Violation of Leggett-Garg-type inequalities in a driven two-level atom interacting with a squeezed thermal reservoir

Javid Naikoo,[1,*] Subhashish Banerjee,[1,†] and Arun M. Jayannavar[2,‡]
[1] *Indian Institute of Technology Jodhpur, Jodhpur 342011, India*
[2] *Institute of Physics, Bhubaneswar 751005, India*

(Received 20 August 2019; published 30 December 2019)

The violation of Leggett-Garg-type inequalities (LGtIs) is studied on a two-level atom, driven by an external field in the presence of a squeezed thermal reservoir. The violations are observed in the underdamped regime where the spontaneous transition rate is much smaller compared to the Rabi frequency. An increase in thermal effects is found to decrease the extent of violation as well as the time over which the violation lasts. With the increase in the value of the squeezing parameter the extent of violation of LGtIs is seen to reduce. The violation of LGtIs is favored by an increase in the driving frequency. Further, the interplay of the degree of violation and strength of the measurements is studied. It is found that the maximum violation occurs for ideal projective measurements.

DOI: 10.1103/PhysRevA.100.062132

I. INTRODUCTION

Quantum mechanics is so far the most elegant interpretation of nature whose predictions have been verified in various experiments. Central to quantum mechanics are the notions like coherence and entanglement arising from the superposition principle [1,2]. Various approaches have been developed for quantification of quantumness leading to computable measures of nonclassicality [3,4]. Another way of assessing the quantum coherent evolution is via inequalities based on the time correlation functions, know as Leggett-Garg inequalities (LGIs).

The LGIs have been developed to test the quantum coherence at a macroscopic level [5,6]. These inequalities are based on the assumptions of *macrorealism* and *noninvasive* measurability. The former assigns well-defined macroscopically distinct states to an observable irrespective of the observation, while the latter ensures that the postmeasurement dynamics is unaffected by the act of measurement. A quantum-mechanical system does not obey these assumptions. The superposition principle violates *macrorealism* and the collapse postulate nullifies the possibility of a noninvasive measurement.

The verification of LGIs involve a single system being measured at different times unlike the Bell inequality which involves multiple parties spatially separated from each other [7]. The simplest Leggett-Garg inequality is the one corresponding to three time measurements made at times t_0, t_1, and t_2 such that $t_0 < t_1 < t_2$. For a dichotomic operator $\hat{\mathcal{M}}(t)$, we define the two time correlation function $C(t_i, t_j) = \langle \hat{\mathcal{M}}(t_i)\hat{\mathcal{M}}(t_j)\rangle = \mathrm{Tr}[\rho \hat{\mathcal{M}}(t_i)\hat{\mathcal{M}}(t_j)]$. For the three time measurement case, we define the following combination of the two time correlation functions $K_3 = C(t_0, t_1) + C(t_1, t_2) - C(t_0, t_2)$, such that the simplest LGI reads

$$-3 \leqslant K_3 \leqslant 1. \tag{1}$$

A violation of either the lower or the upper bound is a signature of the "quantumness" of the system. The two time correlation function can be evaluated as follows:

$$C(t_i, t_j) = \sum_{m,n=\pm} mn\, \mathrm{Tr}\{\Pi^m \mathcal{E}_{t_j \leftarrow t_i}[\Pi^n \rho(t_i) \Pi^n]\}. \tag{2}$$

Here, $\mathcal{E}_{t_b \leftarrow t_a}$ is the map governing the time evolution of the state, i.e., $\rho(t_b) = \mathcal{E}_{t_b \leftarrow t_a}[\rho(t_a)]$. The LGIs have been part of many theoretic [8–19] and experimental [20–27] studies.

In this work, we deviate from the original formulation of LGIs and study instead a variant form of it, known as Leggett-Garg-type inequalities (LGtIs) introduced in [28–30] and experimentally verified in [31,32]. These inequalities were derived to avoid the requirement of noninvasive measurements at intermediate times. This feature makes them more suitable for the experimental verification as compared to LGIs. The assumption of NIM is replaced by a weaker condition known as *stationarity*. This asserts that the conditional probability $p(\phi, t_j|\psi, t_i)$ that the system is found in state ϕ at time t_j given that it was in state ψ at time t_i is a function of the time difference $(t_j - t_i)$. Invoking stationarity leads to the following form of LGtIs:

$$K_\pm = \pm 2C(t_0, t) - C(t_0, 2t) \leqslant 1. \tag{3}$$

Here, $t = t_2 - t_1 = t_1 - t_0$ is the time between two successive measurements. From here on, we will call $K_\pm$ as LG parameter. Though the assumption of stationarity helps to put the inequalities into easily testable forms, it reduces the class of macrorealist theories which are put to the test [28]. The stationarity condition holds provided the system can be prepared in a well-defined state and the system evolves under Markovian dynamics. These conditions are satisfied in the

*naikoo.1@iitj.ac.in
†subhashish@iitj.ac.in
‡jayan@iopb.res.in

model considered in this work. Therefore, for a suitable experimental setup, inequalities (3) provide a tool to quantitatively probe the coherence effects in this system.

Here we study the violation of LGtIs in a driven two-level atom interacting with a squeezed thermal reservoir. Such studies are motivated by the fact that LGtIs provide a way to probe the degree of coherence in a system. Interestingly, the two time correlation functions can be written in terms of experimentally observable quantities. The recent upsurge in the studies of LGtIs has increased considerably leading, for example, to the possible applications of LGtIs violation for ensuring security in quantum key distribution schemes. Further, LGtIs also serve to probe the applicability of the models of unsharp measurements pertaining to nonideal measurement setups [33]. These studies become even more pertinent from a practical perspective when one takes into account open system effects. Thus, for example, the effect of temperature, squeezing and driving frequencies, as well as the role of the strength of measurement on LGtI violation in a paradigm model of quantum optics, as done here, should pave the way for developing our understanding of the multifaceted role of various parameters on the inherent quantumness of the system, thereby helping in *characterizing* the quantumness. This would be particularly relevant from the point of view of applying such systems towards quantum technologies.

The paper is organized as follows. In Sec. (II), we discuss in detail the model considered. Section III is devoted to the description of LGtIs in the context of the model considered. The results and their discussion are given in Sec. IV. We conclude in Sec. V.

II. MODEL: A DRIVEN TWO-LEVEL SYSTEM

Here, we sketch the essential details of a driven two-level system in contact with a squeezed thermal bath [34–38]. The model consists of a two-level system whose Hilbert space is spanned by two states, the ground state $|g\rangle$ and the excited state $|e\rangle$, Fig. 1. The description of such a system is analogous to that of a spin-$\frac{1}{2}$ system. The Pauli operators in terms of these basis vectors are $\sigma_1 = |e\rangle\langle g| + |g\rangle\langle e|$, $\sigma_2 = -i|e\rangle\langle g| + i|g\rangle\langle e|$, and $\sigma_3 = |e\rangle\langle e| - |g\rangle\langle g|$, and satisfy the usual commutation $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ and the anticommutation $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$. The raising and lowering operators can be defined as

$$\sigma_+ = |e\rangle\langle g| = \tfrac{1}{2}(\sigma_1 + i\sigma_2),$$
$$\sigma_- = |g\rangle\langle e| = \tfrac{1}{2}(\sigma_1 - i\sigma_2). \tag{4}$$

With this setting, we can define the system Hamiltonian H_S to be diagonal in basis $\{|e\rangle, |g\rangle\}$. With ω_0 denoting the transition frequency between the two levels (setting $\hbar = 1$), we have

$$H_S = \tfrac{1}{2}\omega_0\sigma_3. \tag{5}$$

A detailed account of two-level systems and their application can be found in [39].

We now consider the case when a two-level atomic transition $|e\rangle \leftrightarrow |g\rangle$ is driven by an external source. The source is assumed to be a coherent single mode field on resonance. Under dipole approximation, the Hamiltonian (in the interaction picture) is given by $H_L = -\vec{E}_L(t)\cdot\vec{D}(t)$. Here, $\vec{E}_L(t) = \vec{\epsilon}e^{-i\omega_0 t} + \vec{\epsilon}^* e^{+i\omega_0 t}$ is the electric-field strength of the driving mode. Also, $\vec{D}(t) = \vec{d}\sigma_- e^{-i\omega_0 t} + \vec{d}^*\sigma_+ e^{+i\omega_0 t}$ is the atomic dipole operator in the interaction picture and $\vec{d} = \langle g|\vec{D}|e\rangle$ is the transition matrix element of the dipole operator. The atom-field interaction can be written in the rotating wave approximation as follows:

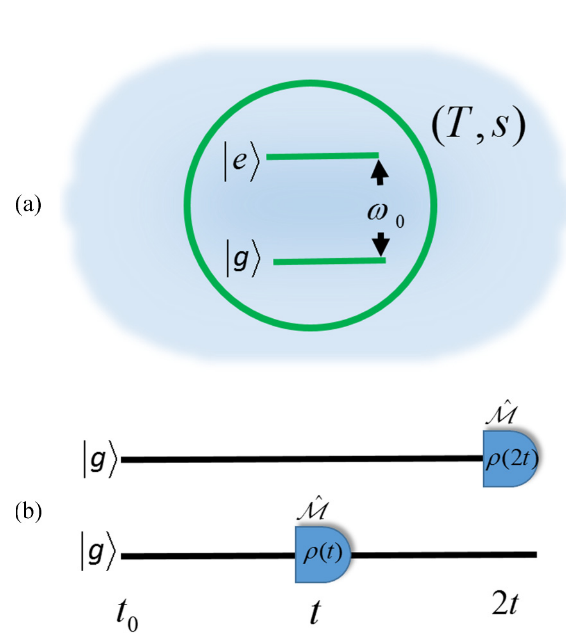

FIG. 1. Schematic diagram for (a) two-level atom interacting with a squeezed thermal bath at temperature T with squeezing parameter s. The transition frequency between the two levels is ω_0. (b) Testing the LGtIs using the statistics of two experiments, with the same preparation state $|g\rangle$ at time $t_0 = 0$. The dichotomic observable $\hat{\mathcal{M}} = |g\rangle\langle g| - |e\rangle\langle e|$ would lead to $+1$ if the atom is found in ground state and -1 otherwise. For example, at t_0, we have $\langle\hat{\mathcal{M}}\rangle = +1$.

$$H_L = -\frac{\Omega}{2}(\sigma_+ + \sigma_-). \tag{6}$$

Here, $\Omega = 2\vec{\epsilon}\cdot\vec{d}^*$ is referred to as the Rabi frequency. Now coupling the system to a thermal reservoir leads to the quantum master equation

$$\begin{aligned}\frac{d\rho(t)}{dt} &= \frac{i\Omega}{2}[\sigma_+ + \sigma_-, \rho(t)]\\ &\quad + \gamma_0 n\big(\sigma_+\rho(t)\sigma_- - \tfrac{1}{2}\sigma_-\sigma_+\rho(t) - \tfrac{1}{2}\rho(t)\sigma_-\sigma_+\big)\\ &\quad + \gamma_0(n+1)\big(\sigma_-\rho(t)\sigma_+ - \tfrac{1}{2}\sigma_+\sigma_-\rho(t) - \tfrac{1}{2}\rho(t)\sigma_+\sigma_-\big)\\ &\quad - \gamma_0 M\sigma_+\rho(t)\sigma_+ - \gamma_0 M^*\sigma_-\rho(t)\sigma_-. \end{aligned} \tag{7}$$

Here, $\gamma = \gamma_0(2n+1)$ is the total transition rate with γ_0 being the spontaneous emission rate. Further,

$$n = n_{th}[\cosh^2(s) + \sinh^2(s)] + \sinh^2(s),$$
$$\text{and } M = -\cosh(s)\sinh(s)e^{i\theta}(2n_{th}+1), \tag{8}$$

where s and θ are the squeezing parameters and $n_{th} = 1/(\exp[\beta\omega_0] - 1)$ is the Plank distribution at transition frequency. In what follows, we will set $\theta = 0$ for the purpose of calculations.

In order to solve Eq. (7), we write the density matrix as

$$\rho(t) = \frac{1}{2}[\mathbf{I} + \vec{v}(t)\cdot\vec{\sigma}] = \begin{pmatrix} \frac{1}{2}(1+\langle\sigma_3\rangle) & \langle\sigma_-\rangle \\ \langle\sigma_+\rangle & \frac{1}{2}(1-\langle\sigma_3\rangle)\end{pmatrix}, \tag{9}$$

with $\vec{v}(t) = \langle\vec{\sigma}(t)\rangle = \mathrm{Tr}[\vec{\sigma}\rho(t)]$, is known as the Bloch vector. With this notation, the master equation (7) becomes

$$\frac{d}{dt}\langle\vec{\sigma}(t)\rangle = \mathcal{G}\langle\vec{\sigma}(t)\rangle + \vec{m}. \tag{10}$$

Here,

$$\mathcal{G} = \begin{pmatrix} -\frac{\gamma}{2} - \gamma_0 M & 0 & 0 \\ 0 & -\frac{\gamma}{2} + \gamma_0 M & \Omega \\ 0 & -\Omega & -\gamma \end{pmatrix}, \tag{11}$$

and $\vec{m} = [0\ \ 0\ \ -\gamma_0]^T$, with T being the transpose operation.

The differential equation (10) has the stationary solution given by

$$\langle\sigma_3\rangle_s = -\frac{\gamma_0(\gamma - 2\gamma_0 M)}{\gamma^2 - 2\gamma\gamma_0 M + 2\Omega^2},$$
$$\langle\sigma_+\rangle_s = -\frac{i\gamma_0\Omega}{\gamma^2 - 2\gamma\gamma_0 M + 2\Omega^2}. \tag{12}$$

Consequently, the stationary population of the excited state $p_e^s = \frac{1}{2}(1+\langle\sigma_3\rangle_s) = \frac{1}{2}[1 - \frac{\gamma_0(\gamma-2\gamma_0 M)}{\gamma^2-2\gamma\gamma_0 M+2\Omega^2}]$.

In the strong driving limit, $\Omega \gg \gamma_s$, we have $p_e^s = 1/2$ and $\langle\sigma_+\rangle_s = -i\gamma_0/2\Omega$.

In order to solve the time-dependent Bloch equation, Eq. (10), it is convenient to introduce the vector

$$\langle\vec{\Sigma}(t)\rangle = \langle\vec{\sigma}(t)\rangle - \langle\vec{\sigma}\rangle_s. \tag{13}$$

This vector satisfies the homogeneous equation

$$\frac{d}{dt}\langle\vec{\Sigma}(t)\rangle = \mathcal{G}\langle\vec{\Sigma}(t)\rangle. \tag{14}$$

This equation can be easily solved by diagonalizing $\mathcal{G}$, which has the eigenvalues

$$\lambda_1 = -\frac{\gamma}{2} - \gamma_0 M,$$
$$\lambda_{2,3} = \frac{\gamma_0 M}{2} - \frac{3\gamma}{4} \pm i\mu_s, \tag{15}$$

where

$$\mu_s = \sqrt{\Omega^2 - \left(\frac{\gamma_s}{4}\right)^2} \quad \text{with} \quad \gamma_s = \gamma + 2\gamma_0 M. \tag{16}$$

Assuming the atom to be initially in the ground state $\rho(0) = |g\rangle\langle g|$, we have

$$\langle\sigma_3(0)\rangle = -1 \quad \text{or} \quad \langle\Sigma_3(0)\rangle = -1 - \langle\sigma_3\rangle_s, \tag{17}$$

and

$$\langle\sigma_\pm(0)\rangle = 0 \quad \text{or} \quad \langle\Sigma_\pm(0)\rangle = -\langle\sigma_\pm\rangle_s. \tag{18}$$

With these initial conditions, the solution of Eq. (14) is given by

$$\langle\vec{\Sigma}(t)\rangle = \begin{pmatrix} e^{-(\gamma+2\gamma_0 M)t/2}\langle\Sigma_1(0)\rangle \\ e^{(-3\gamma+2\gamma_0 M)t/4}\left[\left(\cos(\mu_s t) + \frac{\gamma+3\gamma_0 M}{4\mu_s}\sin(\mu_s t)\right)\langle\Sigma_2(0)\rangle + \frac{\Omega}{\mu_s}\sin(\mu_s t)\langle\Sigma_3(0)\rangle\right] \\ e^{(-3\gamma+2\gamma_0 M)t/4}\left[\left(1 - \frac{\gamma_0 M}{2\mu_s}\right)\cos(\mu_s t) - \frac{\gamma}{4\mu_s}\sin(\mu_s t)\right]\langle\Sigma_3(0)\rangle + \frac{i\Omega}{\mu_s}e^{(-3\gamma+2\gamma_0 M)t/4}\sin(\mu_s t)[\langle\Sigma_+(0)\rangle - \langle\Sigma_-(0)\rangle] \end{pmatrix}. \tag{19}$$

Having obtained the solution, one can calculate the survival probability of the atom being in the ground state $|g\rangle$, as

$$p_g(t) = \frac{1 - [\langle\Sigma_3(t)\rangle + \langle\sigma_3\rangle_s]}{2}. \tag{20}$$

Further, the degree of coherence is proportional to the off-diagonal element

$$\langle\sigma_+(t)\rangle = \frac{\langle\sigma_1(t)\rangle + i\langle\sigma_2(t)\rangle}{2} + \langle\sigma_+\rangle_s. \tag{21}$$

The dynamics is underdamped or overdamped depending on whether μ_s, defined in Eq. (16), is real or imaginary. As a result, in the underdamped regime, the probabilities as well as the coherence exhibit exponentially damped oscillations, while in the overdamped case, they monotonically approach to their stationary values, Fig. 2. Throughout this paper, we work in units with $\hbar = k_B = 1$.

III. LEGGETT-GARG-TYPE INEQUALITY FOR THE TWO-LEVEL DRIVEN SYSTEM

Let $\mathcal{E}_{t_j\leftarrow t_i}$ be the map corresponding to the evolution given by Eq. (7), such that the system in state $\rho(t_i)$ at time t_i evolves to state $\rho(t_j)$ at some later time $t_j > t_i$,

$$\rho(t_j) = \mathcal{E}_{t_j\leftarrow t_i}[\rho(t_i)]. \tag{22}$$

Let at time t_0 the system be in the ground state $|g\rangle$. We define the dichotomic observable $\hat{\mathcal{M}} = |g\rangle\langle g| - |e\rangle\langle e|$. Thus a measurement of this observable leads to +1 or −1 depending on whether the system is in the ground or excited state, respectively, Fig. 1. We introduce the projectors $\Pi^+ = |g\rangle\langle g|$ and $\Pi^- = |e\rangle\langle e|$, such that $O = \Pi^+ - \Pi^-$. Using Eq. (2), with the notation $t_1 - t_0 = t$, the two time correlation $C(t_0, t_1)$ is

$$\begin{aligned} C(t_0,t_1) &= \mathrm{Tr}[\Pi^+\rho(t_0)]\,\mathrm{Tr}\left\{\Pi^+\mathcal{E}_{t_1\leftarrow t_0}\left[\frac{\Pi^+\rho(t_0)\Pi^+}{\mathrm{Tr}[\Pi^+\rho(t_0)]}\right]\right\} \\ &\quad - \mathrm{Tr}[\Pi^+\rho(t_0)]\,\mathrm{Tr}\left\{\Pi^-\mathcal{E}_{t_1\leftarrow t_0}\left[\frac{\Pi^+\rho(t_0)\Pi^+}{\mathrm{Tr}[\Pi^+\rho(t_0)]}\right]\right\} \\ &\quad - \mathrm{Tr}[\Pi^-\rho(t_0)]\,\mathrm{Tr}\left\{\Pi^+\mathcal{E}_{t_1\leftarrow t_0}\left[\frac{\Pi^-\rho(t_0)\Pi^-}{\mathrm{Tr}[\Pi^-\rho(t_0)]}\right]\right\} \\ &\quad + \mathrm{Tr}[\Pi^-\rho(t_0)]\,\mathrm{Tr}\left\{\Pi^-\mathcal{E}_{t_1\leftarrow t_0}\left[\frac{\Pi^-\rho(t_0)\Pi^-}{\mathrm{Tr}[\Pi^-\rho(t_0)]}\right]\right\}, \\ &= p_g(t) - p_e(t) = 2p_g(t) - 1. \end{aligned} \tag{23}$$

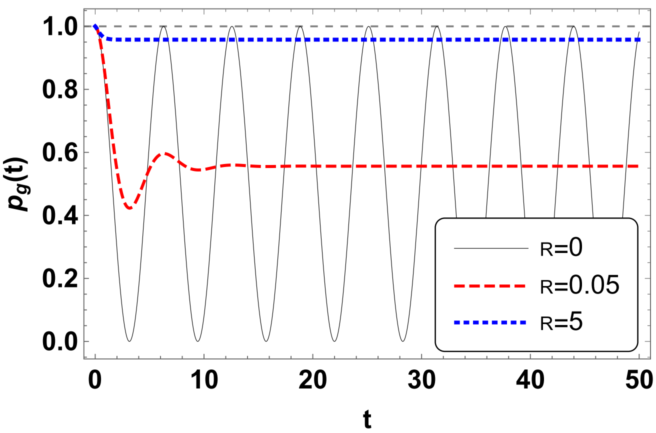

FIG. 2. Probability of finding the atom in ground state at time t, in the units with $\hbar = k_B = 1$. Here, $R = \gamma_0/\Omega$ is the ratio of the spontaneous emission to the Rabi frequency. With squeezing parameter $s = 0$ and transition frequency $\omega_0 = 0.5$, the values $R = 0$, 0.05, and 5 correspond to $\mu_s = 1$, 0.9 (underdamped), and $0.7i$ (overdamped), respectively.

Plugging in the expressions of probabilities, we have

$$K_\pm = \pm 2\mathcal{F}(t) - \mathcal{F}(2t) \mp 1. \tag{24}$$

Here,

$$\mathcal{F}(t) = \mathcal{A}[\mathcal{B} + \mathcal{C}e^{-(3\gamma - 2\gamma_0 M)t/4}\cos(\mu_s t) + \mathcal{D}\sin(\mu_s t)] - 1, \tag{25}$$

with coefficients given by

$$\begin{aligned}
\mathcal{A} &= [4\mu_s(\gamma^2 - 2\gamma\gamma_0 M + 2\Omega^2)]^{-1},\\
\mathcal{B} &= 4(\gamma + \gamma_0)(\gamma - 2\gamma_0 M)\mu_s + 8\mu_s\Omega^2,\\
\mathcal{C} &= -2(\gamma_0 M - 2\mu_s)[(\gamma - \gamma_0)(\gamma - 2\gamma_0 M) + 2\Omega^2],\\
\mathcal{D} &= -\gamma(\gamma - \gamma_0)(\gamma - 2\gamma_0 M) - 2(\gamma - 4\gamma_0)\Omega^2.
\end{aligned} \tag{26}$$

In the strong driving limit, $\Omega \gg \gamma_s$, the coefficients can be approximated as $\mathcal{A} \approx \Omega^{-3}$, $\mathcal{B} \approx \mathcal{C} \approx \Omega^3$, and $\mathcal{D} \approx \Omega^2$, such that in this limit, $\mathcal{F}(t) \propto \cos(\Omega t)$ and therefore

$$K_\pm \approx \pm 2\cos(\Omega t) - \cos(2\Omega t). \tag{27}$$

Effect of weak measurement. The two time correlation function $C(t_0, t)$, Eq. (23), was obtained by assuming that the measurements are ideal or projective. However, it would be interesting to see how weak measurements affect the behavior of $C(t_0, t)$ and thereby of the LG parameters $K_\pm$. The weak measurements are characterized by invoking a parameter ξ [40,41], such that the ideal projectors $\Pi^\pm$ are replaced by the "weak projectors" $W^\pm$ defined as

$$W^\pm = \left(\frac{1 \pm \xi}{2}\right)\Pi^+ + \left(\frac{1 \mp \xi}{2}\right)\Pi^-. \tag{28}$$

Here, $0 < \xi \leqslant 1$, such that when $\xi = 1$, $W^\pm$ reduce to the ideal projection operators $\Pi^\pm$. Invoking weak projectors leads to the following form of the two time correlation function: $C(t_0, t)|_{\text{weak}} = \xi^2 C(t_0, t)$, and consequently

$$K_\pm|_{\text{weak}} = \xi^2 K_\pm. \tag{29}$$

Therefore, the maximum violation of LGtIs occurs for an ideal projective measurement.

IV. RESULTS AND DISCUSSION

The LGtIs given by inequality (3) are studied in the context of a two-level atom with the ground and excited states labeled as $|g\rangle$ and $|e\rangle$, respectively. An external field is driving the transition between the two levels. Further, the atom is allowed to interact with a squeezed thermal bath. The inequalities thus obtained are in terms of experimentally relevant parameters. The violation of LGtIs occurs predominantly in the underdamped regime which is characterized by the real values of parameter μ_s defined in Eq. (16), such that

$$\begin{aligned}
\Omega &> \frac{\gamma_s}{4} = \gamma_0\frac{(2n+1)+2M}{4} \quad \text{underdamped},\\
\Omega &< \frac{\gamma_s}{4} = \gamma_0\frac{(2n+1)+2M}{4} \quad \text{overdamped}.
\end{aligned} \tag{30}$$

Here, the parameter $\gamma_s = \gamma_0[(2n+1)+2M]$, as defined in Eq. (16). Figure 3 depicts the behavior of LG parameters $K_\pm$ with respect to time t, for different values of the ratio $R = \gamma_0/\Omega$. The violations of LGtIs are observed mainly in the underdamped regime and fade quickly with the increase in R. In other words, strong driving favors the violation of LGtIs to their maximum quantum bound. The right most panel of the figure shows coherence paramter C [42,43] which is defined as

$$C = \sum_{i \neq j} |\rho_{ij}|. \tag{31}$$

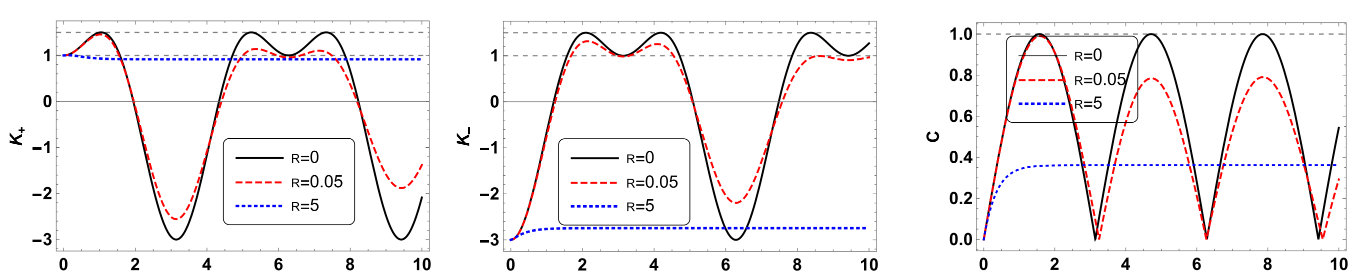

FIG. 3. Evolution of the LG parameters K_+ (left), K_- (middle), and coherence parameter C (right). Here, $\beta = 10$, $\omega_0 = 0.5$, $s = 0$, such that $R = 0$, 0.05, and 5 correspond to $\mu_s = 1$, 0.9 (underdamped), and $0.7i$ (overdamped) cases, respectively. The violation of LGtIs occurs predominantly in underdamped regime such that $K_\pm$ reach their quantum bound 3/2 as $R \to 0$. The coherence parameter shows exponentially damped oscillations in underdamped regime, while in overdamped case, it monotonically saturates to it stationary value.

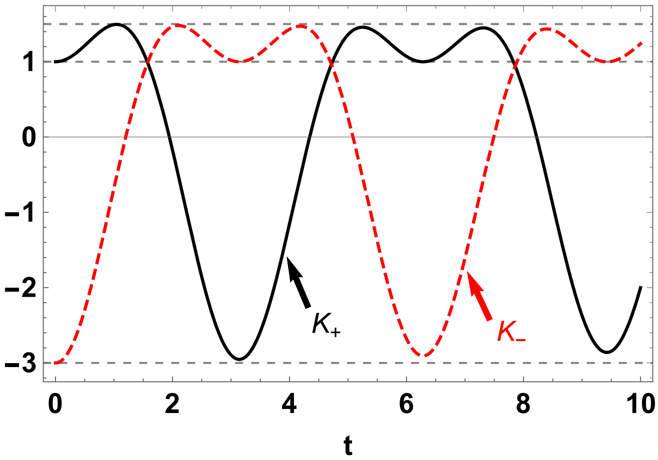

FIG. 4. Complementary behavior of LG parameters $K_{\pm}$ in the strong driving limit. The various parameters used are $\beta = 10$, $\omega_0 = 0.5$, $s = 0$, $R = 0.005$, pertaining to the underdamped regime.

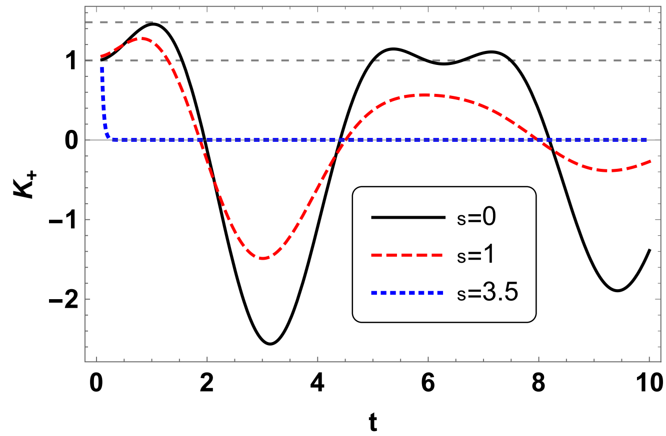

FIG. 6. The LG parameter K_+ for different values of the squeezing parameter s. Here, $\beta = 100$, $\omega_0 = 0.5$, $R = 0.05$. Further, $s = 0$, 1, and 3.5 correspond to $\mu_s = 1$, 0.9 (underdamped), and $6.7i$ (overdamped), respectively.

The extent of violation of LGtIs can be seen as a signature of the degree of coherence in the system.

In the strong driving limit, i.e., $\Omega \gg \gamma_s$, the LG parameters are given by Eq. (27) and are plotted in Fig. 4. The parameters K_+ and K_- show complementary behavior in the sense that when one of these parameters does not show a violation, the other does, together covering the entire parameter range.

The interaction with the squeezed thermal reservoir leads to enhancement in the transition rate which is given by $\gamma = \gamma_0(2n+1)$, where γ_0 is the spontaneous emission rate and $\gamma_0 n$ is the squeezed thermal induced emission and absorption rate. The interactions with the reservoir are expected to decrease the quantumness in the system. This feature is depicted in Fig. 5, where K_+ shows enhanced violations for larger values of the parameter β i.e., for smaller temperature.

The squeezing parameter as defined in Eq. (8) controls the degree of violation of LGtIs, since it affects the total photon distribution. Figure 6 exhibits the variation of the LG parameter K_+ for different values of squeezing parameter s. The increase in s is found to decrease the extent of violation of LGtIs.

The effect of weak measurement on the LG parameters is depicted in Fig. 7. The ideal projective measurements are characterized by $\xi = 1$, while $\xi = 0$ corresponds to no measurement. It is clear from the figure that the maximum violation occurs for ideal projective measurements.

In [44], general evolution of an atom in squeezed vacuum was analyzed and the experimental studies of two-level systems in vacuum were reported in [45,46]. Here, consideration of the effect of various parameters such as temperature and external driving on the LGtI violation helps in developing a better understanding of the quantumness of the system under consideration, under ambient conditions.

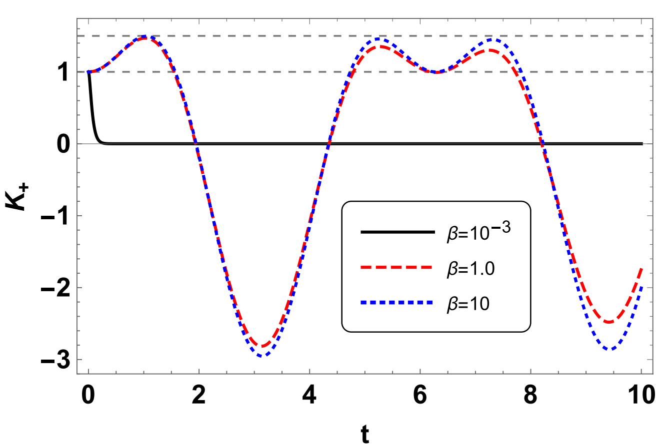

FIG. 5. Temperature dependence of LG parameter K_+. With $\omega_0 = 0.5$, $s = 0$, and $R = 0.005$, the values $\beta = 10$, 1, and 10^{-3} correspond to $\mu_s = 1$, 0.9 (underdamped), and $4.8i$ (overdamped), respectively.

V. CONCLUSION

We studied the violation of Leggett-Garg-type inequalities in a driven two-level atom interacting with a squeezed thermal bath. The effect of various experimentally relevant parameters

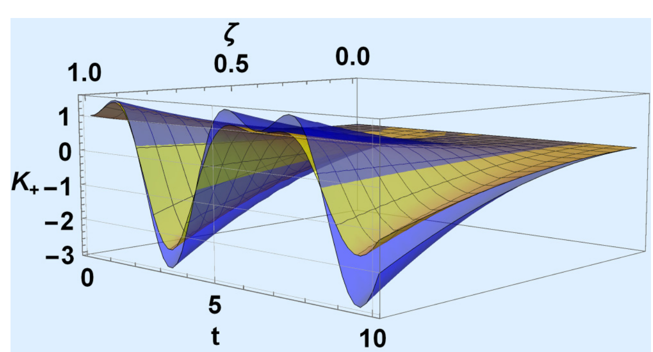

FIG. 7. Variation LG parameter K_+ with respect to t and ξ. With $\beta = 5$, $\omega_0 = 0.5$, and $s = 0$, we have $R = 0$ ($\mu_s \approx 1$) depicted by blue plane surface, and $R = 0.05$ ($\mu_s \approx 0.9$) represented by yellow lined surface. Both these correspond to underdamped case. The maximum violation corresponds to $\xi = 1$, the ideal projective measurement.

on the violation of the inequality were examined carefully. The violations were seen to be prominent in the underdamped case. The increase in temperature was found to decrease the degree of violation as well as the time over which the violation is sustained. Squeezing the thermal state of the reservoir was also found to reduce the violation of LGtIs. Enhanced violations, reaching to the quantum bound, were witnessed in the strong driving limit. Further, we studied the effect of the weak measurements on the extent of violation of LGtI. The weak measurements are characterized by the parameter ξ such that $\xi = 0$ ($\xi = 1$) corresponds to no measurement (ideal projective measurement). The maximum violation was found to occur for the ideal projective measurements. The current study therefore highlights the role of various external parameters on the quantumness of the system.

ACKNOWLEDGMENT

A.M.J. thanks DST India for a J. C. Bose National Fellowship.

[1] E. Schrödinger, Naturwissenschaften **23**, 823 (1935).
[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[3] J. S. Bell, Phys. Phys. Fiz. **1**, 195 (1964).
[4] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[5] A. J. Leggett and A. Garg, Phys. Rev. Lett. **54**, 857 (1985).
[6] C. Emary, N. Lambert, and F. Nori, Rep. Prog. Phys. **77**, 016001 (2013).
[7] I. Chakrabarty, S. Banerjee, and N. Siddharth, Quantum Inf. Comput. **11**, 541 (2011).
[8] M. Barbieri, Phys. Rev. A **80**, 034102 (2009).
[9] D. Avis, P. Hayden, and M. M. Wilde, Phys. Rev. A **82**, 030102(R) (2010).
[10] N. Lambert, C. Emary, Y.-N. Chen, and F. Nori, Phys. Rev. Lett. **105**, 176801 (2010).
[11] N. Lambert, R. Johansson, and F. Nori, Phys. Rev. B **84**, 245421 (2011).
[12] A. Montina, Phys. Rev. Lett. **108**, 160501 (2012).
[13] J. Kofler and Č. Brukner, Phys. Rev. A **87**, 052115 (2013).
[14] C. Budroni, T. Moroder, M. Kleinmann, and O. Gühne, Phys. Rev. Lett. **111**, 020403 (2013).
[15] S. Kumari and A. Pan, Europhys. Lett. **118**, 50002 (2017).
[16] J. Naikoo, A. K. Alok, and S. Banerjee, Phys. Rev. D **97**, 053008 (2018).
[17] J. Naikoo, A. K. Alok, S. Banerjee, and S. U. Sankar, Phys. Rev. D **99**, 095001 (2019).
[18] J. Naikoo, S. Banerjee, and R. Srikanth, arXiv:1806.00537.
[19] J. Naikoo and S. Banerjee, Eur. Phys. J. C **78**, 602 (2018).
[20] A. Palacios-Laloy, F. Mallet, F. Nguyen, P. Bertet, D. Vion, D. Esteve, and A. N. Korotkov, Nat. Phys. **6**, 442 (2010).
[21] J. P. Groen, D. Ristè, L. Tornberg, J. Cramer, P. C. de Groot, T. Picot, G. Johansson, and L. DiCarlo, Phys. Rev. Lett. **111**, 090506 (2013).
[22] M. Goggin, M. Almeida, M. Barbieri, B. Lanyon, J. O'Brien, A. White, and G. Pryde, Proc. Natl. Acad. Sci. USA **108**, 1256 (2011).
[23] J. Dressel, C. J. Broadbent, J. C. Howell, and A. N. Jordan, Phys. Rev. Lett. **106**, 040402 (2011).
[24] Y. Suzuki, M. Iinuma, and H. F. Hofmann, New J. Phys. **14**, 103022 (2012).
[25] V. Athalye, S. S. Roy, and T. S. Mahesh, Phys. Rev. Lett. **107**, 130402 (2011).
[26] A. Souza, I. Oliveira, and R. Sarthour, New J. Phys. **13**, 053023 (2011).
[27] H. Katiyar, A. Shukla, K. R. K. Rao, and T. S. Mahesh, Phys. Rev. A **87**, 052102 (2013).
[28] S. F. Huelga, T. W. Marshall, and E. Santos, Phys. Rev. A **52**, R2497(R) (1995).
[29] S. F. Huelga, T. W. Marshall, and E. Santos, Phys. Rev. A **54**, 1798 (1996).
[30] G. Waldherr, P. Neumann, S. F. Huelga, F. Jelezko, and J. Wrachtrup, Phys. Rev. Lett. **107**, 090401 (2011).
[31] J.-S. Xu, C.-F. Li, X.-B. Zou, and G.-C. Guo, Sci. Rep. **1**, 101 (2011).
[32] Z.-Q. Zhou, S. F. Huelga, C.-F. Li, and G.-C. Guo, Phys. Rev. Lett. **115**, 113002 (2015).
[33] D. Home, Curr. Sci. **109**, 1980 (2015).
[34] R. Srikanth and S. Banerjee, Phys. Rev. A **77**, 012318 (2008).
[35] S. Banerjee and R. Srikanth, Eur. Phys. J. D **46**, 335 (2008).
[36] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
[37] S. Banerjee, *Open Quantum Systems: Dynamics of Nonclassical Evolution* (Springer, New York, 2018), Vol. 20.
[38] S. Omkar, R. Srikanth, and S. Banerjee, Quantum Inf. Process. **12**, 3725 (2013).
[39] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics: The New Millennium Edition: Quantum Mechanics*, The Feynman Lectures on Physics (Basic Books, New York, 2011), Vol. III.
[40] P. Busch, Phys. Rev. D **33**, 2253 (1986).
[41] D. Saha, S. Mal, P. K. Panigrahi, and D. Home, Phys. Rev. A **91**, 032117 (2015).
[42] A. K. Alok, S. Banerjee, and S. U. Sankar, Nucl. Phys. B **909**, 65 (2016).
[43] S. Bhattacharya, S. Banerjee, and A. K. Pati, Quantum Inf. Process. **17**, 236 (2018).
[44] H. J. Carmichael, A. S. Lane, and D. F. Walls, Phys. Rev. Lett. **58**, 2539 (1987).
[45] K. Murch, S. Weber, K. Beck, E. Ginossar, and I. Siddiqi, Nature (London) **499**, 62 (2013).
[46] D. M. Toyli, A. W. Eddins, S. Boutin, S. Puri, D. Hover, V. Bolkhovsky, W. D. Oliver, A. Blais, and I. Siddiqi, Phys. Rev. X **6**, 031004 (2016).