Hyperfine structure of *P* states in muonic ions of lithium, beryllium, and boron

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We calculate hyperfine structure intervals $\Delta E^{hfs}(2P_{1/2})$ and $\Delta E^{hfs}(2P_{3/2})$ for P states in muonic ions of lithium, beryllium, and boron. To construct the particle interaction operator in momentum space we use the tensor method of the projection operators on states with definite quantum numbers of total atomic momentum F and total muon momentum j. We take into account the vacuum polarization, relativistic, quadruple, and structure corrections of order α^4, α^5, and α^6. The obtained numerical values of hyperfine splittings can be used for a comparison with future experimental data of the CREMA Collaboration.

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I. INTRODUCTION

Quantum electrodynamics of the bound states is one of the most successful theories in modern physics and has been checked by very precise measurements for many atoms and molecules. The current experimental program of the Charge Radius Experiments with Muonic Atoms Collaboration (CREMA Collaboration) is directed to study fine and hyperfine energy structure of simple muonic atoms [1–3]. It is realized successfully beginning in 2010, when two transition frequencies $(2S^{F=1}_{1/2} - 2P^{F=2}_{3/2})$ and $(2S^{F=0}_{1/2} - 2P^{F=1}_{3/2})$ in muonic hydrogen were measured with an accuracy that allowed us to obtain one order more precise value of the proton charge radius. The measurement of three transition frequencies $(2S^{F=3/2}_{1/2} - 2P^{F=5/2}_{3/2})$, $(2S^{F=1/2}_{1/2} - 2P^{F=3/2}_{3/2})$, $(2S^{F=1/2}_{1/2} - 2P^{F=1/2}_{3/2})$ in muonic deuterium by means of laser spectroscopy methods was also performed and gave a more precise value of the deuteron charge radius. Laser spectroscopy provides a unique possibility to check and further develop the theoretical models connected with the investigation of the fundamental structure of matter. Experimental investigations with muonic hydrogen gave unexpected results, discovered the problem which was called the "proton radius puzzle" [4–6], and raised the question of a more accurate study of the effects of the nucleus structure [7]. This problem has remained unsolved for a long time while there were many efforts to explain the difference in proton charge radii extracted from electronic and muonic atoms. Among recent papers devoted to the determination of the proton charge radius it is necessary to mention experimental work [8] where the proton charge radius is extracted from the Lamb-shift measurement in ordinary hydrogen with the value $r_p = (0.833 \pm 0.010)$ fm, which is very close to the result obtained by the CREMA Collaboration. Another recent work [9] on the extraction of the proton charge radius from the elastic form factor (FF) data gave the value $r_p = 0.844(7)$ fm, which is consistent with the high-precision muonic-hydrogen results. Finally, a recent high-precision electron-proton scattering experiment was performed at the Jefferson Laboratory [10]. It gave a result for the proton charge radius of $r_p = 0.831 \pm 0.007_{\rm stat} \pm 0.012_{\rm syst}$ fm, which, within the error, is consistent with Refs. [1–3]. Only for simple two-particle bound states are the theoretical methods sufficiently well developed to calculate the energy levels including nuclear effects from first principles. In general, the program for studying the muon systems is gaining momentum: a precision study of the ground-state hyperfine structure (HFS) of muonic hydrogen, the energy interval $(1S\text{-}2S)$, and the processes of the dimuonium production are planned [6,11–13].

One of the future scientific directions of the CREMA collaboration is related with light muonic atoms of lithium, beryllium, and boron [11]. Such experiments could complete existing data and obtain new values of the charge radii of Li, Be, and B. The program of experiments with light muonic atoms was discussed many years ago in Refs. [14,15], where the estimation of different energy intervals to leading order was performed. In our recent papers [16,17] we calculated some corrections to the Lamb shift $(2P\text{-}2S)$ and hyperfine splitting of S states in the muonic lithium, beryllium, and boron and obtain more precise values of these energy intervals. This work continues our investigation in Refs. [16,17] to the case of the P-wave part of the spectrum. The account of the HFS of P levels is also necessary because experimental transition frequencies are measured between different components of $2P$ and $2S$ states. Despite the fact that general theoretical methods for the calculation of the hyperfine structure of P states are well developed, specific numerical calculations of different HFS intervals in the case of P states in muonic lithium, beryllium, and boron were not considered in detail. So, Ref. [14] contains only a general formula for the HFS to leading order. Numerical calculation of the HFS in muonic Li, Be, B represents both pure theoretical interest because we have in this task the nuclei with spins different from $1/2$, and experimental interest connected with experiments of the

TABLE I. Nucleus parameters of lithium, beryllium, and boron [19–22].

Nucleus	Spin	Mass (GeV)	Magnetic dipole moment μ_N (nm)	Charge radius r_N (fm)	Electroquadrupole moment Q (fm^2)	Magnetic octupole moment O (nm fm^2)
$^7_3$Li	3/2	6.53383	3.256427(2)	2.4440 ± 0.0420	−4.06(8)	7.5
$^9_4$Be	3/2	8.39479	−1.177432(3)	2.5190 ± 0.0120	5.29(4)	4.1
$^{11}_5$B	3/2	10.25510	2.6886489(10)	2.4060 ± 0.0294	4.07(3)	7.8

CREMA Collaboration. Therefore, the aim of this work is to calculate the hyperfine splitting intervals for P states in the muonic Li, Be, and B and account for corrections to the vacuum polarization and nuclear structure.

II. GENERAL FORMALISM

Among the nuclei Li, Be, B there are nuclei of different spins. For the nuclei with spin $I = s_2 = 1$ our approach to the calculation of HFS was developed in Ref. [18] by the example of muonic deuterium. All nuclei of lithium, beryllium, and boron have isotopes with the spin $s_2 = 3/2$, so we describe further the HFS of such muonic ions which consists of six states: $2\,^3P_{1/2}$, $2\,^5P_{1/2}$, $2\,^1P_{3/2}$, $2\,^3P_{3/2}$, $2\,^5P_{3/2}$, $2\,^7P_{3/2}$, where the lower index corresponds to the muon total momentum $\mathbf{j} = \mathbf{s}_1 + \mathbf{L}$ and the upper index is the factor $(2F + 1)$ ($\mathbf{F} = \mathbf{j} + \mathbf{s}_2$). Basic nucleus parameters written in Table I are taken from Refs. [19–22]. The contribution of the leading order α^4 to the HFS of P states is determined by the amplitude of the one-photon interaction, which is denoted $T_{1\gamma}$. In this work, we use both the momentum and coordinate representations to describe the interaction of particles. We find it useful to begin with the momentum representation of the interaction amplitude in which the two-particle bound-state wave function of the $2P$ state can be written in tensor form:

$$\psi_{2P}(\mathbf{p}) = (\varepsilon \cdot n_p) R_{21}(p), \tag{1}$$

where ε_δ is the polarization vector of orbital motion, $n_p = (0, \mathbf{p}/p)$, and $R_{21}(p)$ is the radial wave function in momentum space. Then the contribution to the energy spectrum is determined by the integral

$$\Delta E^{\rm hfs} = \int \big(\varepsilon^* \cdot n_q\big) R_{21}(q) \frac{d\mathbf{q}}{(2\pi)^{3/2}} \times \int (\varepsilon \cdot n_p) R_{21}(p) \frac{d\mathbf{p}}{(2\pi)^{3/2}} \Delta V^{\rm hfs}(\mathbf{p}, \mathbf{q}). \tag{2}$$

The hyperfine potential $\Delta V^{\rm hfs}$ can be constructed by means of the one-photon interaction amplitude $T_{1\gamma}$ using the method of projection operators on states with definite quantum numbers [18,23–27]. These projection operators can be written in terms of the particle wave functions in the rest frame in the covariant form. They allow us to avoid direct cumbersome multiplication of different factors in the amplitudes and use the computer methods for calculating amplitudes and energy levels [28]. For their construction we use two different schemes of momentum adding: (1) $\mathbf{J} = \mathbf{s}_1 + \mathbf{L}$, $\mathbf{F} = \mathbf{J} + \mathbf{s}_2$, (2) $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$, $\mathbf{F} = \mathbf{S} + \mathbf{L}$. Taking into account that the nuclei with spin 3/2 are described in the Rarita-Schwinger formalism by the spin-vector $v_\alpha(p)$ we can write the one-photon interaction amplitude in the form

$$\begin{aligned} T_{1\gamma}(\mathbf{p}, \mathbf{q}) &= 4\pi Z\alpha(\varepsilon^* \cdot n_q)\Big[\bar u(q_1)\Big(\frac{(p_1+q_1)_\mu}{2m_1} \\ &+ (1+a_\mu)\sigma_{\mu\epsilon}\frac{k_\epsilon}{2m_1}\Big)u(p_1)\Big](\varepsilon \cdot n_p) D_{\mu\nu}(k) \\ &\times \bar v_\alpha(p_2)\Big\{g_{\alpha\beta}\frac{(p_2+q_2)_\nu}{2m_2}F_1(k^2) + g_{\alpha\beta}\sigma_{\nu\lambda}\frac{k^\lambda}{2m_2}F_2(k^2) \\ &+ \frac{k_\alpha k_\beta}{4m_2^2}\frac{(p_2+q_2)_\nu}{2m_2}F_3(k^2) + \frac{k_\alpha k_\beta}{4m_2^2}\sigma_{\nu\lambda}\frac{k^\lambda}{2m_2}F_4(k^2)\Big\}v_\beta(q_2), \end{aligned} \tag{3}$$

where $p_{1,2} = \frac{m_{1,2}}{(m_1+m_2)}P \pm p$ are four-momenta of initial muon and nuclear, $q_{1,2} = \frac{m_{1,2}}{(m_1+m_2)}Q \pm q$ are four-momenta of final muon and nuclear, and a_μ is the muon anomalous magnetic moment. They are expressed in terms of total two-particle momenta P, Q and relative momenta p, q. $D_{\mu\nu}(k)$ is the photon propagator which is taken to be in the Coulomb gauge. Four form factors which parametrize the nucleus electromagnetic current can be expressed through multipole form factors measured in experiments: charge G_{E0}, electroquadrupole G_{E2}, magnetic dipole G_{M1}, and magnetic octupole G_{M3} form factors. The corresponding equations are the following [29–31]:

$$G_{E0}(k^2) = \big(1 + \tfrac{2}{3}\tau\big)[F_1 + \tau(F_1 - F_2)] - \tfrac{1}{3}\tau(1+\tau)[F_3 + \tau(F_3 - F_4)], \tag{4}$$

$$G_{E2}(k^2) = [F_1 + \tau(F_1 - F_2)] - \tfrac{1}{2}(1+\tau)[F_3 + \tau(F_3 - F_4)], \tag{5}$$

$$G_{M1}(k^2) = \big(1 + \tfrac{4}{5}\tau\big)F_2 - \tfrac{2}{5}\tau(1+\tau)F_4, \tag{6}$$

$$G_{M3}(k^2) = F_2 - \tfrac{1}{2}(1+\tau)F_4, \tag{7}$$

where $\tau = -k^2/4m_2^2$. The calculation of these form factors cannot be carried out with high accuracy in nuclear models. So, we can use for them the experimental data suggesting that they can be improved if necessary. The nucleus spin-vector wave function is described by

$$\psi_{\mu,\alpha}(\mathbf{p}, \sigma) = \sum_{\lambda,\omega}\left\langle \frac{1}{2}\omega; 1\lambda \,\middle|\, \frac{3}{2}\sigma\right\rangle \varepsilon_\mu(\mathbf{p}, \lambda) u_\alpha(\mathbf{p}, \omega), \tag{8}$$

where $\langle \frac{1}{2}\omega; 1\lambda \mid \frac{3}{2}\sigma\rangle$ are the Clebsch-Gordan coefficients. To obtain the contribution to the HFS of order α^4 including recoil

effects we should know also the spin-vector transformation law to the rest frame. The explicit expression of the transformation law of vector wave function $\varepsilon_\sigma(p)$ has the form

$$\varepsilon_\sigma(p_2) = \varepsilon_\sigma(0) - \frac{p_{2,\sigma} + g_{0\sigma} m_2}{\epsilon_2(p) + m_2} \frac{[\varepsilon_\sigma(0) \cdot p_2]}{m_2}. \tag{9}$$

To describe the HFS of state $2P_{1/2}$ we introduce in Eq. (3) on the first stage of the transformation the projection operator on the muon state with $j = 1/2$:

$$\hat{\Pi}_{j=1/2} = [u(0)\varepsilon_\omega(0)]_{j=1/2} = \frac{1}{\sqrt{3}}\gamma_5(\gamma_\omega - v_\omega)\psi(0), \tag{10}$$

where $\psi(0)$ is the Dirac spinor describing the muon state with $j = 1/2$, and $v = (1, 0, 0, 0) = P/(m_1 + m_2)$ is the auxiliary four-vector. On the second stage we should project muon-nucleus pair on the state with total momentum $F = 2$ or $F = 1$. In the case of the state with $F = 2$ the projection operator has the form

$$\hat{\Pi}_{j=1/2}(F = 2) = [\psi(0)\bar{v}_\alpha(0)]_{F=2} = \frac{1 + \hat{v}}{2\sqrt{2}}\gamma_\tau \varepsilon_{\alpha\tau}, \tag{11}$$

where the tensor $\varepsilon_{\alpha\tau}$ describes the state $F = 2$. For a construction of the muon-nucleus interaction operator in this state we make the summation over projections of the total momentum F by using the equation

$$\sum_{M_F=-2}^{2} \varepsilon^*_{\beta\lambda}\varepsilon_{\alpha\rho} = \hat{\Pi}_{\beta\lambda,\alpha\rho} = \left[\frac{1}{2}X_{\beta\alpha}X_{\lambda\rho} + \frac{1}{2}X_{\beta\rho}X_{\lambda\alpha} - \frac{1}{3}X_{\beta\lambda}X_{\alpha\rho}\right], \quad X_{\beta\alpha} = (g_{\alpha\beta} - v_\beta v_\alpha). \tag{12}$$

Then the averaged over the projections M_F amplitude takes the form

$$\begin{aligned}\overline{T_{1\gamma}(\mathbf{p},\mathbf{q})}^{j=1/2}_{F=2} = \frac{Z\alpha}{5} n_q^\delta n_p^\omega \mathrm{Tr}\Bigg\{ & \gamma_\sigma \frac{1 + \hat{v}}{2\sqrt{2}}(\gamma_\delta - v_\delta)\gamma_5 \frac{(\hat{q}_1 + m_1)}{2m_1}\Gamma_\mu \frac{(\hat{p}_1 + m_1)}{2m_1}\gamma_5(\gamma_\omega - v_\omega) \\ & \times \frac{1 + \hat{v}}{2\sqrt{2}}\gamma_\rho \frac{(\hat{p}_2 - m_2)}{2m_2}\Gamma^\nu_{\alpha\beta}\frac{(\hat{q}_2 - m_2)}{2m_2}\Bigg\} D_{\mu\nu}(k)\hat{\Pi}_{\beta_1\sigma,\alpha_1\rho}L_{\alpha\alpha_1}L_{\beta\beta_1},\end{aligned} \tag{13}$$

where we introduce for the convenience short designations of the nucleus vertex function

$$\Gamma^\nu_{\alpha\beta} = \left[g_{\alpha\beta}\frac{(p_2 + q_2)_\nu}{2m_2}F_1(k^2) + g_{\alpha\beta}\sigma_{\nu\lambda}\frac{k^\lambda}{2m_2}F_2(k^2) + \frac{k_\alpha k_\beta}{4m_2^2}\frac{(p_2 + q_2)_\nu}{2m_2}F_3(k^2) + \frac{k_\alpha k_\beta}{4m_2^2}\sigma_{\nu\lambda}\frac{k^\lambda}{2m_2}F_4(k^2)\right], \tag{14}$$

the lepton vertex function

$$\Gamma_\mu = \frac{p_{1,\mu} + q_{1,\mu}}{2m_1} + (1 + a_\mu)\sigma_{\mu\epsilon}\frac{k_\epsilon}{2m_1}, \tag{15}$$

and the Lorentz factors of vector fields,

$$L_{\alpha\alpha_1}L_{\beta\beta_1} = \left[g_{\alpha\alpha_1} - \left(v_\alpha - \frac{p_\alpha}{2m_2}\right)\left(v_{\alpha_1} - \frac{p_{\alpha_1}}{m_2}\right)\right]\left[g_{\beta\beta_1} - \left(v_\beta - \frac{p_\beta}{2m_2}\right)\left(v_{\beta_1} - \frac{p_{\beta_1}}{m_2}\right)\right]. \tag{16}$$

We introduce in Eq. (13) the factor $3/4\pi$ connected with the normalization condition of polarization vector in Eq. (1). The remaining cumbersome part of calculating the trace and numerous convolutions by the Lorentz indices can be performed by means of the package FORM [28]. As a result we obtain the muon-nucleus interaction operator for the state $2\,^5P_{1/2}$ in the form

$$\begin{aligned}V_{1\gamma}(\mathbf{p},\mathbf{q})^{j=1/2}_{F=2} = \frac{2\alpha\mu_N}{27m_1m_p(\mathbf{p}-\mathbf{q})^2}\Bigg\{ & \frac{9}{2}pq + \frac{9m_1}{4m_2}pq - \frac{9}{4}(\mathbf{pq})\left(\frac{p}{q} + \frac{q}{p}\right) + \frac{27m_1}{8m_2}(\mathbf{pq})\left(\frac{p}{q} + \frac{q}{p}\right) \\ & - \frac{9m_1}{m_2}\frac{(\mathbf{pq})^2}{pq} + a_\mu\left[\frac{9}{4}pq - \frac{9}{4}(\mathbf{pq})\left(\frac{p}{q} + \frac{q}{p}\right) + \frac{9}{4}\frac{(\mathbf{pq})^2}{pq}\right] \\ & + \frac{a_\mu}{F_2(0)}\left[-\frac{27}{4}pq\left(1 + \frac{m_2}{m_1}\right) + \frac{27}{4}\frac{(\mathbf{pq})^2}{pq} + \frac{27m_2}{8m_1}(\mathbf{pq})\left(\frac{p}{q} + \frac{q}{p}\right)\right] + \frac{27(\mathbf{pq})(\mathbf{p}^2 - \mathbf{q}^2)^2}{8(\mathbf{p}-\mathbf{q})^2F_2(0)pq} \\ & - \frac{27}{8F_2(0)}\left[pq\left(2 + \frac{m_1}{m_2} + \frac{m_2}{m_1}\right) - \frac{m_2}{m_1}(\mathbf{pq})\left(\frac{p}{q} + \frac{q}{p}\right) + (\mathbf{pq})\left(\frac{p}{q} + \frac{q}{p}\right) + 4m_1m_2\frac{(\mathbf{pq})}{pq} - \frac{2m_1}{m_2}\frac{(\mathbf{pq})^2}{pq}\right]\Bigg\},\end{aligned} \tag{17}$$

where we set $F_1(0) = 1$. This expression clearly shows the general structure of potentials for various states, which we obtain at the exit from the FORM program. Typical momentum integrals that must be calculated in the HFS (2) have the form

$$J_1 = \int R_{21}(q)\frac{d\mathbf{q}}{(2\pi)^{3/2}}\int R_{21}(p)\frac{d\mathbf{p}}{(2\pi)^{3/2}}\frac{pq}{(\mathbf{p}-\mathbf{q})^2} = \left\langle\frac{pq}{(\mathbf{p}-\mathbf{q})^2}\right\rangle = \frac{3}{16},$$

$$(\mathbf{pq})^2 pq(\mathbf{p}-\mathbf{q})^2 = \frac{5}{48}, \quad J_3 = \left\langle\frac{(\mathbf{pq})(p^2+q^2)}{pq(\mathbf{p}-\mathbf{q})^2}\right\rangle = \frac{5}{24}, \quad J_4 = \left\langle\frac{(\mathbf{pq})(\mathbf{p}^2-\mathbf{q}^2)^2}{pq(\mathbf{p}-\mathbf{q})^4}\right\rangle = \frac{1}{6}. \tag{18}$$

It is important to note that, when constructing potentials in this way, we obtain not only the hyperfine part of the potentials, but also the Coulomb contributions and contributions to the fine structure, which are further reduced when considering hyperfine splitting. Let us consider also the construction of the potential in the case of the $2\,^3P_{1/2}$ state. To introduce projection operators for the state $F = 1$, $j = 1/2$, it is necessary to add the spin of the nucleus $s_2 = 3/2$ and the total moment of the muon $j = 1/2$. For this we use a basis transformation of the following form:

$$\Psi_{s_2=3/2,F=1,M_F} = \sqrt{\frac{2}{3}}\Psi_{\tilde{S}=0,F=1,M_F} + \sqrt{\frac{1}{3}}\Psi_{\tilde{S}=1,F=1,M_F}, \tag{19}$$

where the state with $s_2 = 3/2$ is represented as the sum of two moments $\tilde{s}_2 = 1/2$ and $l_2 = 1$, $\tilde{S} = s_1 + \tilde{s}_2$. Furthermore, when working with $\Psi_{\tilde{S}=0,F=1,M_F}$ and $\Psi_{\tilde{S}=1,F=1,M_F}$ we introduce the projection operators on these states, the form of which is well known:

$$\hat{\Pi}_\alpha(\tilde{S}=0,F=1) = \frac{1+\hat{v}}{2\sqrt{2}}\gamma_5\varepsilon_\alpha, \quad \hat{\Pi}_\alpha(\tilde{S}=1,F=1) = \frac{1+\hat{v}}{4}\gamma_\sigma\varepsilon_{\alpha\sigma\rho\omega}v^\rho\varepsilon^\omega, \tag{20}$$

where the polarization vector ε^ω on right side of Eq. (20) describes the state with $F = 1$. When using expansion (16), several contributions to the interaction potential of particles in the state $F = 1$ arise, which are determined by two expressions for $\tilde{S} = 0$ and $\tilde{S} = 1$ with weight factors $2/3$ and $1/3$, respectively:

$$\begin{aligned}\overline{T_{1\gamma}(\mathbf{p},\mathbf{q})}^{j=1/2}_{F=1}(\tilde{S}=0) = \frac{Z\alpha}{3}n_q^\delta n_p^\omega \mathrm{Tr}\Bigg\{&\gamma_5\frac{1+\hat{v}}{2\sqrt{2}}(\gamma_\delta - v_\delta)\gamma_5\frac{(\hat{q}_1+m_1)}{2m_1}\Gamma_\mu\frac{(\hat{p}_1+m_1)}{2m_1}\gamma_5(\gamma_\omega - v_\omega)\frac{1+\hat{v}}{2\sqrt{2}}\gamma_5\frac{(\hat{p}_2-m_2)}{2m_2}\\ &\times\Bigg[g_{\alpha\beta}\frac{(p_2+q_2)_\nu}{2m_2}F_1(k^2) + g_{\alpha\beta}\sigma_{\nu\lambda}\frac{k^\lambda}{2m_2}F_2(k^2) + \frac{k_\alpha k_\beta}{4m_2^2}\frac{(p_2+q_2)_\nu}{2m_2}F_3(k^2)\\ &+\frac{k_\alpha k_\beta}{4m_2^2}\sigma_{\nu\lambda}\frac{k^\lambda}{2m_2}F_4(k^2)\Bigg]\frac{(\hat{q}_2-m_2)}{2m_2}\Bigg\}D_{\mu\nu}(k)\hat{\Pi}_{\beta_1\sigma\alpha_1\rho}L_{\alpha\alpha_1}L_{\beta\beta_1}(g_{\alpha_1\beta_1} - v_{\alpha_1}v_{\beta_1}),\end{aligned} \tag{21}$$

$$\begin{aligned}\overline{T_{1\gamma}(\mathbf{p},\mathbf{q})}^{j=1/2}_{F=1}(\tilde{S}=1) = \frac{Z\alpha}{3}n_q^\delta n_p^\omega \mathrm{Tr}\Bigg\{&\gamma_\rho\frac{1+\hat{v}}{4}(\gamma_\delta - v_\delta)\gamma_5\frac{(\hat{q}_1+m_1)}{2m_1}\Gamma_\mu\frac{(\hat{p}_1+m_1)}{2m_1}\gamma_5(\gamma_\omega - v_\omega)\\ &\times\frac{1+\hat{v}}{4}\gamma_\tau\frac{(\hat{p}_2-m_2)}{2m_2}\Bigg[g_{\alpha\beta}\frac{(p_2+q_2)_\nu}{2m_2}F_1(k^2) + g_{\alpha\beta}\sigma_{\nu\lambda}\frac{k^\lambda}{2m_2}F_2(k^2)\\ &+\frac{k_\alpha k_\beta}{4m_2^2}\frac{(p_2+q_2)_\nu}{2m_2}F_3(k^2) + \frac{k_\alpha k_\beta}{4m_2^2}\sigma_{\nu\lambda}\frac{k^\lambda}{2m_2}F_4(k^2)\Bigg]\frac{(\hat{q}_2-m_2)}{2m_2}\Bigg\}\\ &\times D_{\mu\nu}(k)L_{\alpha\alpha_3}L_{\beta\beta_3}\epsilon_{\rho\beta_3\alpha_1\beta_1}\epsilon_{\tau\alpha_3\rho_1\omega_1}(g_{\omega_1\beta_1} - v_{\omega_1}v_{\beta_1}).\end{aligned} \tag{22}$$

There is also off-diagonal elements of the form $\langle\Psi_{\tilde{S}=0,F=1,M_F}|V^{j=1/2}_{1\gamma,F=1}|\Psi_{\tilde{S}=1,F=1,M_F}\rangle$. Omitting other details of the calculation and using Eqs. (17), (21), and (22), we obtain the hyperfine splitting of the $2P_{1/2}$ state as follows:

$$\begin{aligned}&\Delta E^{\mathrm{hfs}}(2\,^5P_{1/2}\text{-}2\,^3P_{1/2})\\ &= \frac{2\alpha(Z\alpha)^3\mu^3\mu_N}{27m_1m_p}\left[1+\frac{1}{2}a_\mu + \frac{m_1}{2m_2} - \frac{3m_1}{4m_2F_2(0)}\right]\\ &= \begin{cases}^7_3\mathrm{Li}: 210.8960\ \mathrm{meV}\\ ^9_4\mathrm{Be}: -183.2929\ \mathrm{meV}\\ ^{11}_5\mathrm{B}: 818.1086\ \mathrm{meV},\end{cases}\end{aligned} \tag{23}$$

where we take into account only leading-order terms including recoil correction related with nucleus magnetic form factor. At $k^2 = 0$ we have $F_2(0) = m_2\mu_N/Zm_p$.

The calculation of hyperfine splitting for the $2^{(2F+1)}P_{3/2}$ state is a more complicated problem, since it is more complicated to construct the projection operators for these states. The most simple form is the projection operator on the state with $F = 3$. In this case, when the two moments $3/2$ are added, we get the state with the maximum total momentum, which is described by the tensor $\varepsilon_{\alpha\beta\gamma}$. The projection operator on this state is

$$\hat{\Pi}_{\omega\alpha_1} = [u_\omega(0)\bar{v}_{\alpha_1}(0)]^{j=3/2}_{F=3} = \frac{1+\hat{v}}{2\sqrt{2}}\gamma_{\omega_1}\varepsilon_{\omega\alpha_1\omega_1}, \tag{24}$$

and a summation over projections has the form [27]

$$\sum_{M_F=-3}^{3}\varepsilon^{*}_{\alpha\omega\omega_1}\varepsilon_{\beta\delta\delta_1}=\hat{\Pi}_{\alpha\omega\omega_1,\beta\delta\delta_1}=\left[\frac{1}{6}\Omega^{(1)}_{\alpha\omega\omega_1,\beta\delta\delta_1}-\frac{1}{15}\Omega^{(2)}_{\alpha\omega\omega_1,\beta\delta\delta_1}\right],$$

$$\begin{aligned}\Omega^{(1)}_{\alpha\omega\omega_1,\beta\delta\delta_1}&=X_{\alpha\beta}X_{\omega\delta}X_{\omega_1\delta_1}+X_{\alpha\beta}X_{\omega\delta_1}X_{\omega_1\delta}+X_{\alpha\delta}X_{\omega\beta}X_{\omega_1\delta_1}+X_{\alpha\delta}X_{\omega\delta_1}X_{\omega_1\beta}+X_{\alpha\delta_1}X_{\omega\delta}X_{\omega_1\beta}+X_{\alpha\delta_1}X_{\omega\beta}X_{\omega_1\delta},\\ \Omega^{(2)}_{\alpha\omega\omega_1,\beta\delta\delta_1}&=X_{\alpha\omega}X_{\omega_1\delta_1}X_{\beta\delta}+X_{\alpha\omega}X_{\omega_1\delta}X_{\beta_1\delta_1}+X_{\alpha\omega}X_{\omega_1\beta}X_{\delta\delta_1}+X_{\alpha\omega_1}X_{\omega\delta_1}X_{\beta\delta}+X_{\alpha\omega_1}X_{\omega\delta}X_{\beta\delta_1}\\ &\quad+X_{\alpha\omega_1}X_{\omega\beta}X_{\delta\delta_1}+X_{\omega\omega_1}X_{\alpha\delta_1}X_{\beta\delta}+X_{\omega\omega_1}X_{\alpha\delta}X_{\beta\delta_1}+X_{\omega\omega_1}X_{\alpha\beta}X_{\delta\delta_1}.\end{aligned}\tag{25}$$

Then the interaction amplitude averaged over the projections M_F can be represented as follows:

$$\begin{aligned}\overline{T_{1\gamma}(\mathbf{p},\mathbf{q})}^{j=3/2}_{F=3}&=\frac{3Z\alpha}{7}n_q^{\delta}n_p^{\omega}\mathrm{Tr}\left\{\gamma_{\delta_1}\frac{1+\hat{v}}{2\sqrt{2}}\frac{(\hat{q}_1+m_1)}{2m_1}\Gamma_{\mu}\frac{(\hat{p}_1+m_1)}{2m_1}\frac{1+\hat{v}}{2\sqrt{2}}\gamma_{\omega_1}\right.\\ &\quad\left.\times\frac{(\hat{p}_2-m_2)}{2m_2}\Gamma^{\nu}_{\alpha\beta}\frac{(\hat{q}_2-m_2)}{2m_2}\right\}D_{\mu\nu}(k)L_{\alpha\alpha_1}L_{\beta\beta_1}\hat{\Pi}_{\beta_1\delta\delta_1,\alpha_1\omega\omega_1}.\end{aligned}\tag{26}$$

To calculate the interval of the HFS $\Delta E^{\mathrm{hfs}}(2\,^7P_{3/2}\text{-}2\,^5P_{3/2})$, it is also necessary to build the potential for the state with $F=2$, which is obtained by adding two $3/2$ moments. Acting as in Eq. (19), we first represent the state of the nucleus as the result of adding the two moments $\tilde{s}_2=1/2$ and $l_2=1$ and introduce the momentum $\mathbf{j}_1=\mathbf{j}+\tilde{\mathbf{s}}_2$, which takes values 2 and 1:

$$\Psi_{s_2=3/2,j=3/2,F=2}=\frac{1}{\sqrt{2}}\Psi_{j_1=2,l_2=1,F=2}+\frac{1}{\sqrt{2}}\Psi_{j_1=1,l_2=1,F=2}.\tag{27}$$

The projection operator on the state $j_1=2$ has the form

$$\hat{\Pi}^{j=3/2}_{\alpha,j_1=2}=[u_{\alpha}(0)\bar{v}(0)]^{j=3/2}_{j_1=2}=\frac{1+\hat{v}}{2\sqrt{2}}\gamma^{\beta_1}\varepsilon_{\alpha\beta_1}.\tag{28}$$

To write a projection operator on the state $j_1=1$, we will already represent the moment of the muon $j=3/2$ as the result of adding two moments: $\mathbf{j}=\mathbf{s}_1+\mathbf{l}_1$. Given the coefficients of vector addition of moments, we obtain the following expansion:

$$\begin{aligned}&\Psi_{s_2=3/2,j=3/2,F=2}\\ &=\frac{1}{\sqrt{2}}\Psi_{j_1=2,l_2=1,F=2}+\frac{1}{\sqrt{3}}\Psi_{(S=0,l_1=1,j_1=1),l_2=1,F=2}\\ &\quad+\frac{1}{\sqrt{6}}\Psi_{(S=1,l_1=1,j_1=1),l_2=1,F=2}.\end{aligned}\tag{29}$$

Performing the addition of individual moments in $F=2$, we obtain the following result for the projection operator on a state with $F=2$:

$$\begin{aligned}\hat{\Pi}^{j=3/2}_{\alpha\beta,F=2}&=\frac{(1+\hat{v})}{2\sqrt{6}}\gamma_5\varepsilon_{\alpha\beta}+\frac{(1+\hat{v})}{4\sqrt{6}}[-g_{\gamma\alpha}\epsilon_{\beta\alpha_1\alpha_2\alpha_3}\\ &\quad+g_{\gamma\beta}\epsilon_{\alpha\alpha_1\alpha_2\alpha_3}+g_{\gamma\alpha_1}\epsilon_{\alpha\beta\alpha_2\alpha_3}]v^{\alpha_2}\gamma^{\alpha_1}\varepsilon^{\gamma\alpha_3}.\end{aligned}\tag{30}$$

As we see, the tensor on the right-hand side of Eq. (29) contains both the symmetric and antisymmetric parts in the indices α, β. The same decomposition and addition of individual moments is also used to obtain projection operators on other states of the HFS with $F=0$ and $F=1$. They have the form

$$\begin{aligned}\hat{\Pi}^{j=3/2}_{F=0}&=[u_{\omega}\bar{v}_{\alpha}(0)]^{j=3/2}_{F=0}=\frac{1+\hat{v}}{6}\gamma_5(g_{\omega\alpha}-v_{\omega}v_{\alpha})\\ &\quad-\frac{(1+\hat{v})}{12}\gamma^{\lambda}\epsilon_{\lambda\omega\sigma\alpha}v^{\sigma},\end{aligned}\tag{31}$$

$$\begin{aligned}\hat{\Pi}^{j=3/2}_{F=1}&=[u_{\omega}\bar{v}_{\alpha}(0)]^{j=3/2}_{F=1}=\frac{1+\hat{v}}{24\sqrt{5}}\big[-8(g_{\omega\alpha}-v_{\omega}v_{\alpha})\gamma_{\alpha_2}\\ &\quad+2\big(g_{\omega\alpha_2}\gamma_{\alpha}+g_{\alpha\alpha_2}\gamma_{\omega}\big)\\ &\quad-2\big(g_{\alpha_2\omega}v_{\alpha}+g_{\alpha\alpha_2}v_{\omega}\big)-10\epsilon_{\omega\alpha\alpha_3\alpha_2}v^{\alpha_3}\gamma_5\big]\varepsilon^{\alpha_2}.\end{aligned}\tag{32}$$

In the practical use of Eqs. (31) and (32), it is convenient to single out the contributions of the symmetric and antisymmetric parts of the projection operators. The general structure of the amplitudes and interaction potentials of particles in these states has the same form as (13), (17), (21), (22), and the intervals of the HFS themselves are determined by formulas similar to (23):

$$\begin{aligned}&\Delta E^{\mathrm{hfs}}(2\,^7P_{3/2}-2\,^5P_{3/2})\\ &=\frac{\alpha(Z\alpha)^3\mu^3\mu_N}{45m_1m_p}\left[1-\frac{1}{4}a_{\mu}+\frac{5m_1}{4m_2}-\frac{15m_1}{8m_2F_2(0)}\right]\\ &=\begin{cases}^7_3\mathrm{Li}:63.8246\ \mathrm{meV}\\ ^9_4\mathrm{Be}:-55.7466\ \mathrm{meV}\\ ^{11}_5\mathrm{B}:246.6252\ \mathrm{meV},\end{cases}\end{aligned}\tag{33}$$

$$\begin{aligned}&\Delta E^{\mathrm{hfs}}(2\,^5P_{3/2}-2\,^3P_{3/2})\\ &=\frac{2\alpha(Z\alpha)^3\mu^3\mu_N}{135m_1m_p}\left[1-\frac{1}{4}a_{\mu}+\frac{5m_1}{4m_2}-\frac{15m_1}{8m_2F_2(0)}\right]\\ &=\begin{cases}^7_3\mathrm{Li}:42.5497\ \mathrm{meV}\\ ^9_4\mathrm{Be}:-37.1644\ \mathrm{meV}\\ ^{11}_5\mathrm{B}:164.4168\ \mathrm{meV},\end{cases}\end{aligned}\tag{34}$$

$$\begin{aligned}&\Delta E^{\mathrm{hfs}}(2\,^3P_{3/2}-2\,^1P_{3/2})\\ &=\frac{\alpha(Z\alpha)^3\mu^3\mu_N}{135m_1m_p}\left[1-\frac{1}{4}a_{\mu}+\frac{5m_1}{4m_2}-\frac{15m_1}{8m_2F_2(0)}\right]\\ &=\begin{cases}^7_3\mathrm{Li}:21.2932\ \mathrm{meV}\\ ^9_4\mathrm{Be}:-18.5822\ \mathrm{meV}\\ ^{11}_5\mathrm{B}:82.2084\ \mathrm{meV},\end{cases}\end{aligned}\tag{35}$$

The numerical values of the contributions (23) and (33)–(35) are large. Therefore, to increase the accuracy of the

calculations, it makes sense to consider a number of corrections to these formulas, which we do below in other sections.

However, it is useful to consider another approach to solving this problem in the coordinate representation, which is widespread [32–34]. Since in muonic ions we encounter nuclei of different spins, it is necessary to have a two-particle Hamiltonian for electromagnetically interacting particles of arbitrary spin. Some time ago, the task of constructing such an effective Hamiltonian was solved in Refs. [35–39] in connection with the calculation of gyromagnetic factors of bound particles. To calculate the HFS of the spectrum of P levels, it is necessary to use the following term from this Hamiltonian:

$$\Delta H^{\rm hfs} = \frac{Z\alpha g_N}{2m_1m_2r^3}\left[1+\frac{m_1}{m_2}-\frac{m_1}{m_2g_N}\right](\mathbf{L}\mathbf{s}_2) - \frac{Z\alpha(1+a_\mu)g_N}{2m_1m_2r^3}[\mathbf{s}_1\mathbf{s}_2-3(\mathbf{s}_1\mathbf{r})(\mathbf{s}_2\mathbf{r})], \quad (36)$$

where the gyromagnetic factor of the nucleus $g_N = F_1(0)/s_2 = 2m_2\mu_N/3Zm_p$ and nucleus magnetic moment is taken in nuclear magnetons. To calculate the relative level arrangement, a fine part of the Hamiltonian is also necessary:

$$\Delta H^{fs} = \frac{Z\alpha}{m_1m_2r^3}\left[1+\frac{m_2}{2m_1}+a_\mu\left(1+\frac{m_2}{m_1}\right)\right](\mathbf{L}\mathbf{s}_1). \quad (37)$$

Averaging (37) over the wave functions of the $2P$ state, we obtain the main contribution to the fine splitting:

$$E^{fs} = \frac{(Z\alpha)^4\mu^3}{16m_1m_2}\left[1+\frac{m_2}{2m_1}+a_\mu\left(1+\frac{m_2}{m_1}\right)\right] = \begin{cases} {}^7_3\mathrm{Li}: 747.8581\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 2372.2215\ \mathrm{meV} \\ {}^{11}_5\mathrm{B}: 5804.9674\ \mathrm{meV}. \end{cases} \quad (38)$$

While our main goal is the calculation of P-wave hyperfine splittings we estimate here also vacuum polarization correction of order α^5 (leading-order correction) to fine splitting. Using in this case basic relations from Ref. [40] for the corrections in first-order and second-order perturbation theory we obtain the total vacuum polarization contribution as follows:

$$\Delta E^{fs}_{vp} = \begin{cases} {}^7_3\mathrm{Li}: 2.3483\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 10.1158\ \mathrm{meV} \\ {}^{11}_5\mathrm{B}: 30.5417\ \mathrm{meV}. \end{cases} \quad (39)$$

The hyperfine part of the Hamiltonian includes two operators:

$$T_1 = \mathbf{L}\mathbf{s}_2, \quad T_2 = \mathbf{s}_1\mathbf{s}_2 - 3(\mathbf{s}_1\mathbf{n})(\mathbf{s}_2\mathbf{n}). \quad (40)$$

Diagonal in j matrix elements contribute to the HFS in the form

$$E(2^{2F+1}P_j) = \frac{\alpha(Z\alpha)^3\mu^3\mu_N}{72m_1m_p} \times\left[\bar{T}_1+\frac{m_1}{m_2}\bar{T}_1-\frac{3m_1}{2m_2F_2(0)}\bar{T}_1-(1+a_\mu)\bar{T}_2\right]. \quad (41)$$

The calculation of matrix elements $\bar{T}_1$ and $\bar{T}_2$ is carried out by using the basic formulas from Ref. [18] (see Appendix A). As a result, the position of energy levels $2^{2F+1}P_j$ is determined by the following expressions:

$$E(2^7P_{3/2}) = \tilde{E}^{fs} + \frac{\alpha(Z\alpha)^3\mu^3\mu_N}{60m_1m_p}\left[1-\frac{a_\mu}{4}+\frac{5m_1}{4m_2}-\frac{15m_1}{8m_2F_2(0)}\right] = \begin{cases} {}^7_3\mathrm{Li}: 798.0748\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 2340.5274\ \mathrm{meV} \\ {}^{11}_5\mathrm{B}: 6020.4780\ \mathrm{meV}, \end{cases} \quad (42)$$

$$E(2^5P_{3/2}) = \tilde{E}^{fs} - \frac{\alpha(Z\alpha)^3\mu^3\mu_N}{180m_1m_p}\left[1-\frac{a_\mu}{4}+\frac{5m_1}{4m_2}-\frac{15m_1}{8m_2F_2(0)}\right] = \begin{cases} {}^7_3\mathrm{Li}: 734.2502\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 2396.2740\ \mathrm{meV} \\ {}^{11}_5\mathrm{B}: 5773.8528\ \mathrm{meV}, \end{cases} \quad (43)$$

$$E(2^3P_{3/2}) = \tilde{E}^{fs} - \frac{11\alpha(Z\alpha)^3\mu^3\mu_N}{540m_1m_p}\left[1-\frac{a_\mu}{4}+\frac{5m_1}{4m_2}-\frac{15m_1}{8m_2F_2(0)}\right] = \begin{cases} {}^7_3\mathrm{Li}: 691.7005\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 2433.4383\ \mathrm{meV} \\ {}^{11}_5\mathrm{B}: 5609.4360\ \mathrm{meV}, \end{cases} \quad (44)$$

$$E(2^1P_{3/2}) = \tilde{E}^{fs} - \frac{\alpha(Z\alpha)^3\mu^3\mu_N}{36m_1m_p}\left[1-\frac{a_\mu}{4}+\frac{5m_1}{4m_2}-\frac{15m_1}{8m_2F_2(0)}\right] = \begin{cases} {}^7_3\mathrm{Li}: 670.4256\ \mathrm{meV}, \\ {}^9_4\mathrm{Be}: 2452.0205\ \mathrm{meV} \\ {}^{11}_5\mathrm{B}: 5527.2276\ \mathrm{meV}, \end{cases} \quad (45)$$

$$E(2^5P_{1/2}) = \frac{\alpha(Z\alpha)^3\mu^3\mu_N}{36m_1m_p}\left[1+\frac{a_\mu}{2}+\frac{m_1}{2m_2}-\frac{3m_1}{4m_2F_2(0)}\right] = \begin{cases} {}^7_3\mathrm{Li}: 79.0860\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: -68.7348\ \mathrm{meV} \\ {}^{11}_5\mathrm{B}: 306.7907\ \mathrm{meV}, \end{cases} \quad (46)$$

$$E(2^3P_{1/2}) = -\frac{5\alpha(Z\alpha)^3\mu^3\mu_N}{108m_1m_p}\left[1+\frac{a_\mu}{2}+\frac{m_1}{2m_2}-\frac{3m_1}{4m_2F_2(0)}\right] = \begin{cases} {}^7_3\mathrm{Li}: -131.8100\ \mathrm{meV} \\ {}^9_4\mathrm{Be}: 114.5581\ \mathrm{meV} \\ {}^{11}_5\mathrm{B}: -511.3179\ \mathrm{meV}, \end{cases} \quad (47)$$

where we add the sum of Eqs. (38) and (39), $\tilde{E}^{fs} = E^{fs} + \Delta E^{fs}_{vp}$, to fix the relative position of sublevels. The obtained expressions (44)–(47) which contain the factor $1/g_N$ coming from the hyperfine interaction Hamiltonian give the hyperfine splitting coinciding with Eqs. (23) and (33)–(35).

TABLE II. Diagonal matrix elements of hyperfine structure (HFS) of $2P$ states in muonic ions Li, Be, B in first-order perturbation theory.

Contribution	Nucleus	$2^3P_{1/2}$ meV	$2^5P_{1/2}$ meV	$2^1P_{3/2}$ meV	$2^3P_{3/2}$ meV	$2^5P_{3/2}$ meV	$2^7P_{3/2}$ meV	Equation number
Leading order	$^7_3$Li	−131.8100	79.0860	670.4256	691.7005	734.2502	798.0748	(33)–(35),
α^4 correction	$^9_4$Be	114.5581	−68.7348	2452.0205	2433.4383	2396.27400	2340.5274	(42)–(47)
	$^{11}_5$B	−511.3179	306.7907	5527.2276	5609.4360	5773.8528	6020.4780	
Quadrupole	$^7_3$Li	0	0	−186.9598	−37.3920	112.1759	−37.3920	
correction	$^9_4$Be	0	0	583.5774	116.7155	−350.1465	116.7155	(52)
of order α^4	$^{11}_5$B	0	0	882.8935	176.5787	−529.7361	176.5787	
VP correction	$^7_3$Li	−0.5701	0.3420	−0.2784	−0.2042	−0.0557	0.1671	(61)–(63),
of order α^5	$^9_4$Be	0.6710	−0.4026	0.3441	0.2523	0.0689	−0.2065	(65)
	$^{11}_5$B	−3.6909	2.2146	−1.9209	−1.4087	−0.3842	1.1526	
Quadrupole and	$^7_3$Li	0	0	−0.6573	−0.1315	0.3944	−0.1315	
VP correction	$^9_4$Be	0	0	2.7276	0.5455	−1.6365	0.5455	(55)
of order α^5	$^{11}_5$B	0	0	5.0232	1.0046	−3.0140	1.0046	(57)
Relativistic	$^7_3$Li	−0.1289	0.0773	−0.0115	−0.0084	−0.0023	0.0069	
correction	$^9_4$Be	0.1964	−0.1178	0.0176	0.0129	0.0035	−0.0105	(72), (73)
of order α^6	$^{11}_5$B	−1.3686	0.8212	−0.1223	−0.0897	−0.0245	0.0734	
VP correction	$^7_3$Li	−0.0011	−0.0007	−0.0004	−0.0003	−0.0001	0.0002	
of order α^6	$^9_4$Be	0.0011	−0.0007	0.0005	0.0003	0.0001	−0.0003	[18]
	$^{11}_5$B	−0.0054	0.0032	−0.0023	−0.0017	−0.0005	0.0014	
Structure	$^7_3$Li	−0.0784	0.0471	−0.0008	−0.0007	−0.0004	−0.0001	
correction	$^9_4$Be	0.1295	−0.0777	0.0018	0.0015	0.0008	−0.0001	(67)–(70)
of order α^6	$^{11}_5$B	−0.8292	0.4975	−0.0050	−0.0043	−0.0028	−0.0007	
Summary	$^7_3$Li	−132.5885	79.5517	482.5174	653.9634	845.9732	760.7254	
contribution	$^9_4$Be	114.2141	−69.3336	3038.6896	2550.9663	2044.5643	2457.5710	
	$^{11}_5$B	−517.2120	310.3272	6413.0938	5785.8732	5240.6907	6199.2880	

III. CONTRIBUTION OF QUADRUPOLE INTERACTION

To leading order α^4 in the energy spectrum of muonic ions Li, Be, B, there is another important contribution of the quadrupole interaction which must be taken into account. It arises for muon states with $j = 3/2$ due to the fact that the nuclei have a nonspherical shape. The calculation of this contribution to HFS in muonic ions in coordinate space is based on the representation of the quadrupole interaction as a scalar product of two irreducible tensor operators of rank 2. Then the matrix elements of tensor operators are expressed in terms of reduced matrix elements by using the Wigner-Eckart theorem [33,41].

Using the method of projection operators formulated above, we can distinguish in the amplitude of the one-photon interaction a part proportional to the quadrupole form factor $G_{E2}(k^2)$. Its value at zero, $G_{E2}(0) = m_2^2 Q/Z$, and the magnitude of the quadrupole moment of the nucleus Q sets the numerical value of this correction. The averaged amplitudes of quadrupole interaction for different states have the form

$$\overline{T_{1\gamma,Q}(\mathbf{p},\mathbf{q})}^{j=3/2}_{F=3} = \frac{\alpha Q}{20(\mathbf{p}-\mathbf{q})^2}\left[pq - 4(\mathbf{pq})\left(\frac{p}{q}+\frac{q}{p}\right) + 7\frac{(\mathbf{pq})^2}{pq}\right], \quad (48)$$

$$\overline{T_{1\gamma,Q}(\mathbf{p},\mathbf{q})}^{j=3/2}_{F=2} = \frac{\alpha Q}{60(\mathbf{p}-\mathbf{q})^2}\left[9pq + 4(\mathbf{pq})\left(\frac{p}{q}+\frac{q}{p}\right) - 17\frac{(\mathbf{pq})^2}{pq}\right], \quad (49)$$

$$\overline{T_{1\gamma,Q}(\mathbf{p},\mathbf{q})}^{j=3/2}_{F=1} = \frac{\alpha Q}{20(\mathbf{p}-\mathbf{q})^2}\left[pq - 4(\mathbf{pq})\left(\frac{p}{q}+\frac{q}{p}\right) + 7\frac{(\mathbf{pq})^2}{pq}\right], \quad (50)$$

$$\overline{T_{1\gamma,Q}(\mathbf{p},\mathbf{q})}^{j=3/2}_{F=0} = \frac{\alpha Q}{12(\mathbf{p}-\mathbf{q})^2}\left[-3pq + 4(\mathbf{pq})\left(\frac{p}{q}+\frac{q}{p}\right) - 5\frac{(\mathbf{pq})^2}{pq}\right]. \quad (51)$$

You may notice that the amplitudes (48) and (50) coincide. Making momentum integration by means of Eq. (18), we obtain contributions to the energy levels $2^{2F+1}P_{3/2}$:

$$\Delta E_Q^{\rm hfs} = \frac{\alpha Q(\mu Z\alpha)^3}{240}[\delta_{F3} - 3\delta_{F2} + \delta_{F1} + 5\delta_{F0}], \quad (52)$$

where the δ_{Fi} is the Kronecker symbol. The quadruple moments of nuclei are written in Table I. The result (52) coincides exactly with previous calculations made by a different method [41]. Their numerical values are presented in Table II.

The magnitude of the quadrupole contribution is significant; therefore, in the case of quadrupole interaction, it makes sense to consider also different corrections to it. One of the

most important effects leading to the correction of the results obtained to order α^4 is the effect of vacuum polarization (VP). We begin its discussion precisely with the quadrupole interaction, since it can be simply calculated within the formulated method by a small modification of relations (48)–(51) for its calculation in momentum representation. In the first-order perturbation theory we should use the following replacement in the photon propagator of Eqs. (48)–(51):

$$\frac{1}{(\mathbf{p}-\mathbf{q})^2} \to \frac{\alpha}{3\pi}\int_1^\infty \frac{\rho(\xi)d\xi}{(\mathbf{p}-\mathbf{q})^2+4m_e^2\xi^2},$$
$$\rho(\xi) = \sqrt{\xi^2-1}(2\xi^2+1)/\xi^4. \tag{53}$$

Then the correction to vacuum polarization in the quadrupole interaction can be expressed in terms of three-momentum integrals which are calculated analytically:

$$I_1 = \int R_{21}(q)\frac{d\mathbf{q}}{(2\pi)^{3/2}}\int R_{21}(p)\frac{d\mathbf{p}}{(2\pi)^{3/2}}\frac{pq}{\left[(\mathbf{p}-\mathbf{q})^2+4m_e^2\xi^2\right]}$$
$$= \left\langle \frac{pq}{\left[(\mathbf{p}-\mathbf{q})^2+4m_e^2\xi^2\right]}\right\rangle = \frac{a(3a+8)+6}{2(a+2)^4}, \quad a=\frac{4m_e\xi}{\mu Z\alpha},$$
$$I_2 = \left\langle \frac{(\mathbf{pq})^2}{pq\left[(\mathbf{p}-\mathbf{q})^2+4m_e^2\xi^2\right]}\right\rangle = \frac{a(3a+8)+10}{6(a+2)^4},$$
$$I_3 = \left\langle \frac{(\mathbf{pq})(p^2+q^2)}{pq\left[(\mathbf{p}-\mathbf{q})^2+4m_e^2\xi^2\right]}\right\rangle = \frac{2(4a+5)}{3(a+2)^4}. \tag{54}$$

After using Eq. (54) the energy correction becomes a function of the dimensionless parameter $a_1 = \frac{4m_e}{\mu Z\alpha}$:

$$\Delta E^Q_{\rm vp}(2^{(2F+1)}P_{3/2})$$
$$= \frac{\alpha^2(Z\alpha)^3 Q}{90\pi\left(4-a_1{}^2\right)^{5/2}}\left\{2\sqrt{4-a_1^2}\left(a_1{}^2-1\right)\right.$$
$$\left.+\left(5a_1{}^2-8\right)\ln\left[\frac{\left(2-\sqrt{4-a_1{}^2}\right)}{a_1}\right]\right\}$$
$$\times\left[5\delta_{F0}+\delta_{F1}-3\delta_{F2}+\delta_{F3}\right]. \tag{55}$$

This dimensionless parameter is not suitable for expansions, since $a_1({\rm Li}) = 0.89795$, $a_1({\rm Be}) = 0.67109$, $a_1({\rm B}) = 0.53566$. Another contribution of VP plus quadrupole interaction of the same order α^5 comes from second-order perturbation theory. Taking one perturbation as in Eq. (53) but in coordinate representation and the other perturbation as a quadrupole interaction

$$\Delta V_Q(r) = \frac{Z\alpha Q}{6r^3}[\mathbf{s}_2\mathbf{s}_2 - 3(\mathbf{s}_2\mathbf{n})(\mathbf{s}_2\mathbf{n})], \tag{56}$$

we present the necessary contribution in integral form ($b_1 = 2m_e/W$):

$$\Delta E^{\rm hfs}_{Q,{\rm vp,sopt}} = \frac{\alpha^5 Z^3\mu^3 Q}{144\pi}\int_1^\infty \frac{\rho(\xi)d\xi}{(1+b_1)^5}\left[3+11b_1+4b_1^2\right.$$
$$\left.+4(1+b_1)\ln(1+b_1)\right]$$
$$\times\left[\frac{1}{5}\delta_{F3}-\frac{3}{5}\delta_{F2}+\frac{1}{5}\delta_{F1}+\delta_{F0}\right]. \tag{57}$$

The corresponding numerical results of the sum of corrections (55) and (57) for the states $2^{(2F+1)}P_{3/2}$ are included in Table II.

IV. CORRECTIONS TO THE VACUUM POLARIZATION AND NUCLEUS STRUCTURE

The main contribution of the effects of vacuum polarization in the HFS of the energy spectrum of the P states is related to a modification of the particle interaction potential (36), which in turn is determined by the replacement (53). Using the results of the previous section, in which the spin-orbit and spin-spin interaction operator is constructed in the momentum representation, we can present the corrections to vacuum polarization for hyperfine splitting in the form

$$\Delta V^{\rm hfs}_{\rm vp}(2^7P_{3/2}-2^5P_{3/2}) = \frac{\alpha}{135\pi}\int_1^\infty\frac{\rho(\xi)d\xi}{(\mathbf{p}-\mathbf{q})^2+4m_e^2\xi^2}\left\{12pq+15\frac{m_1}{m_2}pq+12(\mathbf{pq})\left(\frac{p}{q}+\frac{q}{p}\right)\right.$$
$$-36\frac{(\mathbf{pq})^2}{pq}-15\frac{m_1}{m_2}\frac{(\mathbf{pq})^2}{pq}+a_\mu\left[-3pq+12(\mathbf{pq})\left(\frac{p}{q}+\frac{q}{p}\right)-21\frac{(\mathbf{pq})^2}{pq}\right]$$
$$\left.-\frac{45m_1}{2m_2F_2(0)}\left[pq-\frac{(\mathbf{pq})^2}{pq}\right]\right\}, \tag{58}$$
$$\Delta V^{\rm hfs}_{\rm vp}(2^5P_{3/2}-2^3P_{3/2}) = \frac{2}{3}\Delta V^{\rm hfs}_{\rm vp}(2^7P_{3/2}-2^5P_{3/2}) = 2\Delta V^{\rm hfs}_{\rm vp}(2^3P_{3/2}-2^1P_{3/2}), \tag{59}$$
$$\Delta V^{\rm hfs}_{\rm vp}\left(2^5P_{1/2}-2^3P_{1/2}\right) = \frac{2\alpha^2}{81\pi}\int_1^\infty\frac{\rho(\xi)d\xi}{(\mathbf{p}-\mathbf{q})^2+4m_e^2\xi^2}\left\{12pq+6\frac{m_1}{m_2}pq-6(\mathbf{pq})\left(\frac{p}{q}+\frac{q}{p}\right)\right.$$
$$\left.-6\frac{m_1}{m_2}\frac{(\mathbf{pq})^2}{pq}+a_\mu\left[6pq-6(\mathbf{pq})\left(\frac{p}{q}+\frac{q}{p}\right)+6\frac{(\mathbf{pq})^2}{pq}\right]-\frac{9m_1}{m_2F_2(0)}\left[pq-\frac{(\mathbf{pq})^2}{pq}\right]\right\}. \tag{60}$$

Further integration over the momentum variables and spectral parameter ξ can be performed analytically by using Eq. (54). But the answer for hyperfine splitting in the energy spectrum is more conveniently written in the integral form

over ξ:

$$\Delta E^{\rm hfs}_{\rm vp}(2\,^7P_{3/2}-2\,^5P_{3/2})=\frac{\alpha^2(Z\alpha)^3\mu^3\mu_N}{135\pi m_1m_p}\int_1^\infty\frac{\rho(\xi)d\xi}{(a+2)^4}\Big[16+20\frac{m_1}{m_2}+a\Big(32+40\frac{m_1}{m_2}\Big)$$
$$+15a^2\frac{m_1}{m_2}-a_\mu(4+8a+15a^2)-\frac{15m_1}{2m_2F_2(0)}\big(4+8a+3a^2\big)\Big], \tag{61}$$

$$\Delta E^{\rm hfs}_{\rm vp}(2\,^5P_{3/2}-2\,^3P_{3/2})=\frac{2}{3}\Delta E^{\rm hfs}_{\rm vp}(2\,^7P_{3/2}-2\,^5P_{3/2})=2\Delta E^{\rm hfs}_{\rm vp}(2\,^3P_{3/2}-2\,^1P_{3/2}), \tag{62}$$

$$\Delta E^{\rm hfs}_{\rm vp}(2\,^5P_{1/2}-2\,^3P_{1/2})=\frac{2\alpha^2(Z\alpha)^3\mu^3\mu_N}{81\pi m_1m_p}\int_1^\infty\frac{\rho(\xi)d\xi}{(a+2)^4}\Big[16+32a+18a^2$$
$$+2\frac{m_1}{m_2}(4+8a+3a^2)+a_\mu(8+16a+12a^2)-\frac{3m_1}{m_2F_2(0)}(4+8a+3a^2)\Big]. \tag{63}$$

In the second order of perturbation theory we also have the VP contribution of order α^5. In this case one perturbation potential is determined by Eq. (36) and other perturbation is the vacuum polarization correction to the Coulomb potential [Eq. (53)]. For the calculation of this type correction it is convenient to use coordinate representation in which the Coulomb Green's function has the form [42]:

$$G_{2P}(\boldsymbol{r},\boldsymbol{r}')=-\frac{\mu^2(Z\alpha)}{36x_1^2x_2^2}\Big(\frac{3}{4\pi}\boldsymbol{nn}'\Big)e^{-(x_1+x_2)/2}g(x_1,x_2),$$
$$g(x_1,x_2)=24x_<^3+36x_<^3x_>+36x_<^3x_>^2+24x_>^3+36x_<x_>^3+36x_<^2x_>^3+49x_<^3x_>^3-3x_<^4x_>^3$$
$$-12e^{x_<}(2+x_<+x_<^2)x_>^3-3x_<^3x_>^4+12x_<^3x_>^3[-2C+Ei(x_<)-\ln x_<-\ln x_>], \tag{64}$$

where $C=0.5772\ldots$ is the Euler constant, $x_1=Wr$, $x_2=Wr'$, $x_<=\min(x_1,x_2)$, and $x_>=\max(x_1,x_2)$. Then making the analytical integration over particle coordinates we obtain the following integral representation for this correction:

$$\Delta E^{\rm hfs}_{\rm vp,sopt}=\frac{\alpha^5Z^3\mu^3\mu_N}{144\pi m_1m_p}\int_1^\infty\frac{\rho(\xi)d\xi}{(1+b_1)^5}\big[3+11b_1+4b_1^2+4(1+b_1)\ln(1+b_1)\big]$$
$$\times\Big[\bar T_1+\frac{m_1}{m_2}\bar T_1-\frac{3m_1}{2m_2F_2(0)}\bar T_1-(1+a_\mu)\bar T_2\Big],\quad b_1=\frac{2m_e}{W}. \tag{65}$$

The summary vacuum polarization correction of order α^5 from first- and second-order perturbation theory is presented in the Table II for separate energy levels. The vacuum polarization correction of order α^6 can be obtained by means of general expressions (62)–(64) and (66), changing spectral function $\rho(\xi)$ [Eq. (53)] on the two-loop spectral function as in Ref. [18].

The nucleus of lithium, beryllium, and boron have sufficiently large size, so the structure effects can be significant. For their estimation in order α^6 we use an expansion of charge, magnetic-dipole, and electric-quadrupole form factors:

$$F_1(k^2)\approx1-\tfrac{1}{6}r_{E0}^2\mathbf{k}^2,\quad F_2(k^2)\approx F_2(0)\big[1-\tfrac{1}{6}r_M^2\mathbf{k}^2\big],$$
$$F_3(k^2)\approx2\big[1-\tfrac{1}{6}r_{E0}^2\mathbf{k}^2\big]-2G_{E2}(0)\big[1-\tfrac{1}{6}r_{E2}^2\mathbf{k}^2\big], \tag{66}$$

and take into account terms proportional to the charge r_E, magnetic dipole r_{M1}, and electric quadrupole r_{E2} radii. Then, in momentum representation, the potentials giving the splitting of the P states are the following:

$$\Delta V_{\rm str}(2\,^7P_{3/2}-2\,^5P_{3/2})=\frac{Z\alpha}{45m_1m_2}\Big\{-r_{E0}^2\frac{15m_1}{4m_2}\Big[pq-\frac{(\mathbf{pq})^2}{pq}\Big]+G_{E2}(0)r_{E2}^2\frac{m_1}{2m_2}\Big[3pq+\frac{(\mathbf{pq})^2}{pq}\Big]$$
$$+F_2(0)r_{M1}^2\Big[2pq-6\frac{(\mathbf{pq})^2}{pq}+\frac{5m_1}{2m_2}\Big(pq-\frac{(\mathbf{pq})^2}{pq}\Big)-\frac{a_\mu}{2}\Big(pq+7\frac{(\mathbf{pq})^2}{pq}\Big)\Big]\Big\}, \tag{67}$$

$$\Delta V_{\rm str}(2\,^5P_{3/2}-2\,^3P_{3/2})=\frac{2Z\alpha}{135m_1m_2}\Big\{-r_{E0}^2\frac{15m_1}{4m_2}\Big[pq-\frac{(\mathbf{pq})^2}{pq}\Big]-G_{E2}(0)r_{E2}^2\frac{m_1}{2m_2}\Big[3pq+\frac{(\mathbf{pq})^2}{pq}\Big]$$
$$+F_2(0)r_{M1}^2\Big[2pq-6\frac{(\mathbf{pq})^2}{pq}+\frac{5m_1}{2m_2}\Big(pq-\frac{(\mathbf{pq})^2}{pq}\Big)-\frac{a_\mu}{2}\Big(pq+7\frac{(\mathbf{pq})^2}{pq}\Big)\Big]\Big\}, \tag{68}$$

$$\Delta V_{\rm str}(2^3P_{3/2}-2^1P_{3/2})=\frac{Z\alpha}{135m_1m_2}\left\{-r_{E0}^2\frac{15m_1}{4m_2}\left[pq-\frac{(\mathbf{pq})^2}{pq}\right]-G_{E2}(0)r_{E2}^2\frac{m_1}{4m_2}\left[3pq+\frac{(\mathbf{pq})^2}{pq}\right]\right.$$
$$\left.+F_2(0)r_{M1}^2\left[2pq-6\frac{(\mathbf{pq})^2}{pq}+\frac{5m_1}{2m_2}\left(pq-\frac{(\mathbf{pq})^2}{pq}\right)-\frac{a_\mu}{2}\left(pq+7\frac{(\mathbf{pq})^2}{pq}\right)\right]\right\}, \quad (69)$$

$$\Delta V_{\rm str}(2^5P_{1/2}-2^3P_{1/2})=\frac{2Z\alpha}{27m_1m_2}\left\{-r_{E0}^2\frac{3m_1}{2m_2}\left[pq-\frac{(\mathbf{pq})^2}{pq}\right]+F_2(0)r_{M1}^2\left[2pq+\frac{m_1}{m_2}pq-\frac{m_1}{m_2}\frac{(\mathbf{pq})^2}{pq}+a_\mu\left(pq+\frac{(\mathbf{pq})^2}{pq}\right)\right]\right\}. \quad (70)$$

The calculation of remaining momentum integrals in Eq. (2) gives $\langle pq\rangle=3/8$, $\langle\frac{(\mathbf{pq})^2}{pq}\rangle=1/8$ and shifts of the energy levels $2^{2F+1}P_J$. The corresponding numerical results are presented in Table II. To obtain them we set approximately $r_{E0}=r_{M1}$ and omit quadruple radius r_{E2}.

Among other important corrections of order α^6 we can distinguish relativistic corrections which can have large numerical values due to the factor Z^5. They can be calculated by means of the Dirac equation [43,44]. The expectation value of hyperfine part of the Dirac Hamiltonian can be expressed in terms of reduced matrix elements by means of the Wigner-Eckart theorem [18]:

$$\Delta E_{\rm rel}^{\rm hfs}=eg_N\tilde{\mu}_N(-1)^{s_2+j'-F}W(js_2j's_2;F1)\langle s_2\|\boldsymbol{s_2}\|s_2\rangle \times\left\langle j'\left\|\frac{[\boldsymbol{r}\times\boldsymbol{\alpha}]}{r^3}\right\|j\right\rangle, \quad (71)$$

where $\tilde{\mu}_N$ is the nuclear magneton, and $W(js_2j's_2;F1)$ are the Racah coefficients. The calculation of reduced matrix elements for the P states with nuclear spin 3/2 gives the following results:

$$E_{\rm rel}^{\rm hfs}(2P_{1/2})=\frac{47Z^5\alpha^6\mu_Nm_1^2}{1296m_p}\left[F(F+1)-\frac{9}{2}\right], \quad (72)$$

$$E_{\rm rel}^{\rm hfs}(2P_{3/2})=\frac{7Z^5\alpha^6\mu_Nm_1^2}{6480m_p}\left[F(F+1)-\frac{15}{2}\right]. \quad (73)$$

Indeed, numerically these corrections are important to achieve high accuracy of the total result (see Table II).

V. NONDIAGONAL MATRIX ELEMENTS

Up to this point we have considered diagonal matrix elements between different states $2^{(2F+1)}P_j$. But the one-photon interaction Hamiltonian leads to the mixing of states $2^3P_{1/2}$, $2^3P_{3/2}$ and $2^5P_{1/2}$, $2^5P_{3/2}$. To calculate transitions between these states, we use the formalism of the projection operators developed in previous sections. Then, the general structure of transition amplitudes between the states $2^3P_{1/2}$, $2^3P_{3/2}$ and $2^5P_{1/2}$, $2^5P_{3/2}$ is the following:

$$\langle 2^3P_{1/2}|T_{1\gamma}|2^3P_{3/2}\rangle=Z\alpha\sqrt{3}n_q^\delta n_p^\omega{\rm Tr}\left\{\left[v_{\beta_1}(0)\bar{\psi}(0)\right]_{j=1/2}^{F=1}(\gamma_\delta-v_\delta)\gamma_5\frac{(\hat{q}_1+m_1)}{2m_1}\Gamma_\mu\frac{(\hat{p}_1+m_1)}{2m_1}\right.$$
$$\left.\times\left[\psi_\omega(0)\bar{v}_{\alpha_1}(0)\right]_{j=3/2}^{F=1}\frac{(\hat{p}_2-m_2)}{2m_2}\Gamma_{\alpha\beta}^\nu\frac{(\hat{q}_2-m_2)}{2m_2}\right\}D_{\mu\nu}(k)L_{\alpha\alpha_1}L_{\beta\beta_1}, \quad (74)$$

$$\langle 2^5P_{1/2}|T_{1\gamma}|2^5P_{3/2}\rangle=Z\alpha\sqrt{3}n_q^\delta n_p^\omega{\rm Tr}\left\{\left[v_{\beta_1}(0)\bar{\psi}(0)\right]_{j=1/2}^{F=2}(\gamma_\delta-v_\delta)\gamma_5\frac{(\hat{q}_1+m_1)}{2m_1}\Gamma_\mu\frac{(\hat{p}_1+m_1)}{2m_1}\right.$$
$$\left.\times\left[\psi_\omega(0)\bar{v}_{\alpha_1}(0)\right]_{j=3/2}^{F=2}\frac{(\hat{p}_2-m_2)}{2m_2}\Gamma_{\alpha\beta}^\nu\frac{(\hat{q}_2-m_2)}{2m_2}\right\}D_{\mu\nu}(k)L_{\alpha\alpha_1}L_{\beta\beta_1}. \quad (75)$$

Using projection operators (11), (28), (30), (32), we obtain contributions to the energy spectrum of order α^4:

$$E^{F=1}(j=1/2;j=3/2)=-\frac{\sqrt{5}\alpha^4Z^3\mu^3\mu_N}{216m_1m_p}\left[1+\frac{2m_1}{m_2}-a_\mu+\frac{m_1}{m_2F_2(0)}\right], \quad (76)$$

$$E^{F=2}(j=1/2;j=3/2)=\frac{\alpha^4Z^3\mu^3\mu_N}{72m_1m_p}\left[1+\frac{2m_1}{m_2}-a_\mu-\frac{3m_1}{m_2F_2(0)}\right]. \quad (77)$$

In the one-photon approximation there exists also another correction of order α^4 connected with the quadrupole electric form factor for the nucleus with spin 3/2. We present the correction to the quadrupole interaction in separate equation as (52)

$$\Delta E_Q^{\rm hfs}(j=1/2;j=3/2)=\frac{\alpha Q(\mu Z\alpha)^3}{60}\left[\delta_{F2}+\frac{\sqrt{5}}{3}\delta_{F1}\right]. \quad (78)$$

The contributions of leading order (76)–(78) must be supplemented by the same vacuum polarization corrections which are calculated above for diagonal matrix elements. Let us write them in integral form over spectral parameter:

$$E^{F=1}(j=1/2;j=3/2)=-\frac{\sqrt{5}\alpha^5Z^3\mu^3\mu_N}{162\pi m_1m_p}\int_1^\infty\frac{\rho(\xi)d\xi}{(a+2)^4}\Bigg[4+8a+9a^2+2\frac{m_1}{m_2}(4+8a+3a^2)$$
$$-a_\mu(4+8a-3a^2)+\frac{m_1}{m_2F_2(0)}(4+8a+3a^2)\Bigg], \tag{79}$$

$$E^{F=2}(j=1/2;j=3/2)=\frac{5\alpha^5Z^3\mu^3\mu_N}{162\pi m_1m_p}\int_1^\infty\frac{\rho(\xi)d\xi}{(a+2)^4}\Bigg[4+8a+9a^2+2\frac{m_1}{m_2}(4+8a+3a^2)$$
$$-a_\mu(4+8a-3a^2)-\frac{3m_1}{m_2F_2(0)}(4+8a+3a^2)\Bigg]. \tag{80}$$

The contribution of vacuum polarization to quadrupole interaction also can be obtained with the use of Eqs. (74) and (75) in the form:

$$E_{Q,vp}(j=1/2;j=3/2)=\frac{\alpha^5Z^3\mu^3Q}{45\pi}\int_1^\infty\frac{\rho(\xi)d\xi}{(a+2)^4}[4+8a+5a^2]\left[\delta_{F2}+\frac{\sqrt{5}}{3}\delta_{F1}\right]. \tag{81}$$

We cannot neglect the relativistic corrections to nondiagonal matrix elements the general expression (71) because its calculation contains the following radial integral with the Dirac wave functions [45]:

$$R_{\frac12\frac32}=\int_0^\infty[g_{1/2}(r)f_{3/2}(r)+g_{3/2}(r)f_{1/2}(r)]dr$$
$$=\frac{(Z\alpha)^3}{48}\left[1+\frac{9(Z\alpha)^2}{16}\right]+O((Z\alpha)^7), \tag{82}$$

where the indexes $1/2$ and $3/2$ designate the total muon momentum. In final form, the relativistic corrections to the nondiagonal matrix elements are determined by the following expressions:

$$E^{F=1}_{\rm rel}(j=1/2;j=3/2)=-\frac{\sqrt{5}\alpha^6Z^5m_1^2\mu_N}{384m_p}, \tag{83}$$

$$E^{F=2}_{\rm rel}(j=1/2;j=3/2)=\frac{\alpha^6Z^5m_1^2\mu_N}{128m_p}. \tag{84}$$

TABLE III. Nondiagonal matrix elements in the HFS of P-wave states of muonic lithium, beryllium, and boron.

Contribution to HFS	Nucleus	$2^3P_{1/2,3/2}$ (meV)	$2^5P_{1/2,3/2}$ (meV)	Equation number
Leading order α^4	$^7_3$Li	−30.2419	40.2378	
correction	$^9_4$Be	25.8382	−35.3158	(76), (77)
	$^{11}_5$B	−116.2435	154.8861	
Leading order α^4	$^7_3$Li	−111.4813	−149.5678	
quadrupole correction	$^9_4$Be	347.9783	466.8619	(78)
	$^{11}_5$B	526.4559	706.3147	
Vacuum polarization	$^7_3$Li	−0.0548	0.1223	
correction of order α^5	$^9_4$Be	0.0566	−0.1272	(79), (80)
	$^{11}_5$B	−0.2842	0.6341	
Quadruple and vacuum	$^7_3$Li	−0.1901	−0.2551	
polarization correction	$^9_4$Be	0.7283	0.9771	(81)
of order α^5	$^{11}_5$B	1.2681	1.7013	
Relativistic correction	$^7_3$Li	−0.0083	0.0111	
of order α^6	$^9_4$Be	0.0126	−0.0169	(83), (84)
	$^{11}_5$B	−0.0879	0.1179	
Vacuum polarization	$^7_3$Li	−0.0008	0.0018	
correction of order α^6	$^9_4$Be	0.0008	−0.0018	[18]
	$^{11}_5$B	−0.0037	0.0082	
Summary contribution	$^7_3$Li	−141.9772	−109.4499	
	$^9_4$Be	374.6148	432.3773	
	$^{11}_5$B	411.1047	863.6623	

TABLE IV. HFS of P states in muonic ions of lithium, beryllium, and boron.

State	$(\mu\mathrm{Li})^{2+}$ energy (meV)	$(\mu\mathrm{Be})^{3+}$ energy (meV)	$(\mu\mathrm{B})^{4+}$ energy (meV)
$2\,^3P_{1/2}$	−157.432	57.912	−543.912
$2\,^5P_{1/2}$	64.228	−154.353	163.415
$2\,^1P_{3/2}$	482.517	3038.062	6413.094
$2\,^3P_{3/2}$	678.806	2607.268	5812.673
$2\,^5P_{3/2}$	861.297	2129.583	5387.603
$2\,^7P_{3/2}$	760.725	2457.571	6199.288

All equations of this section [(76)–(84)] are used for obtaining numerical results which are presented in Table III. We include in Table III an estimate of the vacuum polarization correction of order α^6 obtained as in Ref. [18].

VI. CONCLUSION

HFS of the P-wave energy spectrum plays an important role for precise calculation of transition frequencies between energy levels. We study the HFS of energy levels in muonic ions of lithium, beryllium, and boron on the basis of the quasipotential method in quantum electrodynamics. Our calculation contains the leading order α^4 contribution and corrections of orders α^5 and α^6 to the vacuum polarization, nucleus structure, quadrupole interaction, and relativism. These corrections have significant numerical value because of the factor proportional to the nucleus charge Z. The formalism used allows us to calculate corrections in analytical form, which is demonstrated in different places of the work. In a number of cases we preserve integral representation over some spectral parameter for corrections to the vacuum polarization because it is more compact. Numerical results are given in Tables II–IV.

This work continues our investigation of the P-wave part of the energy spectrum in light muonic atoms which began in Refs. [18,40,46,47]. Taking nuclei of spin 3/2 we obtain the following results in the study of HFS:

(1) We are developing the method of projection operators for nuclei of spin 3/2 in the momentum representation, which allows us to construct the interaction operator of the muon and nucleus for various states. The results of calculating the contributions to the energy spectrum by this method are consistent with the calculation performed in the framework of the coordinate representation.

(2) The vacuum-polarization corrections of the fifth order in α are calculated in the HFS of the P states, including the quadrupole interaction.

(3) An estimate of the sixth-order α contributions to the structure of the nucleus and the vacuum polarization is obtained. Relativistic corrections are also taken into account in the framework of the relativistic Dirac equation.

In recent years, the calculation of different corrections to fine and HFS of muonic atoms was done by several groups. Because of the large number of works we give only references to review articles [4,5,7,32,34,48], which contain many other references regarding the problem. The HFS of P-states in muonic lithium, beryllium, and boron was not calculated directly in these papers. There is the only Ref. [15] in which the basic analytical formula for the contribution of order $O(\alpha^4)$ without recoil was presented. We have improved previous estimates of P-energy levels obtained by authors [15] taking into account new corrections. For definiteness, we present numerical results with an accuracy of four digits after the decimal point. Errors in the determination of fundamental physical constants (fine-structure constant, particle masses, nucleus magnetic moments) are negligible when obtaining the final results. For example, the uncertainty of the magnetic moment of the lithium nucleus leads in the leading order to the error of contributions to the HFS of the spectrum of 0.0001 meV. The accuracy of pure QED theoretical calculation is determined by the omitted corrections of a higher order in α. On the whole, it can be said that we obtained the QED values of the hyperfine splittings of the P levels with an accuracy of 0.001 meV. But the contributions obtained also contain other fundamental parameters of the theory (charge radii of nuclei, quadrupole moments of nuclei), the accuracy of which is not high. So, corrections for the structure of the nucleus [Eqs. (67)–(70)] include the charge radii of the nuclei, which are known with percentage accuracy. Given the approximation made, which we use to estimate contributions (67)–(70), we can say that the error in the results in line 7 of Table II does not exceed 5% . A more significant numerical error is determined by quadrupole-interaction corrections of orders $O(\alpha^4)$, $O(\alpha^5)$, which are proportional to the quadrupole moment of the nucleus Q. Since Q is known with an accuracy of 1% to 2%, these contributions (lines 2 and 4 lines of Tables II and III, respectively) have an error of about 2% . The numerical value of this error can be from 1 to 10 meV for different energy levels and for different nuclei. In this regard, we can say that the precision measurement of various energy intervals in muonic ions of lithium, beryllium, and boron will allow us to obtain more accurate values of r_N, Q. Another not purely QED correction is due to the polarizability effect of the nucleus, which was not considered in this work. The numerical value of such a contribution can be close to the correction for the nucleus structure of order $O(\alpha^6)$. The work in this direction is in progress.

To reduce the error in determining the charge radii of lithium, beryllium, and boron nuclei from experiments with muonic atoms to 0.0005 fm, it is necessary to analyze all possible contributions, which in numerical terms are of the order of 0.1 meV. This estimate is related to the main contribution to the nuclear structure in the Lamb shift ($2P$-$2S$) of the fourth order in α. As shown by the numerical results in Tables II and III, not all sixth-order corrections in α are equally important for achieving high accuracy in obtaining the nuclear charge radii. Among them are, for example, relativistic corrections, which are of order $(Z\alpha)^6$. An increase in the numerical value of this correction due to the nuclear charge factor Z is significant. But for completeness, we also estimated other contributions of order α^6, the numerical values of which turned out to be much smaller, and their role at this stage in obtaining more accurate values of the charge radii Li, Be, B is very small.

Summing all diagonal and off-diagonal matrix elements in the case of muonic lithium, we obtain the following energy

matrix:

$$M = \begin{array}{c} \\ 2^3P_{1/2} \\ 2^5P_{1/2} \\ 2^1P_{3/2} \\ 2^3P_{3/2} \\ 2^5P_{3/2} \\ 2^7P_{3/2} \end{array} \begin{array}{c} \begin{array}{cccccc} 2^3P_{1/2} & 2^5P_{1/2} & 2^1P_{3/2} & 2^3P_{3/2} & 2^5P_{3/2} & 2^7P_{3/2} \end{array} \\ \begin{pmatrix} -132.5885 & 0 & 0 & -141.9772 & 0 & 0 \\ 0 & 79.5517 & 0 & 0 & -109.4499 & 0 \\ 0 & 0 & 482.5174 & 0 & 0 & 0 \\ -141.9772 & 0 & 0 & 653.96340 & 0 & \\ 0 & -109.4499 & 0 & 0 & 845.9732 & 0 \\ 0 & 0 & 0 & 0 & 0 & 760.7254 \end{pmatrix} \end{array} \text{meV}. \tag{85}$$

Its diagonalization leads directly to the position of the energy levels $2P$ (see Table IV) and hyperfine-splitting intervals for muonic lithium which can be measured in experiments. In the same way we obtained HFS of two other ions of muonic beryllium and boron, which are also given in Table IV.

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