Coherent coupling between the motional fluctuation of a mirror and a trapped ion inside an optical cavity: Memory, state transfer, and entanglement

Devender Garg and Asoka Biswas*

Department of Physics, Indian Institute of Technology Ropar, Rupnagar, Punjab 140001, India

(Received 5 April 2019; revised manuscript received 11 August 2019; published 12 November 2019)

We show how the motional fluctuations of two systems of very different sizes—a trapped ion and a mirror—can be coupled coherently, via their common interaction with an optical cavity. Specifically, we show that the fluctuations can be transferred from one system to the other by using pulsed excitation of the cavity mode, and vibrational fluctuations can exhibit entanglement in steady state. We numerically display a sudden death of this entanglement at certain pulse parameters and ambient equilibrium temperature. More interestingly, the state of the driving pulse can also be coherently mapped, in a selective way, into the vibrational mode of the trapped ion or the mirror, which acts as a quantum memory for the optical pulse. We further present a detailed analysis to show that it is also possible to map the state of the pulse into a binomial entangled state of its vibrational fluctuations. We finally investigate how to sympathetically cool the mirror by cooling the ionic vibration, thanks to an effective coupling between the two modes.

DOI: 10.1103/PhysRevA.100.053822

I. INTRODUCTION

In recent times, there has been a major thrust in research on cavity optomechanics. It is posed as a suitable platform to study the interaction of a microscopic system (the electromagnetic field inside a cavity) with a mesoscopic one (a nanometer-sized oscillating mirror of the cavity), with a nontrivial coupling.

The optomechanical coupling, that arises from the radiation pressure force exerted by the cavity field on the mirror, can lead to exchange of quantum states from one system to the other, and also to entanglement between them [1,2]. The state of the cavity field can be coherently transferred, stored, and retrieved from the mirror in an optomechanical setup [3,4]. The mirror can also be used as a switch to control the state transfer between the two cavity modes in a dual-cavity optomechanical system [5] and the cavity mode can also mediate swapping of states between two mirror modes with high efficiency [6]. On the other hand, a movable mirror can be entangled with the cavity mode [7] and the cavity output field [8]. Entanglement in the two-mode driving field can be mapped into two mirrors [9], which can be further enhanced by squeezed light [10–12]. Two mirrors can be entangled in a ring cavity by cooling the mirrors using a phase-sensitive feedback loop [13]. Recently, entanglement between two mechanical oscillators of different frequencies has been experimentally demonstrated [14,15]. Further studies on entanglement between two mirrors can be found in [16–18].

Note that it is demonstrated, in the context of quantum computing, that a trapped ion can also strongly interact with the cavity mode [19,20]. Such an interaction can lead to various quantum effects, namely, transfer of several classes of states of photons, squeezing of the motional state of the ion, and the ion-cavity entanglement [21,22]. Even the motional state of one atom can be transferred to another atom in a two-cavity system [23].

While an ion, an atom, or a mirror can directly couple to a cavity mode, an oscillating mirror does not directly couple to an ion or an atom. In a hybrid cavity optomechanical setup [1], this, however, could be possible via their common coupling to the cavity mode. This idea is quite interesting from the fundamental aspect: one is able to couple a microscopic system (i.e., an ion or an atom) with a mesoscopic mirror (of the size of a few nanometers or more). This leads to coherent control of motion of one system by that of the other. In [24,25], the possibility of strong motional coupling between an atom and a mirror has been explored in a two-mode cavity in a membrane-in-the-middle setup. The authors have shown that it is possible to obtain a linear coupling between them by suitably choosing the location of the atom inside the cavity (see also [26]) and thereby to *dynamically* transfer quadrature squeezed states from the atom to the mirror. On the other hand, in this paper, we use an optomechanical setup with an ion trapped in a cavity with one oscillating mirror at *the end* of the cavity, in which the linear coupling between the cavity mode and the mirror is most dominant. Note that the trapping mechanism of an ion would be very different from that for an atom. For example, in [24,25], the atom is trapped in an *optical lattice* inside a two-mode cavity driven by two lasers of different frequencies, while these lasers drive the atomic transitions as well. In the present case, the ion is trapped by an external Paul trap, within the Lamb-Dicke limit, while the cavity mode is driven by a separate laser (see, e.g., [27]). The center-of-mass motion of the ion near the center of the Paul trap can be modeled as a quantum harmonic oscillation. Clearly, the coupling mechanism between the motional degrees of freedom of an ion and that of an oscillator is quite

*abiswas@iitrpr.ac.in

different from that between the motional degrees of freedom for an atom and an oscillator. In any case, such a coupling is very weak. Here we show how one can indeed achieve strong coupling in motional degrees of freedom between *a single ion* (instead of an atom) and a mirror. This leads to a coherent transfer of average fluctuation energy between them, *in steady state*, unlike in [24,25]. We further investigate the preparation of steady-state entanglement and analyze how to realize a quantum memory of an optical pulse in such a setup.

Note that several authors have suggested how to couple *an atomic ensemble* with the vibrational mode of a mirror [28–31], to achieve cooling of spins, squeezing of light, and electromagnetically induced transparency. In [28], it was shown that these two systems can be prepared in an entangled state, via a quantum nondemolition measurement. Such entanglement can also be generated in steady state [32,33]. We emphasize that our result differs from [28–30,32,33] in which it is the collective *spin* of the atoms, and not their vibration, that couples to the mirror mode, while in [31,34,35] the collective *motion* of the ensemble (not of a single atom) couples to the mirror.

The sympathetic cooling has been used to cool the ion plasma, proteins, and Bose condensates of rubidium with the assistance of cooled atoms and ions [36–38]. In sideband cooling of the mirror, the linewidth of the cavity resonator should be small as compared to the mirror frequency and the frequency of the cavity driving field is so chosen that the corresponding detuning becomes equal to the frequency of the mirror (the so-called red sideband) [39]. Sympathetic cooling of the mirror, on the other hand, using a suitably cooled atomic ensemble [40,41], does not require any specific constraint on the linewidth and the detuning of the cavity mode [34]. Moreover, a temperature of the order of a few hundred micro-Kelvin degrees has been reported [42,43]. In [44], the atoms are embedded in the mirror and cooling is obtained both in resolved and unresolved sidebands. In this paper, we investigate the possibility of sympathetic cooling of a mirror via the interaction with a trapped ion.

Specifically speaking, we study the dynamics of fluctuations of the vibrational mode of the ion and that of the mirror. We show that by using a pulsed excitation of the cavity mode average energy of fluctuation of the mirror can be *deterministically* transferred to the ion at the steady state, in presence of damping. We also show that the mirror can be sympathetically cooled by the ionic vibration at the steady state. Such a state transfer or cooling does not require the atom to be prepared in a specific state, namely, the squeezed state or Fock state, unlike in [24,25]. We further show that these vibrational modes can be entangled *at steady state* by suitably tuning the pulse parameters. Note that this is unlike in [24,25], in which the entanglement between an atom and the mirror changes *dynamically with time*. More interestingly, we show how the state of an input pulse can be *selectively stored* into the *fluctuation states* of the vibrational mode of the ion or of the mirror, which thereby pose as a quantum memory of an optical pulse. Note that the decay rates of the vibrational modes are negligibly smaller than that of the optical mode involved, thereby qualifying these vibrational modes as suitable memory. The vibrational fluctuations of the mirror and the ion can also be prepared in a certain class of entangled state, namely, the binomial state, while the cavity mode remains factorized.

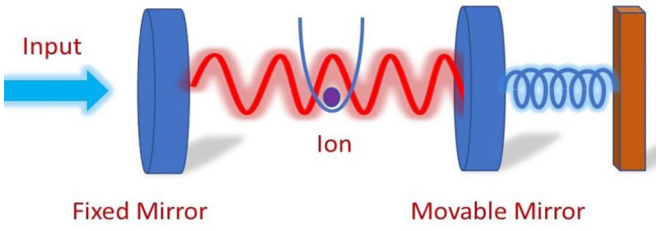

FIG. 1. Schematic diagram depicting a trapped ion inside the cavity optomechanical system.

The paper is organized as follows. In Sec. II, we describe our model with the relevant Hamiltonian. In Sec. III, we obtain an effective Hamiltonian to describe the coupling between the ion and the mirror. We show in Sec. IV how to generate entanglement between them in the bad-cavity limit, by suitable pulse shaping. The details of quantum memory of the input pulse are also described in this section. Next, in Sec. V, we discuss how to achieve the deterministic transfer of fluctuation energy between the relevant vibrational modes. The entanglement between them is analyzed in Sec. VI. In Sec. VII, we have investigated the possibility of cooling the mirror, by cooling the ion. In Sec. VIII, we conclude our paper.

II. MODEL

We consider a single ion of mass m trapped inside an optical cavity with a resonance frequency of ω_a (Fig. 1). One of the cavity mirrors oscillates at a fundamental frequency ω_m, while the other mirror remains fixed. It is assumed that the dipole moment of the two lowest-lying electronic energy states of the ion couples to the cavity mode, so that the ion can be considered as an electronic qubit with a transition frequency $\omega_{\rm el}$. This coupling $g(x_v)$ depends upon the displacement x_v of the ion from its equilibrium position inside the cavity, as $g(x_v) = g_0 \cos(\eta x_v)$, where g_0 is the coupling parameter proportional to ion cavity field interaction strength. Here $\eta = k(\Delta x)$ is the Lamb-Dicke parameter with k as the wave number for the electronic transition of the ion and $\Delta x = \sqrt{\frac{\hbar}{2m\omega_v}}$ as the position uncertainty of the ion in its ground vibrational state in the harmonic potential limit (ω_v as the fundamental vibrational frequency of the ion). The dynamics of the entire system is therefore governed by the following Hamiltonian (with $\hbar = 1$):

$$\begin{aligned} H &= H_0 + H_m + H_c, \\ H_0 &= \omega_a a^\dagger a + \omega_m b_m^\dagger b_m + \omega_v b_v^\dagger b_v + \omega_{\rm el}\sigma_z, \\ H_m &= -g_1 a^\dagger a(b_m + b_m^\dagger), \\ H_c &= g(\hat{x}_v)(a^\dagger \sigma^- + \sigma^+ a), \end{aligned} \tag{1}$$

where H_0 represents the unperturbed Hamiltonian, and H_m and H_c describe the interaction of the cavity mode with the oscillating mirror and the ion, respectively. Here a (b_m) is the annihilation operator for the cavity mode (the mode of oscillation of the mirror). The term g_1 denotes the strength

of the optomechanical coupling between the cavity mode and the oscillating mirror. The operators $\sigma^{\pm}$ and σ_z are usual Pauli spin operators. Note that we have replaced x_v by the relevant displacement operator $\hat{x}_v = (b_v + b_v^\dagger)/\sqrt{2}$, where b_v is the annihilation operator for the vibrational mode of the ion.

We further drive the cavity with a laser field of frequency ω_p. The corresponding coupling strength Ω can be written in terms of the power P of this field as $\Omega = \sqrt{\frac{2\gamma_a P}{\hbar\omega_p}}$, where γ_a is the decay rate of the cavity. This interaction can be described by the following Hamiltonian:

$$H_p = \Omega(a^\dagger e^{-i\omega_p t} + \text{H.c.}). \tag{2}$$

In the rotating frame of the laser frequency, the total Hamiltonian $H + H_p$ takes the following form:

$$\begin{aligned} H_{\text{tot}} = {} & \Delta_a a^\dagger a + \omega_m b_m^\dagger b_m + \omega_v b_v^\dagger b_v + \Delta_{\text{el}}\sigma_z \\ & - g_1 a^\dagger a(b_m + b_m^\dagger) + g(\hat{x}_v)(a^\dagger\sigma^- + \sigma^+ a) \\ & + \Omega(a^\dagger + \text{H.c.}), \end{aligned} \tag{3}$$

where $\Delta_a = \omega_a - \omega_p$ and $\Delta_{\text{el}} = \omega_{\text{el}} - \omega_p$ are the detunings of the cavity mode and the electronic transition of the ion, respectively, with the driving field.

III. EFFECTIVE COUPLING BETWEEN THE MIRROR AND THE IONIC VIBRATION

We will work in the regime where the detuning of the electronic transition of the ion with the laser determines the fastest timescale such that $|\Delta_{\text{el}}| \gg |\Delta_a|$. Further we consider that the decay rates γ_{el}, γ_a, γ_m, and γ_v of the electronic excited state of the ion, the cavity mode, the mirror, and the vibrational mode of the ion, respectively, are much smaller than $|\Delta_{\text{el}}|$. In this limit, the probability that the ion will be prepared in the electronic excited state is negligible and, therefore, one can adiabatically eliminate the electronic degree of freedom of the ion from the dynamics. This refers to the approximation $\dot{\sigma}^+ \approx 0$. Using the Heisenberg equation for σ^+, one obtains $\sigma^+ \approx (\frac{g(\hat{x}_v)a^\dagger}{\Delta_{\text{el}}})$. Note that in the limit $|\Delta_{\text{el}}| \gg |\Delta_a|$ the excitation of the electronic state of the ion will be very small. So the scattering due to spontaneous emission can be neglected in the limit of $\Delta_{\text{el}} \gg \gamma_{\text{el}}$ [45,46]. Replacing $\sigma^{\pm}$ with the above expressions in the Hamiltonian H_{tot}, we obtain the following reduced form of the Hamiltonian:

$$\begin{aligned} H = {} & \Delta a^\dagger a + \omega_m b_m^\dagger b_m + \omega_v b_v^\dagger b_v \\ & - g_1 a^\dagger a(b_m + b_m^\dagger) + \Omega(a^\dagger + \text{H.c.}), \end{aligned} \tag{4}$$

where $\Delta = (\Delta_a + \frac{g^2(\hat{x}_v)}{\Delta_{\text{el}}})$. From the above Hamiltonian, the Langevin equations for the relevant operators can be obtained as

$$\begin{aligned} \dot{a} &= -(\gamma_a + i\Delta)a + ig_1 a(b_m + b_m^\dagger) - i\Omega + \sqrt{2\gamma_a}a_{\text{in}}, \\ \dot{b_m} &= -(\gamma_m + i\omega_m)b_m + ig_1 a^\dagger a + \sqrt{2\gamma_m}b_m^{\text{in}}, \\ \dot{b_v} &= -(\gamma_v + i\omega_v)b_v + i\frac{g_2(\hat{x}_v)}{\sqrt{2}}a^\dagger a + \sqrt{2\gamma_v}b_v^{\text{in}}, \end{aligned} \tag{5}$$

where $g_2(\hat{x}_v) = \frac{\eta g_0^2 \sin(2\eta\hat{x}_v)}{\Delta_{\text{el}}}$, obtained using the commutation property $[b_v, f(\hat{x}_v)] = \frac{\hbar}{\sqrt{2}}f'(\hat{x}_v)$ for any function f and its derivative with respect to $\hat{x}_v$ [47]. Here the corresponding noise operators a_{in} and b_l^{in} ($l \in m, v$) in the above equations satisfy the following correlations [48]:

$$\begin{aligned} \langle a_{\text{in}}^\dagger(t)a_{\text{in}}(t')\rangle &= 0, \\ \langle a_{\text{in}}(t)a_{\text{in}}^\dagger(t')\rangle &= \delta(t-t'), \\ \langle b_l^{\text{in}\dagger}(t)b_l^{\text{in}}(t')\rangle &= n_l\delta(t-t'), \\ \langle b_l^{\text{in}}(t)b_l^{\text{in}\dagger}(t')\rangle &= (n_l+1)\delta(t-t'), \end{aligned} \tag{6}$$

where $n_l = \{\exp[\hbar\omega_l/(k_B T)] - 1\}^{-1}$ is the mean excitation at the lth subsystem ($l \in m, v$) at an equilibrium temperature T and k_B is the Boltzmann constant.

This Hamiltonian is nonlinear in the cavity mode operators a, as evident from the term proportional to g_1. In order to study the dynamics of the optomechanical system, we use the standard linearization procedure [1,32]. We expand the operators a, b_m, and b_v as a sum of their respective means and the fluctuation operators as follows: $a \to \alpha + \delta a$, $b_m \to \beta_m + \delta b_m$, and $b_v \to \beta_v + \delta b_v$, in the limit of large values of the means, i.e., $|\alpha| \gg 1$ and $|\beta_{m,v}| \gg 1$. The time dependence of these means is governed by the following equations, as obtained from Eqs. (5):

$$\begin{aligned} \dot{\alpha} &= -(\gamma_a + i\Delta')\alpha - i\Omega, \\ \dot{\beta_m} &= -(\gamma_m + i\omega_m)\beta_m + ig_1|\alpha|^2, \\ \dot{\beta_v} &= -(\gamma_v + i\omega_v)\beta_v + ig_2(\bar{x}_v)|\alpha|^2, \end{aligned} \tag{7}$$

where $g_2(\bar{x}_v)$ is calculated at the equilibrium position $\bar{x}_v = (\beta_v + \beta_v^*)/\sqrt{2}$ of the ion. Here $\Delta' = (\Delta_a - g_1(\beta_m^* + \beta_m) + \frac{g^2(\bar{x}_v)}{\Delta_{\text{el}}})$ represents the modified cavity detuning, as the cavity resonance frequency is shifted due to the motion of the ion and the mirror [49]. At the steady state, Eqs. (7) can be solved, using $\dot{\alpha}, \dot{\beta_m}, \dot{\beta_v} = 0$. The steady-state values of the means can thereby be obtained as

$$\alpha = \frac{\Omega}{-\Delta' + i\gamma_a}, \quad \beta_m = \frac{g_1|\alpha|^2}{\omega_m - i\gamma_m}, \quad \beta_v = \frac{g_2(\bar{x}_v)|\alpha|^2}{\omega_v - i\gamma_v}. \tag{8}$$

Similarly, the fluctuation operators evolve according to the following Langevin equations:

$$\begin{aligned} \dot{\delta a} = {} & -(\gamma_a + i\Delta')\delta a + ig_1\alpha(\delta b_m + \delta b_m^\dagger) \\ & + i\frac{g_2(\bar{x}_v)}{\sqrt{2}}\alpha(\delta b_v + \delta b_v^\dagger) + \sqrt{2\gamma_a}a_{\text{in}}, \end{aligned} \tag{9}$$

$$\delta\dot{b}_m = -(\gamma_m + i\omega_m)\delta b_m + ig_1\alpha(\delta a^\dagger + \delta a) + \sqrt{2\gamma_m}b_m^{\text{in}}, \tag{10}$$

$$\delta\dot{b}_v = -(\gamma_v + i\omega_v')\delta b_v + i\frac{g_2(\bar{x}_v)}{\sqrt{2}}\alpha(\delta a^\dagger + \delta a) + \sqrt{2\gamma_v}b_v^{\text{in}}, \tag{11}$$

where $\omega_v' = \omega_v + g_2(\bar{x}_v)|\alpha|^2$ is the shifted resonance frequency of the vibrational mode. Here we have made use of the following Taylor expansion of $g^2(\hat{x}_v)$ about the steady-state position $\bar{x}_v$:

$$g^2(\hat{x}_v) = g_0^2\cos^2(\eta\bar{x}_v) - \eta g_0^2\sin(2\eta\bar{x}_v)\delta x_v, \tag{12}$$

where $\delta x_v = (\delta b_v + \delta b_v^\dagger)/\sqrt{2}$. We have considered only up to the first order in η in the Lamb-Dicke limit $\eta \ll 1$.

Clearly, the above set of Langevin's equations (9)–(11) can be derived from the following Hamiltonian:

$$H = \Delta'\delta a^\dagger \delta a + \omega_m \delta b_m^\dagger \delta b_m + \omega_v' \delta b_v^\dagger \delta b_v - G_m(\delta a^\dagger + \delta a)(\delta b_m + \delta b_m^\dagger) - G_v(\delta a^\dagger + \delta a)(\delta b_v + \delta b_v^\dagger), \quad (13)$$

where $G_m = g_1\alpha$ and $G_v = g_2\alpha/\sqrt{2} = \frac{\eta g_0^2 \sin(2\eta\bar{x}_v)\alpha}{\sqrt{2}\Delta_{\rm el}}$. This indicates that two vibrational modes b_m and b_v couple with each other through the cavity mode, with a coupling constant proportional to the mean photon number $|\alpha|^2$ inside the cavity.

Stability analysis

From Eq. (8), we derive the following nonlinear equation for mean photon numbers $|\alpha|^2$ in the cavity in the steady state:

$$\left(-\Delta_a + \frac{2\omega_m g_1^2|\alpha|^2}{\omega_m^2+\gamma_m^2} - \frac{g^2(\bar{x}_v)}{\Delta_{\rm el}}\right)^2 |\alpha|^2 + \gamma_a^2|\alpha|^2 = \Omega^2. \quad (14)$$

To study the bistable behavior, we should have $\frac{\partial|\Omega|^2}{\partial|\alpha|^2} = 0$. From the above equation we have

$$3\left(\frac{2\omega_m g_1^2}{\omega_m^2+\gamma_m^2}\right)^2 |\alpha|^4 + \left[\left(\Delta_a + \frac{g^2(\bar{x}_v)}{\Delta_{\rm el}}\right)^2 + \gamma_a^2\right] - 4\left(\Delta_a + \frac{g^2(\bar{x}_v)}{\Delta_{\rm el}}\right)\frac{2\omega_m g_1^2}{\omega_m^2+\gamma_m^2}|\alpha|^2 = 0. \quad (15)$$

The above equation is quadratic in $|\alpha|^2$ and will have two distinct roots when the discriminant is positive, i.e., when

$$4\left(\frac{2\omega_m g_1^2}{\omega_m^2+\gamma_m^2}\right)^2\left[\left(\Delta_a + \frac{g^2(\bar{x}_v)}{\Delta_{\rm el}}\right)^2 - 3\gamma_a^2\right] > 0. \quad (16)$$

Clearly, a trivial solution for stability corresponds to the absence of the optomechanical coupling. For nonzero optomechanical coupling, the condition for obtaining bistable behavior of the system can be written as

$$\left[\left(\Delta_a + \frac{g^2(\bar{x}_v)}{\Delta_{\rm el}}\right)^2 - 3\gamma_a^2\right] > 0. \quad (17)$$

We can control the bistability behavior by changing the power of the input laser field, the cavity-laser detuning, and the laser-ion detuning. In Figs. 2 and 3 we show how $|\alpha|^2$ varies with the power of the driving laser and the detuning of the cavity mode with the driving laser, respectively. This clearly displays a parameter region for which the dynamics becomes unstable. In this paper, we choose a set of parameters so that we avoid such an unstable region.

IV. SELECTIVE EXCITATION OF THE MIRROR AND THE ION: QUANTUM MEMORY OF THE OPTICAL PULSE

Next, we will describe how one can selectively excite the mirror or the ionic vibration. We will also describe a possible strategy to transfer fluctuation energy between the mirror and the ionic vibration, mediated by the cavity.

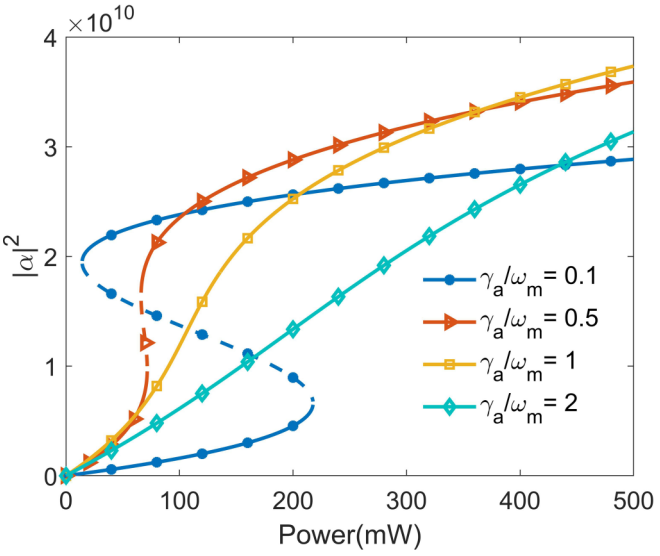

FIG. 2. The variation of the number $|\alpha|^2$ of photons in the cavity with respect to the power P of the laser field. The parameters chosen are $\omega_m = 2\pi \times 10$ MHz, $\omega_p/\omega_m = 351.3 \times 10^6$, $\Delta_a/\omega_m = \omega_v/\omega_m = 1$, $g_1/\omega_m = 125 \times 10^{-6}$, $g/\omega_m = 0.14$, $\gamma_v/\omega_m = 0.001$, $\gamma_m/\omega_m = 0.00001$, $\gamma_{\rm el}/\omega_m = 1.12$, and $\frac{g^2(\bar{x}_v)}{\Delta_{\rm el}}/\omega_m = 0.02$. The dashed portions of the plots represent the unstable regions.

We first start with Eq. (9). In the rotating frame with respect to Δ', ω_m, ω_v', this equation can be rewritten as

$$\delta\dot{\tilde{a}} = -\gamma_a\delta\tilde{a} + iG_m(\delta\tilde{b}_m e^{-i\omega_m t} + \text{H.c.})e^{i\Delta' t} + iG_v(\delta\tilde{b}_v e^{-i\omega_v' t} + \text{H.c.})e^{i\Delta' t} + \sqrt{2\gamma_a}a_{\rm in}e^{i\Delta' t}, \quad (18)$$

where $\delta\tilde{a} = \delta a e^{i\Delta' t}$, $\delta\tilde{b}_m = \delta b_m e^{i\omega_m t}$, and $\delta\tilde{b}_v = \delta b_v e^{i\omega_v' t}$. Integrating the above equation and using the rotating wave approximation at resonance $\Delta' = \omega_m = \omega_v'$, we obtain the

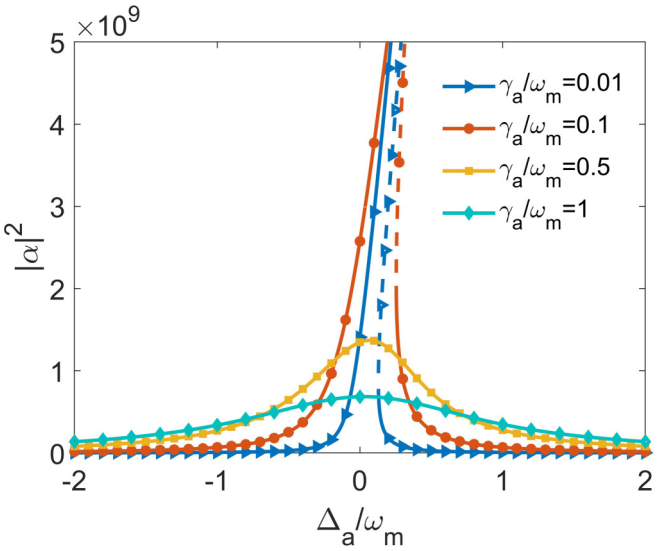

FIG. 3. The variation of the number $|\alpha|^2$ of photons in the cavity with respect to the detuning Δ_a of the cavity with the laser field. We choose the power of the driving laser as $P = 10$ mW, while the other parameters are the same as in Fig. 2. The dashed portions of the plots represent the unstable regions.

following equation:

$$\delta\tilde{a}(t) = \frac{iG_m}{\gamma_a}\delta\tilde{b}_m + \frac{iG_v}{\gamma_a}\delta\tilde{b}_v + \sqrt{2\gamma_a}e^{-\gamma_a t}\int_{t_0}^{t} e^{\gamma_a t'}\tilde{a}_{\rm in}(t')dt', \tag{19}$$

where $\tilde{a}_{\rm in}(t') = a_{\rm in}(t')e^{i\Delta' t'}$, t_0 denotes the initialized time, and we have used the bad-cavity limit $\gamma_a \gg G_m, G_v$. This refers to a timescale during which $\delta\tilde{a}$ decays much faster than the fluctuation terms $\delta\tilde{b}_m$ and $\delta\tilde{b}_v$ evolve. Therefore, the cavity mode fluctuation adiabatically follows the evolution of $\delta\tilde{b}_m$ and $\delta\tilde{b}_v$. Similar approximation can be taken for the input noise $\tilde{a}_{\rm in}$, so that $\tilde{a}_{\rm in}(t')$ can be replaced by $\tilde{a}_{\rm in}(t)$ and we obtain the following form $\delta\tilde{a}$ in terms of the filtered noise $\tilde{a}_{\rm in}(t)$:

$$\delta\tilde{a}(t) = \frac{iG_m}{\gamma_a}\delta\tilde{b}_m + \frac{iG_v}{\gamma_a}\delta\tilde{b}_v + \frac{\sqrt{2}}{\sqrt{\gamma_a}}\tilde{a}_{\rm in}(t). \tag{20}$$

Equations (10) and (11) can also be rewritten in the rotating frame, using the rotating wave approximation, as follows:

$$\begin{aligned}\delta\dot{\tilde{b}}_m &= -\gamma_m\delta\tilde{b}_m + iG_m\delta\tilde{a} + \sqrt{2\gamma_m}\,\tilde{b}_m^{\rm in},\\ \delta\dot{\tilde{b}}_v &= -\gamma_v\delta\tilde{b}_v + iG_v\delta\tilde{a} + \sqrt{2\gamma_v}\,\tilde{b}_v^{\rm in},\end{aligned} \tag{21}$$

where $\tilde{b}_m^{\rm in} = b_m^{\rm in}e^{i\omega_m t}$ and $\tilde{b}_v^{\rm in} = b_v^{\rm in}e^{i\omega_v t}$. Next replacing $\delta\tilde{a}$ using Eq. (20) in the above equations, we have

$$\delta\dot{\tilde{b}}_m = -\gamma_m'\delta\tilde{b}_m - C_{mv}\delta\tilde{b}_v + iG_m\frac{\sqrt{2}}{\sqrt{\gamma_a}}\tilde{a}_{\rm in}(t) + \sqrt{2\gamma_m}\,\tilde{b}_m^{\rm in}, \tag{22}$$

$$\delta\dot{\tilde{b}}_v = -\gamma_v'\delta\tilde{b}_v - C_{mv}\delta\tilde{b}_m + iG_v\frac{\sqrt{2}}{\sqrt{\gamma_a}}\tilde{a}_{\rm in}(t) + \sqrt{2\gamma_v}\,\tilde{b}_v^{\rm in}, \tag{23}$$

where $\gamma_m' = \gamma_m + G_m^2/\gamma_a$ and $\gamma_v' = \gamma_v + G_v^2/\gamma_a$ are the respective modified decay rates of the vibrational modes of the mirror and the ion, modified due to their common coupling to the cavity mode fluctuation. Here $C_{mv} = G_mG_v/\gamma_a$ represents the cavity mode mediated effective coupling strength between the mirror mode and the ionic vibration.

We next integrate Eq. (22) to a time $T_m \gg 1/\gamma_m'$, such that

$$\begin{aligned}\delta\tilde{b}_m(T_m) = &-C_{mv}\int_{t_0}^{T_m} e^{\gamma_m'(t-T_m)}\delta\tilde{b}_v(t)dt\\ &+ iG_m\frac{\sqrt{2}}{\sqrt{\gamma_a}}\int_{t_0}^{T_m} e^{\gamma_m'(t-T_m)}\tilde{a}_{\rm in}(t)dt\\ &+ \sqrt{2\gamma_m}\int_{t_0}^{T_m} e^{\gamma_m'(t-T_m)}\tilde{b}_m^{\rm in}(t)dt.\end{aligned} \tag{24}$$

We here introduce a time-integrated annihilation operator $A_{\rm in}^m$ [50], which is a weighted sum of all time-local noise operators $\tilde{a}_{\rm in}(t)$, as given by

$$A_{\rm in}^m = \frac{1}{N}\left[iG_m\frac{\sqrt{2}}{\sqrt{\gamma_a}}\right]\int_{t_0}^{T_m} e^{\gamma_m' t}\tilde{a}_{\rm in}(t)dt, \tag{25}$$

where N is a normalization factor. This satisfies the usual commutation relation $[A_{\rm in}^m, A_{\rm in}^{m\dagger}] = 1$, while $N = [G_m/\gamma_a]\sqrt{(\gamma_a/\gamma_m')[\exp(2\gamma_m'T_m) - 1]}$. Here we have considered the noise operators to be delta correlated in the time domain. In this way, the operator $A_{\rm in}^m$ can be interpreted as an annihilator operator of a pulse that exists for a time interval $[t_0, T_m]$.

Therefore we can rewrite Eq. (24) in the limit $\gamma_m'T_m \gg 1$ as

$$\delta\tilde{b}_m(T_m) + \frac{C_{mv}}{\gamma_m'}\delta\tilde{b}_v(T_m) = \frac{G_m}{\sqrt{\gamma_m'\gamma_a^2}}A_{\rm in}^m(T_m) + \sqrt{\gamma_m/\gamma_m'}B_{\rm in}^m(T_m), \tag{26}$$

where

$$B_{\rm in}^m(T_m) = \sqrt{\frac{2\gamma_m'}{e^{2\gamma_m'T_m} - 1}}\int_{t_0}^{T_m} e^{\gamma_m' t}\tilde{b}_m^{\rm in}(t)dt \tag{27}$$

is the annihilation operator for structured noise of the mirror mode. Similar solution of the form of Eq. (26) (with m replaced by v) can be obtained for Eq. (23) in the limit $\gamma_v'T_v \gg 1$.

The above solution (26) suggests that for a suitably time-structured input $A_{\rm in}^m$ (in which the input noises at different times add up with a weight factor $e^{\gamma_m' t}$ to make the input pulse) can be mapped into the fluctuation mode $\delta\tilde{b}_m$ at the long-time limit $T_m \gg 1/\gamma_m'$, if $C_{mv} \ll \gamma_m'$. Similarly, if the input field is structured in terms of the weight factor $e^{\gamma_v' t}$, the input pulse could be mapped into the mode $\delta\tilde{b}_v$ at the steady-state limit $T_v \gg 1/\gamma_v'$, if $C_{mv} \ll \gamma_v'$. This clearly implies that if an input pulse is suitably shaped its state can be selectively mapped into the fluctuation of either the mirror or the ionic vibration at a timescale much longer than that of the relevant decay. Thus such state transfer is not affected by the decay of the mirror or the ionic vibration. Note that such a selectivity can be realized if γ_m' and γ_v' are *not* of the same order of magnitude. As the decay rate of the cavity mode is much larger than those of the vibrational modes involved, these modes can be further interpreted as a quantum memory [51] of the optical pulse.

It is important to mention that the transfer of the state in our protocol occurs at steady state, i.e., at a timescale much longer than $1/\gamma_m'$ or $1/\gamma_v'$, depending upon the pulse shaping (i.e., whether the input pulse is modulated by $e^{\gamma_m' t}$ or by $e^{\gamma_v' t}$). In contrast, in [50], such a transfer is prone to the decay of the mirror, as the protocol works only for a timescale less than $1/\gamma_m$. Note that for a Gaussian temporal shape of the input pulse into the cavity, i.e., for $a_{\rm in}(t) \sim \exp(-t^2/\tau^2)$ (τ represents the temporal width of the pulse), the effective shape of the pulse becomes exponentially increasing, when t_0 is a large negative value and $T_m = 0$ (the moment when the interaction is switched on) [see Eq. (25)]. The pulse in this effective exponential shape gets stored into the vibrational mode of the mirror or the ion, at the steady state (large T_m), as discussed above.

On the other hand, if $C_{mv} \sim \gamma_m'$, then the state of the input pulse (shaped with a corresponding weight factor $e^{\gamma_m' t}$) can be

mapped into a collective fluctuation state of the mirror and the ionic vibration, as defined by the joint annihilation operator $\delta\tilde{b}_m(T_m) + \delta\tilde{b}_v(T_m)$. This would refer to a two-mode binomial state (a special class of entangled state) [52] of fluctuations of the two coupled systems, as described below:

$$|\psi\rangle_{\rm in} \otimes |0\rangle_m |0\rangle_v = \sum_n C_n [A_{\rm in}^{m\dagger}(T_m)]^n |0\rangle_{\rm in} \otimes |0\rangle_m |0\rangle_v \rightarrow |0\rangle_{\rm in} \otimes \sum_n C_n [\delta\tilde{b}_m^\dagger(T_m) + \delta\tilde{b}_v^\dagger(T_m)]^n |0\rangle_m |0\rangle_v. \quad (28)$$

Here the state $|\psi\rangle_{\rm in}$ is the initial state of the input pulse. When expressed in the Fock state basis, C_n represents the probability amplitude of the n photons inside the pulse. Note that the state $|n\rangle$ refers to n quanta corresponding to the fluctuation operators $\delta\hat{O}$ ($\hat{O} \equiv \tilde{a}, \tilde{b}_v, \tilde{b}_m$), *in excess of* their respective steady-state number of quanta, namely, $|\alpha|^2$, $|\beta_v|^2$, and $|\beta_m|^2$; e.g., $\langle a^\dagger a\rangle = |\alpha|^2 + \langle \delta a^\dagger \delta a\rangle$ and so on. It must be borne in mind that these fluctuation operators satisfy the usual commutation relation $[\delta O, \delta O^\dagger] = 1$, pertaining to the harmonic oscillator algebra, and hence make a suitable Hilbert space of Fock states. The size of the finite series of the binomial state, as obtained above, depends on the temporal shape of the input pulse.

A similar binomial state can also be obtained if the input pulse is shaped with a weight factor $e^{\gamma_v' t}$ and $C_{mv} \sim \gamma_v'$. Note that C_{mv} can be made of the order of γ_m' or γ_v' by suitable choice of the steady-state photon number $|\alpha|^2$ inside the cavity.

V. FLUCTUATION ENERGY TRANSFER BETWEEN THE MIRROR AND THE ION

We start with Eq. (13), which takes the following form in the rotating wave approximation at resonance $\Delta' = \omega_m = \omega_v'$:

$$H_{\rm RWA} = -G_m(\delta a^\dagger \delta b_m + {\rm H.c.}) - G_v(\delta a^\dagger \delta b_v + {\rm H.c.}). \quad (29)$$

The above Hamiltonian suggests that it is possible to transfer the fluctuation of the mirror into the ionic vibration and vice versa, with the aid of their common coupling to the cavity fluctuation mode. To this end, one could consider adiabatic elimination of the cavity mode δa, by putting $\dot{\delta a} \approx 0$ in the good-cavity limit $\Delta' \gg \gamma_a$. However, such a situation is vulnerable to the cavity decay. We employ an alternative approach, based on pulsed excitation technique. Transfer of population between two states can be obtained by two pulses which are applied in the so-called counterintuitive sequence, as in the stimulated Raman adiabatic passage (STIRAP) technique [53]. Such transfer has been studied using a dark mode in [54,55]. However, in the present case, both G_m and G_v are proportional to α, and thereby to the cavity driving field Ω [see Eq. (8)]. Therefore we can employ a suitable pulse shape $\Omega(t)$ that would *simultaneously* couple the cavity mode to fluctuations of the ionic vibration and the mirror. This would allow us to transfer the fluctuation energy from the mirror to the ionic vibration via the cavity mode and vice versa, in a controlled and deterministic way at long times. Unlike in STIRAP, in this case, the intermediary cavity mode would be populated and so the pulse duration should be well within the decay timescale of the cavity mode. Note that such a method of energy transfer is not truly adiabatic, because it involves exchange of energy between various adiabatic states.

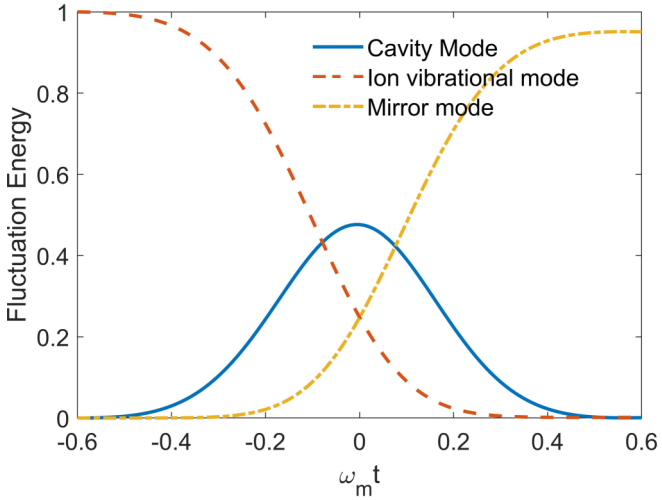

FIG. 4. Fluctuation energy transfer between the vibrational modes of the mirror and the ion. The parameters chosen are $\omega_m = 2\pi \times 10$ MHz, $\Delta'/\omega_m = \omega_v'/\omega_m = 1$, $G_m/\omega_m = G_v/\omega_m = 0.3\exp(-t^2/0.2)$, $\gamma_v/\omega_m = 0.001$, $\gamma_a/\omega_m = 0.01$, $\gamma_m/\omega_m = 0.00001$, and $T = 0.3$ mK.

For suitable numerical simulation, we start with the following Langevin equation for the fluctuation δa in the rotating frame, as obtained using Eq. (29):

$$\dot{\delta\tilde{a}} = -\gamma_a \delta\tilde{a} + iG_m \delta\tilde{b}_m + iG_v \delta\tilde{b}_v + \sqrt{2\gamma_a}\tilde{a}_{\rm in}, \quad (30)$$

while the equations for $\delta\tilde{b}_{v,m}$ are the same as in Eqs. (21). We next derive the equations for quadratic combinations of these fluctuations and solve those equations to study the temporal dynamics of the fluctuation energies $\langle \delta a^\dagger \delta a\rangle$, $\langle \delta b_m^\dagger \delta b_m\rangle$, and $\langle \delta b_v^\dagger \delta b_v\rangle$. In Fig. 4, we show how the fluctuation energies can be transferred from the vibrational mode of the ion to that of the mirror. We consider a pulse with a Gaussian temporal profile, that drives the cavity mode, and eventually make a simultaneous coupling between the cavity mode with the vibrational modes of the other two subsystems, respectively. In presence of the cavity decay (with a rate $\approx 10^5$ Hz), the fluctuations get transferred almost completely. However, It is important to note that, unlike in the case of STIRAP, this nonadiabatic process is vulnerable to the relevant decay rates and the pulse parameters.

VI. ENTANGLEMENT

We next show how fluctuations of various subsystems, namely, the cavity mode and the vibrational modes of the mirror and the ion, can be entangled at the steady state. As these subsystems are primarily bosonic, we prefer to consider entanglement in fluctuations of the relevant continuous variables [56,57], namely, in their position and conjugate momentum quadratures δx_l and δp_l ($l \in a, m, v$),

defined by

$$\delta x_a = \frac{(\delta a^\dagger + \delta a)}{\sqrt{2}}, \quad \delta p_a = \frac{i(\delta a^\dagger - \delta a)}{\sqrt{2}},$$
$$\delta x_m = \frac{(\delta b_m^\dagger + \delta b_m)}{\sqrt{2}}, \quad \delta p_m = \frac{i(\delta b_m^\dagger - \delta b_m)}{\sqrt{2}},$$
$$\delta x_v = \frac{(\delta b_v^\dagger + \delta b_v)}{\sqrt{2}}, \quad \delta p_v = \frac{i(\delta b_v^\dagger - \delta b_v)}{\sqrt{2}}. \tag{31}$$

In the following, we employ the treatment in [16], in which the steady-state entanglement between two bosonic systems, namely, two mirrors, has been studied.

The quantum Langevin equations for these fluctuations can be written in a matrix form:

$$\frac{d}{dt}R(t) = MR(t) + N(t), \tag{32}$$

where $R(t)^T = (\delta x_m, \delta p_m, \delta x_v, \delta p_v, \delta x_a, \delta p_a)$. Here the coefficient matrix M and the noise vector N are given by [58]

$$M = \begin{pmatrix} 0 & \omega_m & 0 & 0 & 0 & 0 \\ -\omega_m & -\gamma_m & 0 & 0 & 2G_m & 0 \\ 0 & 0 & 0 & \omega_v' & 0 & 0 \\ 0 & 0 & -\omega_v' & -\gamma_v & 2G_v & 0 \\ 0 & 0 & 0 & 0 & -\gamma_a & \Delta' \\ 2G_m & 0 & 2G_v & 0 & -\Delta' & -\gamma_a \end{pmatrix},$$

$$N(t) = \begin{pmatrix} 0 \\ \sqrt{2\gamma_m}p_m^{\rm in} \\ 0 \\ \sqrt{2\gamma_v}p_v^{\rm in} \\ \sqrt{2\gamma_a}x_a^{\rm in} \\ \sqrt{2\gamma_a}p_a^{\rm in} \end{pmatrix}. \tag{33}$$

The vibrational modes are usually affected by a Brownian stochastic non-Markovian noise $p_m^{\rm in}$ and $p_v^{\rm in}$, respectively [59,60]. Their two-time correlation functions exhibit a nontrivial dependence on time, so as to ensure that the operators maintain their commutation relation for all the times via the above evolution of R [61]. However, at large temperature ($k_BT \gg \hbar\omega_m, \hbar\omega_v'$) and for large quality factors of the vibrating systems (i.e., $\omega_m \gg \gamma_m$ and $\omega_v' \gg \gamma_v$), these correlation functions represent Markovian dynamics and can be approximated as $\langle p_l^{\rm in}(t)p_l^{\rm in}(t') + p_l^{\rm in}(t')p_l^{\rm in}(t)\rangle \approx (2n_l + 1)\delta(t - t')$ ($l \in m, v$), where n_l is the average number of thermal quanta in the lth subsystem [16,60,62].

The solution of the differential matrix equation for $R(t)$ can be obtained as $R(t) = M(t)R(0) + \int_0^t dsF(s)N(t - s)$, where $F(t) = \exp(Mt)$. Temporal stability of this solution can be obtained in the parameter domain, at which the real parts of all the eigenvalues of the matrix M are negative, which effectively makes sure that $F(t \to \infty) = 0$.

To study the entanglement in a coupled bosonic system, we first obtain the covariance matrix V [56,63,64], which satisfies the following relation [7,65]:

$$MV + VM^T = -D, \tag{34}$$

where the diffusion matrix D is given by

$$D = \text{diag}[0,\ (2n_m + 1)\gamma_m,\ 0,\ (2n_v + 1)\gamma_v,\ \gamma_a,\ \gamma_a]. \tag{35}$$

In steady state, the system attains a Gaussian state [64] and its entanglement properties can be derived from the covariance matrix V at the steady state, when expressed in the following block form:

$$V = \begin{pmatrix} V_m & V_{mv} & V_{ma} \\ V_{mv}^T & V_v & V_{va} \\ V_{ma}^T & V_{va}^T & V_a \end{pmatrix}. \tag{36}$$

Here $V_{ij} = [\langle R_i(\infty)R_j(\infty) + R_j(\infty)R_i(\infty)\rangle]/2$, and $R_i(\infty)$ is the ith element in the matrix R, calculated at the steady state. In Eq. (36) each diagonal element represents a 2×2 matrix for quadratures of the lth subsystem ($l \in m, v, a$), while the off-diagonal elements $V_{ll'}$ represent a 2×2 matrix of intersystem covariance ($l, l' \in m, v, a$). With such a form of the matrix V, the correlation between any two subsystems can be obtained in terms of the corresponding submatrix [59].

To study the entanglement between the vibrational modes of the mirror and the ion, we compute the logarithm negativity E_N [66,67], given by

$$E_N = \max[0, -\ln 2\mu^-], \tag{37}$$

where $\mu^- = 1/\sqrt{2}[A - (A^2 - 4\det V_S)^{1/2}]^{1/2}$. Here, V_S is the relevant submatrix, as given by

$$V_S = \begin{pmatrix} V_m & V_{mv} \\ V_{mv}^T & V_v \end{pmatrix} \tag{38}$$

and $A = \det(V_m) + \det(V_v) - 2\det(V_{mv})$.

A Gaussian state is considered entangled only if $\mu^- < 1/2$. Note that this is equivalent to the necessary and sufficient criterion for entanglement, based on nonpositive partial transpose of Gaussian states [68,69]. We show in Fig. 5 how the entanglement E_N between the fluctuations of the vibrational modes of the mirror and the ion varies with respect to the detuning Δ' for different values of cavity decay rate. Clearly, with the increase in the decay rate, the entanglement decreases, as well as the domain of Δ' over which the two subsystems attain steady-state entanglement decrease. There is an optimal detuning at which the entanglement becomes maximum. At such a detuning, the entanglement also remains robust against the variation of the ambient temperature of the ion and the mirror over a large range (see Fig. 6), while it experiences *sudden death* at a finite temperature. It must be borne in mind that though the mirror and the ion do not directly interact with each other the cavity mode acts as a mediator to produce entanglement between their motional fluctuations.

VII. SYMPATHETIC COOLING

In Sec. V, we have shown that it is possible to transfer the average energy of motional fluctuation from the ion to the mirror. This suggests that it would also be possible to cool the mirror *sympathetically* by the ion. To further investigate this,

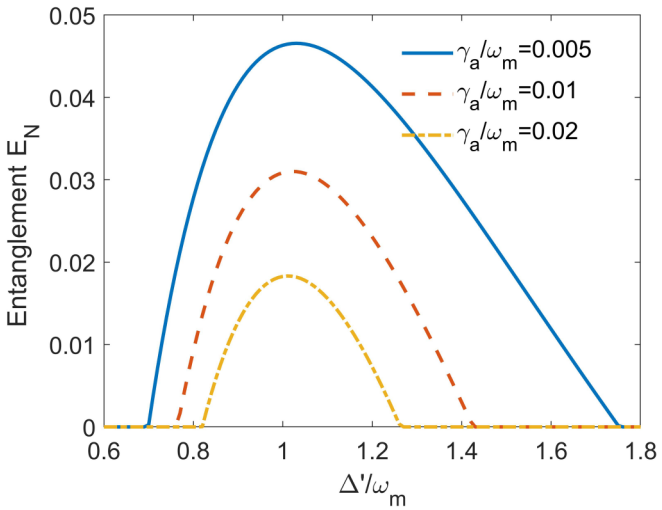

FIG. 5. Variation of the logarithmic negativity between the vibrational fluctuations of the mirror and the ion, with respect to the detuning Δ', for different values of the decay rate γ_a of the cavity mode. Here we have chosen $\omega_v'/\omega_m = 0.88$, $G_m/\omega_m = 0.1$, and $G_v/\omega_m = 0.35$. The other parameters are the same as in Fig. 4.

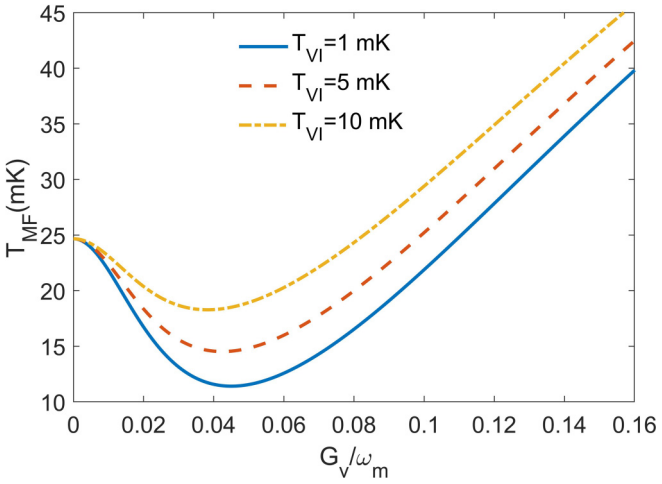

FIG. 7. Variation of the final temperature $T_{\rm MF}$ of the mirror with respect to the cavity-ion coupling G_v, for different values of the initial temperature of the ion. Here we have chosen $\omega_m = 2\pi \times 10$ MHz, $\omega_v'/\omega_m = 1$, $\Delta'/\omega_m = 5$, $G_m/\omega_m = 0.02$, $T_{\rm MI} = 100$ mK, and $\gamma_a/\omega_m = 5$. The other parameters are the same as in Fig. 4.

we choose the cavity detuning and its decay to be very large as compared to all the other frequencies: $\Delta', \gamma_a \gg \omega_m, \omega_v' \gg G_m, G_v$. We also consider the effective resonance frequencies of the ionic vibrational mode and the mirror to be equal, i.e., $\omega_v' = \omega_m$. Under these conditions, the cavity mode can be adiabatically eliminated from the effective dynamics of the system. So, we can write the final Hamiltonian as [see Eq. (13)]

$$H = \omega_m \delta b_m^\dagger \delta b_m + \omega_v' \delta b_v^\dagger \delta b_v + G(\delta b_v^\dagger \delta b_m + \delta b_v \delta b_m^\dagger), \tag{39}$$

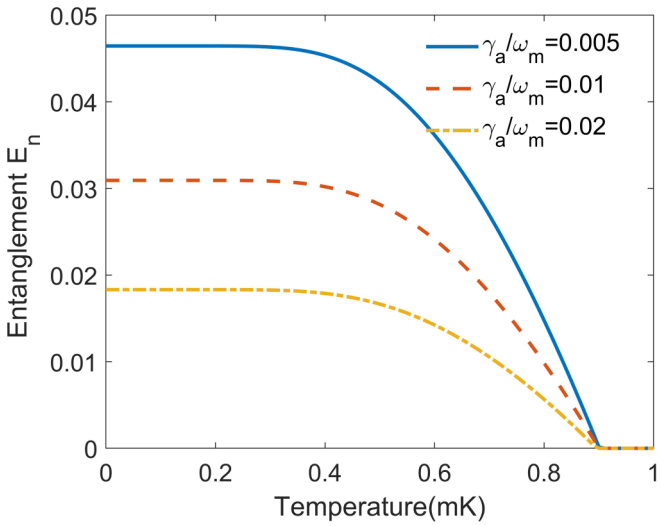

FIG. 6. Variation of the logarithmic negativity between the vibrational fluctuations of the mirror and the ion, with respect to the ambient temperature T, for different values of the decay rate γ_a of the cavity mode. We have chosen $\Delta'/\omega_m = 1$. The other parameters are the same as in Fig. 5.

where $G = \frac{4G_m G_v \Delta'}{\Delta'^2 + \gamma_a^2}$ is the effective coupling strength between the mirror and the ionic vibration. Such a coupling can be employed to sympathetically cool the mirror with the help of the ion.

To demonstrate the cooling, we first calculate the mean numbers of quanta $n_{\rm MF}$ and $n_{\rm VF}$, of the vibrational mode of the mirror and the ion, respectively, at the steady state, using the elements of the covariance matrix V, as follows:

$$\begin{aligned} n_{\rm MF} &= \tfrac{1}{2}\left(\langle \delta x_m^2 \rangle + \langle \delta p_m^2 \rangle\right) = \tfrac{1}{2}(V_{11} + V_{22} - 1), \\ n_{\rm VF} &= \tfrac{1}{2}\left(\langle \delta x_v^2 \rangle + \langle \delta p_v^2 \rangle\right) = \tfrac{1}{2}(V_{33} + V_{44} - 1). \end{aligned} \tag{40}$$

These can be represented in terms of the effective temperatures $T_{\rm MF}$ and $T_{\rm VF}$ of the vibrational modes of the mirror and the ion, respectively, as

$$T_{\rm MF} = \frac{\hbar \omega_m}{k_B \ln\left(1 + \frac{1}{n_{\rm MF}}\right)}, \quad T_{\rm VF} = \frac{\hbar \omega_v'}{k_B \ln\left(1 + \frac{1}{n_{\rm VF}}\right)}. \tag{41}$$

We show in Fig. 7 how the temperature $T_{\rm MF}$ of the mirror varies with respect to the ion-cavity coupling G_v. When the ion is cooled, for an optimal value of G_v, the mirror also gets cooled. We have chosen the mirror to be initially at a temperature of 100 mK. This means that for a large range of values of G_v and T_m the mirror gets cooled. As the temperature of the ion increases, such a sympathetic cooling of the mirror, however, becomes less effective. In absence of the ion, the mirror could be sideband cooled by the cavity mode. In contrast, in the presence of the ion, cooling of the mirror gets enhanced. We further show in Fig. 8 how the temperature $T_{\rm MF}$ of the mirror varies with respect to the cavity detuning Δ'. Clearly, in presence of ion-cavity coupling, the cavity detuning has negligible effect on the cooling as compared to the case when the ion is absent. It is further interesting to note

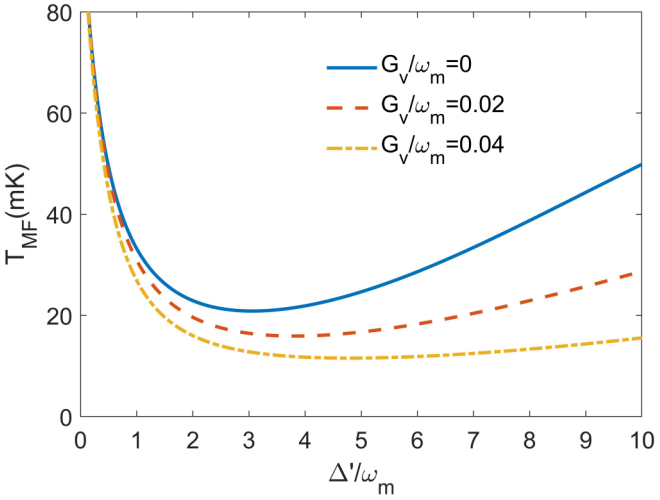

FIG. 8. Variation of the temperature $T_{\rm MF}$ of the mirror with respect to the cavity detuning Δ', for different values of cavity-ion coupling G_v. Here we have chosen $T_{\rm VI} = 1$ mK and $T_{\rm MI} = 100$ mK. The other parameters are the same as in Fig. 7.

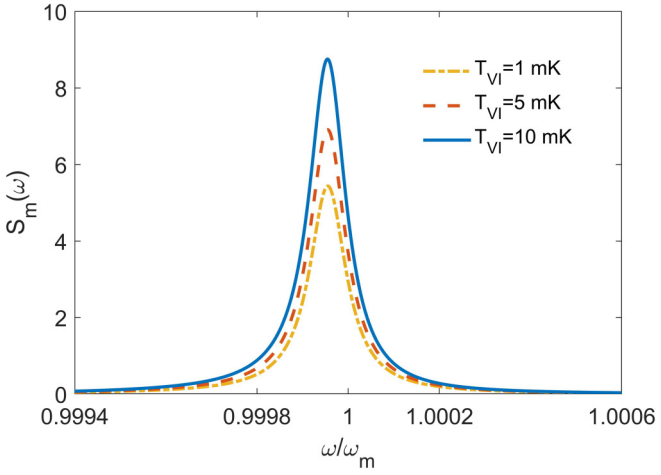

FIG. 9. Spectral noise density (in $\mathrm{m^2Hz^{-1}}$) of the mirror, for different values of the ion temperature. Here we have chosen $G_v/\omega_m = 0.04$. The other parameters are the same as in Fig. 8.

in Fig. 8 that in presence of the ion the mirror gets cooled to a temperature lower than could be obtained with sideband cooling (at $\Delta' = \omega_m$) [42,43].

The effective sympathetic cooling rate for the mechanical oscillator can be written as [34]

$$\Gamma_{\rm eff} = \frac{G^2}{\gamma_v}\frac{1}{1+(\gamma_v/4\omega_{\rm m})^2}. \tag{42}$$

For $\gamma_v \ll \omega_m$, this rate becomes $\Gamma_{\rm eff} = \frac{G^2}{\gamma_v}$. In the limit $n_{\rm VI} \ll n_{\rm MI}$ [where $n_{\alpha i}$ corresponds to the initial average number of thermal quanta of the subsystem α ($\alpha \in v, m$) for the vibrational mode of the ion and of the mirror, respectively], the steady-state occupation number of the mechanical oscillator can be written as [34]

$$n_{\rm MF} \approx \frac{\gamma_m n_{\rm MI}}{\gamma_m + \Gamma_{\rm eff}}. \tag{43}$$

Thus the final temperature of the mechanical oscillator is given by

$$T_{\rm MF} = T_{\rm MI}(1+\Gamma_{\rm eff}/\gamma_m)^{-1} \tag{44}$$

where $T_{\rm MI}$ corresponds to initial temperature of the mechanical oscillator. For the parameters used in Fig. 7, we find that the sympathetic cooling rate is of the order of $\Gamma_{\rm eff} \approx 2\pi \times 0.001$ MHz (for $G_v/\omega_m = 0.04$), so that for $T_{\rm MI} = 100$ mK we get $T_{\rm MF} \approx 9$ mK, which is close to that obtained numerically (see Fig. 7).

The cooling can be further verified from the noise spectrum of the mirror. The spectral noise density of the mirror can be defined as

$$S_m(\omega) \equiv \int_{-\infty}^{+\infty} \langle \delta x_m(t)\delta x_m(0)\rangle e^{i\omega t}dt. \tag{45}$$

In Fig. 9, we display the spectrum of the mirror, for different values of the temperature of the ion. The area of the spectrum represents the final temperature of the mirror. Clearly, as the ion temperature decreases, this area also decreases. Moreover, for lower values of the ion temperature, the sympathetic cooling is more effective.

VIII. CONCLUSION

In conclusion, we have analyzed a hybrid cavity optomechanical setup to obtain a coherent coupling between the vibrational modes of two systems: a mirror which is mesoscopic in size and a trapped ion of microscopic size. Such a coupling is mediated via their common interaction to the cavity field. We show that in the bad-cavity limit the state of the optical pulse driving the cavity can be selectively mapped into the motional fluctuation of either of these two systems. This can be interpreted as a quantum memory of optical information into the vibrational mode of the mirror or the ion, the decay rate of which is negligibly smaller than that of the cavity mode. In this limit, by suitably adjusting the pulse parameters, one can prepare an entangled state of fluctuation of these two systems. On the other hand, in the good-cavity limit, we employ a pulsed excitation technique, to transfer the fluctuation energy of the ionic vibration to the mirror. We further study the possibility of obtaining steady-state entanglement between the two vibrating systems, and the effect of detuning and temperature on this entanglement. Interestingly, the mirror can be sympathetically cooled, when at resonance with the vibrational mode of the ion, by cooling the ion, in the adiabatic limit (i.e., in the bad-cavity limit). We have estimated the effective cooling rate, by taking the heating of the ionic vibration into consideration.

ACKNOWLEDGMENT

We sincerely thank one of the referees for pointing out the possibility of sympathetic cooling.

[1] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Rev. Mod. Phys. **86**, 1391 (2014).
[2] M. Aspelmeyer, P. Meystre, and K. Schwab, Phys. Today **65**(7), 29 (2012).
[3] T. A. Palomaki, J. W. Harlow, J. D. Teufel, R. W. Simmonds, and K. W. Lehnert, Nature (London) **495**, 210 (2013).
[4] A. H. Safavi-Naeini and O. Painter, New J. Phys. **13**, 013017 (2011).
[5] S. S. U and A. Narayanan, Phys. Rev. A **88**, 033802 (2013).
[6] M. J. Weaver *et al.*, Nat. Commun. **8**, 824 (2017).
[7] D. Vitali, S. Gigan, A. Ferreira, H. R. Bohm, P. Tombesi, A. Guerreiro, V. Vedral, A. Zeilinger, and M. Aspelmeyer, Phys. Rev. Lett. **98**, 030405 (2007).
[8] C. Genes, A. Mari, P. Tombesi, and D. Vitali, Phys. Rev. A **78**, 032316 (2008).
[9] J. Zhang, K. Peng, and S. L. Braunstein, Phys. Rev. A **68**, 013808 (2003).
[10] M. Pinard *et al.*, Europhys. Lett. **72**, 747 (2005).
[11] S. Mancini, V. Giovannetti, D. Vitali, and P. Tombesi, Phys. Rev. Lett. **88**, 120401 (2002).
[12] S. Huang and G. S. Agarwal, New J. Phys. **11**, 103044 (2009).
[13] D. Vitali, S. Mancini, L. Ribichini, and P. Tombesi, J. Opt. Soc. Am. B **20**, 1054 (2003).
[14] S. Gröblacher *et al.*, Nature (London) **556**, 473 (2018).
[15] M. A. Sillanpää *et al.*, Nature (London) **556**, 478 (2018).
[16] D. Vitali, S. Mancini, and P. Tombesi, J. Phys. A **40**, 8055 (2007).
[17] J. Q. Liao, Q. Q. Wu, and F. Nori, Phys. Rev. A **89**, 014302 (2014).
[18] J. Li *et al.*, Europhys. Lett. **110**, 64004 (2015).
[19] C. Monroe and J. Kim, Science **339**, 1164 (2013).
[20] J. Q. You and F. Nori, Nature (London) **474**, 589 (2011).
[21] E. Massoni and M. Orszag, Opt. Commun. **179**, 315 (2000).
[22] R. L. Rodrigues, M. H. Y. Moussa, and C. J. Villas-Boas, Phys. Rev. A **74**, 063811 (2006).
[23] A. S. Parkins and H. J. Kimble, J. Opt. B **1**, 496 (1999).
[24] K. Hammerer, M. Wallquist, C. Genes, M. Ludwig, F. Marquardt, P. Treutlein, P. Zoller, J. Ye, and H. J. Kimble, Phys. Rev. Lett. **103**, 063005 (2009).
[25] M. Wallquist, K. Hammerer, P. Zoller, C. Genes, M. Ludwig, F. Marquardt, P. Treutlein, J. Ye, and H. J. Kimble, Phys. Rev. A **81**, 023816 (2010).
[26] H. K. Cheung and C. K. Law, Phys. Rev. A **84**, 023812 (2011).
[27] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, Rev. Mod. Phys. **75**, 281 (2003).
[28] K. Hammerer, M. Aspelmeyer, E. S. Polzik, and P. Zoller, Phys. Rev. Lett. **102**, 020501 (2009).
[29] C. Genes, H. Ritsch, and D. Vitali, Phys. Rev. A **80**, 061803(R) (2009).
[30] C. Genes, H. Ritsch, M. Drewsen, and A. Dantan, Phys. Rev. A **84**, 051801(R) (2011).
[31] D. Meiser and P. Meystre, Phys. Rev. A **73**, 033417 (2006).
[32] C. Genes, D. Vitali, and P. Tombesi, Phys. Rev. A **77**, 050307(R) (2008).
[33] M. Abdi and A. R. Bahrampour, Phys. Lett. A **376**, 2955 (2012).
[34] B. Vogell, K. Stannigel, P. Zoller, K. Hammerer, M. T. Rakher, M. Korppi, A. Jockel, and P. Treutlein, Phys. Rev. A **87**, 023816 (2013).
[35] S. Camerer, M. Korppi, A. Jockel, D. Hunger, T. W. Hansch, and P. Treutlein, Phys. Rev. Lett. **107**, 223001 (2011).
[36] C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell, and C. E. Wieman, Phys. Rev. Lett. **78**, 586 (1997).
[37] D. Offenberg, C. B. Zhang, Ch. Wellers, B. Roth, and S. Schiller, Phys. Rev. A **78**, 061401(R) (2008).
[38] D. J. Larson, J. C. Bergquist, J. J. Bollinger, Wayne M. Itano, and D. J. Wineland, Phys. Rev. Lett. **57**, 70 (1986).
[39] F. Marquardt, J. P. Chen, A. A. Clerk, and S. M. Girvin, Phys. Rev. Lett. **99**, 093902 (2007).
[40] A. Jöckel, A. Faber, T. Kampschulte, M. Korppi, M. T. Rakher, and P. Treutlein, Nat. Nanotechnol. **10**, 55 (2015).
[41] P. Christoph *et al.*, New J. Phys. **20**, 093020 (2018).
[42] J. D. Teufel *et al.*, Nature (London) **475**, 359 (2011).
[43] J. B. Clark, F. Lecocq, R. W. Simmonds, J. Aumentado, and J. D. Teufel, Nature (London) **541**, 191 (2017).
[44] A. Dantan, B. Nair, G. Pupillo, and C. Genes, Phys. Rev. A **90**, 033820 (2014).
[45] C. Maschler and H. Ritsch, Opt. Commun. **243**, 145 (2004).
[46] Y.-H. Ma and L. Zhou, J. Appl. Phys. **111**, 103109 (2012).
[47] M. K. Transtrum and J. S. van Huele, J. Math. Phys. **46**, 063510 (2005); for the single-mode binomial state, see D. Stoler, B. E. A. Saleh, and M. C. Teich, Opt. Acta **32**, 345 (1985); M. Moussa and B. Baseia, Phys. Lett. A **238**, 223 (1998).
[48] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 2008).
[49] P. Domokos and H. Ritsch, Phys. Rev. Lett. **89**, 253003 (2002).
[50] K. Hammerer *et al.*, in *Cavity Optomechanics*, edited by M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt (Springer-Verlag, Berlin, 2014), Chap. 3.
[51] A. E. Kozhekin, K. Mølmer, and E. Polzik, Phys. Rev. A **62**, 033809 (2000); B. J. Brown, D. Loss, J. K. Pachos, C. N. Self, and J. R. Wootton, Rev. Mod. Phys. **88**, 045005 (2016).
[52] X.-G. Wang, B. Sanders, and S.-h. Pan, J. Phys. A **33**, 7451 (2000).
[53] K. Bergmann, H. Theuer, and B. W. Shore, Rev. Mod. Phys. **70**, 1003 (1998).
[54] Y.-D. Wang and A. A. Clerk, Phys. Rev. Lett. **108**, 153603 (2012); L. Tian, *ibid.* **108**, 153604 (2012).
[55] D. Garg, A. K. Chauhan, and A. Biswas, Phys. Rev. A **96**, 023837 (2017).
[56] S. M. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).
[57] G. Adesso and F. Illuminati, J. Phys. A **40**, 7821 (2007); C. Weedbrook, S. Pirandola, R. Garcia-Patron, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. **84**, 621 (2012).
[58] G. S. Agarwal and S. Huang, Phys. Rev. A **93**, 043844 (2016).
[59] M. Abdi, Sh. Barzanjeh, P. Tombesi, and D. Vitali, Phys. Rev. A **84**, 032325 (2011).
[60] C. W. Gardiner and P. Zoller, *Quantum Noise*, 3rd ed. (Springer-Verlag, Berlin, 2004).
[61] V. Giovannetti and D. Vitali, Phys. Rev. A **63**, 023812 (2001).
[62] G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 504 (1965); R. Benguria and M. Kac, Phys. Rev. Lett. **46**, 1 (1981).
[63] O. Gittsovich and O. Guhne, Phys. Rev. A **81**, 032333 (2010).

[64] R. F. Werner and M. M. Wolf, Phys. Rev. Lett. **86**, 3658 (2001).
[65] P. C. Parks and V. Hahn, *Stability Theory* (Prentice-Hall, Englewood Cliffs, NJ, 1993).
[66] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[67] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318 (2004).
[68] L. M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).
[69] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).