


Closing the detection loophole in nonlinear entanglement witnessesKornikar Sen, Sreetama Das, and Ujjwal Sen *Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhansi, Allahabad 211 019, India*

(Received 27 June 2019; published 26 December 2019)

Detectors in the laboratory are often unlike their ideal theoretical cousins. They have nonideal efficiencies, which may then lead to nontrivial implications. We show how it is possible to predict correct answers about whether a shared quantum state is entangled in spite of finite detector efficiencies, when the tool for entanglement detection is a nonlinear entanglement witness. We first consider the detection loophole for shared quantum states with nonpositive partial transpose. We subsequently find nonlinear witness operators for bound entangled states with positive partial transpose, and show how the detection loophole can be closed also in such instances.

DOI: [10.1103/PhysRevA.100.062333](https://doi.org/10.1103/PhysRevA.100.062333)**I. INTRODUCTION**

Entanglement is a useful resource in quantum tasks [1,2], including quantum teleportation [3], quantum dense coding [4], and entanglement-based quantum cryptography [5]. It is therefore important to find out whether a shared quantum state is entangled. There are several methods known for detection of entanglement, including the positive partial transpose (PPT) criterion [6,7], entropic criteria [8,9], Bell inequalities [10], and entanglement witnesses [7,11]. A necessary and sufficient criterion that is analytically tractable or numerically efficient remains elusive. There have been significant advances in experimental detection of entanglement by using the above criteria [12,13].

Whatever is the approach for detecting entanglement, it will of course involve measurements on the shared quantum state. The devices that are used for such measurements are typically assumed, in theoretical discussions, to be ideal.

From an experimental perspective, a useful method for detecting entanglement is by using entanglement witnesses, which are linear operators on the space of quantum states (density matrices) and which provide a sufficient condition for detecting entanglement. The criterion is based on the Hahn-Banach separation theorem on normed linear spaces [14]. A large number of experiments have utilized entanglement witnesses for detecting entanglement [13].

Bell inequality violation for a shared quantum state implies that the state cannot be described by a local hidden variable model. It also implies that the state is entangled. Indeed, a typical Bell inequality, e.g., the Clauser-Horne-Shimony-Holt inequality [15], is a nonoptimal entanglement witness. There exists a series of works on the detection loophole for Bell inequality violations [16] (see also [17]), where the theoretical discussion allows the detectors to have nonideal efficiencies. Experimental violation of Bell inequalities, while acknowledging nonideal detector efficiencies, has been explored in several works [18]. Reference [19] considered implications of the detection loophole in experiments for entanglement detection via entanglement witnesses.

Entanglement witnesses predicted by the Hahn-Banach theorem are linear operators. For every entangled state, there

always exists an entanglement witness that can detect it, as well as some—but not all—other entangled states. However, it is possible to add nonlinear terms to linear witness operators that detect the entangled states that are detected by the linear parent witness, as well as some more entangled states [20–22].

There are two results obtained in this paper, and in the first one we find limits on the threshold efficiency of detectors for implementing nonlinear entanglement witnesses, for entangled states with a nonpositive partial transpose (NPPT).

The second one relates to bound entangled states, which, in the two-party case, are shared quantum states that are entangled but not distillable, i.e., it is not possible to obtain singlets, even asymptotically, from the shared state by local quantum operations and classical communication [23]. In this part, we begin by constructing nonlinear entanglement witnesses for bound entangled states with positive partial transpose. As a particular example, we consider nonlinear witnesses for the family of bound entangled states given in Ref. [24]. We subsequently provide bounds on the threshold efficiency of detectors for detecting the bound entangled state by utilizing the nonlinear witness.

The paper is arranged as follows. In Sec. II, we briefly discuss certain general aspects of linear and nonlinear entanglement witnesses. The detection loophole for linear witnesses is reviewed in Sec. III, which also sets up the notations for the succeeding sections. We present our results on the detection loophole for nonlinear entanglement witnesses for entangled states with a NPPT in Sec. IV. Bound entangled states with PPT are considered in Sec. V, where we first present nonlinear witnesses for them, and then consider the limits on detection efficiencies for their detection using nonlinear witnesses. We present a conclusion in Sec. VI.

II. LINEAR AND NONLINEAR ENTANGLEMENT WITNESSES

Among the various methods for detecting entangled states, there are a few which can be realized experimentally without going through an entire state tomography. One of them is by using witness operators. The concept of the entanglement witness is based on the Hahn-Banach theorem [14]. It states

that if S is a closed and convex set in a normed linear space L , and $x \in L \setminus S$, then there exists a continuous functional $f : L \rightarrow \mathbb{R}$ such that $f(s) < r \leq f(x)$ for all $s \in S$ where $r \in \mathbb{R}$. The space of density matrices on a given Hilbert space forms a normed linear space for the norm, $\|\rho\| = \sqrt{\text{tr}(\rho\rho^\dagger)}$, of a density matrix ρ . This remains valid for density matrices on the tensor products of several Hilbert spaces, and in particular for the tensor product, of two Hilbert spaces H_A and H_B . We now identify S with the set of separable states on $H_A \otimes H_B$, and x with an entangled state [1,2] on the same bipartite system. We note that separable states form a closed and convex set in the space of density matrices. The Hahn-Banach theorem, therefore, guarantees the existence of a functional which separates the set of separable states with the entangled state. This functional is called a witness operator [11] and is defined as an operator W which satisfies the following conditions:

$$\text{tr}(W\rho_s) \geq 0 \text{ for all } \rho_s \in S,$$

$$\text{tr}(W\rho) < 0 \text{ for at least one entangled state } \rho.$$

Note that if for any state ρ one gets $\text{tr}(W\rho) < 0$ one can surely conclude that it is entangled. Moreover, since the set of nonseparable states is open, there will always exist an open ball, in a suitable metric, the entanglement of every state of which will be detected by the same witness. This is a useful fact for experimental implementation of the witness, as small and often inevitable errors in the preparation of the state can then be nullified. Furthermore, for every entangled state, ρ , there always exists a witness that detects it. An example of a witness operator for an NPPT state ρ_ϕ is $W_\phi = |\phi\rangle\langle\phi|^{T_B}$ [11], where $|\phi\rangle$ is an eigenvector corresponding to a negative eigenvalue of $\rho_\phi^{T_B}$. Here, one can easily check that the expectation value of W_ϕ is positive for all separable states and negative for ρ_ϕ , i.e., it can detect the entanglement of ρ_ϕ . But such witness operators can only detect NPPT states. Witness operators for detecting PPT bound entangled states are discussed in Sec. V.

The operator W is a “linear” operator, in the sense that it acts linearly on the space of density matrices. One can get more efficient witness operators by adding nonlinear terms to linear witness operators in such a way that the new “nonlinear witness operator” can detect the entangled states that can be detected by the parent linear witness operator, as well as additional ones. We will introduce nonlinear witness operators more formally in Sec. IV.

III. DETECTION LOOPHOLE

In this section, we briefly recapitulate the implications of a finite (i.e., nonzero) efficiency for linear entanglement witnesses [19]. While we consider only the two-qubit case, the methods work also in higher dimensions and higher number of parties. A decomposition of the witness operator, W , in the two-qubit case, is given by

$$\begin{aligned} W &= C_{00}I \otimes I + \sum_{i=1}^3 C_{0i}I \otimes \sigma_i \\ &+ \sum_{i=1}^3 C_{i0}\sigma_i \otimes I + \sum_{i,j=1}^3 C_{ij}\sigma_i \otimes \sigma_j \\ &= C_{00}I \otimes I + \sum_{k=1}^{15} C_k S_k, \end{aligned} \quad (1)$$

where S_k 's are tensor products of all combinations of two σ_i ($i = 0, 1, 2, 3$) except $I \otimes I$, with $\sigma_0 = I$, σ_i for $i = 1, 2, 3$ being the Pauli matrices. Here, I is the identity operator on the qubit Hilbert space. C_{ij} and C_k are real numbers. To detect the entanglement of a two-qubit state through the expectation value of W in that state, one has to measure these S_k 's for that state. Since there could be errors in these measurements, the status of a state—with respect to whether or not it is entangled—found by using the value of a witness operator could have a “loophole” in the argument. We want to find the condition for overcoming such a loophole. The measured expectation value of S_k for a certain two-qubit state, ρ , is given by $\langle S_k \rangle_m = \frac{\sum n_i \lambda_i}{N} = \frac{\sum (\tilde{n}_i + \epsilon_{+i} - \epsilon_{-i}) \lambda_i}{\tilde{N} + \epsilon_+ - \epsilon_-}$. Here, n_i denotes the number of times that the i th eigenvalue λ_i of S_k has clicked in experiment, and \tilde{n}_i denotes the number of times the same should have clicked in case of perfect detectors. Also, $N = \sum_i n_i$ and $\tilde{N} = \sum_i \tilde{n}_i$. ϵ_{+i} are the number of additional events and ϵ_{-i} are the number of lost events at the i th outlet. The total number of additional and lost events are, respectively, given by $\epsilon_+ = \sum_i \epsilon_{+i}$ and $\epsilon_- = \sum_i \epsilon_{-i}$, and the corresponding detection efficiencies are defined as $\eta_+ = \frac{\tilde{N}}{\tilde{N} + \epsilon_+}$ (equal to additional event efficiency) and $\eta_- = \frac{\tilde{N} - \epsilon_-}{\tilde{N}}$ (equal to lost event efficiency). In this paper, we assume that the additional event efficiency $\eta_+ = 1$, i.e., $\epsilon_+ = 0$, and that the ϵ_{-i} 's are equal for all i 's and the value is, say, ϵ . With additional notations and algebra, those assumptions can of course be lifted. With these assumptions, we have $\langle S_k \rangle_m = \frac{\sum \tilde{n}_i \lambda_i - \epsilon \sum \lambda_i}{\tilde{N} - \epsilon}$. Since S_k 's are tensor products of the Pauli matrices (which are all traceless matrices) with each other or with the identity matrix, the traces of S_k 's are zero. Hence, we get $\langle S_k \rangle_m = \frac{1}{\eta_-} \frac{\sum \tilde{n}_i \lambda_i}{\tilde{N}} = \frac{1}{\eta_-} \langle S_k \rangle_t$. Here, $\langle S_k \rangle_t$ denotes the true value of S_k , i.e., the expectation value of S_k when measured with ideal detectors, for the state ρ . Now, from Eq. (1), we have $\langle W \rangle_m = C_{00} + \frac{1}{\eta_-} \sum_k \langle S_k \rangle_m = C_{00}(1 - \frac{1}{\eta_-}) + \frac{1}{\eta_-} \langle W \rangle_t$. An entangled state would be detected when $\langle W \rangle_t < 0$, so that we need

$$\langle W \rangle_m < C_{00} \left(1 - \frac{1}{\eta_-} \right).$$

Now, for a particular detector, the value of η_- is known, or can be estimated, usually, by independent means. If the measured value of the witness satisfies the above inequality for some state ρ , then, in spite of the inefficiencies of the detectors, we can conclude that the state ρ is entangled. We can see from the relation that if one uses a witness such that in its decomposition the coefficient $C_{00} = 0$ then the loophole in the detection cannot affect the result.

Let us now take a particular witness operator, given by $W_{\phi^+} = |\phi^+\rangle\langle\phi^+|^{T_B}$ (where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$). In an ideal scenario, this witness operator will be able to detect the entanglement in any two-qubit state ρ_{ϕ^+} that has $|\phi^+\rangle$ as the eigenvector corresponding to the negative eigenvalue of $\rho_{\phi^+}^{T_B}$. An exemplary family of such states is the Werner family [25], $\rho_p = p|\psi^-\rangle\langle\psi^-| + (1-p)\frac{1}{2}I \otimes \frac{1}{2}I$, for $\frac{1}{3} < p \leq 1$. See [6,7] in this regard. Here, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Note here that two-qubit states can have at most a single negative eigenvalue after being partially transposed [26]. If we repeat the above

calculation with this witness operator, we will get the following condition:

$$\langle W_{\phi^+} \rangle_m < \frac{1}{4} \left(1 - \frac{1}{\eta_-} \right).$$

Now, for example, if the lost event efficiency $\eta_- > \frac{1}{3}$, then to overcome the loophole and detect an entangled state one needs $\langle W_{\phi^+} \rangle_m < -\frac{1}{2}$.

IV. DETECTION LOOPHOLE IN NONLINEAR WITNESS OPERATORS

As we have mentioned before, one can improve a linear witness operator by adding nonlinear terms to the linear witness operator, such that it “bends towards negativity.” If we consider the witness operator which witnesses NPPT states and is given by $|\phi\rangle\langle\phi|^{T_b}$ then one can add a nonlinear term in the following way [20]:

$$F = \langle |\phi\rangle\langle\phi|^{T_b} \rangle - \frac{1}{s(\psi)} \langle X^{T_b} \rangle \langle (X^{T_b})^\dagger \rangle,$$

where the expectations are for the state ρ , the entanglement of which we wish to detect. Here, X is given by $|\phi\rangle\langle\psi|$, where $|\psi\rangle$ is an arbitrary but fixed state and $s(\psi)$ is the square of the largest Schmidt decomposition coefficient of $|\psi\rangle$. It is shown in [20] that $F \geq 0$ if the expectations in F are for a separable state, and that when the expectations are for an entangled state $F < 0$. Moreover, $F < 0$ is true for more entangled states than for which $\langle W_\phi \rangle < 0$. Here we wish to find the limits on the measured values of F such that we can still correctly predict whether ρ is entangled, in the case when the detectors are nonideal. To do this, one needs to find F , and hence has to measure W and X^{T_b} , while acknowledging that the detectors are not ideal. Although X^{T_b} is not Hermitian, we can decompose X^{T_b} into Hermitian and anti-Hermitian parts as $X^{T_b} = H + iA$, where H and A are Hermitian, so that we get $\langle X^{T_b} \rangle \langle (X^{T_b})^\dagger \rangle = \langle H \rangle^2 + \langle A \rangle^2$. Since H and A are Hermitian, we can measure them. Here we have considered the case where all the operators are measured by using similarly engineered detectors so that the η_- are the same for all the measurements. Just like W , the H and A can also be decomposed in terms of tensor products of the Pauli matrices and the identity matrix, and we obtain

$$\langle H \rangle_m = C_{0H} \left(1 - \frac{1}{\eta_-} \right) + \frac{1}{\eta_-} \langle H \rangle_t, \tag{2}$$

$$\langle A \rangle_m = C_{0A} \left(1 - \frac{1}{\eta_-} \right) + \frac{1}{\eta_-} \langle A \rangle_t. \tag{3}$$

The suffixes m and t indicate, respectively, the measured and true values, and $C_{0H} = \frac{1}{4} \text{tr}(H)$ and $C_{0A} = \frac{1}{4} \text{tr}(A)$. Hence, the measured value of the nonlinear witness operator is

$$\begin{aligned} \langle F \rangle_m &= \langle W_\phi \rangle_m - \frac{1}{s(\psi)} [\langle H \rangle_m^2 + \langle A \rangle_m^2] \\ &= C_{00} \left(1 - \frac{1}{\eta_-} \right) + \frac{1}{\eta_-} \langle W \rangle_t \\ &\quad - \frac{1}{s(\psi)} [\langle H \rangle_m^2 + \langle A \rangle_m^2] \end{aligned}$$

$$\begin{aligned} &= C_{00} \left(1 - \frac{1}{\eta_-} \right) \\ &\quad + \frac{1}{\eta_-} \left[\langle F \rangle_t + \frac{1}{s(\psi)} (\langle H \rangle_t^2 + \langle A \rangle_t^2) \right] \\ &\quad - \frac{1}{s(\psi)} [\langle H \rangle_m^2 + \langle A \rangle_m^2]. \end{aligned}$$

Putting the value of $\langle H \rangle_t$ and $\langle A \rangle_t$ in terms of $\langle H \rangle_m$ and $\langle A \rangle_m$ from (2) and (3), we get

$$\begin{aligned} \langle F \rangle_m &= C_{00} \left(1 - \frac{1}{\eta_-} \right) + \frac{1}{\eta_-} \langle F \rangle_t \\ &\quad + \frac{\eta_-}{s(\psi)} (\langle H \rangle_m^2 + k_H^2 - 2\langle H \rangle_m k_H) \\ &\quad + \frac{\eta_-}{s(\psi)} (\langle A \rangle_m^2 + k_A^2 - 2\langle A \rangle_m k_A) \\ &\quad - \frac{1}{s(\psi)} [\langle H \rangle_m^2 + \langle A \rangle_m^2], \end{aligned}$$

where $k_H = C_{0H} (1 - \frac{1}{\eta_-})$ and $k_A = C_{0A} (1 - \frac{1}{\eta_-})$. This will detect an entangled state when $\langle F \rangle_t < 0$. Putting this in the above equation, we get

$$\begin{aligned} \langle F \rangle_m &< C_{00} \left(1 - \frac{1}{\eta_-} \right) \\ &\quad + \frac{\eta_-}{s(\psi)} (\langle H \rangle_m^2 + k_H^2 - 2\langle H \rangle_m k_H) \\ &\quad + \frac{\eta_-}{s(\psi)} (\langle A \rangle_m^2 + k_A^2 - 2\langle A \rangle_m k_A) \\ &\quad - \frac{1}{s(\psi)} [\langle H \rangle_m^2 + \langle A \rangle_m^2]. \tag{4} \end{aligned}$$

Writing F in terms of the linear witness operator and the nonlinear terms, we get

$$\begin{aligned} \langle W_\phi \rangle_m &< C_{00} \left(1 - \frac{1}{\eta_-} \right) \\ &\quad + \frac{\eta_-}{s(\psi)} (\langle H \rangle_m^2 + k_H^2 - 2\langle H \rangle_m k_H) \\ &\quad + \frac{\eta_-}{s(\psi)} (\langle A \rangle_m^2 + k_A^2 - 2\langle A \rangle_m k_A). \end{aligned}$$

The values of $\langle W_\phi \rangle_m$, $\langle H \rangle_m$, and $\langle A \rangle_m$ which will satisfy the above inequality for a given η_- will detect an entangled state, and for that state the loophole would be closed. Although we have derived the condition for loophole closure for a particular case, the method can also be utilized for deriving conditions for other nonlinear entanglement witnesses.

Now if we consider the linear witness operator W_{ϕ^+} and add the nonlinear term $X = |\phi^+\rangle\langle\phi^-|$, the condition for closing the loophole will be

$$\langle W_{\phi^+} \rangle_m < \frac{1}{4} \left(1 - \frac{1}{\eta_-} \right) + 2\eta_- (\langle H \rangle_m^2 + \langle A \rangle_m^2).$$

Here, $|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. The region above the curved surface in Fig. 1 shows the range of $\langle W_{\phi^+} \rangle_m$, X_{nl} , and η_- , for which the loophole would be closed and an entangled state would be detected. Here, $X_{nl} = (\langle H \rangle_m^2 + \langle A \rangle_m^2)^{\frac{1}{2}}$. The

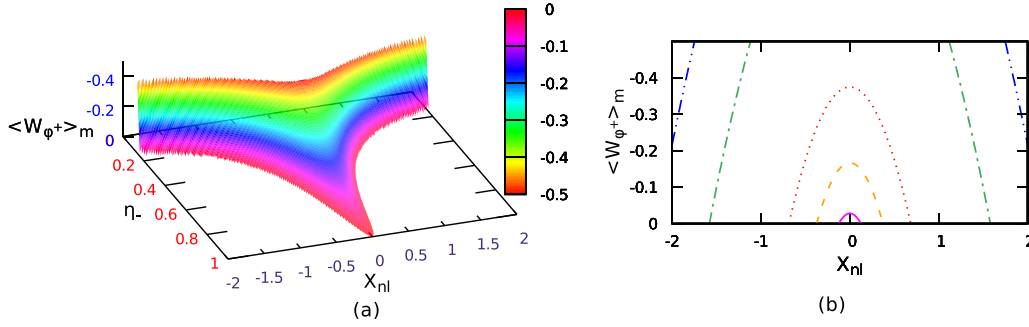


FIG. 1. Closing the detection loophole for a nonlinear entanglement witness. The values of the triad, $\langle W_{\phi^+} \rangle_m$, X_{nl} , and η_- , ascertain whether the entanglement in the state ρ_{ϕ^+} is detected, where ρ_{ϕ^+} is any state the partial transpose of which has $|\phi^+\rangle$ as the eigenvector for its negative eigenvalue. In spite of the nonideal detector efficiency, the entanglement in ρ_{ϕ^+} is detected whenever the triad lies in the region above the surface plotted in panel (a). Here the different colors denote different ranges of values of $\langle W_{\phi^+} \rangle_m$ as indicated in the colorbox. In panel (b), values of X_{nl} and $\langle W_{\phi^+} \rangle_m$ are shown for some fixed values of η_- . The blue double-dot-dashed, green dot-dashed, red dotted, orange dashed, and magenta continuous lines are, respectively, for $\eta_- = 0.15, 0.2, 0.4, 0.6$, and 0.9 . Each curve, therefore, is a cross section of the surface in panel (a) for different values of η_- . The region outside each η_- curve shows the values of X_{nl} and $\langle W_{\phi^+} \rangle_m$ for which entanglement can be detected for that value of efficiency. It can be seen that as the efficiency increases there is an increase in the region for which successful detection of entanglement is possible. Note that $X_{nl} = (\langle H_m \rangle^2 + \langle A_m \rangle^2)^{\frac{1}{2}}$. All quantities are dimensionless.

figure shows that the condition for detecting entanglement becomes progressively better as the value of the nonlinear term X_{nl}^2 increases. More precisely, for a given value of $\langle W_{\phi^+} \rangle_m$, an increase in the nonlinear contribution due to X_{nl} allows for the detection of entanglement with lower η_- .

V. NONLINEAR WITNESS OPERATORS FOR BOUND ENTANGLED STATES

In this section, we begin by identifying nonlinear witness operators for bound entangled states with positive partial transpose. We subsequently show how one can deal with the detection loophole also in this case.

A map M , on the space of operators on a Hilbert space, \mathbb{C}^{d_2} , which has the property $M(X^\dagger) = M(X)^\dagger$, and which preserves positivity [i.e., if eigenvalues of X are positive, then eigenvalues of $M(X)$ will also be positive], is called a positive map. If we apply $I_{d_1} \otimes M_{d_2}$ on operators on the Hilbert space $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and if the positivity is still preserved, for all d_1 , then the map is called completely positive. All positive maps behave as completely positive if we restrict their action to separable states on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, and corresponding to every entangled state (say ρ) there exists some positive map (say M_1) for which $I_{d_1} \otimes M_1(\rho)$ will have at least one negative eigenvalue, for some d_1 [7]. Here, M_1 is a map on the space of operators on \mathbb{C}^{d_2} , and I_{d_1} is the identity map on the space of operators on \mathbb{C}^{d_1} . If an eigenvector corresponding to a negative eigenvalue of $I_{d_1} \otimes M_1(\rho)$ is $|\phi\rangle$, then $\tilde{W}_\phi = (I_{d_1} \otimes M_1)^\dagger |\phi\rangle\langle\phi|$ will satisfy the conditions of a witness operator and can detect the state ρ . Here, $(I_{d_1} \otimes M_1)^\dagger$ is defined by the equation $\text{tr}[(I_{d_1} \otimes M_1)^\dagger(O_1)O_2] = \text{tr}[O_1(I_{d_1} \otimes M_1)(O_2)]$, for all operators O_1 and O_2 on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. We can now construct a corresponding nonlinear witness operator as

$$F = (I \otimes M_1)^\dagger |\phi\rangle\langle\phi| - \frac{1}{s(\psi)} \langle X^{T_B} \rangle \langle (X^{T_B})^\dagger \rangle, \quad (5)$$

with $X = |\phi\rangle\langle\psi|$, where again $|\psi\rangle$ is an arbitrary but fixed vector and $s(\psi)$ is the square of the largest Schmidt coefficient

of $|\psi\rangle$. Let us consider a particular bound entangled state [24],

$$\rho_B = \frac{2}{7} |\tilde{\psi}\rangle\langle\tilde{\psi}| + \frac{a}{7} \sigma_+ + \frac{5-a}{7} \sigma_-,$$

where

$$\begin{aligned} |\tilde{\psi}\rangle &= \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + |22\rangle), \\ \sigma_+ &= \frac{1}{3} (|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|), \\ \sigma_- &= \frac{1}{3} (|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|). \end{aligned}$$

One can easily check that, for $a \leq 4$, ρ_B is PPT. Now, if we use the map [27],

$$\begin{aligned} A &\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right) \\ &= \begin{bmatrix} a_{11} + a_{33} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} + a_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} + a_{22} \end{bmatrix}, \end{aligned}$$

and find eigenvalues of $I \otimes A(\rho_B)$, we can see that it has a negative eigenvalue value for $a > 3$. So the state is bound entangled for $3 < a \leq 4$.

The eigenvector corresponding to the negative eigenvalue is $|\phi\rangle = \frac{1}{\sqrt{3}} [|00\rangle + |11\rangle + |22\rangle]$. To construct the nonlinear witness operator as given in (5) let us take $|\psi\rangle = \frac{1}{2} [|01\rangle + |10\rangle + |12\rangle + |21\rangle]$. Then the corresponding C_{0H} and C_{0A} can be evaluated to be zero and hence k_H and k_A are also zeros. Now, decomposition of the linear term, $(I \otimes M_1)^\dagger |\phi\rangle\langle\phi|$, in terms of Gell-Mann matrices, is given by

$$(I \otimes M_1)^\dagger |\phi\rangle\langle\phi| = \sum_{i,j=0}^8 C_{ij} \lambda_i \lambda_j,$$

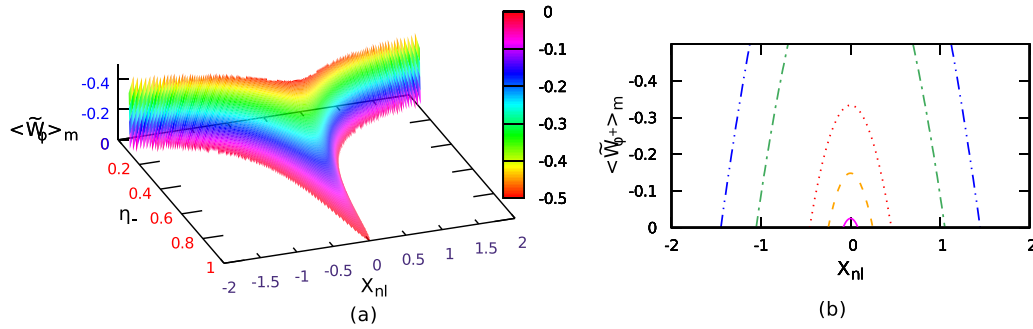


FIG. 2. Closing the detection loophole for a nonlinear witness to detect entanglement in a bound entangled state. Just like in Fig. 1, the detection of entanglement is determined by the triad, $\langle \tilde{W}_\phi \rangle_m$, X_{nl} , and η_- . The entanglement detected is of the state ρ_B , or of one in a small neighborhood of the same. Despite a possible nonideal detector efficiency, the entanglement of any state in this neighborhood is detected whenever $\langle \tilde{W}_\phi \rangle_m$, X_{nl} , and η_- lie in the region above the plotted surface in panel (a), or $\langle \tilde{W}_\phi \rangle_m$ and X_{nl} lie outside a plotted curve in panel (b) for the corresponding fixed value of η_- . In panel (a), different values of $\langle \tilde{W}_\phi \rangle_m$ have been indicated using different colors as shown in the colorbox. The curves in panel (b) have been plotted for some particular values of η_- . The blue double-dot-dashed, green dot-dashed, red dotted, orange dashed, and magenta continuous lines are, respectively, for $\eta_- = 0.15, 0.2, 0.4, 0.6,$ and 0.9 . Here again, $X_{nl} = (\langle H \rangle_m^2 + \langle A \rangle_m^2)^{\frac{1}{2}}$. All quantities are dimensionless.

where λ_i 's are the 3×3 identity operator [for $i = 0$] and Gell-Mann matrices [for $i = 1, \dots, 8$]. C_{ij} can be evaluated from the relation

$$\begin{aligned} \text{tr}[(I \otimes M_1)^+ |\phi\rangle\langle\phi| (\lambda_i \otimes \lambda_j)] &= C_{ij} \text{tr}(\lambda_i^2) \text{tr}(\lambda_j^2) \\ \Rightarrow \text{tr}[|\phi\rangle\langle\phi| (I \otimes M_1) (\lambda_i \otimes \lambda_j)] &= C_{ij} \text{tr}(\lambda_i^2) \text{tr}(\lambda_j^2). \end{aligned} \quad (6)$$

Now, M_1 maps $\lambda_0 \rightarrow 2\lambda_0$. Hence putting $i = j = 0$ in (6), we get $C_{00} = \frac{2}{9}$. Then from the inequality (4), we get

$$\langle F \rangle_m \leq \frac{2}{9} \left(1 - \frac{1}{\eta_-}\right) + 4(\eta_- - 1) [\langle H \rangle_m^2 + \langle A \rangle_m^2].$$

Representing the above relation in terms of the linear witness operator, we get

$$\langle \tilde{W}_\phi \rangle_m \leq \frac{2}{9} \left(1 - \frac{1}{\eta_-}\right) + 4\eta_- [\langle H \rangle_m^2 + \langle A \rangle_m^2]. \quad (7)$$

This is the condition for detecting bound entangled states in real experiments in which the detector does not work ideally. The boundary beyond which the nonlinear witness operator can detect entanglement of ρ_B is shown in Fig. 2. We can see that as the measured value of X_{nl}^2 increases from 0 to 4, i.e., the value of X_{nl} increases from 0 to 2 or decreases from 0 to -2 , the chance of detecting entanglement increases, i.e., it increases with increase in the measured value of nonlinear terms. For example, suppose that the value of $\langle \tilde{W}_\phi \rangle_m$ is zero. Then, if the nonlinear term X_{nl} is 0.5, detection of entanglement is possible for $\eta_- \gtrsim 0.37$. For the same value of $\langle \tilde{W}_\phi \rangle_m$, if the nonlinear term X_{nl} attains a higher value of, say,

1.0, the same detection is possible for the larger range of the efficiency, viz., $\eta_- \gtrsim 0.21$.

We can also conclude by observing the figures or from relation (7) that the nonlinear witness constructed for detecting the bound entangled state is better than its corresponding linear witness for any nonzero value of the lost event efficiency.

VI. CONCLUSION

We found conditions for detection of entanglement in bipartite quantum states using nonlinear witness operators in situations where the detectors have nonideal, but known, efficiency. The method is related to the way that the detection loophole is dealt with in experiments looking for violation of Bell inequalities, and for detection of entanglement using linear entanglement witnesses. While the method followed can be generalized to several other situations, we have first dealt with the case of detecting entangled states with a nonpositive partial transpose by using a nonlinear witness operator related to the positive partial transpose criterion. We have then found nonlinear entanglement witnesses for a bound entangled state, and have subsequently derived conditions for it to perform the detection in presence of errors. In both the cases, the nonlinear witnesses turn out to be more efficient in detecting entanglement, even for nonideal efficiencies, than their linear counterparts.

ACKNOWLEDGMENT

We acknowledge useful discussions with Aditi Sen(De).

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