Information-theoretic approach to atomic spin nonclassicality

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Nonclassicality of light has been studied, both extensively and intensively, in terms of optical coherent states, and various quantifiers for nonclassicality of light have been introduced in recent decades. In contrast, the corresponding quantification issue for atomic spin systems is relatively less studied, though there has been some remarkable progress. In this paper, by virtue of the Wigner-Yanase skew information and working in the framework of atomic coherent states, we introduce a quantifier for nonclassicality of atomic spin states from an information-theoretic perspective and reveal its fundamental properties. This quantifier leads to a simple yet powerful criterion for detecting nonclassicality via convexity of the skew information. This criterion is sufficient, but not necessary. The concept is illustrated through several paradigms. We further elucidate some intrinsic relations between nonclassicality and entanglement and indicate that the nonclassicality quantifier may be transferred to an entanglement quantifier.

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I. INTRODUCTION

The departure of the quantum from the classical description of reality is usually signified by certain nonclassicality. Depending on different contexts, various meanings have been attributed to nonclassicality, such as the nonclassicality of light (related to the particle aspect of photons) [1–3], nonlocality [4,5], quantum entanglement [6–8], negativity of Wigner functions [9–11], quantum correlations [12–21], and quantumness [22–25]. In this paper, we will be mainly concerned with the well-established meaning of nonclassicality in the sense of quantum optics and atomic spin systems defined via coherent states and will quantify atomic spin nonclassicality from an information-theoretic perspective.

Recall that, in quantum optics, the optical coherent states $|\alpha\rangle$ (as eigenstates of the bosonic annihilation operator a) are the closest to classical states among pure states and they form an overcomplete family [26,27]. All statistical mixtures of coherent states are called classical, while other states are nonclassical. This dichotomy is usually formulated via the Glauber-Sudarshan representation $\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha$ in terms of the P function $P(\alpha)$, which is a quasiprobability and may be negative or highly singular [1–3]. Those states possessing non-negative P functions are regarded as classical states; otherwise they are nonclassical. This kind of nonclassicality of optical states has been widely studied from both theoretical and experimental perspectives [28–32], and quantification of nonclassicality has been pursued from various angles [33–47], though there is no universally accepted measure for nonclassicality.

Coherent states have been generalized extensively, in particular to the atomic (spin) systems [48–51]. The diversity of terminologies for coherent states in atomic systems, such as atomic coherent states, coherent spin states, angular momentum coherent states, Bloch coherent states, and SU(2) coherent states, indicates the wide relevance and usage of these distinguished states. In spite of extensive studies of atomic coherent states, the quantification issue of nonclassicality of atomic states is relatively less studied [52–57]. Classicality and nonclassicality measures of quantum states were studied in Ref. [52] in terms of the maximum of Hilbert-Schmidt scalar products between the state concerned and all displaced thermal states. The concept of classicality in quantum optics was extended to atomic spin states, and nonclassicality witnesses were studied in Ref. [53]. Some distance-based nonclassicality measures were proposed in Ref. [54]. A simple method based on easily accessible collective atomic quadrature measurements to certify nonclassicality was introduced in Ref. [55]. A criterion of nonclassicality in terms of expectation values was developed in Ref. [56]. The quantumness of spin-1 states defined as the Hilbert-Schmidt distance to the convex hull of atomic coherent states was investigated in Ref. [57]. All these studies provide considerable insight into the nature of classicality and nonclassicality of atomic spin states. In this paper, we will employ the Wigner-Yanase skew information to quantify the nonclassicality of atomic spin states.

To motivate our study and to highlight the basic idea of our approach, recall that, usually, the ubiquitous variance $V(\rho, K) = \mathrm{tr}\rho K^2 - (\mathrm{tr}\rho K)^2$ is widely used in characterizing the uncertainty of an observable K in a state ρ (pure or mixed). This is a kind of total uncertainty including both classical and quantum uncertainties [58]. In sharp contrast, an information-theoretic refinement of the variance is the skew information [59]

$$I(\rho, K) = -\frac{1}{2}\mathrm{tr}[\sqrt{\rho}, K]^2$$

introduced by Wigner and Yanase in their seminal study of information content of the state ρ skew to the observable K.

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For pure states, the skew information coincides with the variance, while for any mixed state ρ, $I(\rho, K) \leqslant V(\rho, K)$. The difference and similarity between variance and skew information is more transparent if we recast them as

$$V(\rho, K) = \mathrm{tr}\rho K^2 - \mathrm{tr}\rho K \mathrm{tr}\rho K,$$
$$I(\rho, K) = \mathrm{tr}\rho K^2 - \mathrm{tr}\sqrt{\rho}K\sqrt{\rho}K.$$

respectively.

Now it is well recognized that the skew information is a kind of quantum Fisher information [60,61], with operational significance as quantum uncertainty of the observable K in the state ρ [62,63]. It can be used to quantifying correlations [64,65], coherence [66–68], and asymmetry [69,70]. In other words, the skew information is playing an increasingly interesting and significant role in quantum information theory.

This paper is devoted to quantifying nonclassicality of atomic spin states in terms of the skew information and investigating its consequences. The remainder of this paper is structured as follows. In Sec. II we review briefly atomic coherent states and associated notions of classicality and nonclassicality. In Sec. III we quantify the nonclassicality of atomic states (pure as well as mixed) in terms of a certain average of the Wigner-Yanase skew information, reveal its physical meaning, develop a convenient nonclassicality criterion, and indicate potential applications. We illustrate the nonclassicality quantifier through some prototypical examples in Sec. IV. We further elucidate some intrinsic links between nonclassicality and entanglement in Sec. V. Finally, we conclude with a summary and discussion in Sec. VI.

II. ATOMIC COHERENT STATES AND CLASSICAL STATES

Consider an atomic spin system with spin j (an integer or half-integer) with system Hilbert space $\mathbb{C}^{2j+1}$. Here we will be concerned only with the spin observables and ignore other structural and spatial freedoms of atoms. The spin vector observables (angular momenta) $\mathbf{J} = (J_x, J_y, J_z)$ relative to an xyz coordinate satisfy the commutation relations [26,27]

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y.$$

If we let $J_\pm = J_x \pm iJ_y$ be the angular momentum ladder (non-Hermitian) operators, then

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z.$$

The simultaneous eigenstates of the total spin observable (Casimir operator) $\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2$ and the z component J_z are the Dicke states $|j, m\rangle$, $m = -j, -j+1, \dots, j-1, j$, which form an orthonormal basis of the spin-j system space $\mathbb{C}^{2j+1}$. The operations of $\mathbf{J}^2$, $J_\pm$, and J_z on these basis states are

$$\begin{aligned}
\mathbf{J}^2|j, m\rangle &= j(j+1)|j, m\rangle,\\
J_+|j, m\rangle &= \sqrt{(j-m)(j+m+1)}|j, m+1\rangle,\\
J_-|j, m\rangle &= \sqrt{(j+m)(j-m+1)}|j, m-1\rangle,\\
J_z|j, m\rangle &= m|j, m\rangle,
\end{aligned}$$

with the extremal conditions $J_+|j, j\rangle = 0$ and $J_-|j, -j\rangle = 0$. Moreover, the Dicke states can be recast as

$$\begin{aligned}
|j, m\rangle &= \sqrt{\frac{(j-m)!}{(j+m)!(2j)!}}J_+^{j+m}|j, -j\rangle\\
&= \sqrt{\frac{(j+m)!}{(j-m)!(2j)!}}J_-^{j-m}|j, j\rangle.
\end{aligned}$$

For fixed spin number j, the atomic coherent states are defined as [48–50]

$$|\zeta\rangle = e^{\zeta J_+ - \zeta^* J_-}|j, j\rangle,$$

which can be expanded in terms of the Dicke states as

$$|\zeta\rangle = \sum_{m=-j}^{+j} \binom{2j}{j+m}^{1/2} \left(\cos\frac{\theta}{2}\right)^{j+m} \left(e^{i\phi}\sin\frac{\theta}{2}\right)^{j-m} |j, m\rangle,$$

where $\zeta = \frac{\theta}{2}e^{-i\phi}$, $0 \leqslant \theta \leqslant \pi$, and $0 \leqslant \phi < 2\pi$.

Geometrically, there is a one-to-one correspondence between atomic coherent states $|\zeta\rangle$ and points on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ [or, equivalently, SU(2)/U(1)] with the correspondence

$$\zeta = \frac{\theta}{2}e^{-i\phi} \in \mathbb{C} \leftrightarrow \mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in \mathbb{S}^2.$$

Consequently, we may also parametrize the atomic coherent states by $|\mathbf{n}\rangle \in \mathbb{S}^2$. More explicitly,

$$|\mathbf{n}\rangle = e^{-i\theta(J_x\sin\phi - J_y\cos\phi)}|j, j\rangle.$$

A state ρ is called classical if it can be represented as a convex combination of the atomic coherent states; otherwise it is called nonclassical. By the Carathédory theorem on convex sets [71], for the $(2j+1)^2$-dimensional spin operator space, every classical state ρ can be represented as a finite convex sum of atomic coherent states in the form

$$\rho = \sum_{i=1}^{(2j+1)^2} p_i|\mathbf{n}_i\rangle\langle\mathbf{n}_i|,$$

where $p_i \geqslant 0$, $\sum_i p_i = 1$, and $\mathbf{n}_i \in \mathbb{S}^2$.

III. QUANTIFYING NONCLASSICALITY

With the preparation in the preceding section, we now proceed to introduce our key character which serves as a quantifier for nonclassicality of atomic spin states [see the subsequent Eq. (2)].

Consider spin-j systems. For each point $\mathbf{n} = (n_x, n_y, n_z) \in \mathbb{S}^2$ on the unit sphere, let

$$J_\mathbf{n} = \mathbf{n}\cdot\mathbf{J} = n_xJ_x + n_yJ_y + n_zJ_z$$

be the corresponding angular momentum observable along the direction $\mathbf{n}$. For any state (pure or mixed), consider the Wigner-Yanase skew information

$$I(\rho, J_\mathbf{n}) = -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho}, J_\mathbf{n}]^2.$$

Simple manipulation leads to

$$I(\rho, J_\mathbf{n}) = \mathbf{n}\Gamma\mathbf{n}^\mathsf{T}, \tag{1}$$

where the superscript T denotes the transpose of vectors and the 3×3 matrix $\Gamma = (\gamma_{ij})$ has matrix entries

$$\gamma_{ij} = \mathrm{tr}\rho J_i J_j - \mathrm{tr}\sqrt{\rho}J_i\sqrt{\rho}J_j, \quad i, j = x, y, z.$$

In particular,

$$I(\rho, J_{(1,0,0)}) = I(\rho, J_x) = -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho}, J_x]^2,$$
$$I(\rho, J_{(0,1,0)}) = I(\rho, J_y) = -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho}, J_y]^2,$$
$$I(\rho, J_{(0,0,1)}) = I(\rho, J_z) = -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho}, J_z]^2.$$

For any state ρ we define

$$N(\rho) = I(\rho, J_x) + I(\rho, J_y) + I(\rho, J_z) \tag{2}$$

as a quantifier for nonclassicality of ρ. This quantity will play a central role in this paper. It is remarkable that the above quantifier has the following three equivalent forms.

(a) For any three mutually orthogonal directions $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ which form an orthonormal basis for $\mathbb{R}^3$ we have

$$N(\rho) = I(\rho, J_{\mathbf{n}_1}) + I(\rho, J_{\mathbf{n}_2}) + I(\rho, J_{\mathbf{n}_3}). \tag{3}$$

(b) $N(\cdot)$ can be expressed as the average

$$N(\rho) = \frac{3}{4\pi}\int_{\mathbb{S}^2} I(\rho, J_{\mathbf{n}})d\mathbf{n}, \tag{4}$$

where $d\mathbf{n}$ denotes the canonical Haar measure on the unit sphere $\mathbb{S}^2$ with the normalization $\int_{\mathbb{S}^2} d\mathbf{n} = 4\pi$.

(c) Recall that $J_\pm = J_x \pm iJ_y$. Then

$$N(\rho) = -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho}, J_+][\sqrt{\rho}, J_-] + I(\rho, J_z).$$

To establish item (a), from Eq. (1) we have

$$I(\rho, J_{\mathbf{n}_1}) + I(\rho, J_{\mathbf{n}_2}) + I(\rho, J_{\mathbf{n}_3}) = \sum_{i=1}^{3} \mathbf{n}_i\Gamma\mathbf{n}_i^{\mathsf{T}} = \mathrm{tr}\Gamma.$$

The last equality follows from the fact that $\{\mathbf{n}_i : i = 1, 2, 3\}$ constitutes an orthonormal basis for $\mathbb{R}^3$.

To prove item (b), noting that

$$\frac{3}{4\pi}\int_{\mathbb{S}^2} \mathbf{n}^{\mathsf{T}}\mathbf{n}\, d\mathbf{n} = \mathbf{1},$$

with $\mathbf{1}$ the identity operator acting on $\mathbb{R}^3$, it follows that

$$\frac{3}{4\pi}\int_{\mathbb{S}^2} \mathbf{n}^{\mathsf{T}}\mathbf{n}\Gamma\, d\mathbf{n} = \Gamma.$$

By taking the trace and using the cyclic property of the trace, we have

$$\frac{3}{4\pi}\int_{\mathbb{S}^2} \mathbf{n}\Gamma\mathbf{n}^{\mathsf{T}} d\mathbf{n} = \mathrm{tr}\Gamma.$$

Now the desired result follows from Eq. (1) as

$$\frac{3}{4\pi}\int_{\mathbb{S}^2} I(\rho, J_{\mathbf{n}})d\mathbf{n} = \frac{3}{4\pi}\int_{\mathbb{S}^2} \mathbf{n}\Gamma\mathbf{n}^{\mathsf{T}} d\mathbf{n} = \mathrm{tr}\Gamma.$$

Item (c) follows from

$$\begin{aligned}
&-\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho}, J_+][\sqrt{\rho}, J_-]\\
&= -\tfrac{1}{2}\mathrm{tr}([\sqrt{\rho}, J_x] + i[\sqrt{\rho}, J_y])([\sqrt{\rho}, J_x] - i[\sqrt{\rho}, J_y])\\
&= -\tfrac{1}{2}(\mathrm{tr}([\sqrt{\rho}, J_x]^2 + [\sqrt{\rho}, J_y]^2)\\
&= I(\rho, J_x) + I(\rho, J_y).
\end{aligned}$$

In the following, we will fix the spin number j [thus $\mathbf{J}^2 = j(j+1)$] and work in the system Hilbert space $\mathbb{C}^{2j+1}$ of an irreducible representation of the SU(2) angular momentum algebra. In this context, $N(\rho)$ has the following attractive and desirable properties.

(i) $N(\cdot)$ is non-negative and, in particular, $N(\rho) = 0$ if and only if $\rho = \mathbf{1}/(2j+1)$, the maximally mixed state.

(ii) $N(\cdot)$ is convex.

(iii) $N(\cdot)$ is invariant under rotations in the sense that

$$N(e^{-i\theta\mathbf{n}\cdot\mathbf{J}}\rho e^{i\theta\mathbf{n}\cdot\mathbf{J}}) = N(\rho)$$

for any $\theta \in \mathbb{R}$ and any vector $\mathbf{n} \in \mathbb{S}^2$. Geometrically, $e^{-i\theta\mathbf{n}\cdot\mathbf{J}}$ is a rotation of angle θ with rotating axis $\mathbf{n}$ in the Bloch sphere (as well as the Bloch ball).

(iv) $N(|\mathbf{n}\rangle\langle\mathbf{n}|) = j$ for any spin-j atomic coherent state $|\mathbf{n}\rangle$. Moreover, among all spin-j pure states, the atomic coherent states have the minimal nonclassicality, i.e.,

$$N(|\psi\rangle\langle\psi|) \geqslant N(|\mathbf{n}\rangle\langle\mathbf{n}|) = j,$$

for any pure state $|\psi\rangle$ and any atomic coherent state $|\mathbf{n}\rangle$ of spin-j systems.

(v) Among all spin-j pure states $|\psi\rangle$, the nonclassicality $N(|\psi\rangle\langle\psi|)$ achieves its maximal value $j(j+1)$ if and only if the spin expectation (mean spin) vector vanishes in the sense that $\langle\mathbf{J}\rangle = (\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle) = 0$, where $\langle J_x\rangle = \langle\psi|J_x|\psi\rangle$, etc.

(vi) For any classical spin-j state ρ, we have $N(\rho) \leqslant j$. Consequently, if a spin-j state σ satisfies $N(\sigma) > j$, then σ is nonclassical. This is a criterion for detecting nonclassicality, though it is only a sufficient but not necessary condition for a state to be nonclassical in the sense that there exist nonclassical states ρ satisfying $N(\rho) < j$.

(vii) Among all spin-j states (pure or mixed), the maximal value $j(j+1)$ can be achieved by any pure state with vanishing spin expectation.

Now we proceed to prove the above statements.

For item (i), $N(\cdot) \geqslant 0$ is apparent by definition. If $N(\rho) = 0$, then $I(\rho, J_x) = I(\rho, J_y) = I(\rho, J_z) = 0$, which implies that ρ commutes with J_x, J_y, and J_z simultaneously, that is, $[\rho, J_x] = [\rho, J_y] = [\rho, J_z] = 0$. Since we are working in an irreducible representation space $\mathbb{C}^{2j+1}$ of the angular momentum algebra, it follows that ρ is proportional to the identity operator on $\mathbb{C}^{2j+1}$, and thus by normalization, $\rho = \mathbf{1}/(2j+1)$.

Item (ii) follows from the celebrated convexity of the Wigner-Yanase skew information [59,72].

Item (iii) follows readily from the equivalent definitions of $N(\cdot)$, Eq. (3) or (4), and the unitary covariance of the skew information

$$I(e^{-i\theta\mathbf{n}\cdot\mathbf{J}}\rho e^{i\theta\mathbf{n}\cdot\mathbf{J}}, K) = I(\rho, e^{i\theta\mathbf{n}\cdot\mathbf{J}}Ke^{-i\theta\mathbf{n}\cdot\mathbf{J}}).$$

To establish item (iv), let

$$\langle\mathbf{J}\rangle = (\langle\psi|J_x|\psi\rangle, \langle\psi|J_y|\psi\rangle, \langle\psi|J_z|\psi\rangle)$$

be the spin expectation vector. Then direct evaluation yields

$$N(|\psi\rangle\langle\psi|) = \langle\psi|\mathbf{J}^2|\psi\rangle - |\langle\mathbf{J}\rangle|^2 = j(j+1) - |\langle\mathbf{J}\rangle|^2. \tag{5}$$

Assuming the mean spin $\langle\mathbf{J}\rangle = |\langle\mathbf{J}\rangle|\mathbf{n}_3$ with $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ forming an orthonormal basis of $\mathbb{R}^3$, then $\langle J_{\mathbf{n}_1}\rangle = \langle J_{\mathbf{n}_2}\rangle = 0$.

By the rotation invariance, we have

$$\begin{aligned}|\langle \mathbf{J}\rangle|^2 &= \langle J_x\rangle^2 + \langle J_y\rangle^2 + \langle J_z\rangle^2\\ &= \langle J_{\mathbf{n}_1}\rangle^2 + \langle J_{\mathbf{n}_2}\rangle^2 + \langle J_{\mathbf{n}_3}\rangle^2,\end{aligned}$$

from which we obtain $|\langle \mathbf{J}\rangle|^2 = \langle J_{\mathbf{n}_3}\rangle^2 \leqslant j^2$, and the equality is achieved only for atomic coherent states. Now the desired result follows from Eq. (5).

Item (v) follows readily from Eq. (5).

For item (vi), we note that any classical state ρ can be expressed as a probabilistic mixture of the atomic coherent states in the sense that

$$\rho = \int_{\mathbb{S}^2} P(\mathbf{n})|\mathbf{n}\rangle\langle\mathbf{n}|d\mathbf{n}, \quad P(\mathbf{n}) \geqslant 0.$$

Consequently, by the convexity of the skew information, we have

$$\begin{aligned}N(\rho) &= N\left(\int_{\mathbb{S}^2} P(\mathbf{n})|\mathbf{n}\rangle\langle\mathbf{n}|d\mathbf{n}\right)\\ &\leqslant \int_{\mathbb{S}^2} P(\mathbf{n})N(|\mathbf{n}\rangle\langle\mathbf{n}|)d\mathbf{n}\\ &= \int_{\mathbb{S}^2} P(\mathbf{n})j\, d\mathbf{n}\\ &= j.\end{aligned}$$

Item (vii) follows from the convexity of $N(\cdot)$ (which implies that the maximal value is achieved by pure states) and Eq. (5).

To gain more insight into our quantifier for nonclassicality $N(\cdot)$, here we compare it with some popular quantifiers for nonclassicality in the literature. In Ref. [52] a simple quantifier for classicality of quantum states is defined as the maximum of the Hilbert-Schmidt scalar products between the renormalized operators of the states and all displaced thermal states. Furthermore, the quantifier for nonclassicality

$$A(\rho) = \max_n \langle n|\rho|n\rangle$$

is also introduced. Here the maximum is over all Fock states. These quantities capture certain aspects of classicality and nonclassicality, but their informational meaning remains to be investigated. More relevant to the quantification of spin nonclassicality, the measure for nonclassicality

$$Q(\rho) = \min_{\rho_c} \|\rho - \rho_c\|,$$

based on the Hilbert-Schmidt distance, was introduced in Ref. [54]. Here the minimum is over all classical spin states ρ_c and $\|X\| = \sqrt{\mathrm{tr}X^\dagger X}$ is the Hilbert-Schmidt norm. This quantifier is a genuine measure for nonclassicality with apparent intuition; however, it is incomputable due to the formidable optimization, even for spin-1 systems [57]. In sharp contrast, our quantifier can be evaluated straightforwardly, though it is not a genuine measure for nonclassicality; it has metrological significance since the Wigner-Yanase skew information is a particular kind of quantum Fisher information.

IV. ILLUSTRATIVE EXAMPLES

In this section we evaluate the nonclassicality of several important and typical quantum states. The detailed calculations are in the Appendix.

A. Spin $\frac{1}{2}$

For $j = \frac{1}{2}$ in spin-$\frac{1}{2}$ systems, both Dicke states $|\frac{1}{2}, -\frac{1}{2}\rangle$ and $|\frac{1}{2}, \frac{1}{2}\rangle$ are atomic coherent states, and actually all pure states in this case are atomic coherent states and thus classical. Indeed, we have $N(|\psi\rangle\langle\psi|) = \frac{1}{2}$ for any $|\psi\rangle = \cos\frac{\theta}{2}|\frac{1}{2}, -\frac{1}{2}\rangle + e^{i\phi}\sin\frac{\theta}{2}|\frac{1}{2}, \frac{1}{2}\rangle$.

For general (mixed) states

$$\rho = \frac{1}{2}\left(\mathbf{1} + \sum_{i=1}^{3} r_i\sigma_i\right),$$

with $\mathbf{r} = (r_1, r_2, r_3) \in \mathbb{R}^3$, $|\mathbf{r}| \leqslant 1$, and σ_i, $i = 1, 2, 3$, the vector of Pauli spin matrices, we have

$$N(\rho) = \tfrac{1}{2}(1 - \sqrt{1 - |\mathbf{r}|^2}), \tag{6}$$

which is an increasing function of $|\mathbf{r}|$, consistent with our intuition. The case $|\mathbf{r}| = 0$ corresponds to the maximally mixed state, while $|\mathbf{r}| = 1$ corresponds to pure states.

B. Spin 1

For $j = 1$, we have three Dicke states $|1, -1\rangle$, $|1, 0\rangle$, and $|1, 1\rangle$, among which $|1, -1\rangle$ and $|1, 1\rangle$ are atomic coherent states (thus classical). We have $N(|1, \pm1\rangle\langle 1, \pm1|) = 1$, while $N(|1, 0\rangle\langle 1, 0|) = 2$ for the nonclassical state $|1, 0\rangle$. Moreover, for the superposition

$$|\psi_\lambda\rangle = \frac{1}{\sqrt{2+\lambda}}(|1, -1\rangle + \sqrt{\lambda}|1, 0\rangle + |1, 1\rangle), \quad \lambda \geqslant 0,$$

we have

$$N(|\psi_\lambda\rangle\langle\psi_\lambda|) = 2 - \frac{8\lambda}{(2+\lambda)^2}.$$

It is interesting to note that the minimal value of $N(|\psi_\lambda\rangle\langle\psi_\lambda|)$ is 1, which is achieved when $\lambda = 2$ with the state $|\psi_2\rangle = (|1, -1\rangle + \sqrt{2}|1, 0\rangle + |1, 1\rangle)/2$, while the maximal value 2 is achieved in two extreme cases for $\lambda = 0$ with the cat state $|\psi_0\rangle = (|1, -1\rangle + |1, 1\rangle)/\sqrt{2}$ or $\lambda \to \infty$ with the Dicke state $|\psi_\infty\rangle = |1, 0\rangle$.

C. Dicke states

For the Dicke states $\rho = |j, m\rangle\langle j, m|$ in spin-j systems, we have

$$N(\rho) = j^2 + j - m^2, \quad m = -j, \ldots, j.$$

Consequently, when $m \neq \pm j$, the Dicke states are nonclassical, and for $m = \pm j$, the corresponding states $|j, \pm j\rangle$ are atomic coherent states.

For integer j, the maximal value $j(j+1)$ is achieved by $|j, 0\rangle$ or any superposition $(|j, m\rangle + |j, -m\rangle)/\sqrt{2}$, while for half-integer j, the maximal value $j(j+1)$ can be achieved by $(|j, m\rangle + |j, -m\rangle)/\sqrt{2}$. In particular, we see that superradiance in the Dicke states corresponds to large nonclassicality.

D. Schrödinger's cat states

For the cat states

$$|\mathrm{cat}_j\rangle = \frac{1}{\sqrt{2}}(|j,-j\rangle + |j,j\rangle), \quad j > \frac{1}{2},$$

direct evaluation yields $\langle \mathbf{J}\rangle = 0$; consequently, by Eq. (5) we have

$$N(|\mathrm{cat}_j\rangle\langle \mathrm{cat}_j|) = j(j+1),$$

the maximally possible value. In this context, it is interesting to compare the superposition states with their constituent states $|j,\pm j\rangle$, which are atomic coherent states and have the minimal nonclassicality $N(|j,\pm j\rangle\langle j,\pm j|) = j$. Here we observe a remarkable feature: A superposition of two minimally nonclassical states yields a maximally nonclassical state.

E. Mixed states

For the maximally mixed state $\rho = \mathbf{1}/(2j+1)$ on $\mathbb{C}^{2j+1}$, we have $N(\rho) = 0$. From the resolution of identity in terms of the atomic coherent states

$$\mathbf{1} = \frac{2j+1}{4\pi}\int_{\mathbb{S}^2} |\mathbf{n}\rangle\langle \mathbf{n}| d\mathbf{n}$$

we know that $\rho = \mathbf{1}/(2j+1)$ is classical, as it should be.

Now consider the mixed state

$$\rho = p|j,m\rangle\langle j,m| + (1-p)\frac{\mathbf{1}}{2j+1}, \quad m \neq \pm j, \tag{7}$$

where $0 \leqslant p \leqslant 1$. After somewhat tedious calculation, we have

$$N(\rho) = [p + 2q - 2\sqrt{q(p+q)}](j^2 - m^2 + j),$$

where $q = (1-p)/(2j+1)$. For illustrative simplicity, we consider the case $j = 1$ and $m = 0$; then $N(\rho) > j = 1$ if $p > \frac{5}{6}$, which implies nonclassicality of ρ in this situation.

To emphasize the fact that for a spin-j state σ, $N(\sigma) > j$ is only a sufficient but not necessary condition for nonclassicality of σ, here we present a simple example illustrating that there exist nonclassical states ρ satisfying $N(\rho) < j$. Consider the spin-j state ρ defined by Eq. (7) with $j = 1$; then $N(\rho) < j = 1$ when $p < \frac{5}{6}$. On the other hand, the above state ρ is nonclassical for $p > \frac{1}{4}$, which can be proved as follows. Since any classical state can be represented as $\rho_c = \sum_i p_i|\zeta_i\rangle\langle\zeta_i|$ with $p_i \geqslant 0$ and $|\zeta_i\rangle$ atomic coherent states, it follows that

$$\mathrm{tr}\rho_c|1,0\rangle\langle 1,0| = \sum_i p_i|\langle\zeta_i|1,0\rangle|^2 \leqslant \max_i|\langle\zeta_i|1,0\rangle|^2 \leqslant \tfrac{1}{2}.$$

However, $\mathrm{tr}\rho|1,0\rangle\langle 1,0| = (2p+1)/3 > \frac{1}{2}$ for $p > \frac{1}{4}$. This implies that ρ cannot be classical when $\frac{1}{4} < p$. Consequently, the nonclassicality criterion $N(\rho) > j$ is only a sufficient but not necessary condition for nonclassicality, since when $\frac{1}{4} < p < \frac{5}{6}$, the state ρ is nonclassical, yet it satisfies $N(\rho) < j = 1$.

V. NONCLASSICALITY VERSUS ENTANGLEMENT

Both nonclassicality and entanglement are quantum resources in some sense, and a natural question arises: What are the relations between them? Some intrinsic connections between nonclassicality and entanglement and their interconversion are discussed in the context of bosonic fields [73–80]. Here, working in different yet equivalent representations of spin-j systems, we establish a quantitative correspondence between nonclassicality and entanglement through some typical examples.

A. NOON states as the most nonclassical states

Consider two-mode bosonic fields consisting of modes a and b. Then it is widely recognized that the NOON states

$$|\mathrm{NOON}\rangle = \frac{1}{\sqrt{2}}(|n\rangle_a|0\rangle_b + |0\rangle_a|n\rangle_b)$$

are maximally entangled. Here $|n\rangle_a$ and $|n\rangle_b$ are the Fock states for modes a and b, respectively.

It is well known that two-mode bosonic fields have intrinsic connections with spin systems due to the Schwinger representation [81], and accordingly the above states can be equivalently regarded as atomic spin states. Then one may enquire what the amount of their nonclassicality is. It turns out that they are the most nonclassical states in spin-j systems with $j = n/2$. To evaluate the nonclassicality of the above state, recall that in the Schwinger representation, an angular momentum system can be equivalently described by two optical modes a and b satisfying the canonical commutation relations $[a, a^\dagger] = \mathbf{1}$, $[b, b^\dagger] = \mathbf{1}$, and $[a, b] = 0$ and being restricted to a fixed total photon number subspace. The spin operators are realized as

$$J_x = \tfrac{1}{2}(a^\dagger b + ab^\dagger), \quad J_y = \tfrac{1}{2i}(a^\dagger b - ab^\dagger),$$
$$J_z = \tfrac{1}{2}(a^\dagger a - b^\dagger b), \quad J_0 = \tfrac{1}{2}(a^\dagger a + b^\dagger b).$$

The ladder operators are realized as

$$J_+ = J_x + iJ_y = a^\dagger b, \quad J_- = J_x - iJ_y = ab^\dagger.$$

Now the nonclassicality of the NOON states can be evaluated as

$$N(|\mathrm{NOON}\rangle\langle \mathrm{NOON}|) = j(j+1),$$

which is the maximally possible value.

Actually, via the Schwinger representation, the NOON states correspond to the spin-j cat states with $j = n/2$,

$$|\mathrm{cat}_j\rangle = \frac{1}{\sqrt{2}}(|j,-j\rangle + |j,j\rangle),$$

whose nonclassicality is

$$N(|\mathrm{cat}_j\rangle\langle \mathrm{cat}_j|) = j(j+1).$$

In this context, $N(\cdot)$ may be regarded as an entanglement quantifier for two-mode optical states. Furthermore, $N(\cdot)$ is a well-defined quantity for any mixed state.

B. Greenberger-Horne-Zeilinger states as the most nonclassical states

Consider a composite system consisting of $2j$ spin-$\frac{1}{2}$ systems. The generalized Greenberger-Horne-Zeilinger (GHZ) states

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1 \cdots |0\rangle_{2j} + |1\rangle_1 \cdots |1\rangle_{2j}) \tag{8}$$

are widely recognized as maximally entangled states in some sense. Here $|0\rangle_k = |\frac{1}{2}, -\frac{1}{2}\rangle$ and $|1\rangle_k = |\frac{1}{2}, \frac{1}{2}\rangle$ refer to the kth spin-$\frac{1}{2}$ system. It is well known that symmetric states of $2j$ spin-$\frac{1}{2}$ systems are in natural correspondence with single-system spin-j states through the Majorana representation, and thus one may wonder what the amount of nonclassicality of the GHZ states is. It turns out they are the most nonclassical.

To elucidate this, recall that in his pioneering study of orientation of atoms in magnetic fields [82] Majorana initiated a representation which maps a spin-j system to $2j$ spin-$\frac{1}{2}$ systems with permutation symmetry. The state correspondence is

$$\sum_{m=-j}^{j} a_m|j, m\rangle \leftrightarrow \sum_{m=-j}^{j} a_m|D(j, m)\rangle,$$

where

$$|D(j, m)\rangle = c\sum_{\tau} |0\rangle_{\tau(1)}\cdots|0\rangle_{\tau(j-m)}|1\rangle_{\tau(j-m+1)}\cdots|1\rangle_{\tau(2j)},$$

in which c is a normalization constant and the summation is over all permutations on the set $\{1, \dots, 2j\}$. Thus, a spin-j system can be formally regarded as a fictitious ensemble of $2j$ spin-$\frac{1}{2}$ systems with exchange symmetry. In this context, atomic coherent states are mapped to separable (product) states in the Majorana representation because they are obtained by rotations of $|j, -j\rangle$, which is a product state.

For any state τ of composite systems consisting of $2j$ spin-$\frac{1}{2}$ systems, its nonclassicality is defined as

$$N(\tau) = I(\tau, \mathcal{J}_x) + I(\tau, \mathcal{J}_y) + I(\tau, \mathcal{J}_z), \tag{9}$$

where

$$\mathcal{J}_\mu = \sum_{k=1}^{2j} \frac{1}{2}\sigma_\mu^{(k)}, \quad \mu = x, y, z,$$

are the collective spin operators, with $\sigma_\mu^{(k)}$ denoting the Pauli spin matrices for system k in the ensemble. For $j > \frac{1}{2}$, we have

$$N(|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|) = j(j+1),$$

which is the maximally possible value. In fact, the GHZ states are mapped to the cat state $|\mathrm{cat}_j\rangle = (|j, -j\rangle + |j, j\rangle)/\sqrt{2}$ via the Majorana representation, and the result is just as expected. In this context, $N(\cdot)$ may be regarded as an entanglement quantifier for symmetric multiqubit states, pure or mixed.

Incidentally, we note that via the Majorana representation, a measure for noncoherence (nonclassicality) and a corresponding entanglement measure are introduced for spin-j states on $\mathbb{C}^{2j+1}$ and symmetric $2j$-qubit states, respectively, in Ref. [83]. It may be interesting to extend these results to mixed states in some sense. In this context, we point out that the barycentric measure used in the above approach is related to a kind of variance [83], while the Wigner-Yanase skew information employed in our approach to nonclassicality is a kind of measure of quantum uncertainty, which refines the variance in an informational fashion [58,62].

C. Nonclassicality versus entanglement for mixed states

For pure states, the relation between certain symmetric composite states and single-system spin states is well understood due to the Schwinger representation and the Majorana construction. Our results in Secs. V A and V B further illustrate this connection from an informational perspective by showing that the maximal entanglement corresponds to maximal nonclassicality, as quantified by the total skew information. Since for mixed states our quantifier is also easily computable with information-theoretic significance, it may be useful in characterizing some connection between nonclassicality and entanglement in the mixed state case.

To illustrate this, let us derive a simple entanglement criterion in terms of nonclassicality. Recall that for any n-qubit state τ its nonclassicality is defined by Eq. (9). Noting that by Eq. (6),

$$N(\rho) = \tfrac{1}{2}(1 - \sqrt{1 - |\mathbf{r}|^2}) < \tfrac{1}{2}$$

for any qubit (spin-$\frac{1}{2}$) state ρ, it follows that for any n-qubit separable state

$$\rho_{\mathrm{sep}} = \sum_i p_i\rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(n)},$$

with $\rho_i^{(k)}$ the qubit of system k, we have, by the convexity of $N(\cdot)$,

$$\begin{aligned} N(\rho_{\mathrm{sep}}) &\leqslant \sum_i p_i N\big(\rho_i^{(1)} \otimes \rho_i^{(2)} \otimes \cdots \otimes \rho_i^{(n)}\big) \\ &= \sum_i p_i\big[N\big(\rho_i^{(1)}\big) + \cdots + N\big(\rho_i^{(n)}\big)\big] \\ &\leqslant \frac{n}{2}\sum_i p_i \\ &= \frac{n}{2}. \end{aligned}$$

Consequently, if $N(\tau) > n/2$, then the state τ must be entangled. This provides an entanglement criterion in terms of nonclassicality. In particular, for the GHZ states defined by Eq. (8) with $n = 2j$, we have $N(|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|) = n(n+2)/4 > n/2$, which indicates entanglement.

While nonclassicality and entanglement are both manifestations of quantumness and are intrinsically connected, they capture different aspects of quantumness and have a subtle relation, in particular in the context of mixed states. Here we present an example illustrating the similarity and difference between nonclassicality and entanglement. Consider the spin-j state with $j = 1$,

$$\rho_t = (1-t)\tfrac{\mathbf{1}}{3} + t|\mathrm{cat}_j\rangle\langle\mathrm{cat}_j|, \quad 0 \leqslant t \leqslant 1,$$

where $|\mathrm{cat}_j\rangle = (|1, 1\rangle + |1, -1\rangle)/\sqrt{2}$. The nonclassicality can be directly evaluated as

$$N(\rho_t) = \frac{4 + 2t - 4\sqrt{(1-t)(1+2t)}}{3}.$$

When $t > c_1 := \frac{5}{6}$, we have $N(\rho_t) > 1$, which implies that the state ρ_t is nonclassical. Moreover, the nonclassicality achieves the maximal value $2j = 2$ when $t = 1$.

Under the Majorana mapping, which establishes the one-to-one correspondence between spin-j states and symmetric $2j$-qubit states, the corresponding (symmetric) state of the spin-1 state ρ_t in the two-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be represented as

$$\begin{aligned}\hat{\rho}_t = &\frac{2+t}{6}(|00\rangle\langle 00| + |11\rangle\langle 11|)\\ &+ \frac{1-t}{6}(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|)\\ &+ \frac{t}{2}(|11\rangle\langle 00| + |00\rangle\langle 11|).\end{aligned}$$

Its entanglement, as quantified by the celebrated concurrence [84], can be evaluated as

$$C(\hat{\rho}_t) = \max\left\{\frac{4t-1}{3}, 0\right\}.$$

When $t > c_2 := \frac{1}{4}$, the state is entangled and when $t = 1$, the state is maximally entangled. In view of the difference between c_1 and c_2, we see that the nonclassicality, as quantified by $N(\cdot)$, exhibits a different feature from the entanglement, as quantified by the concurrence: Nonclassicality and entanglement are not in perfect correspondence for mixed states. The interplay between nonclassicality and entanglement is important and subtle, which calls for further investigation.

VI. CONCLUSION

We have introduced a quantifier for nonclassicality of atomic spin states in terms of the Wigner-Yanase skew information, which is a bona fide measure of quantum uncertainty. This quantifier is conceptually simple, physically meaningful, and easy to compute. It induces a natural and convenient criterion for detecting nonclassicality, though it is only a sufficient but not necessary condition for detecting nonclassicality. We have revealed a variety of fundamental properties of this nonclassicality quantifier and illustrated the concept via various examples. In particular, we have identified the minimal as well as the maximal value of nonclassicality. The atomic coherent states are characterized as those states with minimal value among pure states. Our quantifier differs from the conventional approach to nonclassicality in terms of variance and is of information-theoretic type.

We emphasize that the Wigner-Yanase skew information is only a special version of quantum Fisher information, and it will be interesting to employ other quantum Fisher information to synthesize quantum uncertainty and to quantify nonclassicality. A remarkable generalization of the Wigner-Yanase skew information is the Wigner-Yanase-Dyson skew information [59,72,85–87], which plays an interesting role in quantum information theory. Our nonclassicality quantifier can be straightforwardly generalized in terms of the Wigner-Yanase-Dyson skew information.

It would be desirable to further investigate operational significance and physical meaning of the nonclassicality quantifier and to find applications to experimental scenarios involving information tasks or protocols.

We have further revealed certain connection between nonclassicality and entanglement. In this context, the interplay between coherent states, nonclassicality, and entanglement arises in a broader scenario involving symplectic geometry. In a series of studies [56,88–90], the geometry of nonclassicality and entanglement was pursued in the framework of Lie algebras and generalized coherent states. The total variance there coincides with our quantifier for nonclassicality for pure states, but our quantifier applies to any mixed state as well and captures genuine quantum uncertainty of any state (with respect to the relevant Lie algebra). It is worth exploring further the geometry of generalized coherent states and nonclassicality from an information-theoretical perspective and to pursue a unified framework of nonclassicality and entanglement based on group representation theory.

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APPENDIX

In this Appendix we present some details for the evaluation of nonclassicality in Sec. IV.

1. Spin $\frac{1}{2}$

For $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$, where $|0\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$ and $|1\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$, we have

$$J_-|\psi\rangle = e^{i\phi}\sin\frac{\theta}{2}|0\rangle, \quad J_+|\psi\rangle = \cos\frac{\theta}{2}|1\rangle,$$

$$J_+J_-|\psi\rangle = e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \quad J_-J_+|\psi\rangle = \cos\frac{\theta}{2}|0\rangle,$$

$$J_z|\psi\rangle = \frac{1}{2}\cos\frac{\theta}{2}|0\rangle - \frac{1}{2}e^{i\phi}\sin\frac{\theta}{2}|1\rangle,$$

$$J_z^2|\psi\rangle = \frac{1}{4}\cos\frac{\theta}{2}|0\rangle + \frac{1}{4}e^{i\phi}\sin\frac{\theta}{2}|1\rangle,$$

from which we have

$$I(\rho, J_x) + I(\rho, J_y) = \tfrac{1}{2} - \tfrac{1}{4}\sin^2\theta,$$

$$I(\rho, J_z) = \tfrac{1}{4}\sin^2\theta.$$

Therefore, $N(\rho) = I(\rho, J_x) + I(\rho, J_y) + I(\rho, J_z) = \frac{1}{2}$.

For $\rho = \frac{1}{2}(\mathbf{1} + \sum_{i=1}^{3} r_i\sigma_i)$, in the standard base, we have [87]

$$\sqrt{\rho} = c\begin{pmatrix} 1 + \sqrt{1-|\mathbf{r}|^2} + r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 + \sqrt{1-|\mathbf{r}|^2} - r_3 \end{pmatrix},$$

with

$$c = \frac{1}{\sqrt{2}(\sqrt{1+|\mathbf{r}|} + \sqrt{1-|\mathbf{r}|})}, \qquad \mathbf{r} = (r_1, r_2, r_3).$$

In spin-$\frac{1}{2}$ systems we have

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By straightforward calculation we have

$$
\begin{aligned}
&\tfrac{1}{2}\mathrm{tr}J_-J_+\rho + \tfrac{1}{2}\mathrm{tr}J_+J_-\rho = \tfrac{1}{2},\\
&\mathrm{tr}J_+\sqrt{\rho}J_-\sqrt{\rho} = c^2(1+\sqrt{1-|\mathbf{r}|^2})^2 - c^2 r_3^2,\\
&\mathrm{tr}J_z^2\rho = \tfrac{1}{4},\\
&\mathrm{tr}J_z\sqrt{\rho}J_z\sqrt{\rho} = \frac{c^2}{2}\big[(1+\sqrt{1-|\mathbf{r}|^2})^2 + r_3^2 - r_1^2 - r_2^2\big],
\end{aligned}
$$

from which we obtain

$$N(\rho) = \tfrac{1}{2}(1-\sqrt{1-|\mathbf{r}|^2}).$$

2. Spin 1

Recalling that here $j=1, J_+|j,j\rangle = 0, J_-|j,-j\rangle = 0$, and

$$
\begin{aligned}
J_+|j,m\rangle &= \sqrt{(j-m)(j+m+1)}|j,m+1\rangle,\\
J_-|j,m\rangle &= \sqrt{(j+m)(j-m+1)}|j,m-1\rangle,\\
J_z|j,m\rangle &= m|j,m\rangle
\end{aligned}
$$

for $|\psi_\lambda\rangle = \frac{1}{\sqrt{2+\lambda}}(|1,-1\rangle + \sqrt{\lambda}|1,0\rangle + |1,1\rangle)$, we have

$$
\begin{aligned}
J_+|\psi_\lambda\rangle &= \frac{\sqrt{2}}{\sqrt{2+\lambda}}(|1,0\rangle + \sqrt{\lambda}|1,1\rangle),\\
J_-|\psi_\lambda\rangle &= \frac{\sqrt{2}}{\sqrt{2+\lambda}}(\sqrt{\lambda}|1,-1\rangle + |1,0\rangle),\\
J_+^2|\psi_\lambda\rangle &= \frac{2}{\sqrt{2+\lambda}}|1,1\rangle,\\
J_-^2|\psi_\lambda\rangle &= \frac{2}{\sqrt{2+\lambda}}|1,-1\rangle,\\
J_+J_-|\psi_\lambda\rangle &= \frac{2}{\sqrt{2+\lambda}}(\sqrt{\lambda}|1,0\rangle + |1,1\rangle),\\
J_-J_+|\psi_\lambda\rangle &= \frac{2}{\sqrt{2+\lambda}}(|1,-1\rangle + \sqrt{\lambda}|1,0\rangle),\\
J_z|\psi_\lambda\rangle &= \frac{1}{\sqrt{2+\lambda}}(-|1,-1\rangle + |1,1\rangle),\\
J_z^2|\psi_\lambda\rangle &= \frac{1}{\sqrt{2+\lambda}}(|1,-1\rangle + |1,1\rangle).
\end{aligned}
$$

Now

$$
\begin{aligned}
&I(|\psi_\lambda\rangle\langle\psi_\lambda|, J_x) + I(|\psi_\lambda\rangle\langle\psi_\lambda|, J_y)\\
&\quad= \tfrac{1}{2}\langle J_-J_+\rangle + \tfrac{1}{2}\langle J_+J_-\rangle - \langle J_+\rangle\langle J_-\rangle\\
&\quad= 2 - \frac{2}{2+\lambda} - \frac{8\lambda}{(2+\lambda)^2}
\end{aligned}
$$

and $I(|\psi_\lambda\rangle\langle\psi_\lambda|, J_z) = \langle J_z^2\rangle - \langle J_z\rangle^2 = 2/(2+\lambda)$. Consequently,

$$
\begin{aligned}
&N(|\psi_\lambda\rangle\langle\psi_\lambda|)\\
&\quad= I(|\psi_\lambda\rangle\langle\psi_\lambda|, J_x) + I(|\psi_\lambda\rangle\langle\psi_\lambda|, J_y) + I(|\psi_\lambda\rangle\langle\psi_\lambda|, J_z)\\
&\quad= 2 - \frac{8\lambda}{(2+\lambda)^2}.
\end{aligned}
$$

3. Dicke states

For $\rho = |j,m\rangle\langle j,m|$ we have

$$
\begin{aligned}
J_+\rho &= \sqrt{(j-m)(j+m+1)}|j,m+1\rangle\langle j,m|,\\
J_-\rho &= \sqrt{(j+m)(j-m+1)}|j,m-1\rangle\langle j,m|,\\
J_z\rho &= m|j,m\rangle\langle j,m|
\end{aligned}
$$

and moreover

$$
\begin{aligned}
J_-J_+\rho &= (j-m)(j+m+1)|j,m\rangle\langle j,m|,\\
J_+J_-\rho &= (j+m)(j-m+1)|j,m\rangle\langle j,m|,\\
J_z\rho &= m|j,m\rangle\langle j,m|,\\
J_z^2\rho &= m^2|j,m\rangle\langle j,m|.
\end{aligned}
$$

These imply that

$$
\begin{gathered}
\langle j,m|J_+|j,m\rangle = \langle j,m|J_-|j,m\rangle = 0,\\
I(\rho, J_z) = \langle J_z^2\rangle - \langle J_z\rangle^2 = 0,
\end{gathered}
$$

and

$$
\begin{aligned}
&I(\rho, J_x) + I(\rho, J_y)\\
&\quad= \tfrac{1}{2}\langle J_-J_+\rangle + \tfrac{1}{2}\langle J_+J_-\rangle - \langle J_+\rangle\langle J_-\rangle\\
&\quad= \tfrac{1}{2}(j-m)(j+m+1) + \tfrac{1}{2}(j+m)(j-m+1)\\
&\quad= j^2 + j - m^2.
\end{aligned}
$$

Therefore,

$$N(\rho) = I(\rho, J_x) + I(\rho, J_y) + I(\rho, J_z) = j^2 + j - m^2.$$

4. Schrödinger cat states

Here we show that for the cat states, $\langle\mathbf{J}\rangle = 0$. Indeed, from

$$J_x|\mathrm{cat}_j\rangle = \frac{1}{2}(J_+ + J_-)|\mathrm{cat}_j\rangle = \frac{\sqrt{j}}{2}(|j,-j+1\rangle + |j,j-1\rangle)$$

we know that $\langle J_x\rangle = \langle\mathrm{cat}_j|J_x|\mathrm{cat}_j\rangle = 0$. Similarly, from

$$
\begin{aligned}
J_y|\mathrm{cat}_j\rangle &= \frac{1}{2i}(J_+ - J_-)|\mathrm{cat}_j\rangle\\
&= \frac{\sqrt{j}}{2i}(|j,-j+1\rangle - |j,j-1\rangle)
\end{aligned}
$$

we know that $\langle J_y\rangle = \langle\mathrm{cat}_j|J_y|\mathrm{cat}_j\rangle = 0$. Finally,

$$\langle J_z\rangle = \langle\mathrm{cat}_j|J_z|\mathrm{cat}_j\rangle = \tfrac{1}{2}(j-j) = 0.$$

Combining the above, we have

$$\langle\mathbf{J}\rangle = (\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle) = 0.$$

5. Mixed states

For the mixed state

$$\rho = p|j,m\rangle\langle j,m| + \frac{1-p}{2j+1}\mathbf{1}, \quad m \neq j, -j,$$

we have the spectral decomposition

$$\rho = \left(p + \frac{1-p}{2j+1}\right)|j,m\rangle\langle j,m| + \frac{1-p}{2j+1}\sum_{n\neq m}|j,n\rangle\langle j,n|,$$

which implies that

$$\sqrt{\rho} = \sqrt{p + \frac{1-p}{2j+1}}|j,m\rangle\langle j,m| + \sqrt{\frac{1-p}{2j+1}}\sum_{n\neq m}|j,n\rangle\langle j,n|.$$

Now the desired result follows from straightforward calculation.

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