Out-of-time-ordered correlation functions in open systems: A Feynman-Vernon influence functional approach

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Recent theoretical and experimental studies have shown the significance of quantum information scrambling (i.e., the spread of quantum information over a system's degrees of freedom) for problems encountered in high-energy physics, quantum information, and condensed matter. Due to the complexity of quantum many-body systems it is plausible that new developments in this field will be achieved by experimental explorations. Since noise effects are inevitably present in experimental implementations, a better theoretical understanding is needed of quantum information scrambling in systems affected by noise. To address this problem we study indicators of quantum scrambling—out-of-time-ordered correlation functions (OTOCs) in open quantum systems. As most experimental protocols for measuring OTOCs are based on backward time evolution, we consider two possible scenarios of joint system-environment dynamics reversal: In the first one the evolution of the environment is reversed, whereas in the second it is not. We derive general formulas for OTOCs in those cases and study in detail the model of a spin chain coupled to the environment of harmonic oscillators. In the latter case we derive expressions for open-system OTOCs in terms of the Feynman-Vernon influence functional. Subsequently, assuming that dephasing dominates over dissipation, we provide bounds on open-system OTOCs and illustrate them for a spectral density known from the spin-boson problem. In addition to being significant for quantum information scrambling, the results also advance the understanding of decoherence in processes involving backward time evolution.

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I. INTRODUCTION

Dynamics of a quantum many-body system leads to a spread of quantum information across its degrees of freedom. As a consequence, localized states become inaccessible to local measurements. This phenomenon, refereed to as "scrambling," has recently become a vivid area of research joining different fields of physics such as quantum information theory [1,2], quantum field theory [3,4], and condensed matter [5,6]. Studies of quantum information scrambling allowed us to gain new insights into problems such as thermalization (see e.g., Ref. [7]) or many-body chaos [8,9], in quantum systems. Scrambling can be diagnosed by unusual correlation functions called out-of-time-ordered correlators (OTOCs), which for two operators V and W read

$$F_t(V, W) = \langle W_t^\dagger V^\dagger W_t V \rangle_\rho = \mathrm{Tr}(W_t^\dagger V^\dagger W_t V \rho), \qquad (1)$$

where $W_t = e^{itH} W e^{-itH}$ and H is the Hamiltonian of the system under consideration. Contrary to standard correlators, a measurement of OTOCs involves backward time evolution that must be applied twice to the investigated system. Backward time evolution makes OTOCs similar to the Loschmidt echo (LE) [10], however in the latter only one imperfect reversal of dynamics is applied, and no measurements in between are made. The aim of OTOCs is to measure how quickly two initially commuting operators W and V cease to commute (OTOCs can be seen as a state-dependent version of Lieb-Robinson bounds [4]), whereas LE aims to capture the sensitivity of a system's evolution to perturbations. For a more detailed discussion regarding relations between OTOCs and LEs, see Ref. [11]. The crucial feature of OTOCs is their time dependence: the faster an OTOC decays, the shorter is the scrambling time, which indicates the onset of quantum chaos in the considered system.

So far, quantum information scrambling has been investigated mostly in the isolated system setting. OTOCs were used to characterize chaotic behavior of several types of systems [5–7,12–15] and a bound on their decay rate was conjectured [3]. Several experimental scenarios to measure OTOCs were proposed [16,17] and the results of the first experiments were reported [18–22]. Moreover, links between OTOCs and thermodynamics [23–25], quasiprobabilities [26], and quantum information [1,2] were investigated.

Although very convenient, the notion of an isolated system constitutes an idealization. In real situations, such as experimental apparatus, all systems are open—due to interaction with the environment they are influenced by external noise. The physics of open quantum systems is qualitatively different from that of closed systems: Open systems decohere, losing their quantum coherence as well as energy to the environment. Therefore, one expects that an interaction with the environment will affect the spread of quantum information in

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the considered system. Yet the issue of how the coupling to the environment influences quantum information scrambling remains vastly unexplored.

Few works address scrambling in open quantum systems: a master equation was derived [27], a measurement protocol was proposed [28], and numerical studies were performed [29,30]. Thus, the aim of this work is to advance the understanding of open-system OTOCs. The main tool used is the Feynman-Vernon influence functional [31], which allows us to capture the influence of the environment on the system under study. This approach proved useful both from practical [32–34] as well as fundamental points of view [35–37] and still provides insights into problems encountered in fields such as open systems [38], quantum thermodynamics [39–45], or quantum computing [46]. As we show here, it encapsulates the decoherence of OTOCs in terms of the microscopic parameters of the model under consideration, allows us to gain a better insight into differences between the two considered backward time evolution schemes, and has useful applications; e.g., it can be used to bound the difference between OTOCs in isolated systems and their open-system counterparts. The results of this paper not only contribute to the particular problem of open-system OTOCs but also advance our understanding of decoherence in processes involving backward time evolution.

The paper is organized as follows: In Sec. II possible schemes of backward time evolution are discussed and the corresponding expressions for open-system OTOCs are derived. Subsequently, in Sec. III, we provide expressions for open-system OTOCs in terms of the Feynman-Vernon influence functional for a class of spin chain systems interacting with a bosonic environment. In Sec. IV, the obtained results are analyzed and some applications of the proposed approach are discussed. In particular, a lower bound on open-system OTOCs is provided. A summary and open questions are provided in Sec. V. Details of derivations are presented in the Appendix.

II. OUT-OF-TIME-ORDERED CORRELATORS IN OPEN SYSTEMS

In the standard treatment of open systems it is assumed that environmental degrees of freedom are out of control, so that they should be traced out from the description of the problem. This, together with the fact that the environment couples to the system, leads to the well-known phenomenon of decoherence [37]. Application of the open-system paradigm to the quantities involving backward time evolution, such as OTOCs, leads to two possibilities regarding behavior of the environment. In the first case one assumes that the joint dynamics of the system and the environment can be reversed perfectly. Then OTOCs will be affected by decoherence, which is caused by limitations of a measurement apparatus that is not capable of measuring all environmental degrees of freedom, which interacted with the system, and backward time evolution is simply $U_{SE}^{\dagger} = e^{iH_{SE}t}$, with a joint Hamiltonian $H_{SE} = H_S + H_E + H_{S:E}$ describing an evolution of the system, H_S, the environment, H_E, and an interaction between them, $H_{S:E}$. Because negation of the total Hamiltonian effectively implements reversal of the evolution, we refer to this case as to the full backward time evolution (FBTE) case. However, due to the complexity of the process, it may be impossible to reverse the dynamics of the environment. In such a case only backward time evolution of the system can be implemented, which reads $U_{S^{\dagger}E} = e^{i(H_S - H_E - H_{S:E})t}$ (for a detailed discussion of possible backward time evolution schemes for open systems, see Ref. [47]). We refer to this case as to the partial backward time evolution (PBTE) case (this corresponds to the canonical time reversal in stochastic thermodynamics [48]).

To derive expressions for OTOCs in both cases considered, let us analyze the following steps of a protocol measuring OTOCs, involving backward time evolution:

(1) Apply V to ρ_{SE} and perform forward time evolution: In the FBTE and PBTE cases given by U_{SE},

$$U_{SE} V \rho_{SE} U_{SE}^{\dagger}.$$

Here, and in the following, $V = V_S \otimes I_E$ denotes an observable acting trivially on the environment (similarly for W) and ρ_{SE} is an initial system-environmental state.

(2) Apply W and perform backward time evolution:

$$\text{FBTE: } U_{SE}^{\dagger} W U_{SE} V \rho_{SE} U_{SE}^{\dagger} U_{SE} = U_{SE}^{\dagger} W U_{SE} V \rho_{SE},$$

$$\text{PBTE: } U_{S^{\dagger}E} W U_{SE} V \rho_{SE} U_{SE}^{\dagger} U_{SE^{\dagger}}.$$

At this point the characteristic distinction between the FBTE and PBTE cases can be seen: in the former, unitary operators on the right side of ρ_{SE} form the identity operator, in contrary to the latter. Repetition of the above steps for $V^{\dagger}$ and $W^{\dagger}$ leads to

$$\begin{aligned}\text{FBTE: } & U_{SE}^{\dagger} W^{\dagger} U_{SE} V^{\dagger} U_{SE}^{\dagger} W U_{SE} V \rho_{SE} U_{SE}^{\dagger} U_{SE} \\ & = U_{SE}^{\dagger} W^{\dagger} U_{SE} V^{\dagger} U_{SE}^{\dagger} W U_{SE} V \rho_{SE},\end{aligned} \tag{2}$$

$$\text{PBTE: } U_{S^{\dagger}E} W^{\dagger} U_{SE} V^{\dagger} U_{S^{\dagger}E} W U_{SE} V \rho_{SE} U_{SE}^{\dagger} U_{SE^{\dagger}} U_{SE}^{\dagger} U_{SE^{\dagger}}. \tag{3}$$

By taking the trace of Eqs. (2) and (3) we obtain

$$F_t^{OS}(V, W) = \text{Tr}(U_{SE}^{\dagger} W^{\dagger} U_{SE} V^{\dagger} U_{SE}^{\dagger} W U_{SE} V \rho_{SE}) \tag{4}$$

for the FBTE case, and

$$F_t^{OS}(V, W) = \text{Tr}(U_{SE}^{\dagger} U_{SE^{\dagger}} U_{SE}^{\dagger} W^{\dagger} U_{SE} V^{\dagger} U_{S^{\dagger}E} W U_{SE} V \rho_{SE}) \tag{5}$$

for the PBTE case. Note that the PBTE scheme has been considered previously only for pure states [29], so Eq. (5) is the first result of this paper. Note also that the reasoning presented can be applied straightforwardly to higher-order OTOCs, i.e., those involving more stages, at which evolution of the system or the environment is reversed and more measurements in between are made.

III. A SPIN CHAIN CASE STUDY

In what follows we focus on a model of a spin-1/2 chain, whose sites couple linearly to environments consisting of harmonic oscillators. We assume that the environments are independent so that there is no coupling between them. Therefore, the considered class of Hamiltonians is of a form $H_{SE} = H_S + H_E + H_{S:E}$, where H_S is a general N-site spin-1/2-chain Hamiltonian, H_E describes the environment, which consists of independent collections of harmonic oscillators (one group for each site of the chain):

$$H_E = \sum_{k=0}^{N-1}\sum_{j=0}^{M_N-1} \omega_{k,j} a_{k,j}^\dagger a_{k,j}, \tag{6}$$

and $H_{S:E}$ describes a linear coupling between each site and its environment,

$$H_{S:E} = \sum_{k=0}^{N-1} \lambda_k \sigma_{z,k} \otimes \sum_{j=0}^{M_N-1} C_{k,j}(a_{k,j}^\dagger + a_{k,j}). \tag{7}$$

With the help of spin-coherent states [49–52], one can apply the usual path-integral reasoning and formulate an expression for OTOCs as a sum over all possible trajectories in the spin phase space. The details can be found in the Appendix; here only the main steps of the derivation are presented. We start by using the resolution of identity in the basis of spin-coherent states, which in the case of an N-site chain reads

$$I = \prod_{k=0}^{N-1} \int \frac{dz_k dz_k^*}{\pi(1+|z_k|^2)^2} |z_k\rangle\langle z_k| \equiv \int d(z,z^*)|z\rangle\langle z|, \tag{8}$$

where $|z_k\rangle$ is the spin coherent state of the kth site of the chain,

$$|z_k\rangle \equiv \frac{1}{\sqrt{1+|z_k|^2}} e^{\sigma_k^+ z_k}|0_k\rangle,$$

and $\sigma_k^+ = \sigma_{x,k} + i\sigma_{y,k}$ [49]. This resolution of identity is inserted before and after each unitary operator in Eqs. (4) and (5). As a result, in those expressions one needs to deal with terms of the form $\langle z_{F_i}|U_{SE}|z_{I_i}\rangle$, which act on the Hilbert space of the environment. In the Appendix we show that they may be represented as

$$\langle z_{F_i}|U_{SE}|z_{I_i}\rangle = \int_{z_{I_i}}^{z_{F_i}} d(z,z^*) e^{\Gamma[z,z^*]+iS[z,z^*]} \times e^{-i\sum_k H_{E,k}t} e^{-i\sum_k \xi_{k;t}[z]} D\left(\sum_k \chi_{k,t}[z]\right), \tag{9}$$

where the precise form of the action and surface terms is provided by Eqs. (A8) and (A9), respectively, $D(\sum_k \chi_k) \equiv e^{\sum_k \chi_k a_k^\dagger - \chi_k^* a_k}$ is the displacement operator, whose argument is given by Eq. (A10), and the phase factor $\xi_{k;t}t[z]$ is given by Eq. (A11). Subsequently, we assume a product initial system-environment state i.e., $\rho_{SE} = \rho_S \otimes \bigotimes_{k=0}^{N-1} \rho_{E,k}$, where $\rho_{E,k} = e^{-\beta H_{E,k}}/\mathrm{Tr}(e^{-\beta H_{E,k}})$. The environmental degrees of freedom can be traced out analytically; for details of the derivation we refer the interested reader to the Appendix. One arrives at the expression in which the interaction of the environment on the system is captured by the Feynman-Vernon influence functional

$$F_t^{OS}(V,W) = \int d(\boldsymbol{Z},\boldsymbol{Z}^*) e^{\Gamma[\boldsymbol{Z},\boldsymbol{Z}^*]+iS[\boldsymbol{Z},\boldsymbol{Z}^*]} \times F_t[\boldsymbol{Z},\boldsymbol{Z}^*] e^{-\Phi_t[\boldsymbol{Z},\boldsymbol{Z}^*]}, \tag{10}$$

where, in order to keep formulas concise, $\boldsymbol{Z}$ is an abbreviation for all variables of the problem, i.e., $\boldsymbol{Z} \equiv \boldsymbol{z_1}, \boldsymbol{z_2}, \boldsymbol{z_3}, \boldsymbol{z_3}$, bold stands for a vector, e.g., $\boldsymbol{z_1} \equiv (z_{\boldsymbol{1},1}, \dots, z_{\boldsymbol{1},N})$, formulas for action $S[\boldsymbol{Z},\boldsymbol{Z}^*]$ and surface $\Gamma[\boldsymbol{Z},\boldsymbol{Z}^*]$ terms are presented in the Appendix because they are not important in further considerations, and $F_t[\boldsymbol{Z},\boldsymbol{Z}^*]$ denotes elements of a closed-system OTOC expressed in the basis of coherent states

$$F_t[\boldsymbol{Z},\boldsymbol{Z}^*] = W^*(z_{I_4}{}^*, z_{F_3}) V^*(z_{I_3}{}^*, z_{F_2}) \times W(z_{I_2}{}^*, z_{F_1})(V\rho_S)(z_{I_1}{}^*, z_{F_4}), \tag{11}$$

with $W(\boldsymbol{z^*},\boldsymbol{z'}) \equiv \langle z|W|z'\rangle$ and $\Phi_t[\boldsymbol{Z},\boldsymbol{Z}^*]$ is the influence phase, whose explicit form will be presented shortly. Due to the form of the system-environment Hamiltonian, the influence phase consists of contributions coming from individual baths,

$$\Phi_t[\boldsymbol{Z},\boldsymbol{Z}^*] = \sum_k \Phi_{k;t}[\boldsymbol{Z},\boldsymbol{Z}^*]. \tag{12}$$

It will prove useful to express the results in terms of the usual influence functional obtained for a bosonic thermal bath [53]:

$$\Phi_t^B[z,z'] = \int_0^t dt' \int_0^{t'} dt''[z(t') - z'(t')] \times [\xi_J(t'-t'')z(t'') - \xi_J^*(t'-t'')z'(t'')]. \tag{13}$$

For convenience, we pass to the description of the environment in terms of the bath correlation function:

$$\xi_J(t) = \int_0^\infty d\omega J(\omega)\left[\coth\left(\frac{\beta\omega}{2}\right)\cos(\omega t) + i\sin(\omega t)\right], \tag{14}$$

which is expressed with the help of a spectral density $J(\omega) = \sum_j C_j^2 \delta(\omega - \omega_j)$ encapsulating details of the coupling between the system and the environment.

In the FBTE case, when the system and the environment jointly undergo forward and backward time evolution, calculation (see Appendix for details) results in the following influence phase:

$$\begin{aligned}\Phi_{k;t}[\boldsymbol{Z},\boldsymbol{Z}^*] &= \Phi_t^B[n_z[z_{\boldsymbol{1},k}], n_z[z_{\boldsymbol{2},k}]] + \Phi_t^B[n_z[z_{\boldsymbol{3},k}], n_z[z_{\boldsymbol{4},k}]] \\ &+ \int_0^t dt' \int_0^{t'} dt''\{n_z[z_{\boldsymbol{1},k}(t')] - n_z[z_{\boldsymbol{2},k}(t')]\} \\ &\times \xi_{J_k}(t'-t'')\{n_z[z_{\boldsymbol{3},k}(t'')] - n_z[z_{\boldsymbol{4},k}(t'')]\} \\ &+ \int_0^t dt' \int_0^{t'} dt''\{n_z[z_{\boldsymbol{3},k}(t')] - n_z[z_{\boldsymbol{4},k}(t')]\} \\ &\times \xi_{J_k}(t'-t'')\{n_z[z_{\boldsymbol{1},k}(t'')] - n_z[z_{\boldsymbol{2},k}(t'')]\},\end{aligned} \tag{15}$$

where $n_z[z_k] = (1-|z_k|^2)/(1+|z_k|^2)$. The above expression can be understood in the following way: Two first terms of the influence phase are essentially identical to the standard influence phase for the spin-boson problem. They stem from two pairs of forward-backward time branches (the first and the last one, respectively) in the left panel of Fig. 1. However, those branches are not independent, which is manifested by the last two terms of the influence phase.

In the PBTE case we find (see Appendix for details) that the influence phase is of the form

$$\Phi_{k;t}[\boldsymbol{Z},\boldsymbol{Z}^*] = \Phi_{k;3t}^B\left[n_z\left[\sum_{r=1}^3 \Pi_{(r-1)t,rt}\boldsymbol{z_r},_k\right], n_z[z_{\boldsymbol{4},k}]\right], \tag{16}$$

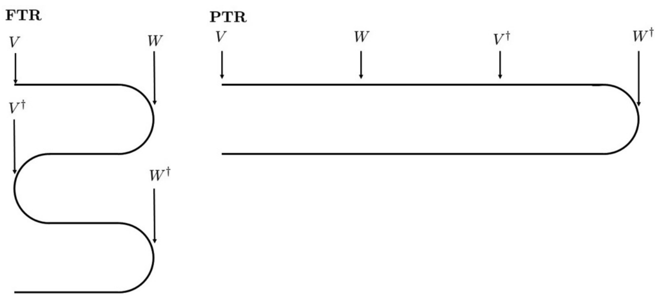

FIG. 1. Schematic representation of the two considered environment evolutions: the full (left panel) and the partial (right panel) time-reversal case (for details see text). In the FBTE case measurements performed on the spin chain, apart from the first one, happen at points, at which the evolution of the environment is reversed. In the PBTE case, kets of the environmental density matrix evolve forward in time (bras backward); all measurements take place on the forward-time branch. The time between subsequent measurements is t.

where is Π_{t_i,t_f} a window function defined as a difference between Heaviside step functions, $\Pi_{t_i,t_f} f(t') = [\theta(t'-t_i) - \theta(t'-t_f)]f(t')$. Comparing the above expression with the standard influence phase for the spin-boson problem we see that, on the forward-time branch, the external force is composed of three independent terms, which correspond to a paths taken by the spin-chain between measurements (cf. right panel of Fig. 1).

The influence functionals for the FBTE and PBTE cases have the same proprieties as in the standard Feynman-Vernon formalism [31,37]. Most importantly, for both cases considered, the absolute value of the influence functional is bounded from above by one; $|e^{-\Phi_t[\mathbf{Z},\mathbf{Z}^*]}| \leqslant 1$, what is most easily seen from Eqs. (A16) and (A21) in the Appendix. This implies that the absolute value of the open-system OTOCs will be in general smaller than that of the corresponding closed quantum systems. One can understand this fact in the following way: In an open system, spread of the quantum information, as measured by OTOCs, is faster than that in a corresponding closed system. This is because, in the former, there are additional degrees of freedom provided by the environment, which become correlated with the system degrees of freedom via system-environment interactions.

Equations (15) and (16) are the main contribution of this paper. They allow us to compare decoherence scenarios corresponding to FBTE and PBTE schemes and describe these processes in terms of the microscopic parameters of a model.

IV. APPLICATIONS

In this section, for the sake of presentation clarity, we assume a uniform coupling strength between sites of the chain and their respective environments as well as the same spectral density for all environments. Our aim is to obtain a bound on open-system OTOCs. Let us assume that we are in a regime, in which dephasing dominates over dissipation. In such a case the imaginary part of the influence functional can be neglected [37], which results in a purely dephasing channel. The rate of dephasing can be related to the microscopic description of the model under consideration. First of all, noting that $|n_z[\mathbf{z}_k(t')]| \leqslant \frac{1}{2}$, one sees that, in the most destructive case, the difference between the respective spin trajectories in Eqs. (15) and (16) is equal to 1. As a result, for the FBTE, the following bound applies:

$$\left|F_t^{OS}\right| \geqslant |F_t| e^{-4\lambda^2 N \int_0^t dt' \int_0^{t'} dt'' \mathrm{Re}\xi_J(t'-t'')}, \tag{17}$$

whereas for the PBTE case it reads

$$\left|F_t^{OS}\right| \geqslant |F_t| e^{-\lambda^2 N \int_0^{3t} dt' \int_0^{t'} dt'' \mathrm{Re}\xi_J(t'-t'')}. \tag{18}$$

As an illustration, let us consider a spectral density of the form $J(\omega) = \frac{\omega^s}{\Lambda^{s-1}} e^{-\omega/\Lambda}$, known from the spin-boson problem [33]. For $s = 1$ the relevant integrals can be evaluated if we assume the low-temperature limit, which is determined by the cutoff: $k_B T \ll \Lambda$. In the FBTE case, we find

$$\int_0^t dt' \int_0^{t'} dt'' \mathrm{Re}\xi_J(t'-t'') = \ln\left[\sqrt{1+\Lambda^2 t^2}\frac{\sinh(t/\tau_T)}{t/\tau_T}\right], \tag{19}$$

where $\tau_T = \frac{1}{\pi k_B T}$ is thermal time. In the case $s > 1$, the integrals can be computed analytically for all temperature regimes:

$$\begin{aligned}
&\frac{\beta^{s-1}\Lambda^{s-1}}{\Gamma(s-1)} \int_0^t dt' \int_0^{t'} dt'' \mathrm{Re}\xi_J(t'-t'') \\
&= \zeta\left(s-1, \frac{1}{\Lambda\beta}\right) + \zeta\left(s-1, 1+\frac{1}{\Lambda\beta}\right) \\
&\quad - \frac{1}{2}\left[\zeta\left(s-1, \frac{1}{\Lambda\beta} + i\frac{t}{\beta}\right)\right. \\
&\quad \left. + \zeta\left(s-1, 1+\frac{1}{\Lambda\beta} + i\frac{t}{\beta}\right) + \mathrm{c.c.}\right],
\end{aligned} \tag{20}$$

for $s \neq 2$, where $\zeta(p,a)$ is Hurwitz zeta function [54]

$$\zeta(p,q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^p}. \tag{21}$$

$\Gamma(s)$ stands for the Euler Gamma function, $\beta \equiv \frac{1}{k_B T}$, and c.c. denotes complex conjugation. The case $s = 2$ requires a separate treatment; the resulting expression is similar to that above but with the Hurwitz zeta functions replaced by digamma functions [54] $\psi(q) = \frac{d}{dq} \ln \Gamma(q)$. The analytical expressions in the PBTE case are obtained by the substitution $t \rightarrow 3t$ in Eqs. (19) and (20).

The performance of the bound for the FBTE and PBTE cases for different values of temperature T and the Ohmicity parameter s is illustrated in Fig. 2. In the pure dephasing scenario, the PBTE case leads to a tighter bound on open-system OTOCs, especially in the case of the super-Ohmic spectral density (i.e., for $s > 1$). The possible explanation of this fact may be related to the phenomenon of non-Markovianity (for a general introduction, see, e.g., Ref. [55]). Similarly to the classical case, a quantum evolution is said to be Markovian if it is memoryless. Then the reduced dynamics of the open system can be modeled by means of a dynamical semigroup with a corresponding time-independent generator in Lindblad form. In such a case, certain quantum features of the open system, e.g., coherences or quantum correlations, are typically

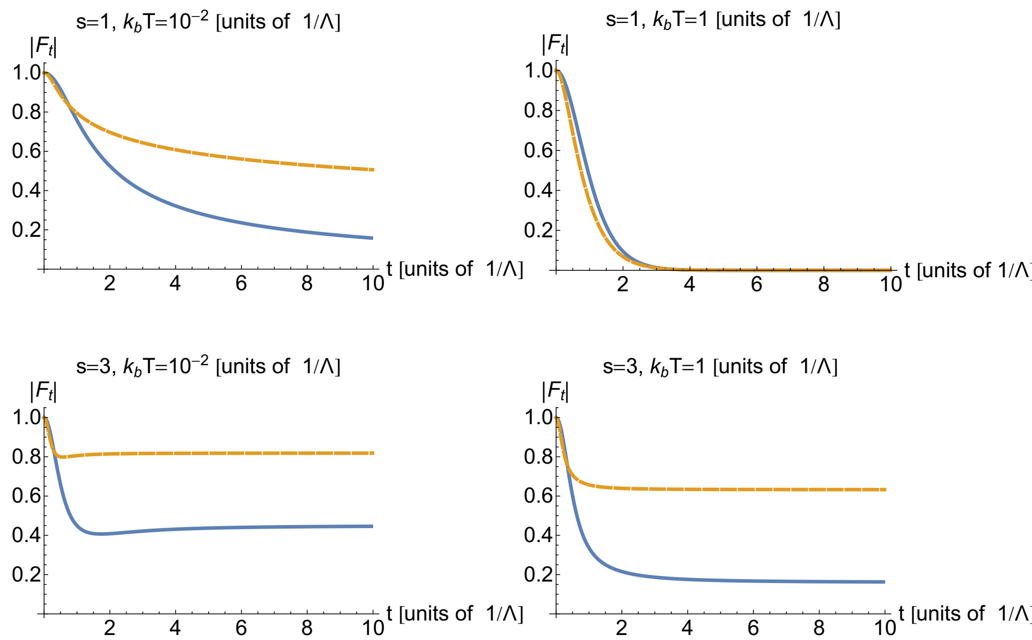

FIG. 2. Lower bound on open-system OTOC in the pure dephasing regime for FBTE case [cf. Eq. (17), solid line] and PBTE case [cf. Eq. (18); dashed line]. The left panels show results for the low-temperature regime $k_BT = 10^{-2}\Lambda$ ($s = 1$ in the upper panel and $s = 3$ in the lower panel). The right panels show results for the intermediate-temperature regime $k_BT = \Lambda$ ($s = 1$ in the upper panel and $s = 3$ in the lower panel). The plots were done for a chain consisting of $N = 20$ sites with $\lambda = 0.1$.

irreversibly lost to the environment. On the other hand, in non-Markovian evolution, memory effects are present and those properties can be regained, at least to some extent, by the open system. For the spin-boson model it is known that super-Ohmic spectral densities lead to non-Markovian evolution [56,57]: There is a revival of qubit coherences previously lost to the environment. OTOCs aim to measure the spread of quantum correlations across system degrees of freedom, which in open systems is enhanced by the presence of the environment. This unwanted enhancement can be suppressed if some quantum information lost to the environment will flow back to the system because of non-Markovian memory effects. It is plausible that such a back flow may be more significant in PBTE compared with the FBTE case, in which evolution of the environment is also reversed. It would be interesting to further investigate this issue.

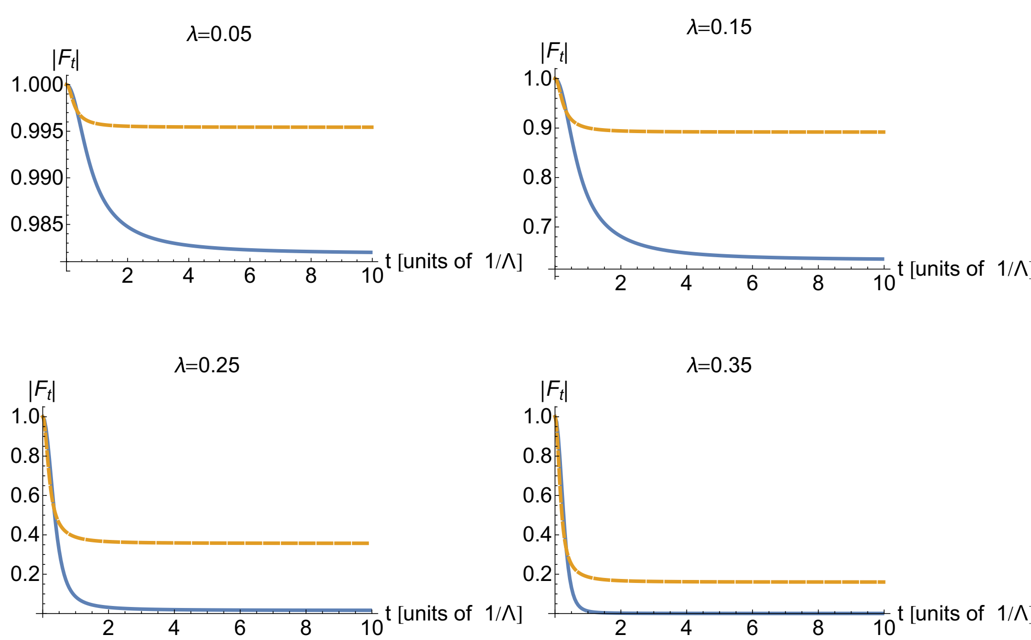

FIG. 3. Lower bound on open-system OTOC in the pure dephasing regime for FBTE case [cf. Eq. (17), solid line] and PBTE case [cf. Eq. (18), dashed line] for different values of the coupling strength λ. The plots were done for a chain consisting of $N = 20$ sites, $k_BT = \Lambda$, and $s = 3$.

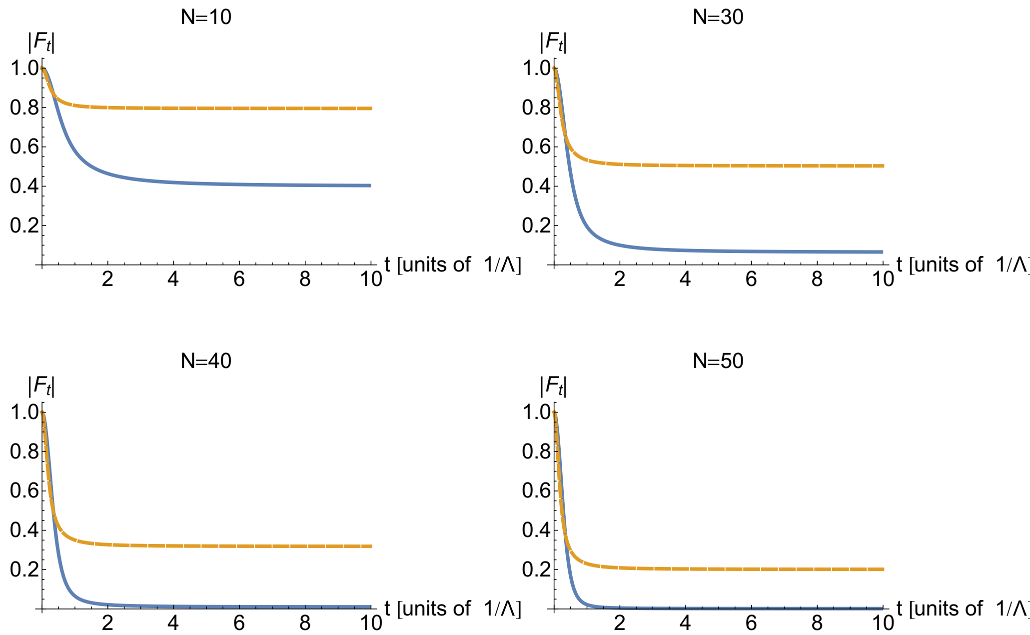

FIG. 4. Lower bound on open-system OTOC in the pure dephasing regime for FBTE case [cf. Eq. (17), solid line], and PBTE case [cf. Eq. (18), dashed line] for different numbers N of sites in a chain. The plots were done for $\lambda = 0.1$, $k_B T = \Lambda$, and $s = 3$.

Next we analyze the influence of the coupling strength λ and the number N of sites in the chain on the bound; see Figs. 3 and 4, respectively (because the results are qualitatively the same for all values of T and s considered, we decided to present only those for $k_B T = \Lambda$ and $s = 3$). One sees that the PBTE case is less affected by decoherence than FBTE also with growing λ and N.

It would be desirable to generalize the above bound in order to account for dissipation. In the most straightforward way this can be done by application of a similar reasoning to that presented in Ref. [38]. Estimation of the subsequent terms in the Taylor expansion of the influence functional leads to a bound on $|\Delta F_t(V, W)| \equiv |F_t(V, W) - F_t^{OS}(V, W)|$, i.e., the difference between an OTOC and its open-system counterpart. It reads

$$\frac{|\Delta F_t(V, W)|}{|F_t(V, W)|} \leqslant e^{-4\lambda^2 N \int_0^t dt' \int_0^{t'} dt'' |\xi_J(t'-t'')|} - 1 \tag{22}$$

for the FBTE case (a similar expression can be found for the PBTE case). However, in general we have $|\Delta F_t(V, W)|/|F_t(V, W)| \leqslant 1$, and numerical simulations show that the right-hand side of the above inequality quickly exceeds 1, which makes the above bound not a useful one. To improve tightness of the bound a more careful treatment is required, e.g., one using a suitably modified version of the noninteracting blip approximation [33].

V. SUMMARY AND OUTLOOK

In this work we applied the Feynman-Vernon influence functional technique to study open-system OTOCs. We considered two possible backward time evolution schemes—in the first one the evolution of the environment was reversed whereas in the second is was not. We derived expressions for open-system OTOCs in both cases. Subsequently, we considered the model of a one-dimensional spin-1/2 chain interacting with a bosonic environment and computed the influence phase for both scenarios. The influence phase was used to derive bounds on open-system OTOCs. The behavior of the bounds was analyzed for the spectral density known from the spin-boson model.

It would be interesting to extend the present study for higher spins. This requires careful treatment because it has been shown that, for spins $s > \frac{1}{2}$, the spin-coherent path integrals are not well defined [58]. However, the resolution of this problem has also been proposed [59], which may open a path for higher-spin extension. Moreover, a potential future research direction concerns deriving a master equation for open-system OTOCs using the calculated influence phase. Although at present no closed expression for a master equation corresponding to the spin-boson problem is known [60], the results obtained here will have a very similar structure to those for a bosonic central system. In such a case the standard techniques of deriving a master equation from the influence phase should apply. This problem will be studied elsewhere.

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APPENDIX: EVOLUTION OF THE ENVIRONMENT

Here we derive an expression for the evolution operator of the environmental degrees of freedom. We start by expressing the full evolution operator in the basis of spin-coherent states:

$$\langle z_{F_i}|U_{SE}|z_{I_i}\rangle = \langle z_{F_i}|\lim_{N\to\infty}\prod_{n=1}^{N} e^{-i(H_S+H_E+H_{S:E})\Delta t}|z_{I_i}\rangle, \tag{A1}$$

where $\Delta t \equiv \frac{t}{N}$. In the next step, we insert the resolution of identity expressed in terms of spin coherent states,

$$I = \prod_k \int \frac{dz_k dz_k^*}{\pi(1+|z_k|^2)^2} |z_k\rangle \langle z_k| \equiv \int d(z, z^*) |z\rangle \langle z|, \tag{A2}$$

between subsequent terms of the product

$$\langle \boldsymbol{z_{F_i}} | U_{SE} | \boldsymbol{z_{I_i}} \rangle = \langle \boldsymbol{z_{F_i}} | \lim_{N\to\infty} \prod_{n=1}^{N} \left[\iint d(\boldsymbol{z_{n+1}}, \boldsymbol{z_{n+1}}^*) d(\boldsymbol{z_n}, \boldsymbol{z_n}^*) |\boldsymbol{z_{n+1}}\rangle \langle \boldsymbol{z_{n+1}} | e^{-i(H_S+H_E+H_{S:E})\Delta t} |\boldsymbol{z_n}\rangle \langle \boldsymbol{z_n}| \right] |\boldsymbol{z_{I_i}}\rangle. \tag{A3}$$

We focus on a single term:

$$\begin{aligned} \langle \boldsymbol{z_{n+1}} | e^{-i(H_S+H_E+H_{S:E})\Delta t} |\boldsymbol{z_n}\rangle &\approx \langle \boldsymbol{z_{n+1}} | (1 - i(H_S + H_E + H_{S:E})\Delta t) |\boldsymbol{z_n}\rangle \\ &= \langle \boldsymbol{z_{n+1}} | \boldsymbol{z_n}\rangle \{1 - i[H_S(\boldsymbol{z_{n+1}}, \boldsymbol{z_n}) + H_E + H_{S:E}(\boldsymbol{z_{n+1}}, \boldsymbol{z_n})]\Delta t\} \\ &\approx \langle \boldsymbol{z_{n+1}} | \boldsymbol{z_n}\rangle e^{-iH_S(\boldsymbol{z_{n+1}}, \boldsymbol{z_n})\Delta t} e^{-i[H_E+H_{S:E}(\boldsymbol{z_{n+1}}, \boldsymbol{z_n})]\Delta t}. \end{aligned} \tag{A4}$$

Our final aim is to take the limit $N \to \infty$, when the states will become close to each other, i.e., $\Delta \boldsymbol{z_{n}}_{,k} \equiv \boldsymbol{z_{n+1}}_{,k} - \boldsymbol{z_n}_{,k} = O(\Delta t)$. In such a case the scalar product of spin-coherent states becomes [51]

$$\begin{aligned} \langle \boldsymbol{z_{n+1}} | \boldsymbol{z_n}\rangle &= \prod_k \frac{1 + \boldsymbol{z_{n+1}}_{,k} \boldsymbol{z_n}^*_{,k}}{\sqrt{1+|\boldsymbol{z_{n+1}}_{,k}|^2}\sqrt{1+|\boldsymbol{z_n}_{,k}|^2}} \\ &\approx \prod_k \exp\left[\frac{\boldsymbol{z_n}_{,k} \frac{\Delta \boldsymbol{z_n}^*_{,k}}{\Delta t} - \boldsymbol{z_n}^*_{,k} \frac{\Delta \boldsymbol{z_n}_{,k}}{\Delta t}}{1+|\boldsymbol{z_n}_{,k}|^2} \Delta t \right] \\ &\equiv \exp\left[\frac{\boldsymbol{z_n} \frac{\Delta \boldsymbol{z_n}^*}{\Delta t} - \boldsymbol{z_n}^* \frac{\Delta \boldsymbol{z_n}}{\Delta t}}{1+|\boldsymbol{z_n}|^2} \Delta t \right], \end{aligned} \tag{A5}$$

and the elements of spin operators entering the Hamiltonian are replaced by [52]

$$\begin{aligned} \frac{\langle \boldsymbol{z_{n+1}}_{,k} | \sigma_{x,k} | \boldsymbol{z_n}_{,k}\rangle}{\langle \boldsymbol{z_{n+1}}_{,k} | \boldsymbol{z_n}_{,k}\rangle} &= \frac{\mathrm{Re}\boldsymbol{z_n}_{,k}}{1+|\boldsymbol{z_n}_{,k}|^2} \equiv n_x[\boldsymbol{z_n}_{,k}], \\ \frac{\langle \boldsymbol{z_{n+1}}_{,k} | \sigma_{y,k} | \boldsymbol{z_n}_{,k}\rangle}{\langle \boldsymbol{z_{n+1}}_{,k} | \boldsymbol{z_n}_{,k}\rangle} &= \frac{Im\boldsymbol{z_n}_{,k}}{1+|\boldsymbol{z_n}_{,k}|^2} \equiv n_y[\boldsymbol{z_n}_{,k}], \\ \frac{\langle \boldsymbol{z_{n+1}}_{,k} | \sigma_{z,k} | \boldsymbol{z_n}_{,k}\rangle}{\langle \boldsymbol{z_{n+1}}_{,k} | \boldsymbol{z_n}_{,k}\rangle} &= \frac{|\boldsymbol{z_n}_{,k}|^2 - 1}{1+|\boldsymbol{z_n}_{,k}|^2} \equiv n_z[\boldsymbol{z_n}_{,k}]. \end{aligned} \tag{A6}$$

Using Eqs. (A5), (A4), and (A6), we can formally take the limit of Eq. (A3):

$$\int_{\boldsymbol{z_{I_i}}}^{\boldsymbol{z_{F_i}}} d(\boldsymbol{z_i}, \boldsymbol{z_i}^*) e^{\Gamma[\boldsymbol{z_i}, \boldsymbol{z_i}^*] + iS[\boldsymbol{z_i}, \boldsymbol{z_i}^*]} \mathcal{T} e^{-i\int_0^t dt' \{H_E + H_{S:E}[\boldsymbol{z_i}(t')]\}} = \int_{\boldsymbol{z_{I_i}}}^{\boldsymbol{z_{F_i}}} d(\boldsymbol{z_i}, \boldsymbol{z_i}^*) e^{\Gamma[\boldsymbol{z_i}, \boldsymbol{z_i}^*] + iS[\boldsymbol{z_i}, \boldsymbol{z_i}^*]} e^{-i\sum_k H_{E,k} t} e^{-i\sum_k \xi_{k;t}[\boldsymbol{z_i}]} D\left(\sum_k \chi_{k,t}[\boldsymbol{z_i}] \right). \tag{A7}$$

The action $S[\boldsymbol{z_i}, \boldsymbol{z_i}^*]$ and the boundary term $\Gamma[\boldsymbol{z_i}, \boldsymbol{z_i}^*]$ are given by [51]

$$S[\boldsymbol{z_i}, \boldsymbol{z_i}^*] \equiv \int_0^t dt' \left\{ \frac{i}{2} \frac{\boldsymbol{z_i}^*(t') \dot{\boldsymbol{z_i}}(t') - \dot{\boldsymbol{z_i}}^*(t') \boldsymbol{z_i}(t')}{1+|\boldsymbol{z_i}(t')|^2} - H\{\boldsymbol{z_i}(t'), \boldsymbol{z_i}^*(t')\} \right\}, \tag{A8}$$

and

$$\Gamma[\boldsymbol{z_i}, \boldsymbol{z_i}^*] \equiv \frac{1}{2} \ln \left(\frac{[1 + \boldsymbol{z_i}^*(0) \boldsymbol{z_{I_i}}][1 + \boldsymbol{z_{F_i}}^* \boldsymbol{z_i}(t)]}{(1+|\boldsymbol{z_{I_i}}|^2)(1+|\boldsymbol{z_{F_i}}|^2)} \right), \tag{A9}$$

respectively. Moreover, in the above expression $D(\sum_k \chi_k) \equiv e^{\sum_k \chi_k a_k^\dagger - \chi_k^* a_k}$ is a multimode displacement operator whose argument reads

$$\chi_{k;t}[\boldsymbol{z}] = -i \sum_j \int_0^t dt' C_{k,j} e^{i\omega t'} n_z[z_k(t')], \tag{A10}$$

where $n_z[z_k] = (1 - |z_k|^2)/(1 + |z_k|^2)$, and the phase is

$$\xi_{k;t}[\boldsymbol{z}] = \int_0^\infty J_k(\omega) \int_0^t dt' \int_0^{t'} dt'' n_z[z_k(t')] n_z[z_k(t'')] \sin\{\omega(t' - t'')\}. \tag{A11}$$

The above expressions were written by using the spectral density $J(\omega) = \sum_j C_j^2 \delta(\omega - \omega_j)$.

The evolution operator (A7) may now be used to obtain the path integral representation for the open-system OTOCs. In the FBTE case we have that

$$
\begin{aligned}
F_t^{OS}(V,W) &= \int d(\mathbf{Z},\mathbf{Z}^*)\mathrm{Tr}_E\big(\langle z_{F_4}|U_{SE}^\dagger|z_{I_4}\rangle\langle z_{I_4}|W^\dagger|z_{F_3}\rangle\langle z_{F_3}|U_{SE}|z_{I_3}\rangle\\
&\quad\times\langle z_{I_3}|V^\dagger|z_{F_2}\rangle\langle z_{F_2}|U_{SE}^\dagger|z_{I_2}\rangle\langle z_{I_2}|W|z_{F_1}\rangle\langle z_{F_1}|U_{SE}|z_{I_1}\rangle\langle z_{I_1}|V\rho_{SE}|z_{F_4}\rangle\big)\\
&= \int d(\mathbf{Z},\mathbf{Z}^*)F_t[\mathbf{Z},\mathbf{Z}^*]\mathrm{Tr}_E\big(\langle z_{F_4}|U_{SE}^\dagger|z_{I_4}\rangle\langle z_{F_3}|U_{SE}|z_{I_3}\rangle\langle z_{F_2}|U_{SE}^\dagger|z_{I_2}\rangle\langle z_{F_1}|U_{SE}|z_{I_1}\rangle\rho_E\big)\\
&= \int d(\mathbf{Z},\mathbf{Z}^*)e^{\Gamma[\mathbf{Z},\mathbf{Z}^*]+iS[\mathbf{Z},\mathbf{Z}^*]}F_t[\mathbf{Z},\mathbf{Z}^*]e^{-\Phi[\mathbf{Z},\mathbf{Z}^*]},
\end{aligned}
\tag{A12}
$$

where $\mathbf{Z}$ is an abbreviation for all variables of the problem, i.e., $\mathbf{Z} \equiv \mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3}, \mathbf{z_3}$, bold stands for a vector, e.g., $\mathbf{z_1} \equiv (\mathbf{z_1}_{,1}, \dots, \mathbf{z_1}_{,N})$, and

$$
F_t[\mathbf{Z},\mathbf{Z}^*] = W^*(z_{F_3}, z_{I_4}{}^*)V^*(z_{F_2}, z_{I_3}{}^*)W(z_{F_1}, z_{I_2})(V\rho_S)(z_{F_4}, z_{I_1}{}^*),
\tag{A13}
$$

with $W(z', z^*) \equiv \langle z|W|z'\rangle$. We proceed by inserting Eq. (A7) into Eq. (A12). We find

$$
F_t^{OS}(V,W) = \int_{Z_I}^{Z_F{}^*} d(\mathbf{Z},\mathbf{Z}^*)e^{\Gamma[\mathbf{Z},\mathbf{Z}^*]+iS[\mathbf{Z},\mathbf{Z}^*]}F_t[\mathbf{Z},\mathbf{Z}^*]e^{-\Phi_t[\mathbf{Z},\mathbf{Z}^*]},
\tag{A14}
$$

where

$$
\begin{aligned}
S[\mathbf{Z},\mathbf{Z}^*] &= S[z_1, z_1{}^*] - S[z_2{}^*, z_2] + S[z_3, z_3{}^*] - S[z_4{}^*, z_4],\\
\Gamma[\mathbf{Z},\mathbf{Z}^*] &= \Gamma[z_1, z_1{}^*] + \Gamma[z_2, z_2{}^*] + \Gamma[z_3, z_3{}^*] + \Gamma[z_4, z_4{}^*],
\end{aligned}
\tag{A15}
$$

and the influence functional is

$$
e^{-\Phi[\mathbf{Z},\mathbf{Z}^*]} = \prod_k e^{i(\xi_{k,t}[z_1]-\xi_{k,t}[z_2]+\xi_{k,t}[z_4]-\xi_{k,t}[z_4])}\mathrm{Tr}[D^\dagger(\chi_{k,t}[z_4])D(\chi_{k,t}[z_3])D^\dagger(\chi_{k,t}[z_2])D(\chi_{k,t}[z_1])\rho_{E,k}].
\tag{A16}
$$

The initial state of the environment is represented in terms of the Glauber-Sudarshan P function $\rho_{E,k} = \int d\gamma_k d\gamma_k^* P(\gamma_k)|\gamma_k\rangle\langle\gamma_k|$. Then a straightforward calculation gives

$$
\begin{aligned}
e^{-\Phi[\mathbf{Z},\mathbf{Z}^*]} = &\int d\gamma_k d\gamma_k^* P(\gamma_k)e^{-|\sum_{i=1,3}\Delta\chi_{k,t}[z_i,z_{i+1}]|^2/2}e^{2i\sum_{k=1,3}\mathrm{Im}\Delta\chi_{k,t}[z_i,z_{i+1}]\gamma_k^*}\\
&\times e^{i\sum_{k=1,3}(\Delta\xi_{k,t}[z_i,z_{i+1}]+\mathrm{Im}\chi_{k,t}[z_i]\chi_{k,t}[z_{i+1}]^*)}e^{i\mathrm{Im}\Delta\chi_{k,t}[z_3,z_4]\Delta\chi_{k,t}[z_1,z_2]^*},
\end{aligned}
\tag{A17}
$$

where $\Delta\chi_{k,t}[z_i, z_{i+1}] \equiv \chi_{k,t}[z_i] - \chi_{k,t}[z_{i+1}]$ and similarly $\Delta\xi_{k,t}[z_i, z_{i+1}] \equiv \xi_{k,t}[z_i] - \xi_{k,t}[z_{i+1}]$. Assuming that that environment is initialized as a thermal state, with the corresponding P function of the form $P(\gamma) = e^{-|\gamma|^2/\bar{n}}$ with $\bar{n}$ being the mean photon number, $\bar{n} = \frac{1}{1-e^{-\beta\omega}}$. In such a case the integral in Eq. (A17) can be computed analytically and the resulting expression is Eq. (15) of the main text.

The FBTE case is more familiar from the point of view of standard open-system theory: environmental ket states evolve forward in time, whereas environmental bra states evolve backward in time. All the measurements take place on the forward-time branch. As a result one finds that there is one displacement operator corresponding to the forward path, for which the driving force changes at the measurement times, as well as one corresponding to the backward path. More precisely, one inserts the resolution of identity into Eq. (5):

$$
\begin{aligned}
F_t^{OS}(V,W) &= \int d(\mathbf{Z},\mathbf{Z}^*)\mathrm{Tr}_E\big(\langle z_{F_4}|U_{SE}^\dagger|z'\rangle\langle z'|U_{SE^\dagger}|z\rangle\langle z|U_{SE}^\dagger|z_{I_4}\rangle\langle z_{I_4}|W^\dagger|z_{F_3}\rangle\\
&\quad\times\langle z_{F_3}|U_{SE}|z_{I_3}\rangle\langle z_{I_3}|V^\dagger|z_{F_2}\rangle\langle z_{F_2}|U_{S^\dagger E}|z_{I_2}\rangle\langle z_{I_2}|W|z_{F_1}\rangle\langle z_{F_1}|U_{SE}|z_{I_1}\rangle\langle z_{I_1}|V\rho_{SE}|z_{F_4}\rangle\big)\\
&= \int d(\mathbf{Z},\mathbf{Z}^*)F_t[\mathbf{Z},\mathbf{Z}^*]\mathrm{Tr}_E\big(\langle z_{F_4}|U_{SE}^\dagger|z'\rangle\langle z'|U_{SE^\dagger}|z\rangle\langle z|U_{SE}^\dagger|z_{I_4}\rangle\langle z_{F_3}|U_{SE}|z_{I_3}\rangle\langle z_{F_2}|U_{S^\dagger E}|z_{I_2}\rangle\langle z_{F_1}|U_{SE}|z_{I_1}\rangle\rho_E\big)\\
&= \int d(\mathbf{Z},\mathbf{Z}^*)e^{\Gamma[\mathbf{Z},\mathbf{Z}^*]+iS[\mathbf{Z},\mathbf{Z}^*]}F_t[\mathbf{Z},\mathbf{Z}^*]e^{-\Phi[\mathbf{Z},\mathbf{Z}^*]},
\end{aligned}
\tag{A18}
$$

where

$$
S[\mathbf{Z},\mathbf{Z}^*] = S[z_1, z_1{}^*] + S[z_2, z_2{}^*] + S[z_3, z_3{}^*] + \bar{S}[z_4, z_4{}^*],
\tag{A19}
$$

and

$$\bar{S}[z_4, z_4^*] = \int_0^{3t} dt' \left[\frac{i}{2} \frac{z_i^*(t')\dot{z}_i(t') - \dot{z}_i^*(t')z_i(t')}{1 + |z_i(t')|^2} + (\Pi_{0,t} + \Pi_{2t,3t}) H\{z_i(t'), z_i^*(t')\} - \Pi_{t,2t} H\{z_i(t'), z_i^*(t')\} \right],$$
$$\Gamma[Z, Z^*] = \Gamma[z_1, z_1^*] + \Gamma[z_2, z_2^*] + \Gamma[z_3, z_3^*] + \Gamma^*[z_4, z_4^*]. \tag{A20}$$

The expression for the influence functional is

$$e^{-\Phi[Z,Z^*]} = \prod_k e^{i(\xi_{k;3t}[z_1+z_2+z_3] - \xi_{k;t}[z_4])} \mathrm{Tr}\{\rho_{E,k} D^\dagger(\chi_{k,t}[z_4]) D(\chi_{k,3t}[z_1 + z_2 + z_3])\}, \tag{A21}$$

where the argument of the forward displacement operator reads

$$\chi_{k;3t}[z_1 + z_2 + z_3] = -i \sum_j \int_0^{3t} dt' C_{k,j}(\omega) e^{i\omega t'} n_z \left[\sum_{r=1}^{3} \Pi_{(r-1)t,rt} z_k(t') \right], \tag{A22}$$

and the corresponding phase is

$$\xi_{k;3t}[z_1 + z_2 + z_3] = \int_0^{3t} dt' \int_0^{t'} dt'' \int_0^{\infty} J_k(\omega) n_z \left[\sum_{r=1}^{3} \Pi_{(r-1)t,rt} z_{r,k}(t') \right] n_z \left[\sum_{r'=1}^{3} \Pi_{(r'-1)t,r't} z_{r',k}(t'') \right] \sin\{\omega(t' - t'')\}. \tag{A23}$$

Evaluation of Eq. (A21) leads to the following expression:

$$\int d\gamma_k d\gamma_k^* P(\gamma_k) e^{-|\Delta\chi_{k,t}[z_1+z_2+z_3,z_4]|^2/2} e^{2i\mathrm{Im}\Delta\chi_{k,t}[z_1+z_2+z_3,z_4]\gamma_k^*} e^{i\Delta\xi_{k,t}[z_1+z_2+z_3,z_4] + \mathrm{Im}\chi_{k,t}[z_1+z_2+z_3,z_4]\chi_{k,t}[z_4]^*}, \tag{A24}$$

which, as in the previous case, can be computed for thermal states of the environment, leading to Eq. (16) of the main text.

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