Atom-photon bound states and non-Markovian cooperative dynamics in coupled-resonator waveguides

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We study the properties of atom-photon bound states and single-photon cooperative dynamics in a waveguide system which consists of a finite-bandwidth channel with model dispersion and an ensemble of two-level atoms whose size is ignorable when compared with the lattice constant. The bound states are formed by all atoms and a localized photonic excitation. We find that the effect of atomic collection is equivalent to the case of one atom by rescaling the coupling strength with the square root of the atom number, as far as the eigenenergy equation is concerned. Besides, it is found there is a quantum phase transition when more than one type of atom are present. The characteristic lengths and wave functions are analyzed near the phase transition point. The exact analytical results for the cooperative dynamics at the single excitation level are obtained and we point out the dark state in this system leads to a universal population trapping in the time evolution process. This type of trapping obeys a simple law that is only associated with the atom number. A direct conclusion that results from the trapping law is that the single-photon cooperative emission is suppressed when the number of atoms is large enough.

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I. INTRODUCTION

With the experimental progress in quantum communication, quantum computation, and quantum information processes, there has been a growing interest in the study of light-matter interaction in waveguiding structures in recent years [1–7]. Because of the dimensional reduction and the confined modal dispersion, the photon-qubit interaction can be significantly increased while the system's dissipation can be decreased in a waveguide geometry [8,9]. Strong photon-atom coupling has been achieved by restricting the photons in a high-quality microcavity in the past decade [10,11]. Recently, the systems consisting of quantum emitters and one-dimensional waveguides such as optical fibers, microwave transmission lines, and coupling-cavity arrays have been widely studied [12–25] and many quantum devices have been proposed based on the rich physics in these systems, such as the controllable light quantum switches [2,14], photon memory device [13], controllable single-photon routers [18,19], and frequency converters [20].

The inhibition of electromagnetic wave propagation in waveguides provides us a way to control the time evolution of excited atoms. For that the coupled-resonator waveguide (CRW) with good scalability and integrability has become available [26–28], the interest in this type of waveguide systems has been growing. This system provides an appropriate platform for photon manipulation and storage, as well as the quantum simulation of many-body systems. The nearest-neighbor coupling structure gives rise to a finite-energy band. In a one-dimensional CRW coupled by a two-level quantum emitter (atom for brevity), which is known as one of the general Fano-Anderson models, it has been demonstrated that there are localized photon states around the emitter whose energies go beyond the band [29,30]. These localized bound states lead to the fractional trapping of incident photons in the scattering process [31–34] and also result from the spontaneous emission of atom experiences from exponential decay to Rabi oscillations as the coupling strength goes from weak to strong [35,36]. Besides, in a two-dimensional coupled-resonator system, exotic quantum dynamics from both individual and collective behavior emerges, such as exponential relaxation followed by overdamped oscillations, directional spontaneous emission, and remarkable superradiant and subradiant behavior [37,38]. The cooperative dynamics of atoms with one excitation were early investigated in a cavity [39–42] and in free space [43]. It has been re-explored in free space with spatial anisotropy of the emitted radiation in recent theoretical and experimental studies [44–51].

In this paper, we study the CRW, which is coupled to an ensemble of two-level atoms. The size of the atomic ensemble is much smaller than the resonator lattice constant. The atom-photon bound states in this system are analyzed in the presence of both one and two types of atoms based on the time-independent Schrödinger equation. Specifically, the eigenvalue equation, position population, and characteristic length are calculated. As far as the eigenvalue equation is concerned, we find the effect of a collection of identical atoms with atomic number M is equivalent to that of one atom by rescaling the coupling strength V with $\sqrt{M}$ times. Unlike the case of one type of atom where there are always two bound states, there is a change of the energy level structure in the presence of two types of atoms when the coupling strengths and relative positions of the atom's transition frequencies from

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the band become different, which reveals there is a quantum phase transition in this process. We also use a nonperturbative method to analyze the single-photon cooperative dynamics of the system and found that the atomic population in the excited state can exhibit a nonzero decay caused by system's dark states. Unlike the population trapping regime caused by atom-photon bound states [52,53], this type of trapping takes place only when more than one atom is present.

The paper is organized as follows. In Sec. II, we introduce the model of coupled-resonator waveguide coupled by an atomic ensemble. In Sec. III, we calculate the system's atom-photon bound states and single-photon cooperative dynamics in the presence of one type of atom. In Sec. IV, the bound states and dynamics are analyzed in the case of two types of atoms. Finally, we summarize the results and give the conclusions in Sec. IV.

II. MODEL

The system we consider consists of a coupled-resonator waveguide with different types of two-level atoms coupled at site $x = x_0$. The jth atom in type i is characterized by ground state $|g_j^i\rangle$ and excited state $|e_j^i\rangle$. Denote a_x ($a_x^\dagger$) as the bosonic annihilation (creation) operator for the cavity field at position x, and then the Hamiltonian of the system can be written as (with $\hbar = 1$) [2]

$$\begin{aligned} H = &\sum_x \omega a_x^\dagger a_x - \sum_x J(a_{x+1}^\dagger a_x + a_x^\dagger a_{x+1}) \\ &+ \sum_i \sum_j \Omega_i |e_j^i\rangle\langle e_j^i| \\ &+ \sum_i \sum_j V_i(\sigma_j^{i+} a_{x_0} + \sigma_j^{i-} a_{x_0}^\dagger), \end{aligned} \tag{1}$$

where the first two terms describe the waveguide field and ω is the on-site energy of each resonator. J is the hopping energy of photon between two neighbor lattices. The third term describes the two-level atoms. Here, we set the ground energy of atoms to be zero as reference and the resonant transition frequency of type-i atoms are represented by Ω_i. The last term is the interaction between field and atoms. $\sigma_j^{i+} = |e_j^i\rangle\langle g_j^i|$ ($\sigma_j^{i-} = |g_j^i\rangle\langle e_j^i|$) is the raising (lowering) operator acting onto the jth atom and V_i is the coupling strength between field modes and type-i atoms. Here, l is the lattice constant so that the system's total length is $L = Nl$. For simplicity, we assume l to be unity in the following. Such coupled-resonator devices are now realizable [54–57]. In these experiments, typical values for the coupling strength V_i and hopping energy J go up to a few hundred megahertz. The photon dissipative rate γ_c and atomic dissipative rate γ_a are in the kilohertz regime and are thus very small in comparison with V_i, J, Ω_i, and ω [57], so we neglect the system's dissipation.

The first line of Eq. (1) represents the free photon Hamiltonian which can be diagonalized by imposing periodic boundary conditions. We introduce the Fourier transform

$$a_k = \frac{1}{\sqrt{N}} \sum_x e^{-ikx} a_x, \tag{2}$$

where k is the wave number within the first Brillouin zone and $k \in [-\pi, \pi]$ by assuming $N \to \infty$, the free photon Hamiltonian is written as $\sum_k \omega_k a_k^\dagger a_k$ with the dispersion relation

$$\omega_k = \omega - 2J\cos(k). \tag{3}$$

This mode frequency captures the feature of the energy band with a finite bandwidth which is centered at the on-site energy ω, i.e., $\omega_k \in [\omega - 2J, \omega + 2J]$. Close to the band edges $\omega \pm 2J$, ω_k can be approximately expanded as the quadratic relation. For example, $\omega_k \approx \omega - (2J - Jk^2)$ for the lower edge when $|k| << 1$. In the middle, ω_k can be written as the linear relation, for example, for $|k - \pi/2| << 1$, $\omega_k \approx \omega + 2J(k - \pi/2)$. Photons propagate in the waveguide with group velocity $v_g(k) = 2J\sin(k)$, which gets to its maximum value at the middle of band and becomes zero at the band edges.

In the momentum space, the Hamiltonian in Eq. (1) can be rewritten as

$$H = H_0 + H_I \tag{4}$$

with

$$H_0 = \sum_k \omega_k a_k^\dagger a_k + \sum_i \sum_j \Omega_i |e_j^i\rangle\langle e_j^i| \tag{5}$$

and

$$H_I = \sum_i \sum_{j,k} \frac{V_i}{\sqrt{N}} (\sigma_j^{i+} a_k + \sigma_j^{i-} a_k^\dagger). \tag{6}$$

Here, for simplicity, we take the site x_0 as zero point. The finite-width energy band allows one to explore the situation in which the transition frequencies of atoms are in or out of the band. In this paper, the physics we investigate are restricted to the single excitation subspace.

III. CASE OF ONE TYPE OF ATOMS

We first consider the situation in which there is one type of atoms. According to the Fano-Anderson model, it is known that the system has two types of excitation spectra: (1) continuous spectra corresponding to the scattering states whose energies lie inside the finite band and (2) discrete excitation spectra corresponding to the atom-field dressed states whose energies lie outside the band. In this section, we investigate the discrete atom-field dressed states and dynamics when one type of atoms whose number is M are present.

A. Atom-photon bound states

Since the Hamiltonian in Eq. (4) commutes with the total excitation number of photons and excited atoms $N_e = \sum_k a_k^\dagger a_k + \sum_j |e_j\rangle\langle e_j|$ (here the atom-type index i is omitted), the eigenstate of H can be discussed separately within the subspace of each given excitation number. For the single excitation case, i.e., $N_e = 1$, the eigenstate can be written as

$$|\phi\rangle = \sum_j \alpha_j |e_j, 0\rangle + \sum_k \beta_k a_k^\dagger |g, 0\rangle, \tag{7}$$

where the state $|g, 0\rangle$ indicates that there is no photon in the waveguide and all the atoms are in their ground states while $|e_j, 0\rangle$ means that jth atom stays in its excited state, others stay in ground states, and no photon exists. By taking this

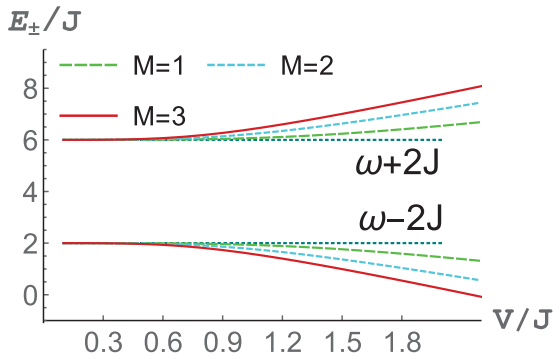

FIG. 1. Spectrum of atom-photon bound states as a function of the atom-photon coupling strength V with different atom number M. The on-site photon energy $\omega/J = 4$, and the atom's transition frequency $\Omega/J = 4$.

ansatz into the Schrödinger equation $H|\phi\rangle = E|\phi\rangle$, one can obtain the equations for the amplitude coefficients α_j and β_k

$$\Omega\alpha_j + \frac{V}{\sqrt{N}}\sum_k \beta_k = E\alpha_j, \tag{8}$$

$$\omega_k\beta_k + \sum_j \frac{V}{\sqrt{N}}\alpha_j = E\beta_k. \tag{9}$$

By using the second equation to eliminate β_k in the first one, we acquire the energy equation for atom-photon bound state

$$E - \Omega - \frac{MV^2}{(E-\omega)\sqrt{1-\frac{4J^2}{(E-\omega)^2}}} = 0. \tag{10}$$

Here, the condition $|E-\omega| > 2J$ for the atom-photon bound states is used in the calculation [35]. The result shows that there are only two bound states in this system. One's energy E_+ is above the scattering band $[\omega-2J,\ \omega+2J]$, and the other's E_- is below it. In Fig. 1, we plot the coupling strength V dependence of bound-state energy $E_\pm$ with different atom number M by numerically solving Eq. (10). It is shown that as M or V increases, the E_+ and E_- move away from the scattering band. Besides, from Eq. (10), it can be seen that the effect of M atoms is equivalent to that of one atom by rescaling the coupling strength V with $\sqrt{M}$ times. However, this relation is not satisfied in the wave functions (see the following).

If we take E_+ and E_- into Eqs. (8) and (9), then the corresponding bound-state wave functions can be derived as

$$|\phi^\pm\rangle = \alpha(E_\pm)\left(\sum_j \sigma_j^+ + \sum_k \frac{\frac{MV}{\sqrt{N}}}{E_\pm-\omega_k}a_k^\dagger\right)|g,0\rangle, \tag{11}$$

where $\alpha(E_\pm) \equiv \{M + V^2M^2/[(E_\pm-\omega)^2(1-4J^2/(E_\pm-\omega)^2)^{3/2}]\}^{-1/2}$ is the normalized factor. With the Fourier transform $a_k^\dagger = (1/\sqrt{N})\sum_x e^{ikx}a_x^\dagger$, the expressions of the two wave functions are transformed into the positional space

$$|\phi^\pm\rangle = \sum_x \frac{\alpha(E_\pm)MV\varphi(E_\pm)e^{-|x|/\lambda^\pm}}{(E_\pm-\omega)\sqrt{1-\left(\frac{2J}{E_\pm-\omega}\right)^2}}|g,x\rangle + \sum_j \alpha(E_\pm)|e_j,0\rangle, \tag{12}$$

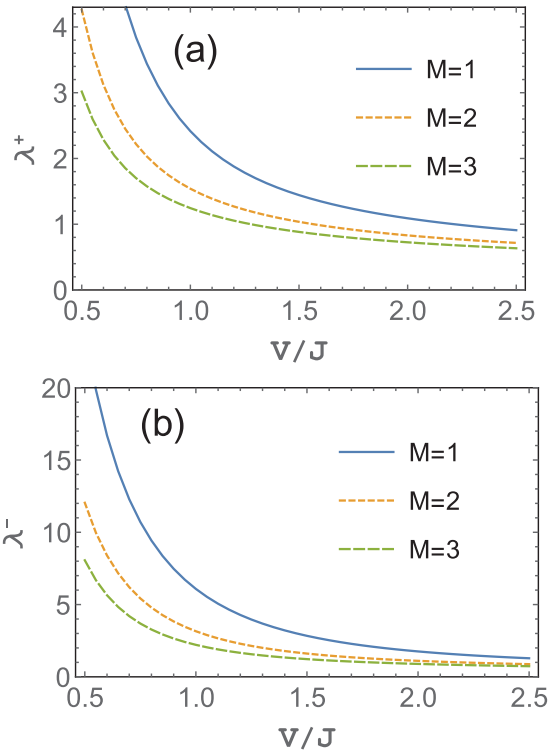

FIG. 2. Coupling strength dependence of the lengths λ^+ (a) and λ^- (b) with different numbers of atoms. Other parameters are $\Omega/J = 4$, $\omega/J = 3$.

where $\varphi(E_\pm) = (-1)^{|x|\theta(E_\pm-\omega)}$ and $\theta(y)$ is the step function which is one when $y > 0$ and is zero when $y < 0$. The amplitudes of the wave functions in positional space drop exponentially as $e^{-|x|/\lambda^\pm}$ when the distance from the atoms increases. The size of the localized amplitude is reflected by the length $\lambda^\pm = -1/\log(\{1-\sqrt{1-[2J/(E_\pm-\omega)]^2}\}|E_\pm-\omega|/2J)$. As the energy difference $|E_\pm-\omega|$ decreases, the length $\lambda^\pm$ increases and the amplitudes become more nonlocalized. When the value of $|E_\pm-\omega|$ decreases to $2J$, $\lambda^\pm$ tends to infinity and the wave functions are no longer localized. In Fig. 2, we plot the length $\lambda^\pm$ as a function of coupling strength V with different atom number M. It can be seen that as V decreases, both λ^+ and λ^- become large. Further research shows that λ^+ and λ^- will be infinite when V tends to zero. Comparing with the case of one atom, the localization of the bound states is enhanced due to the presence of more than one atom. Besides, as long as the frequency Ω is greater than the band's center ω and no matter how many atoms are present, $|\phi^+\rangle$ will be more localized than $|\phi^-\rangle$ and vice versa. In Fig. 3, we show the probability distribution of bound states as a function of position with different detuning $\delta = \Omega-\omega$ and atom number M. In the case of resonance, $|\langle x|\phi^+\rangle|$ and $|\langle x|\phi^-\rangle|$ have the symmetric distribution. As the detuning δ becomes large, $|\phi^+\rangle$ gets more localized and $|\phi^-\rangle$ gets more nonlocalized, while both $|\phi^+\rangle$ and $|\phi^-\rangle$ get more localized when M increases. Furthermore, the total position probability $\sum_x |\langle x|\phi^+\rangle|^2$ will become small and the total probability on excited atoms $M|\alpha(E_+)|^2$ will increase with δ as increases, which is contrary to the case of $|\phi^-\rangle$.

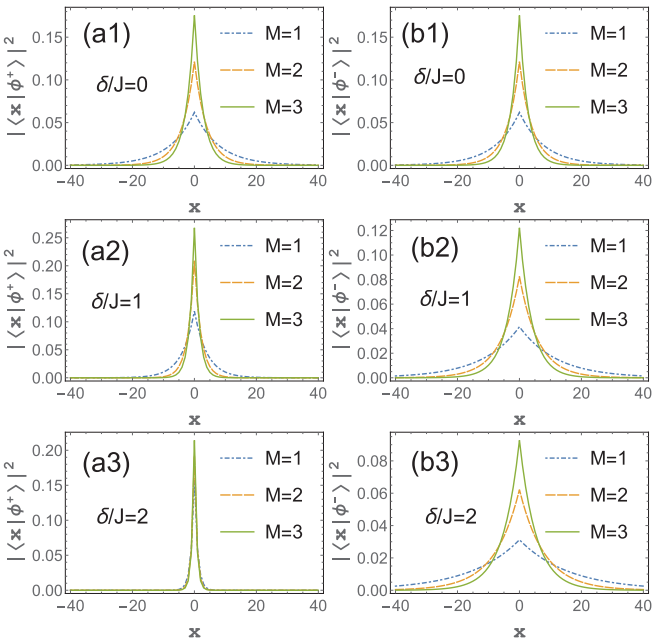

FIG. 3. Position populations of wave functions $|\phi^+\rangle$ for (a1), (a2), (a3) and $|\phi^-\rangle$ for (b1), (b2), (b3). The coupling strength $V/J = 0.5$.

B. Universal trapping induced by dark states

To investigate the dynamics of the system when one of the atoms is initially excited and the radiation field is in the vacuum state, we start from the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{13}$$

The wave function $|\psi(t)\rangle$ of the system at time t has the form in single excitation subspace $|\psi(t)\rangle = \sum_j A_j(t)|e_j, 0\rangle + \sum_k C_k(t)a_k^\dagger|g, 0\rangle$, where $A_j(t)$ $(j = 1, 2, \ldots, M)$ and $C_k(t)$ are the probability amplitudes for $|e_j, 0\rangle$ and $a_k^\dagger|g, 0\rangle$, respectively. Taking it into the Schrödinger equation, we obtain the relations about $A_j(t)$ and $C_k(t)$:

$$i\frac{\partial A_j(t)}{\partial t} = \Omega A_j(t) + \frac{V}{\sqrt{N}}\sum_k C_k(t), \tag{14}$$

$$i\frac{\partial C_k(t)}{\partial t} = \omega_k C_k(t) + \frac{V}{\sqrt{N}}\sum_j A_j(t). \tag{15}$$

To go beyond the Wigner-Weisskopf approximation and Markovian perturbation theory, which leads to the result that the atomic population reveals exponential decay or the population decay is complete [38], we take a Laplace transform of Eqs. (14) and (15). It gives

$$i[-A_j(0) + s\bar{A}_j(s)] = \Omega\bar{A}_j(s) + \frac{V}{\sqrt{N}}\sum_k \bar{C}_k(s), \tag{16}$$

$$i[-C_k(0) + s\bar{C}_k(s)] = \omega_k\bar{C}_k(s) + \frac{V}{\sqrt{N}}\sum_j \bar{A}_j(s). \tag{17}$$

By denoting the initial excited atom as j_0, i.e., with the initial amplitudes $A_{j_0}(0) = 1$, $A_j(0) = 0$ $(j \neq j_0)$, and $C_k(0) = 0$, we derive the expression of $\bar{A}_{j_0}(s)$

$$\bar{A}_{j_0}(s) = i\frac{is - \Omega - (M-1)V^2 f(s)}{(is-\Omega)[is - \Omega - MV^2 f(s)]}, \tag{18}$$

where $f(s) \equiv (1/N)\sum_k 1/(is - \omega_k)$. The amplitude on the excited atom is given by the inverse Laplace transform $A_{j_0}(t) = (1/2\pi i)\int_{\sigma-i\infty}^{\sigma+i\infty}\bar{A}_{j_0}(s)e^{st}ds$. To evaluate this integral, we consider the analytic behavior of $\bar{A}_{j_0}(s)$ in the whole complex plane except a branch cut along the imaginary axis from $-i(2J+\omega)$ to $i(2J-\omega)$. With the residue theorem, we obtain the final result after some complex but exact calculations

$$A_{j_0}(t) = \frac{M-1}{M}e^{-i\Omega t} + \sum_m \frac{e^{st}}{M[G(s)]'}|_{s=x_m^{(1)}} + \frac{J}{\pi}\int_{-1}^{1}\frac{4J\sqrt{1-y^2}V^2e^{i2Jyt}}{K(y)+M^2V^4}dy, \tag{19}$$

where $G(s) \equiv s + i\Omega + iMV^2 f(s)$ and $K(y) \equiv 4J^2(1-y^2)(2Jy+\Omega)^2$. Here, $[G(s)]'$ means the derivative of $G(s)$ with respect to s. $x_m^{(1)}$ is the roots of the equation $G(s) = 0$. Compared with Eq. (10), it can be seen that $G(-iE) = 0$ is nothing but the energy equation of atom-photon bound state. In fact, the equation $G(s) = 0$ only has pure imaginary roots whose imaginary parts correspond to the system's eigenenergies. According to the analysis based on Green's function method [35,58], it is known that the second term in Eq. (19) comes from the contribution of system's atom-photon bound states with photonic excitation, and the third term which vanishes when time tends to infinity comes from the contribution of system's scattering states. When only one atom is present, $A_{j_0}(\infty) = \sum_m e^{st}/\{M[G(s)]'\}|_{s=x_m^{(1)}}$. These pure imaginary roots in equation $G(s) = 0$ show that the population on the excited atom is fractionally trapped when time goes long enough. This type of trapping caused by atom-photon dressed states was pointed out in photonic crystal systems with quadratic dispersion relation [52,53]. In Fig. 4, we plot the time evolution of $|A_{j_0}(t)|$ with $M = 1$. It is seen that more of the population on the excited atom is trapped as the resonance frequency Ω is far away from the center of energy band when time is long enough. Besides, it

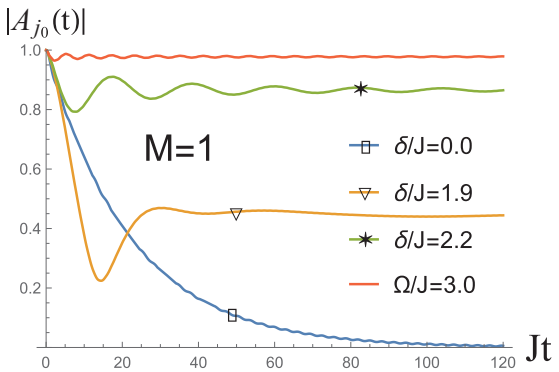

FIG. 4. Time evolution of the population on the excited atom with $M = 1$. The time is in units of $1/J$. The coupling strength $V/J = 0.3$.

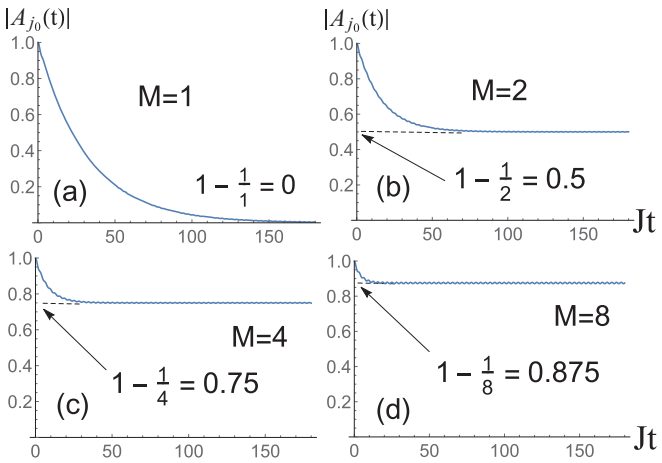

FIG. 5. Time evolution of the population on the excited atom with different atom numbers. The coupling strength $V/J = 0.25$.

can be seen that when Ω lies in the vicinity of band center, the decay is nearly complete.

Now we analyze the case of more than one atom when the system's parameters satisfy the condition that the value of $1/\{M[G(s)]'\}|_{s=x_m^{(1)}}$ is small enough that the second term in Eq. (19) can be ignored. The final result of the amplitude $A_{j_0}(t)$ in this case is

$$|A_{j_0}(\infty)| = 1 - \frac{1}{M}, \tag{20}$$

which is only related with the number of atoms. Unlike the atom-photon bound state trapping regime, here we point out this type of trapping is caused by the system's dark state in which all the excitation numbers focus on the atoms and the populations of field modes are zero, which also satisfies the time-independent Schrödinger equation with eigenenergy $E = \Omega$ [59,60]. This type of trapping takes place when more than one atom is present. It is associated with the collective coherence of atomic clouds. In Fig. 5, we show the time evolution of $|A_{j_0}(t)|$ with different number of atoms when $\delta = \Omega - \omega = 0$. One sees that $|A_{j_0}(\infty)|$ tends to the value $1 - 1/M$ as time goes infinite.

IV. CASE OF TWO TYPES OF ATOMS

We now study the atom-field dressed states and dynamics of the waveguide system when two types of atoms are present. Unlike the case of one type of atoms in which there are always two atom-field bound states whose energies are above the top and below the bottom of the scattering energy band, the case for two types of atoms is different. In this section, we will show that how the quantum phase transition happens when the coupling strengths V_A and V_B of the two types of atoms change with different atom numbers M_A and M_B.

A. Bound states and quantum phase transition

In single excitation subspace, the system's eigenstate is assumed to be

$$|\tilde{\phi}\rangle = \sum_i \sum_j \alpha_j^i |e_j^i, 0\rangle + \sum_k \tilde{\beta}_k a_k^\dagger |g, 0\rangle, \tag{21}$$

where α_j^i $(i = A, B)$ is the probability amplitude for the jth atom in type i to be in an excited state with no photons in the waveguide field and $\tilde{\beta}_k$ is the probability amplitude for all atoms to be in their ground state with a photon having frequency ω_k. By taking $|\tilde{\phi}\rangle$ into the Schrödinger equation $H|\tilde{\phi}\rangle = E|\tilde{\phi}\rangle$, we acquire

$$\Omega_i \alpha_j^i + \frac{V_i}{\sqrt{N}} \sum_k \tilde{\beta}_k = E\alpha_j^i, \tag{22}$$

$$\omega_k \tilde{\beta}_k + \sum_i \sum_j \frac{V_i}{\sqrt{N}} \alpha_j^i = E\tilde{\beta}_k. \tag{23}$$

With these equations, the system's eigenenergy equation is obtained

$$(E - \Omega_A)(E - \Omega_B) = \frac{M_A V_A^2 (E - \Omega_B) + M_B V_B^2 (E - \Omega_A)}{(E - \omega)\sqrt{1 - \frac{4J^2}{(E-\omega)^2}}}. \tag{24}$$

It can be seen that the effect of M_i atoms is equivalent to that of one atom in type i by rescaling the coupling strength V_i with $\sqrt{M_i}$ times. The number of roots in Eq. (24) depends on the relative positions between Ω_i and scattering energy band. Without loss of generality, we assumed $\Omega_A > \Omega_B$ in the following. First, when the two resonance frequencies Ω_A and Ω_B lie outside the energy band and are at the same side, i.e., Ω_A, $\Omega_B < \omega - 2J$ or Ω_A, $\Omega_B > \omega + 2J$, Eq. (24) has three roots. One is above the band and two are below, which corresponds to the case Ω_A, $\Omega_B < \omega - 2J$, while two are above the band and one is below, which corresponds to the case Ω_A, $\Omega_B > \omega + 2J$. However, if Ω_A, Ω_B lie outside the band and are at the different sides, i.e., $\Omega_B < \omega - 2J$ and $\Omega_A > \omega + 2J$, there will be two or three roots in the energy equation. It depends on the value of $Y(M_A, M_B) \equiv (M_A V_A^2 \Omega_B + M_B V_B^2 \Omega_A)/(M_A V_A^2 + M_B V_B^2)$. When $|Y(M_A, M_B) - \omega| < 2J$, two roots exist in Eq. (24), or else there will be three roots. Second, if one of the frequencies Ω_A, Ω_B lies inside the band and the other lies outside, for example, $\Omega_B \in [\omega - 2J, \omega + 2J]$ and $\Omega_A > \omega + 2J$, there are also two different cases. When $Y(M_A, M_B) < \omega + 2J$, two roots exist; otherwise, the equation has three roots. Finally, we analyze the situation in which both Ω_A and Ω_B are inside the band. In this case, because the values of $Y(M_A, M_B)$ are always in the range of scattering band, only two roots exist.

In Fig. 6(a), we plot the system's eigenvalues as a function of the coupling strength V_B with different atom number M_B in the case where Ω_B is in the range $[\omega - 2J, \omega + 2J]$ and Ω_A is out. It can be seen that as the coupling strength V_B varies from small to large, E_1 (E_3) moves away from the top (bottom) of the band from the beginning, but E_2 appears only when V_B is greater than a fixed value which is determined by the equation $Y(M_A, M_B) = \omega + 2J$. This means that the number of atom-photon bound states of the system changes from two to three in this process. The change in the energy level structure reveals that there is a quantum phase transition [61]. In Fig. 6(b), we plot the system's eigenvalues as a function of the atom number M_A. One can see that as M_A increases, E_2 gets closer to the top of the band and disappears when M_A is greater than four while E_1 and E_3 get away from the band.

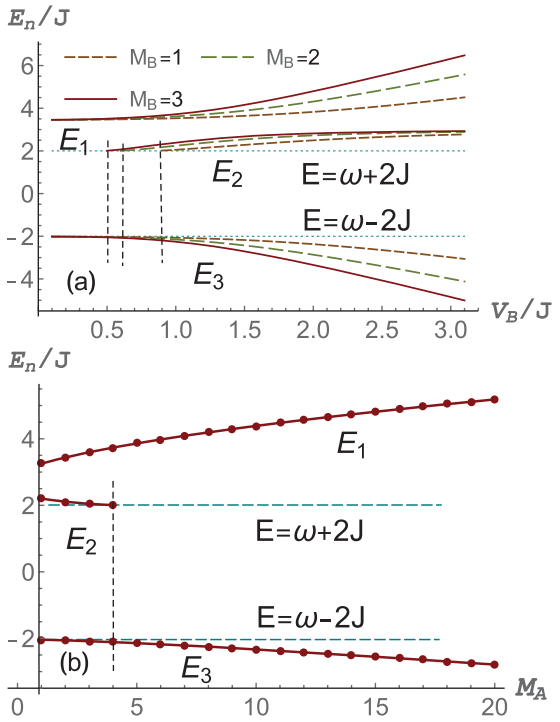

FIG. 6. (a) Energies of atom-photon bound states as a function of the coupling strength V_B with different atom number M_B, $V_A/J = 0.8$, and $M_A = 2$. (b) Energies as a function of the atom number M_A, $V_A/J = 0.7$, $V_B/J = 0.6$, and $M_B = 3$. Other parameters are $\Omega_A/J = 3$, $\Omega_B/J = 1.5$, and $\omega/J = 0$.

Take the eigenvalues E_n into Eqs. (22) and (23), the corresponding bound-state wave functions can be obtained as

$$|\tilde{\phi}_n\rangle = \alpha_n^A \sum_j |e_j^A, 0\rangle + \alpha_n^B \sum_j |e_j^B, 0\rangle + \frac{1}{\sqrt{N}} \sum_k \frac{M_A V_A \alpha_n^A + M_B V_B \alpha_n^B}{E_n - \omega_k} a_k^\dagger |g, 0\rangle, \quad (25)$$

where $\alpha_n^A = \{M_A + M_B V_B^2 (E_n - \Omega_A)^2/[V_A^2 (E_n - \Omega_B)^2] + [M_A V_A + M_B V_B^2 (E_n - \Omega_A)/(V_A E_n - V_A \Omega_B)]^2 / [(E_n - \omega)^2 (1 - 4J^2/(E_n - \omega)^2)^{3/2}]\}^{-1/2}$ and $\alpha_n^B = \alpha_n^A V_B (E_n - \Omega_A)/[V_A (E_n - \Omega_B)]$. With the Fourier transform $a_k^\dagger = (1/\sqrt{N}) \sum_x e^{ikx} a_x^\dagger$, the wave functions are transformed into the positional space

$$|\tilde{\phi}_n\rangle = \alpha_n^A \sum_j |e_j^A, 0\rangle + \alpha_n^B \sum_j |e_j^B, 0\rangle + \sum_x \frac{\tilde{\varphi}(E_n) e^{-|x|/\lambda_n}}{(E_n - \omega)\sqrt{1 - \left(\frac{2J}{E_n - \omega}\right)^2}} |g, x\rangle, \quad (26)$$

where $\tilde{\varphi}(E_n) = (M_A V_A \alpha_n^A + M_B V_B \alpha_n^B)(-1)^{|x|\theta(E_n - \omega)}$ and the characteristic length $\lambda_n = -1/\log\{(1 - \sqrt{1 - [2J/(E_n - \omega)]^2})|E_n - \omega|/2J\}$. In Figs. 7(a), 7(b), and 7(c), we show the V_B dependence of the length λ_n with different atom numbers M_B. As V_B decreases to zero, λ_1 and λ_3 tend to be finite, but λ_2 gets to infinity when V_B decreases to a fixed value which corresponds to the disappearance of the bound state $|\tilde{\phi}_2\rangle$. As M_B becomes large, this fixed value gets small. In Fig. 7(d), it is shown the probability distribution of bound state $|\tilde{\phi}_2\rangle$ as a function of position with different atom numbers M_A. It can be seen that $|\tilde{\phi}_2\rangle$ becomes nonlocalized as M_A increases and finally $|\tilde{\phi}_2\rangle$ will disappear when M_A reaches a certain point.

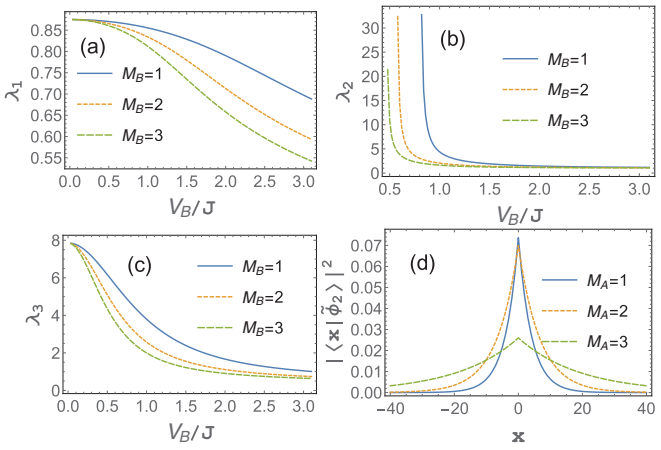

FIG. 7. [(a)–(c)] Coupling strength V_B dependence of the length λ_n with different atom number M_B, $V_A/J = 0.8$, and $M_A = 2$. (d) Position populations of wave function $|\tilde{\phi}_2\rangle$ with different atom number M_A, $V_A/J = 0.32$, $V_B/J = 0.3$, and $M_B = 2$. Other parameters are $\Omega_A/J = 3$, $\Omega_B/J = 1.5$, and $\omega/J = 0$.

B. Dynamics and trapping by same type of atoms

We now consider the dynamics of the system with two types of atoms. The wave function at time t in this situation can be written as

$$|\tilde{\psi}(t)\rangle = \sum_i \sum_j b_j^i(t) |e_j^i, 0\rangle + \sum_k \tilde{C}_k(t) a_k^\dagger |g, 0\rangle. \quad (27)$$

With the Schrödinger equation $i\partial|\tilde{\psi}(t)\rangle/\partial t = H|\tilde{\psi}(t)\rangle$, the relations about $b_j^i(t)$ and $\tilde{C}_k(t)$ are acquired:

$$i\frac{\partial}{\partial t} b_j^i(t) = \Omega_i b_j^i(t) + \frac{V_i}{\sqrt{N}} \sum_k \tilde{C}_k(t), \quad (28)$$

$$i\frac{\partial}{\partial t} \tilde{C}_k(t) = \omega_k \tilde{C}_k(t) + \sum_i \sum_j \frac{V_i}{\sqrt{N}} b_j^i(t). \quad (29)$$

Here we denote the initial excited atom with j_0 and it is assumed to belong to type A, i.e., $b_{j_0}^A(0) = 1$, $b_j^A(0) = 0$ $(j \neq j_0)$, and $b_j^B(0) = C_k(0) = 0$. By taking a Laplace transform of Eqs. (28) and (29) and use the inverse formula $b_{j_0}^A(t) = (1/2\pi i) \int_{\sigma - i\infty}^{\sigma + i\infty} \tilde{b}_{j_0}^A(s) e^{st} ds$, which are similar to the steps in the case of one type of atoms, we finally derive the expression of $b_{j_0}^A(t)$

$$b_{j_0}^A(t) = \frac{M_A - 1}{M_A} e^{-i\Omega_A t} + \sum_m \frac{i f(s)(is - \Omega_B) V_A^2 e^{st}}{(is - \Omega_A)[F(s)]'}\Big|_{s = x_m^{(2)}} + \int_{-1}^{1} \frac{4J^2\sqrt{1 - y^2}(2Jy + \Omega_B)^2 V_A^2 e^{i2Jyt}}{\pi L(y)} dy, \quad (30)$$

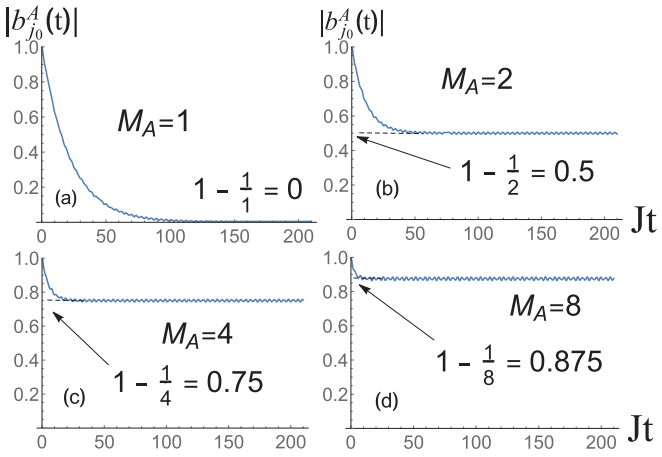

FIG. 8. Time evolution of the population on the excited atom with different atom numbers M_A when $\delta_A/J = (\Omega_A - \omega)/J = 0.4$ and $\delta_B/J = (\Omega_B - \omega)/J = 1.5$. Other parameters are $V_A/J = 0.3$, $V_B/J = 0.25$, and $M_B = 2$.

where $F(s) \equiv (is - \Omega_A)(is - \Omega_B) - f(s)[(is - \Omega_A)M_B V_B^2 + (is - \Omega_B)M_A V_A^2]$ and $L(y) \equiv 4J^2(1 - y^2)(2Jy + \Omega_A)^2(2Jy + \Omega_B)^2 + [(2Jy + \Omega_A)M_B V_B^2 + (2Jy + \Omega_B)M_A V_A^2]^2$. Here, $x_m^{(2)}$ is the roots of the equation $F(s) = 0$. It can be seen that $F(-iE) = 0$ is the energy equation of atom-photon bound states. The second term in Eq. (30) comes from the contribution of atom-photon bound states which results in the population trapping in the time evolution. The third term which comes from the contribution of the scattering states will become zero at last. If the value of $if(s)(is - \Omega_B)V_A^2/(is - \Omega_A)[F(s)]'|_{s=x_m^{(2)}}$ is small enough, which usually happens when both Ω_A and Ω_B are inside the scattering band, $b^A_{j_0}(t)$ will tend to

$$\left|b^A_{j_0}(\infty)\right| = 1 - \frac{1}{M_A}, \tag{31}$$

which is only related with the number of type-A atoms that is the same as the type of the initial excited atom. In Fig. 8, we plot the time evolution of $|b^A_{j_0}(t)|$ with different numbers of atoms M_A. It can be seen that $|b^A_{j_0}(\infty)|$ tends to the value of $1 - 1/M_A$ when time is infinite. The small oscillation in $|b^A_{j_0}(t)|$ after a sufficiently long time results from the contribution atom-photon bound states which is small but not zero. Here, the trapping is caused by the system's dark state and the excitation number focuses on the type-A atoms, and the populations of type-B atoms and field modes are zero. The dark state satisfies the time-independent Schrödinger equation with eigenenergy $E = \Omega_A$. This type of trapping is associated with the collective coherence of type-A atoms.

V. DISCUSSION AND CONCLUSION

In this paper, we have studied the energy structure and single-photon cooperative dynamics of a one-dimensional waveguide with model dispersion which is locally coupled by ensembles of two-level atoms. While the bound states and dynamics have been investigated in the condition that the atoms are placed in different resonators [30,62], here we focus on the situation in which the atomic ensemble is much smaller than the resonator lattice constant, which is achieved by putting atoms in the same resonator. Because of the different distribution of atoms, the energy structure and single-photon cooperative dynamics are very different. In this system, the discrete bound-state eigenvalues are analyzed by solving the time-independent Schrödinger equation. Both the cases, where single and multiple atomic ensembles are present, are discussed. While there are always two bound states in the single-ensemble case, the number of bound states in the two-ensemble case is not fixed and depends on the coupling strengths and the relative positions of the atom's transition frequencies from the band. The change of energy structure reveals that there is a quantum phase transition. The characteristic length and wave function near the phase transition are analyzed. By exactly solving the time-dependent Schrödinger equation, an analytical expression for the system's single-photon cooperative dynamics is provided. Unlike the trapping regime caused by atom-photon bound states, we first point out the atomic population in the excited state cannot decay to zero because of the existence of a dark state which is particularly evident when the bound state trapping regime is ignorable.

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