeq -(1 + 2\Phi_N/c^2)$, which follows from a nonrelativistic limit of the geodesics equation [74].

In analogy to the above, we consider a photon fluid with zero flow and a spatially localized inhomogeneity in the density, i.e., $E_0 = \rho_\infty^{1/2}[1 + u(r)]$ with $u \ll 1$ and $u \to 0$ at infinity. We thus assume that only a small region of the fluid deviates from the constant asymptotic value of the density ρ_∞. This implies a rescaling of the speed of sound $c_s^2 = c_\infty^2[1 + 2u(r)]$ and thus of the 00 component of the acoustic metric (16). On the basis of the hydrodynamic equations (3) and (4), a density inhomogeneity would also imply an inhomogeneous flow. However, as demonstrated in [47], the velocity perturbations do not contribute *at first order* to the anolog gravitational potential. In other words, at the first order all the information about the gravitational potential is encoded in the density perturbation. This result is general in acoustic models and does not depend on the specific fluid under consideration. Therefore, for the sake of simplicity and without loss of generality, here we assume deviations in the density only.

B. Nonrelativistic phonon dynamics

In order to show the emergence of a gravitational potential term we should derive the equation of motion for excitations in the nonrelativistic regime of the Bogoliubov spectrum, i.e., when $p \ll mc_\infty$, and for a homogeneous background except for the small density inhomogeneity $u(r)$.

To this end, we start directly from the nonlocal wave equation (18) for the complex excitation field ε. Setting $\rho_0 = \rho_\infty[1 + 2u(r)]$ and $\mathbf{v_0} = 0$ we get

$$\Box\varepsilon = c_\infty^2\frac{\gamma}{n_2}\nabla^2 R(r) * \{[1 + 2u(r)]\varepsilon\}. \tag{24}$$

In deriving (24) we disregarded terms containing the spatial derivatives of $u(r)$ and the products $u(r)\nabla^2\varepsilon$. The former are negligible in the asymptotic region and the latter are suppressed both by the smallness of u and by the fact that we are interested in the nonrelativistic regime $p \ll mc_\infty$. Restricting ourselves to the case of an infinite-range thermo-optical nonlinearity, Eq. (24) further simplifies and reads

$$\left(\partial_{tt}^2 - c_\infty^2 \nabla^2\right)\varepsilon + \Omega_\infty^2[1+2u(r)]\varepsilon = 0 \tag{25}$$

where we have defined the asymptotic rest frequency $\Omega_\infty = c_\infty\sqrt{\alpha|\beta|/\kappa n_2}$.

The nonrelativistic limit $p \ll mc_\infty$ (or, equivalently, $c_\infty \to \infty$) means that the kinetic energy of the particle should be small with respect to its mass energy $\hbar\Omega_\infty = mc_\infty^2$. Making the ansatz $\varepsilon = \varphi\exp(-i\Omega_\infty t)$ to factor out the rest frequency (i.e., the contribution to the total energy due to the rest energy of the particle) we can approximate [75]

$$\partial_{tt}^2\varepsilon \simeq \left(-2i\Omega_\infty\partial_t\varphi - \Omega_\infty^2\varphi\right)e^{-i\Omega_\infty t}.$$

Substituting the above expression into Eq. (25) we get the Schrödinger equation for a particle of mass m

$$i\hbar\partial_t\varphi = -\frac{\hbar^2}{2m}\nabla^2\varphi + mc_\infty^2 u(r)\varphi \tag{26}$$

subject to an external potential proportional to $u(r)$. The latter can be formally identified as a gravitational potential defining $\Phi_G = c_\infty^2 u(r)$. We finally remark that for finite-range thermo-optical nonlinearities Eq. (26) would remain approximately valid in the momentum range $p_{\rm nl} \ll p \ll mc_\infty$.

C. Modified Poisson equation

In the previous section we have found that the low-energy evolution of massive phonons obeys the Schrödinger equation for a nonrelativistic quantum particle in an external potential Φ_G. Our interpretation of Φ_G as a gravitational potential is based on the way it enters in the Schrödinger equation and because it is related to the 00 component of the acoustic metric, similarly to the Newtonian potential in the weak-field approximation of GR. In this framework we should find that in the appropriate limits Φ_G also obeys a kind of Poisson's equation [76].

Following the same argument of [47], since the Newtonian potential is the manifestation of small deviations of the order parameter E from perfect homogeneity, the corresponding Laplace equation should be encoded in the nonlinear evolution equation (1). On the other hand, we expect the source term to be directly related to the phononic fluctuations. Indeed in Newtonian gravity the only source for the gravitational field is a mass-density distribution and in our system this can originate only from the massive elementary excitations. As a result, the nonrelativistic massive phonons should experience a kind of gravitational potential generated by themselves, i.e., they should feel their own gravity.

This self-interaction can thus be derived from the nonlinear equation Eq. (1), but adding the corrections to the mean-field dynamics induced by the fluctuations, in order to see how the phonons backreact over the background fluid.

Backreaction effects can be calculated expanding Eq. (1) up to second order, $E = E_0 + \eta_1 + \eta_2 = E_B + \eta_1$: here η_1 and η_2 are linear and quadratic quantities in the fluctuation amplitude, respectively, and we introduced new variable E_B including the modifications to the zeroth-order dynamics, that is to say, the backreaction [77].

Substituting the above ansatz in Eq. (1) with $\Delta n = n_2|E|^2 + \gamma R * |E|^2$ we obtain

$$\partial_t(E_B+\eta_1) = \frac{ic}{2kn_0}\nabla^2(E_B+\eta_1) - \frac{i\omega}{n_0}(E_B+\eta_1)\Delta n_B \tag{27}$$

where $\Delta n_B = (n_2 + \gamma R*)[|E_B|^2 + 2\mathrm{Re}(E_B^*\eta_1) + |\eta_1|^2]$ and we used the effective time coordinate $t = (n_0/c)z$.

Since E_B consists only of zeroth-order and second-order terms in the fluctuation amplitude, all linear quantities in η_1 must vanish. The zeroing of the linear fluctuation terms in (27) leads to the Bogoliubov–de Gennes equations (5) and (6) upon substituting $\eta_1 = E_0\varepsilon$ and $E_B = E_0$. What remains is a nonlinear evolution equation for E_B in which the fluctuations appear quadratically:

$$\begin{aligned}\partial_t E_B = {} & \frac{ic}{2kn_0}\nabla^2 E_B - \frac{i\omega}{n_0}E_B[(n_2+\gamma R*)|E_B|^2 + 2n_2|\eta_1|^2] \\ & - \frac{i\omega}{n_0}\left\{n_2E_B^*\eta_1^2 + \gamma[E_BR*|\eta_1|^2 + 2\eta_1R*\mathrm{Re}(E_B\eta_1^*)]\right\}.\end{aligned} \tag{28}$$

Here, η_1 is in general time dependent, and for the purpose of simulating the effects of quantum and/or thermal fluctuations on the mean-field dynamics in analogy to real quantum gases it could be taken as a stochastic variable. However, for our purposes we are now interested in calculating the backreaction effects on stationary background solutions, i.e., stationary spacetime geometries. The corresponding mean-field equation is then obtained by time averaging Eq. (28) and replacing the fluctuating quantities with their mean values. For $\gamma = 0$, the averaged (28) closely resembles the modified Gross-Pitaevskii equation with beyond-mean-field corrections due to fluctuations [78–80]. In this context the quadratic terms $\mathfrak{n}(\mathbf{r}) = \langle|\eta_1|^2\rangle$ and $\mathfrak{m}(\mathbf{r}) = \langle\eta_1^2\rangle$ play the role of the density of noncondensed particles (i.e., the "out-of-condensate" photons) and of the anomalous density in the Bogoliubov-Popov-Beliaev approximation. The quantities $\gamma\langle\eta_1R*\mathrm{Re}(E_B^*\eta_1)\rangle$ and $\gamma R*\langle|\eta_1|^2\rangle$ provide further corrections due to nonlocality.

Setting $\langle E_B\rangle = \rho_\infty^{1/2}[1+u(r)]$ with $u \ll 1$ in the averaged Eq. (28) yields

$$\begin{aligned}&\frac{c}{2kn_0}\nabla^2 u - \frac{\omega}{n_0}\rho_\infty(n_2+\gamma R*)(1+2u) \\ &\quad = \frac{\omega}{n_0}\{n_2[2\mathfrak{n}(\mathbf{r}) + \mathfrak{m}(\mathbf{r})] + \gamma[R*\mathfrak{n}(\mathbf{r}) + 2\mathfrak{g}(\mathbf{r})]\}\end{aligned} \tag{29}$$

where $\mathfrak{g}(\mathbf{r}) = \langle\eta_1[R*\mathrm{Re}(\eta_1^*)]\rangle$.

In the ideal case of our interest of infinite-range thermo-optical nonlinearity, and applying the operator $c^2/(2kn_0)\nabla^2$ to both sides of Eq. (29), we get

$$\begin{aligned}&\frac{1}{K_c}\nabla^4\Phi_G - \nabla^2\Phi_G + \frac{1}{\lambda\!\!\!^{-}_C{}^2}\Phi_G + \frac{c_\infty^2}{2\lambda\!\!\!^{-}_C{}^2} \\ &\quad = \frac{c_\infty^2}{2\lambda\!\!\!^{-}_C{}^2\rho_\infty}\left\{\lambda\!\!\!^{-}_C{}^2\nabla^2[2\mathfrak{n}(\mathbf{r}) + \mathfrak{m}(\mathbf{r})] - \mathfrak{n}(\mathbf{r}) + 2\nabla^2\mathfrak{g}(\mathbf{r})\right\}\end{aligned} \tag{30}$$

where in the definition of the critical wave vector $K_c = 2\pi/\xi_\infty$ we used the asymptotic healing length $\xi_\infty = \lambda/2\sqrt{n_0n_2\rho_\infty}$.

Equation (30) can be interpreted as a modified fourth-order Poisson equation for the potential Φ_G, provided that we are able to identify the right-hand side with a genuine source term.

To this end, we remark that $\mathfrak{n}$, $\mathfrak{m}$, and $\mathfrak{g}$ are quadratic terms in η_1, i.e., they have the dimension of a photon-fluid mass density and are related to the phononic excitations through the relation $\eta_1 = E_0\varepsilon$. Therefore, they can be safely interpreted as a source of the gravitational field.

In contrast to Newtonian gravity, Eq. (30) contains the additional terms $\nabla^4\Phi_G/K_c^2$ and $\Phi_G/\lambda\!\!\!^{-}_C{}^2$. Owing to the K_c^2 coefficient, the first term suggests modifications of the Poisson equation which would become increasingly important as the analog of the Planck scale ξ_∞ is approached. Since in the nonrelativistic limit here considered we are dealing with long-wavelength modes with $K \ll K_c$, the variations of Φ_G over spatial scales of order ξ_∞ can be neglected.

The second term $\Phi_G/\lambda\!\!\!^{-}_C{}^2$ denotes a finite range for the gravitational interaction, with a characteristic length scale given by $\lambda\!\!\!^{-}_C$. Such a finite interaction scale for gravity would translate into a massive graviton with a mass that, in our model, corresponds to that of the massive phonons. A further comparison with the Newtonian limit of Einstein equations allows us to identify the quantity $1/2\lambda\!\!\!^{-}_C{}^2$, which indeed has the dimension of the square of an inverse length, with a cosmological constant Λ. Finally, in the right-hand side of Eq. (18) it is natural to define the analogy of the universal gravitational constant $G = \frac{c_\infty{}^2}{2\lambda\!\!\!^{-}_C{}^2\rho_\infty}$.

In light of the above considerations, Eq. (30) takes the more meaningful form

$$\nabla^2\Phi_G - \frac{1}{\lambda\!\!\!^{-}_C{}^2}\Phi_G = c_s^2\Lambda + G\varrho_{\text{matter}}(\mathbf{r}) \tag{31}$$

where we introduced the mass-density distribution

$$\varrho_{\text{matter}}(\mathbf{r}) = -\lambda\!\!\!^{-}_C{}^2\nabla^2[2\mathfrak{n}(\mathbf{r}) + \mathfrak{m}(\mathbf{r})] + \mathfrak{n}(\mathbf{r}) - 2\nabla^2\mathfrak{g}(\mathbf{r}). \tag{32}$$

The mass density (32) is a complicated function of quadratic fluctuation terms, which deserves further analysis. We first note that in the nonrelativistic limit, $\lambda\!\!\!^{-}_C{}^2K^2 \ll 1$, the first Laplacian term is less relevant with respect to the others and can thus be neglected. Moreover, applying the product rule to $\nabla^2\mathfrak{g}(\mathbf{r})$ we find

$$\begin{aligned}\nabla^2\mathfrak{g}(\mathbf{r}) = {} & \langle[R * \text{Re}(\eta_1^*)]\nabla^2\eta_1\rangle + 2\langle\nabla\eta_1 \cdot \nabla[R * \text{Re}(\eta_1^*)]\rangle \\ & - \tfrac{1}{2}[\mathfrak{m}(\mathbf{r}) + \mathfrak{n}(\mathbf{r})]\end{aligned} \tag{33}$$

where we used the previously defined quantities $\mathfrak{n}(\mathbf{r}) = \langle|\eta_1|^2\rangle$ and $\mathfrak{m}(\mathbf{r}) = \langle\eta_1^2\rangle$ and the relation $\nabla^2R(\mathbf{r}) = -\delta(\mathbf{r} - \mathbf{r}')$, valid in the limit of infinite-range nonlocality. While the first two terms in (33) are difficult to handle in general, we observe that their contribution becomes less important at very low wave vectors, in which case the mass-density distribution approximates

$$\varrho_{\text{matter}}(\mathbf{r}) \approx 2\mathfrak{n}(\mathbf{r}) + \mathfrak{m}(\mathbf{r}). \tag{34}$$

The above expression coincides with the density distribution obtained in [47] [see Eq. (43)] where a Newtonian-like gravity has been shown to emerge in a BEC model modified with a U(1) symmetry-breaking term. The distribution (34) has an immediate physical interpretation: in analogy to BECs, the two terms $\mathfrak{n}(\mathbf{r})$ and $\mathfrak{m}(\mathbf{r})$ indeed correspond to the so-called normal and anomalous density encoding the effects of "non-condensed particles" on the mean-field dynamics [78–80].

We finally remark that, similarly to the rest energy, also the Newton constant can be expressed solely in terms of the nonlocal coefficients, $G = (c^2/n_0^3)(\alpha|\beta|/\kappa)$. The nonlocal nonlinearity is thus responsible for the emergent gravitational interaction.

To conclude, Eqs. (26) and (31), mutually coupled via the relations $\eta_1 = E_0\varepsilon$ and (34), actually give rise to an effective Schrödinger-Newton dynamics describing the evolution of a quantum mass density experiencing its own gravitational field [66], although here the source is a more complicated function of the mass-density distribution and the gravitational interaction is characterized by a short interaction range and a nonzero cosmological constant.

VI. CONCLUSIONS AND FUTURE PERSPECTIVES

Quantum fluids of light such as exciton-polariton BECs and more recently photon fluids have offered alternative platforms for fundamental studies of quantum many-body physics. Recent experiments in these systems provided evidence of collective many-photon phenomena, such as the emergence of a phonon regime in the Bogoliubov dispersion [34,35], superfluidity and nucleation of quantized vortices in the flow past a physical obstacle [30,31], and classical wave condensation [32].

Here, we have theoretically investigated a photon fluid with both local and nonlocal interactions from the analog gravity perspective. We have found that collective excitations in this system display a gapped Bogoliubov spectrum which at low energies corresponds to that of massive phonons with a relativistic energy-momentum relation. In the presence of an inhomogeneous flow the dynamics of the density fluctuations is equivalent to that of a massive scalar field propagating in a curved spacetime the geometry of which is specified by the acoustic metric. This generalizes previous studies in local fluids to the case of massive phonons and provides a quite natural setting for analog simulations of quantum gravity phenomenology.

The massive nature of the elementary excitations allows us to study their nonrelativistic dynamics in a nearly homogeneous background that, as explained, corresponds to the case of a weak gravitational field. In this limit we find that the phonon modes behave as a *self-gravitating* quantum system. The evolution equations are indeed the Schrödinger equation for a massive quantum particle, including a term that represents the interaction of the particle with its own gravitational field, and a kind of Poisson equation with a source depending on the phononic mass-density distribution. In analogy to the Newtonian limit of GR, the potential in the Poisson equation is related to the background geometry (namely, to the 00 component of the metric) experienced by the particles propagating on it. Since most analog models are dealing with massless excitations that in the framework of Newtonian gravity cannot act as sources of a gravitational field, our system is one of the very few in which a form of semiclassical gravitational dynamics can emerge.

One of the next stages of this investigation will focus on the design of realistic experimental schemes for the implementation of such photon fluids. Apart from the analog-gravity side, we expect these experiments to be interesting also from the perspective of the quantum fluids of light, as the interplay between local and nonlocal nonlinearities with different—possibly tunable—kernels could unveil new collective many-photon phenomena and hydrodynamic phase transitions.

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[76] We remark that the weak-field limit of GR in $(2+1)$ dimensions (i.e., the dimensionality of our system) does not correspond to 2D Newtonian gravity. In two space dimensions all vacuum solutions of Einstein's equations are flat, and even in the presence of matter the spacetime curvature remains null everywhere except in the interior of the matter distribution. Similarly to other analog gravity models, such a restriction does not apply to our system, the spacetime metrics of which are not solutions of the Einstein field equations. As a consequence, the Newtonian-like dynamics here discussed should thus be hypothetically considered as the limit of a theory that strongly differs from Einstein's gravity.

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