Optimal basis for the generalized Dicke model

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A methodology is devised for building optimal bases for the generalized Dicke model based on the symmetry adapted variational solution to the problem. At order zero, the matter sector is constructed by distributing N_a particles in all the possible two-level subsystems connected with electromagnetic radiation; the next order is obtained when the states of $N_a - 1$ particles are added and distributed again into the two-level subsystems and so on. In the electromagnetic sector, the order zero for each mode is the direct sum of the Fock spaces, truncated to a value of the corresponding constants of motion of each two-level subsystem; by including contributions of the other modes, the next orders are obtained. As an example of the procedure, we consider four atoms in the Ξ configuration interacting dipolarly with two modes of electromagnetic radiation. The results may be applied to situations in quantum optics, quantum information, and quantum computing.

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I. INTRODUCTION

The study of a finite number of n-level matter systems, be they atoms, artificial atoms, spin systems, or molecules, interacting with an electromagnetic field of ℓ modes, has regained importance as applications in quantum optics, quantum gates, and quantum information have become realistic. In particular, electromagnetic induced transparency, population trapping, and quantum memories require the presence of, at least, three atomic (matter) levels [1–3].

Dynamically driven quantum coherence in qubit systems, which are made to cross the quantum phase transition into the superradiant region, has been shown [4–6] as well as the generation of field-matter entanglement in the system by varying the light-matter coupling parameter [7].

Even if one restricts the number of photons in the radiation field with some upper bound, a strong limitation in these studies is that the dimension of the Hilbert space becomes unwieldy as the number of atoms and total excitations grow.

In this paper, we build a sequence of ever-approximating bases for the infinite-dimensional Hilbert space $\mathcal{H}$ of matter interacting with radiation in order to carry out a complete study for noninteracting particles exchanging energy with ℓ modes of electromagnetic radiation. In particular, an upper bound is placed on the total number of excitations of the system, essentially limiting the number of photons in order to obtain a finite-dimensional Hilbert space to work on. This upper bound is chosen in such a way that the ground state (which is the one to be studied) obtained in this truncated space differs from the exact ground state by no more than a certain allowed error e_{rr} as measured by the fidelity F between the two states. We, here, show examples for both $e_{\mathrm{rr}} = 10^{-10}$ and $e_{\mathrm{rr}} = 10^{-15}$. The value for e_{rr} is, of course, arbitrary and will be demanded by the type of application to be given. For $e_{\mathrm{rr}} = 10^{-10}$, for example, the error in the energy surface and observables is less than 10^{-8}.

The fact that we have an iterative method for reducing a system of n-level atoms interacting with radiation to a system of $n - 1$ levels [8] plus the result that the polychromatic phase diagram divides itself into monochromatic subregions [8–10] where only one of the radiation modes strongly dominates suggest a methodology for reducing the space dimension even further. This methodology is used to build a sequence of bases for the Hilbert space which approximates better the exact results as we move along the sequence. Previously untractable problems may be tackled in this way, and, depending on the desired approximation, the appropriate basis may be chosen.

The iterative method just mentioned allows the study of the ground state of a very general atomic system of n levels in the presence of an electromagnetic field of ℓ modes even in the case where each mode produces transitions between more than one pair of levels to be carried out by studying subsystems consisting of two atomic levels interacting dipolarly with one radiation mode.

The investigation of the four-level N and λ atomic configurations interacting with two radiation modes has been shown to present qualitatively different quantum phase diagrams [9]. While the collective superradiant regime in the latter divides itself into two subregions, corresponding to each of the modes, that of the former may be divided into two or three subregions depending on how the field modes divide the atomic system into two-level subsystems. Furthermore, a four-level Josephson circuit shows the dynamics of two-qubit systems [11]. This shows the importance of studying two-level atomic systems under the influence of one-mode radiation fields (for a review, cf. Ref. [12]). Recently, the importance of adding unitary invariant phase factors in the matter-field interactions of two- and three-level particles has been established, which can be seen as a canonical transformation represented as a unitary transformation [13]. They found that the phase factors affect the intrinsic symmetry of the two- and three-level Dicke

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models interacting with one mode of radiation [14]. However, we want to stress that the ground-state phase diagram presented is very similar to the case of the Dicke model but with two modes of electromagnetic radiation [8].

As mentioned above, in Ref. [9], we have studied the quantum phase diagrams for the four-level systems associated with the λ and N atomic configurations by means of a variational procedure. Our present approach allows one to investigate these systems without approximations and in different ways:

(i) By considering the optimal basis presented here to study the corresponding eigensystems for the different parity symmetries together with transitions between different levels, preserving or not the parity symmetry.

(ii) By studying the entanglement properties between the matter and the field sectors together with the statistical properties of light, such as squeezing, delocalization, and the Mandel parameter.

(iii) By allowing subsystems of two and three levels to be considered as open quantum systems where the remaining part of the system behaves as an environment and consider the dynamics through a Gorini-Kossakowski-Sudarshan-Lindblad type of master equation. This approach considers the dynamics of the environment to be similar to the dynamics of the system under study as opposed to other approximation methods, such as the quantum trajectory approach [15–17] or the master equation [18,19].[1] For the reader interested in two-level systems, an efficient numerical simulation using permutational invariance that allows one to study the evolution mainly of the superradiant emission for different initial states and under different local and collective processes, may be found in Ref. [20].

(iv) Another specific problem would be the determination of universal parametric curves for three- and four-level systems following the procedure indicated in Ref. [21] and establishing the properties of light when the transitions in the finite quantum phase diagrams are crossed.

Our methodology shows how to study the properties of the ground state by reducing the full system to two-level subsystems and has been tested in the particular case of one three-level atom [22] and in the existence of universal parametric curves [21]. Here, we show that the method is generalizable to any finite number of atoms.

After presenting the general methodology, the study of four atoms in the Ξ configuration in the presence of two electromagnetic modes is given in full as an example. The energy surface and the photon number fluctuations are also calculated.

This paper is organized as follows: In Sec. II, the generalized Dicke model is presented. Section III builds the full basis of the Hilbert space as well as a criterion to obtain convergence of the solution based on the fidelity of states. We show that this procedure yields the minimum number of excitations needed to obtain convergence in a related two-level system, and we discuss a method to obtain the convergence in the general three-level case. We also calculate the minimum energy surface for the exact quantum solution of a three-level system. In Sec. IV, a reduction method is presented which results in a sequence of ever-approximating bases for the Hilbert space. Section V presents the results for a three-level system interacting with two modes of electromagnetic field, obtained from the exact solution and from the reduced bases. These results are discussed and compared. Finally, in Sec. VI, some concluding remarks are given.

II. GENERALIZED DICKE MODEL

A. Preliminaries

In a noninteracting gas of atoms or molecules, the spontaneous emission of a two-level system leads to the emission of coherent radiation called superradiant emission. This yields a completely different behavior if the atoms or molecules are considered independent. It is a collective performance induced by the interaction with the electromagnetic field.

In the Dicke description of an N molecule gas, it is assumed that each molecule has two internal degrees of freedom and the Hamiltonian is written in two parts, one related to the translational degrees of freedom $\boldsymbol{H}_0$ and a second one related to the two internal degrees of freedom $\boldsymbol{\sigma}_{kz}$, i.e.,

$$\boldsymbol{H} = \boldsymbol{H}_0 + \hbar\omega \boldsymbol{J}_z, \tag{1}$$

where $\boldsymbol{J}_z = (1/2)\sum_k \boldsymbol{\sigma}_{kz}$. As $\boldsymbol{H}_0$ commutes with the internal degrees of freedom $\boldsymbol{J}_z$, the eigenstates of the Hamiltonian are the direct product of the center-of-mass degree of freedom $|g\rangle$ times the so-called Dicke states $|rm\rangle$,

$$\boldsymbol{J}^2|rm\rangle = \hbar r(r+1)|rm\rangle, \quad \boldsymbol{J}_z|rm\rangle = \hbar m|rm\rangle, \tag{2}$$

where $|m| \leqslant r \leqslant N/2$ because $\boldsymbol{J}_z$ is a constant of motion of the Hamiltonian. For the case of identical molecules, when $r = m = N/2$, the eigenstate is not degenerate, and it is straightforward to show that when all the molecules are excited the spontaneous decay probability is $I \propto N$, whereas for $m = 0$ one obtains $I \propto N^2$ (cf. Ref. [23]).

In what follows, we will consider a generalization of the Dicke studies in that the internal degrees of freedom for each atom or molecule are constituted by n levels, which interact dipolarly with ℓ modes of electromagnetic radiation in a cavity. We are, here, interested in the determination of the quantum phase diagrams or quantum phase transitions for a finite number of particles (sometimes called precursors of quantum phase transitions). However, for any quantum-mechanical system described by an effective Hamiltonian of the form $\boldsymbol{H} = \boldsymbol{H}_a + \boldsymbol{H}_f + \boldsymbol{H}_{af}$, the results obtained in this contribution can be applied.

B. Generalized model

Let us consider N_a atoms of n levels interacting dipolarly with ℓ modes of an electromagnetic field where the transition between any given pair of atomic levels is promoted only by one mode of the field. The Hamiltonian is composed of two terms: a diagonal part $\boldsymbol{H}_D$ containing the field and matter sectors and a nondiagonal $\boldsymbol{H}_{\rm int}$ containing the matter and field dipolar interactions. So we can write ($\hbar = 1$) [10]

$$\boldsymbol{H} = \boldsymbol{H}_D + \sum_{s=1}^{\ell} \boldsymbol{H}_{\rm int}^{(s)}, \tag{3}$$

[1] This approach is under current consideration.

with

$$\boldsymbol{H}_D = \sum_{s=1}^{\ell} \Omega_s \boldsymbol{\nu}_s + \sum_{k=1}^{n} \omega_k \boldsymbol{A}_{kk}, \tag{4}$$

where Ω_s denotes the s-mode field frequency, ω_k the frequency of the atomic level k, $\boldsymbol{\nu}_s$ the bosonic field operator $\boldsymbol{\nu}_s = \boldsymbol{a}_s^\dagger \boldsymbol{a}_s$ of mode s with $\boldsymbol{a}_s^\dagger$ and $\boldsymbol{a}_s$ the creation and annihilation operators, and $\boldsymbol{A}_{kk}$ and $\boldsymbol{A}_{jk}$ are atomic weight and transition operators, respectively, obeying the unitary algebra $U(n)$ in n dimensions,

$$[\boldsymbol{A}_{jk}, \boldsymbol{A}_{lm}] = \delta_{kl}\boldsymbol{A}_{jm} - \delta_{jm}\boldsymbol{A}_{lk}. \tag{5}$$

For the totally symmetric irreducible representation of $U(n)$, the generators have a bosonic representation as $\boldsymbol{A}_{jk} = \boldsymbol{b}_j^\dagger \boldsymbol{b}_k$ and first-order Casimir operator,

$$\sum_{k=1}^{n} \boldsymbol{A}_{kk} = N_a \mathbf{1}_{\text{matt}}, \tag{6}$$

with $\mathbf{1}_{\text{matt}}$ the identity operator in the matter sector of the Hilbert space.

The second contribution term in (3) reads

$$\boldsymbol{H}_{\text{int}}^{(s)} = -\frac{1}{\sqrt{N_a}} \sum_{j<k}^{n} \mu_{jk}^{(s)} (\boldsymbol{A}_{jk} + \boldsymbol{A}_{kj})(\boldsymbol{a}_s^\dagger + \boldsymbol{a}_s), \tag{7}$$

where $\mu_{jk}^{(s)}$ is the matter-field coupling parameter and denotes the dipolar intensity. Since we have assumed that transitions between a pair of atomic levels are promoted only by one mode of electromagnetic field, say Ω_s, one has as a condition that, if $\mu_{jk}^{(s)} \neq 0$, then $\mu_{jk}^{(s')} = 0$ for all $s' \neq s$.

The adopted convention $\omega_j \leqslant \omega_k$ for $j < k$ on the atomic levels allows us to refer to a particular atomic configuration by the appropriate choice of vanishing dipolar strengths $\mu_{jk}^{(s)}$. Also, fixing values $\omega_1 = 0$ and $\omega_n = 1$, one may to refer to all energy (and frequency) quantities in units of $\hbar\omega_n$ (and ω_n).

For each mode s, the interaction term (7) has the form $\boldsymbol{H}_{\text{int}}^{(s)} = \boldsymbol{R}_{\text{int}}^{(s)} + \boldsymbol{C}_{\text{int}}^{(s)}$ with $\boldsymbol{R}_{\text{int}}^{(s)}$ the rotating and $\boldsymbol{C}_{\text{int}}^{(s)}$ the counterrotating terms. The rotating term preserves the total number of excitations,

$$\boldsymbol{R}_{\text{int}}^{(s)} = -\frac{1}{\sqrt{N_a}} \sum_{j<k}^{n} \mu_{jk}^{(s)} (\boldsymbol{A}_{jk}\boldsymbol{a}_s^\dagger + \boldsymbol{A}_{kj}\boldsymbol{a}_s), \tag{8}$$

because a decrease (or increase) in an atomic excitation involves an increase (or decrease) in the photon number. The counterrotating term does not preserve the total number of excitations and is given by

$$\boldsymbol{C}_{\text{int}}^{(s)} = -\frac{1}{\sqrt{N_a}} \sum_{j<k}^{n} \mu_{jk}^{(s)} (\boldsymbol{A}_{jk}\boldsymbol{a}_s + \boldsymbol{A}_{kj}\boldsymbol{a}_s^\dagger). \tag{9}$$

The Hamiltonian in the rotating-wave approximation (RWA), obtained when the counterrotating term is neglected, would be called the *generalized Tavis-Cummings model* (GTCM).

III. FULL BASIS

A complete basis for the Hamiltonian (3) is formed by the direct product of the Hilbert spaces of the field and matter sectors. An element is of the form

$$|\vec{\nu}; \vec{n}\rangle := |\nu_1, \ldots, \nu_\ell; a_1, \ldots, a_n\rangle, \tag{10}$$

which satisfies the eigenvalue equations,

$$\boldsymbol{\nu}_s|\vec{\nu}; \vec{n}\rangle = \nu_s|\vec{\nu}; \vec{n}\rangle, \quad \boldsymbol{A}_{kk}|\vec{\nu}; \vec{n}\rangle = a_k|\vec{\nu}; \vec{n}\rangle \tag{11}$$

for the number of photon operator $\boldsymbol{\nu}_s$ of mode Ω_s and the particle number operator $\boldsymbol{A}_{kk}$ for the atomic level k.

Denoting the Fock space of each mode Ω_s by

$$\mathcal{F}_s := \{|\nu_s\rangle | \nu_s = 0\text{–}2, \ldots\}, \tag{12}$$

with infinite dimension and the matter space by

$$\mathcal{M} := \left\{ |a_1, \ldots, a_n\rangle \,\middle|\, \sum_{k=0}^{n} a_k = N_a,\ a_k \geqslant 0 \right\}, \tag{13}$$

with finite dimension given by $\binom{N_a+n-1}{n-1}$ because the number of particles is preserved (6), the full basis is then

$$\mathcal{B} := \bigotimes_{s=1}^{\ell} \mathcal{F}_s \otimes \mathcal{M}. \tag{14}$$

A. Parity adapted basis

When the rotating-wave approximation is considered, operators of the form

$$\boldsymbol{K}_\zeta = \sum_{s=1}^{\ell} \eta_s^{(\zeta)} \boldsymbol{\nu}_s + \sum_{k=1}^{n} \lambda_k^{(\zeta)} \boldsymbol{A}_{kk} \tag{15}$$

commute with the Hamiltonian for certain values of the coefficients $\eta_s^{(\zeta)}$, $\lambda_k^{(\zeta)}$. These operators $\boldsymbol{K}_\zeta$ play the role of constants of motion of the system.

For the generalized Dicke Hamiltonian, only the *parity* of $\boldsymbol{K}_\zeta$ is preserved, i.e., the full Hamiltonian commutes with operators,

$$\boldsymbol{\Pi}_\zeta = e^{i\pi \boldsymbol{K}_\zeta}, \quad \zeta = 1, 2, \ldots, \zeta_0, \tag{16}$$

with ζ_0 denoting the number of parity operators that commute with the Hamiltonian. Its value depends on the particular atomic configuration and is given, in general, by [10]

$$\zeta_0 + 1 = \ell + n - R, \tag{17}$$

where R is the rank of the system of algebraic equations,

$$\mu_{jk}^{(s)}(\eta_s + \lambda_j - \lambda_k) = 0, \tag{18}$$

in which one needs to take into account all the modes $s = 1, 2, \ldots, \ell$ and the connected pairs for each mode. Note that when each mode connects only one pair of levels, the number of constants of motion is $\zeta_0 = n - 1$. On the other hand, when only one mode is responsible for all dipolar transitions, one gets $\zeta_0 = 1$.

Table I shows the values of the coefficients $\eta_s^{(\zeta)}$, $\lambda_k^{(\zeta)}$ for the particular case of three-level atoms interacting dipolarly with two modes of the electromagnetic field.

The basis in the RWA approximation can be characterized in terms of the eigenvalues $\kappa = \{k_1, k_2, \ldots, k_{\zeta_0}\}$ of the constants of motion $\boldsymbol{K}_\zeta$ and can be written as

$$\mathcal{B}_{\text{RWA}}^{(\kappa)} := \{|\vec{\nu}; \vec{n}\rangle | \boldsymbol{K}_\zeta |\vec{\nu}; \vec{n}\rangle = k_\zeta |\vec{\nu}; \vec{n}\rangle \text{ for } \zeta = 1, 2, \ldots, \zeta_0\}. \tag{19}$$

TABLE I. Coefficients corresponding to the $\boldsymbol{K}_\zeta$ operators in Eq. (15) for the Λ, Ξ, and V atomic configurations. Subscripts s, s', and s'' correspond to the modes of the transitions $1 \rightleftharpoons 2$, $1 \rightleftharpoons 3$, and $2 \rightleftharpoons 3$, respectively.

Configuration	$\boldsymbol{K}_\zeta$	$\eta_s^{(\zeta)}$	$\eta_{s'}^{(\zeta)}$	$\eta_{s''}^{(\zeta)}$	$\lambda_1^{(\zeta)}$	$\lambda_2^{(\zeta)}$	$\lambda_3^{(\zeta)}$
Λ							
	$\boldsymbol{K}_1$	0	1	1	0	0	1
	$\boldsymbol{K}_2$	0	0	1	1	0	1
Ξ							
	$\boldsymbol{K}_1$	1	0	1	0	1	2
	$\boldsymbol{K}_2$	0	0	1	0	0	1
V							
	$\boldsymbol{K}_1$	1	1	0	0	1	1
	$\boldsymbol{K}_2$	0	1	0	0	0	1

For three-level atoms interacting with two modes of the electromagnetic field, the dimension of this basis can be obtained in analytic form (cf. the Appendix).

When the counterrotating terms are included in the Hamiltonian, only the parity of the constants of motion is preserved. The Hilbert space then takes into account the direct sum of all the subbases $\mathcal{B}_{\mathrm{RWA}}^{(\kappa)}$ for which the parity of each element k_ζ in κ is preserved.

The full basis is then divided in blocks as

$$\mathcal{B} = \bigoplus_\sigma \mathcal{B}_\sigma, \tag{20}$$

where $\sigma = \mathrm{parity}(\kappa)$, $\mathcal{B}_\sigma$ is given by

$$\mathcal{B}_\sigma := \bigoplus_{j_1=0}^{\infty} \cdots \bigoplus_{j_{\zeta_0}=0}^{\infty} \mathcal{B}_{\mathrm{RWA}}^{(\kappa_\sigma + 2\{j_1, j_2, \ldots, j_{\zeta_0}\})}, \tag{21}$$

and κ_σ is the set of minimum values of the elements of κ with the desired parity. The expression $\kappa_\sigma + 2\{j_1, j_2, \ldots, j_{\zeta_0}\}$ denotes element by element addition, i.e., $\{k_1^{\min} + 2j_1, \ldots, k_{\zeta_0}^{\min} + 2j_{\zeta_0}\}$.

The full Hamiltonian (3) may then be rewritten as $\boldsymbol{H} = \bigoplus_\sigma \boldsymbol{H}_\sigma$, and the minimum energy surface is given by

$$\mathcal{E}_g = \min\{E_{g\sigma}\}, \tag{22}$$

at each point in parameter space, where $E_{g\sigma}$ is the eigenvalue of $\boldsymbol{H}_\sigma$ for the ground state: $\boldsymbol{H}_\sigma |\Psi_{g\sigma}\rangle = E_{g\sigma} |\Psi_{g\sigma}\rangle$ for each parity σ.

B. Truncated basis via fidelity

In practice, the exact quantum ground-state $|\Psi_{g\sigma}\rangle$ is obtained to an approximate precision $|\psi_{g\sigma}^\kappa\rangle$ by using the cutoff basis with fixed parity,

$$\mathcal{B}_\sigma^\kappa := \bigoplus_{j_1=0}^{J_1} \bigoplus_{j_2=0}^{J_2} \cdots \bigoplus_{j_{\zeta_0}=0}^{J_{\zeta_0}} \mathcal{B}_{\mathrm{RWA}}^{(\kappa_\sigma + 2\{j_1, j_2, \ldots, j_{\zeta_0}\})}, \tag{23}$$

where J_i is the minimum value of j_i required for convergence to the desired precision of the ground-state solution in the Hilbert space.

In order to calculate the minimum values κ in (23) which provide a good approximation to the ground state, one may take the variational solution of the problem [8,10] and propose as minimum values $k_\zeta = \langle \boldsymbol{K}_\zeta \rangle + 3(\Delta K_\zeta)$. This proposal, however, provides a good value of k_ζ only when $\boldsymbol{K}_\zeta$ obeys a Gaussian distribution and except in the normal region. Another approach is to use a criterion based on the fidelity between two states.

In this paper, we use the fidelity criterion to get an approximate ground state. Noting that the error between the exact and the approximate quantum ground states,

$$e_\sigma^\kappa := 1 - \left| \langle \psi_{g\sigma}^\kappa | \Psi_{g\sigma} \rangle \right|^2 \tag{24}$$

vanishes in the limit $\kappa \to \infty$ and for values of $\kappa, \kappa + 2, \ldots$, one has $e_\sigma^\kappa > e_\sigma^{\kappa+2} > \cdots$, one may cut the full basis to a desirable error e_{rr} by imposing the condition,

$$1 - F_\sigma^\kappa \leqslant e_{\mathrm{rr}}, \quad F_\sigma^\kappa := \left| \langle \psi_{g\sigma}^\kappa | \psi_{g\sigma}^{\kappa+2} \rangle \right|^2. \tag{25}$$

This criterion is more general in that it does not depend on the particular distribution of the values of $\boldsymbol{K}_\zeta$, and an iterative method allows us to evaluate the value of κ for the desired approximation.

In order to illustrate the method, we next consider the particular case of a two-level system.

1. Two-level system

We, here, consider N_a atoms of two levels ($\omega_j < \omega_k$) interacting with a single-mode Ω_s of an electromagnetic field. The Hamiltonian is

$$\begin{aligned} \boldsymbol{H} &= \Omega_s \boldsymbol{\nu}_s + \omega_j \boldsymbol{A}_{jj} + \omega_k \boldsymbol{A}_{kk} \\ &\quad - \frac{1}{\sqrt{N_a}} \mu_{jk}^{(s)} (\boldsymbol{A}_{jk} + \boldsymbol{A}_{kj})(\boldsymbol{a}_s^\dagger + \boldsymbol{a}_s). \end{aligned} \tag{26}$$

This possesses only one parity operator, namely,

$$\boldsymbol{\Pi}_{jk}^{(s)} = e^{i\pi \boldsymbol{M}_{jk}^{(s)}}, \quad \text{with } \boldsymbol{M}_{jk}^{(s)} = \boldsymbol{\nu}_s + \boldsymbol{A}_{kk}. \tag{27}$$

Here, $\boldsymbol{M}_{jk}^{(s)}$ stands for the total number of excitations operator (with eigenvalues $m_{jk}^{(s)} = 0, 1, \ldots$, if the rotating-wave approximation were considered). From the variational calculation, one finds that this system presents a phase transition at the critical point,

$$\bar{\mu}_{jk}^{(s)} := \frac{1}{2} \sqrt{\Omega_s \omega_{jk}}; \quad \omega_{jk} := |\omega_j - \omega_k|,$$

which allows us to write the Hamiltonian in terms of the dimensionless dipolar intensity $x_{jk}^{(s)}$ and the detuning parameter $\Delta_{jk}^{(s)}$, given by

$$x_{jk}^{(s)} := \frac{\mu_{jk}^{(s)}}{\bar{\mu}_{jk}^{(s)}}, \quad \Delta_{jk}^{(s)} := \frac{\Omega_s}{\omega_{jk}} - 1, \tag{28}$$

so that all the two-level systems with the same detuning values $\Delta_{jk}^{(s)}$ have the same quantum phase diagram as function of $x_{jk}^{(s)}$, i.e., all of these systems are equivalent in these variables.

We then calculate, iteratively, a value of $\kappa = \{\overline{m}_{jk}^{(s)}\}$ for a fixed parity which will fulfill the inequality in Eq. (25). This value depends of the number of particles N_a, the dimensionless dipolar intensity $x_{jk}^{(s)}$, the detuning parameter $\Delta_{jk}^{(s)}$, and the error value e_{rr}.

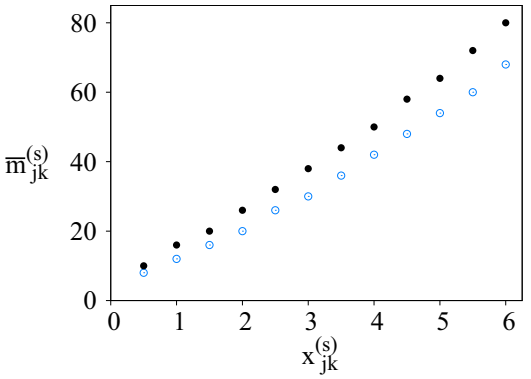

FIG. 1. For $N_a = 3$ atoms with $\Delta^{(s)}_{jk} = 0$, the value of $\overline{m}^{(s)}_{jk}$ as a function of $x^{(s)}_{jk}$ is shown for an error of $e_{\rm rr} = 10^{-10}$ (circles) and $e_{\rm rr} = 10^{-15}$ (dots) in the fidelity F [Eq. (25)].

In Fig. 1, the value of $\overline{m}^{(s)}_{jk}$ for the even solution $\sigma = e$ is displayed as a function of the dimensionless dipolar intensity $x^{(s)}_{jk}$ for the case of $N_a = 3$ particles, zero detuning, and error values $e_{\rm rr} = 10^{-10}$ (circles) and $e_{\rm rr} = 10^{-15}$ (dots). As the error becomes smaller, $\overline{m}^{(s)}_{jk}$ grows considerably and, hence, the corresponding dimension of the truncated basis. In fact, it diverges as $e_{\rm rr} \to 0$. For the case of the odd-parity $\sigma = o$, the number of excitations is given by $\overline{m}^{(s)}_{jk} + 1$ with $\overline{m}^{(s)}_{jk}$ the number of excitations for the even solution.

Fixing the error value to $e_{\rm rr} = 10^{-10}$ and taking the even-parity $\sigma = e$, Fig. 2(a) shows the number of excitations $\overline{m}^{(s)}_{jk}$ as a function of the number of particles N_a for a fixed value of $x^{(s)}_{jk} = 3/2$ and for different detuning values $\Delta^{(s)}_{jk} = -1/2, 0, 1/2$. In a similar way, for $N_a = 4$ atoms, the number of excitations is shown as a function of $x^{(s)}_{jk}$ in Fig. 2(b). The calculation for $\Delta^{(s)}_{jk} = 0$ overestimates the values of $\overline{m}^{(s)}_{jk}$ when $\Delta^{(s)}_{jk} > 0$. Note also that, for small values of $x^{(s)}_{jk}$, $\overline{m}^{(s)}_{jk}$ does not depend on the detuning value.

2. General case

In order to justify the general procedure, it is convenient to first look at a specific case. We take that of three-level atoms in the Ξ configuration interacting dipolarly with two modes of an electromagnetic field. The modes Ω_1, Ω_2 promote transitions $\omega_1 \rightleftharpoons \omega_2$ and $\omega_2 \rightleftharpoons \omega_3$, respectively. The coefficients of the two operators $\boldsymbol{K}_\zeta$ Eq. (15) are given in Table I, identifying the number of excitations $\boldsymbol{M}^{(s)}_{jk}$ of each subsystem and may be written as

$$\boldsymbol{K}_1 = \boldsymbol{M}^{(1)}_{12} + \boldsymbol{M}^{(2)}_{23} + \boldsymbol{A}_{33}, \tag{29}$$

$$\boldsymbol{K}_2 = \boldsymbol{M}^{(2)}_{23}. \tag{30}$$

The Hilbert space then divides itself into four subspaces according to the parity of the eigenvalues of $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$:{ee, eo, oe, oo}. For each of these subspaces, the minimum values for k_1, k_2 that satisfy the convergence criterion (25) are calculated by using the iterative method described earlier. These values are determined from those of $\overline{m}^{(1)}_{12}, \overline{m}^{(2)}_{23}$ of the two-level subsystems and from the total number of particles N_a.

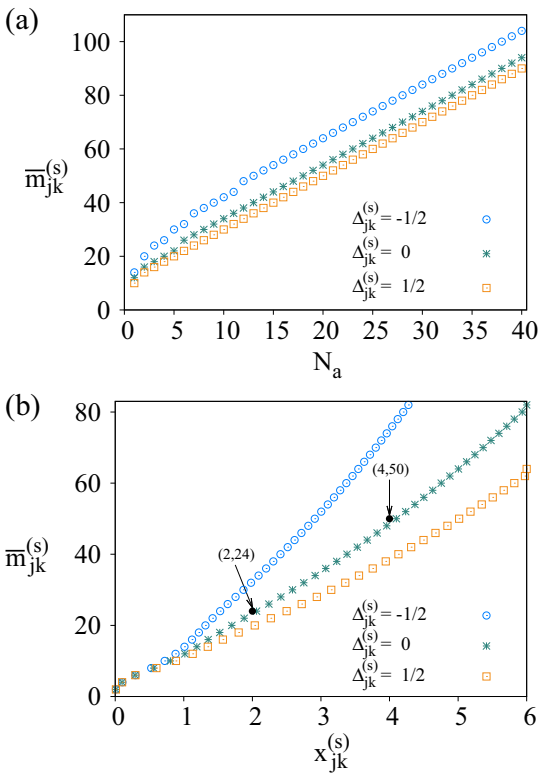

FIG. 2. Number of excitations $\overline{m}^{(s)}_{jk}$ as a function of: (a) the number of particles N_a with $x^{(s)}_{jk} = 3/2$ and (b) the dipolar intensity $x^{(s)}_{jk}$ for $N_a = 4$. In both cases, the detuning values used are indicated, and we have taken $e_{\rm rr} = 10^{-10}$.

Therefore, the maximum eigenvalues that the operators $\boldsymbol{M}^{(1)}_{12}$ and $\boldsymbol{M}^{(2)}_{23}$ will take are precisely $\overline{m}^{(1)}_{12}$ and $\overline{m}^{(2)}_{23}$. Using these and the value of N_a for the operator $\boldsymbol{A}_{33}$, we arrive at

$$k_1 = \overline{m}^{(1)}_{12} + \overline{m}^{(2)}_{23} + N_a, \tag{31}$$

$$k_2 = \overline{m}^{(2)}_{23}. \tag{32}$$

Consequently, imposing the condition that the number of excitations of each two-level subsystem $m^{(s)}_{jk}$ satisfies the inequality $m^{(s)}_{jk} \leqslant \overline{m}^{(s)}_{jk}$ for each state of the basis (23), the criterion (25) must also be fulfilled.

For this example, Table II shows the dimension of the basis (23) at fixed values of the dimensionless dipolar strength. One may observe how this dimension grows as the number of particles and dipolar strengths increases.

For the general case of n-level atoms interacting with ℓ modes, we first identify, in the RWA, the number of operators that commute with the Hamiltonian. These are rewritten in terms of the two-level subsystem operators $\boldsymbol{M}^{(s)}_{jk}$ and the weight operators $\boldsymbol{A}_{kk}$. Their maximum eigenvalues $\overline{m}^{(s)}_{jk}$ are calculated via the iterative procedure described.

For the full Hamiltonian, only the parities of the operators $\boldsymbol{K}_\zeta$ are preserved, and these tell us the number of parity subsystems into which the whole Hilbert space divides. For each one of these parities, we substitute the value of the operators $\boldsymbol{M}^{(s)}_{jk}$ for $\overline{m}^{(s)}_{jk}$ and the value of the weight operators

TABLE II. Dimension of the truncated basis $\mathcal{B}^{\kappa}_{ee}$ Eq. (23) for three-level atoms in the Ξ configuration interacting with two modes of the electromagnetic field under resonant condition $\Delta^{(s)}_{jk} = 0$. The number of allowed photons is restricted by $\nu_s \leqslant \overline{m}^{(s)}_{jk}$. This basis permits us to approximate the quantum ground state with a desirable error $e_n := 10^{-n}$ at two different values of the dimensionless dipolar strength (x_{12}, x_{23}) shown.

N_a	$e_{10}(1.5, 1.5)$	$e_{10}(3, 3)$	$e_{15}(1.5, 1.5)$	$e_{15}(3, 3)$
1	91	271	169	397
2	330	925	532	1426
3	664	2295	1030	3667
4	1222	4876	2170	7956
5	2017	9090	3442	13985

$\boldsymbol{A}_{kk}$ for N_a. These yield the values for k_ζ, and for all $m^{(s)}_{jk} \leqslant \overline{m}^{(s)}_{jk}$, the convergence criterion (25) will be fulfilled.

A good estimate for the dimension of the basis is given by

$$\frac{1}{\zeta_0}\prod_{s=1}^{\ell}(\overline{m}^{(s)}_{jk}+1)\binom{N_a+n-1}{n-1}, \tag{33}$$

where ζ_0 is the number of parities preserved, which, in our example of Table II, is $\zeta_0 = 4$ together with $\ell = 2$ and $n = 3$.

C. Minimum energy surface

The minimum energy surface of physical systems lets us determine the quantum phase transitions at zero temperature [24]. In the Dicke model, the quantum phase transitions were determined by Hepp and Lieb [25], and the free energy of the system in the thermodynamic limit was calculated by Wang and Hioe [26]. A review of the dynamics of matter-field interactions of two- and three-level systems was performed by Yoo and Eberly [27]. A procedure to determine the quantum phase transitions was proposed by Gilmore [28], which uses a variational test function together with the catastrophe formalism. Another possibility to determine the quantum phase transitions (also called crossovers) for a finite number of particles is by means of the fidelity concept of quantum information [21,22,29,30]. Here, we illustrate how to build the ground-state energy surface of $N_a = 4$ atoms of three levels interacting dipolarly with two modes of an electromagnetic field together with the determination of the quantum phase diagram through the calculation of the fidelity.

We choose the parameters in the Hamiltonian to be as follows: atomic levels $\omega_1 = 0$, $\omega_2 = 1/4$, and $\omega_3 = 1$; field frequencies $\Omega_1 = 1/4$ and $\Omega_2 = 3/4$; and, as phase-space parameters, the dimensionless dipolar strengths $x^{(1)}_{12}$ and $x^{(2)}_{23}$. The superscripts indicate that modes Ω_1 and Ω_2 promote transitions between the atomic levels $\omega_1 \rightleftharpoons \omega_2$ and $\omega_2 \rightleftharpoons \omega_3$, respectively. Note that the system is in double resonance, i.e., the case of zero detuning.

Recall that the basis $\mathcal{B}^{(\kappa)}_{\mathrm{RWA}}$ in Eq. (19) allows us to calculate the ground-state energy surface of the generalized Tavis-Cumming model where the operators $\boldsymbol{K}_\zeta$ are constants of motion. In this case, one can evaluate the ground-state energy surfaces for fixed values of κ and, then, take the minimum value as a function of the control parameters. On the other hand, the minimum energy surface for the generalized Dicke model requires the use of the basis $\mathcal{B}^{\kappa}_{\sigma}$ in Eq. (23). One, then, evaluates the minimum energy surfaces for every fixed parity of κ and takes the one which is minimum at every point in phase space. For atoms in the Ξ configuration, the ground-state energy surface has even-even-parity $\sigma = ee$ when the number of particles is even, whereas it is composed of the parities $\sigma = ee$ and $\sigma = oe$ for an odd number of particles (depending on the region of phase space). A similar situation occurs for the one atom Λ configuration [22].

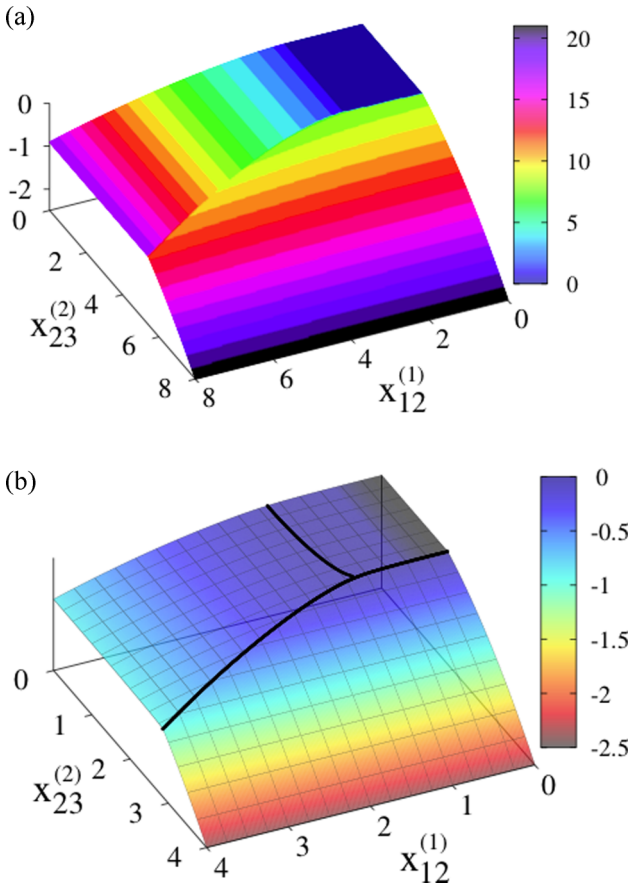

FIG. 3. Quantum ground energy surface for $N_a = 4$ atoms in the Ξ configuration. (a) Generalized Tavis-Cumming model, where k_1 is a constant; the color indicates the value of the total number of excitations in the system. (b) Generalized Dicke model where only the parity of k_1 is conserved; the black lines define the separatrix where a minimum in the fidelity occurs. Parameters used are discussed in the text.

In the Tavis-Cumming model, we have two constants of motion, viz., the total number of excitations k_1 and the number of excitations of the $2 \rightleftharpoons 3$ subsystem k_2. Its energy surface is plotted in Fig. 3(a) for $N_a = 4$, and we see that the phase diagram is divided into three regions: a normal region where $\mathcal{E}_g = 0$ and a collective region showing a separatrix between its two subregions. The height indicates the energy value (in units of $\hbar\omega_3$), and the color gives the value of k_1; in this approximation, $k_2 = 0$ for the smaller values of x_{23} where there are only photons of mode Ω_1. For the collective region where there are only photons of mode Ω_2, k_2 increases gradually from three in steps of the unity. Note that all the transitions

are discontinuous because the ground state changes from one subspace to another as k_1 changes. The transitions where k_1 is increased (or decreased) by a value greater than one remain as discontinuous transitions in the thermodynamic limit. When leaving the normal region as $x_{12}^{(1)}$ increases, the transition is continuous; when leaving it as $x_{23}^{(2)}$ increases, the transition is discontinuous.

For the generalized Dicke model, k_1 and k_2 are no longer constants of motion, only their parity is preserved, and they distribute about their corresponding GTCM values taking into account that the dimensionless coupling strengths x_{ij} scale by a factor of $1/2$. The ground-state energy surface is plotted in Fig. 3(b). Since N_a is even, the ground-state energy surface has an even-even parity. We also show the separatrix (black points), obtained from the local minima in the fidelity between neighboring points. Here, one finds a second-order transition from the normal region (the enclosed region around the origin) to the collective region that we reach by increasing the value of $x_{12}^{(1)}$; all other transitions are first-order discontinuous transitions.

IV. REDUCED BASES

We have shown how the dimension of the truncated basis $\mathcal{B}_\sigma^\kappa$ Eq. (23) grows quickly as both the number of particles and the dimensionless parameter control $x_{jk}^{(s)}$ increase. However, for a fixed desirable error $e_{\rm rr}$ (as, for example, $e_{\rm rr} = 10^{-10}$), it is clear that any value less than $e_{\rm rr}$ in the calculation is negligible. Thus, in principle, one may discard all the weakly coupled states in the ground state with a joint probability less than $e_{\rm rr}$ defining in this way a reduced basis and obtaining a good approximation for the quantum ground state.

The variational solution is used as a criterion that allows us to discard weakly coupled states of the basis. The variational solution of this kind of system shows that the collective region is divided into subregions where only one kind of photon contributes to the ground state, whereas the other ones remain in the vacuum state. In fact, in each subregion, the full system has a behavior similar to a subsystem with a single mode [8–10] except in a small vicinity of the separatrix. This behavior was exhibited for the case of a single particle [22].

In order to discard states in the full basis, we consider the two sectors, matter and field, separately.

A. Matter sector

The procedure to extract the significant states from the matter sector is based on the following statement: By considering the case where a single-mode Ω_s promotes the transitions between a pair of atomic levels, that is, $x_{jk}^{(s)} \neq 0$, we have $x_{jk}^{(s')} = 0$ for $s' \neq s$. In this case, the variational solution shows that the collective region divides itself into ℓ_0 subregions (here, $\ell_0 = \ell$, but, in general, $\ell_0 \geqslant \ell$) where, in each of them, a two-level Hamiltonian of the form given in Eq. (26) dominates [8]. For the variational solution, one finds that the parity of the operator $\boldsymbol{M}_{jk}^{(s)}$ in Eq. (27) is preserved and also $\boldsymbol{A}_{jj} + \boldsymbol{A}_{kk} = N_a \mathbf{1}_{\rm matt}$ is fulfilled. Therefore, we define the number of particles of each subsystem as

$$N_{jk}^{(s)} := \langle a_1, \ldots, a_n | \boldsymbol{A}_{jj} + \boldsymbol{A}_{kk} | a_1, \ldots, a_n \rangle. \tag{34}$$

Since the variational solution of the matter sector has contributions of states where, at least, one $N_{jk}^{(s)}$ takes the value N_a, one may classify the matter subbasis by the set of states $\mathcal{M}_r$ where $N_{jk}^{(s)} = N_a - r$ is satisfied, at least, for one subsystem, i.e.,

$$\mathcal{M}_r := \left\{ |a_1, \ldots, a_n\rangle \middle| \bigvee_s N_{jk}^{(s)} = N_a - r \right\}, \tag{35}$$

where $\bigvee_s$ is the logical "or" operator. Note that $\mathcal{M}_0$ is the matter contribution according to the variational solution and, hence, this contribution remains in the thermodynamic limit, whereas the other ones (for $r > 0$) vanish as N_a grows. Also note that the full matter basis (13) is given by

$$\mathcal{M} = \bigoplus_{r=0}^{O_1} \mathcal{M}_r, \tag{36}$$

where O_1 is the maximum number of matter subbases and it is given, in general, by the floor of (greatest integer less than or equal to) N_a/ℓ_0,

$$O_1 = \left\lfloor \frac{N_a}{\ell_0} \right\rfloor, \tag{37}$$

relationship that is obtained from the inequalities,

$$N_a \leqslant N_{jk}^{(1)} + \cdots + N_{j'k'}^{(\ell_0)} \leqslant \ell_0 N_a.$$

From expression (36), one may consider different orders to the matter contribution,

$$\mathcal{M}[o_1] := \bigoplus_{r=0}^{o_1} \mathcal{M}_r, \quad o_1 \leqslant O_1. \tag{38}$$

As an example of the division of the matter sector, consider four three-level atoms in the Ξ configuration interacting dipolarly with two modes of the electromagnetic field. In this case, we have two two-level subsystems as described above $N_{12}^{(1)} = a_1 + a_2$, $N_{23}^{(2)} = a_2 + a_3$, and

$$\mathcal{M}_r := \{|a_1, a_2, a_3\rangle | a_1 + a_2 = 4 - r \vee a_2 + a_3 = 4 - r\},$$

which gives rise to

Subbasis	States
$\mathcal{M}_0$	$\lvert a_1, a_2, 0\rangle$, $\lvert 0, a_2, a_3\rangle$
$\mathcal{M}_1$	$\lvert a_1, a_2, 1\rangle$, $\lvert 1, a_2, a_3\rangle$
$\mathcal{M}_2$	$\lvert a_1, a_2, 2\rangle$, $\lvert 2, a_2, a_3\rangle$

Here, $a_1 + a_2 + a_3 = 4$, the number of particles.

In Fig. 4, we show schematically the states in each $\mathcal{M}_r$. The variational solution for the ground state has matter sector $\mathcal{M}_0$ [8,10]. Hence, $\mathcal{M}[o_1]$ with $o_1 = 1, 2$ provides the corrections in the matter sector due to the entanglement between the subsystems.

In the general case, when a single mode may promote transitions of two or more atomic level pairs, Eqs. (35)–(38) are the same, but the expression of $N_{jk}^{(s)}$ is replaced by $N^{(s)}$, which takes the form of the sum of the matter weight operators $\boldsymbol{A}_{kk}$ that describe the sth subsystem. One may find the different subsystems by breaking down the full atomic configuration in

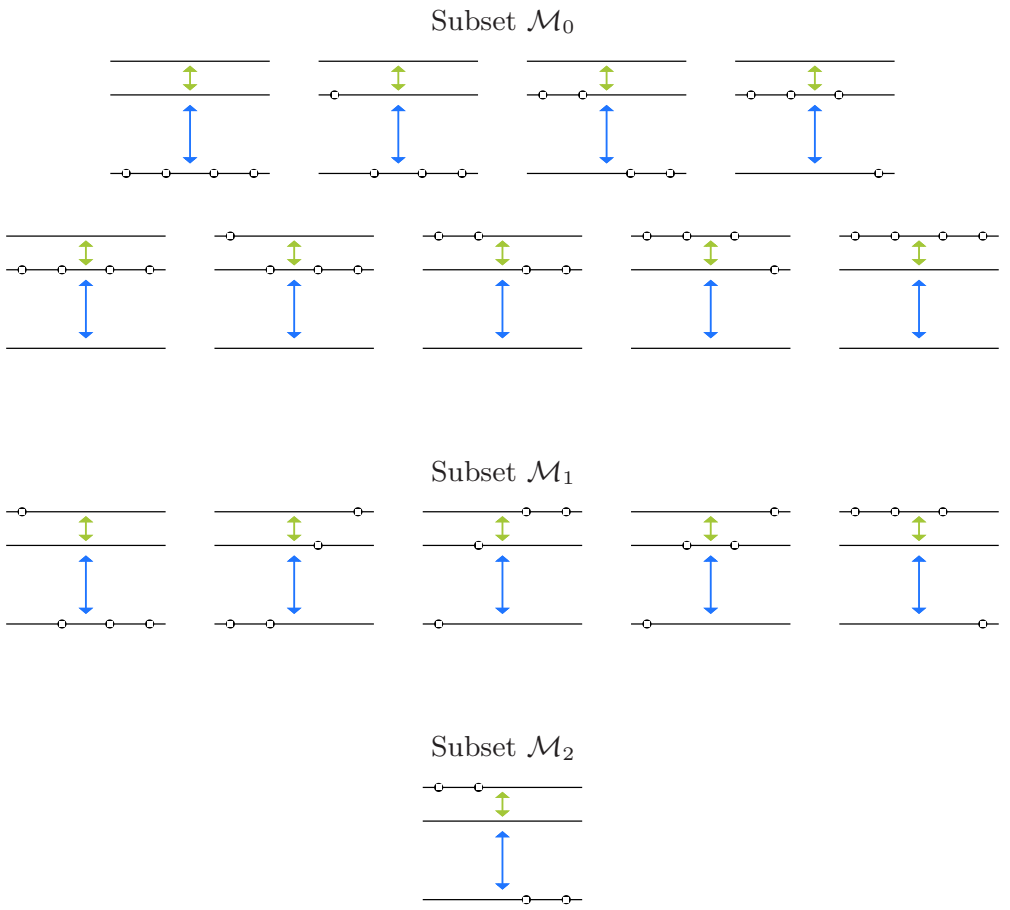

FIG. 4. For $N_a = 4$ atoms in the Ξ configuration, the matter states are shown for each defined matter subspace $\mathcal{M}_r$. Horizontal lines denote atomic levels, and circles denote atomic occupations.

parts where only one mode connects the atomic levels as was shown in Refs. [9,10].

B. Field sector

In the truncated basis of $\mathcal{B}_\sigma^\kappa$ Eq. (23), the maximum photon contribution $\tilde{\nu}_s$ for each mode Ω_s depends on the value of κ. Since, in general, there is no simple relationship between κ and $\tilde{\nu}_s$, we take without loss of generality $\tilde{\nu}_s := \nu_0 = \max(\kappa)$. In the end, we eliminate the states that do not satisfy the parity and upper limits of the truncated basis. So, for the value ν_0, the truncated Fock basis of each mode is

$$\mathcal{F}_s[\nu_0] := \{|\nu_s\rangle | \nu_s \leqslant \nu_0\}, \tag{39}$$

and the truncated field sector is thereby given, in general, as

$$\mathcal{F} = \bigotimes_{s=1}^{\ell} \mathcal{F}_s[\nu_0]. \tag{40}$$

We next want to subdivide the different photon contributions. Since the Hamiltonian interaction (3) connects state $|\nu_s\rangle$ with states $|\nu_s + 1\rangle$ and $|\nu_s - 1\rangle$ and using the fact that in the variational solution for each subsystem $\boldsymbol{H}_s$ the contribution of mode $\Omega_{s'}$ $(s' \neq s)$ is negligible, one may truncate the contribution of negligible photons by taking

$$\mathcal{F}[o_2] = \bigoplus_{s=1}^{\ell} \left[\bigotimes_{s'}^{\ell} \mathcal{F}_{s'}[\zeta_{ss'}] \right], \tag{41}$$

with

$$\zeta_{ss'} = \begin{cases} \nu_0, & s = s', \\ 2o_2 + 1, & s \neq s'. \end{cases} \tag{42}$$

Here, o_2 is the order in the field sector, which can take the maximum value,

$$O_2 := \left\lfloor \frac{\nu_0}{2} \right\rfloor. \tag{43}$$

As an example of how to truncate the field sector, we consider as before four atoms in the Ξ configuration with two photon modes, one for each two-level subsystem. By considering $x_{12}^{(1)} = 2$ and $x_{23}^{(2)} = 4$, one determines (see Fig. 2)

$$\overline{m}_{12}^{(1)} = 24, \quad \overline{m}_{23}^{(2)} = 50.$$

Therefore, the minimum values for the constants of motion to achieve convergence to the required value of $e_{\text{rr}} = 10^{-10}$ are given by

$$k_1 = \overline{m}_{12}^{(1)} + \overline{m}_{23}^{(2)} + 4 = 78, \quad k_2 = \overline{m}_{23}^{(2)} = 50.$$

The basis states are given by

$$\mathcal{F}[o_2] = \mathcal{F}_1[\nu_0] \otimes \mathcal{F}_2[2o_2+1] \oplus \mathcal{F}_1[2o_2+1] \otimes \mathcal{F}_2[\nu_0],$$

where $\nu_0 = 78$, $0 \leqslant o_2 \leqslant 39$ and the dimension is given by

$$\begin{aligned} \mathrm{Dim}(F[o_2]) &= 4(\nu_0+1)(o_2+1) - 4(o_2+1)^2 \\ &= 4(\nu_0 - o_2)(o_2+1), \end{aligned} \tag{44}$$

in comparison with the dimension of the full field basis given by $\mathrm{Dim}[\mathcal{F}] = (\nu_0+1)^2$. Note the difference in the cardinality of the different bases,

Subbasis	Dimension
$\mathcal{F}[0]$	312
$\mathcal{F}[1]$	616
$\mathcal{F}[2]$	912
$\mathcal{F}$	6241

C. Matter-field sector

For the combined matter-field system, using the definition of the truncated basis in the matter and field sectors, we take the reduced basis to be

$$\mathcal{B}^{\kappa}_{\sigma}[o_1, o_2] := \left[\bigoplus_{r_1=0}^{o_1} \bigoplus_{r_2=0}^{o_2} \mathcal{F}[r_2] \otimes \mathcal{M}[r_1] \right]^{\kappa}_{\sigma}. \tag{45}$$

Indices κ and σ indicate that states that do not preserve the parity, σ and states with values $\kappa' > \kappa$ are eliminated. Note that one has the identity

$$\mathcal{B}^{\kappa}_{\sigma} = \mathcal{B}^{\kappa}_{\sigma}[O_1, O_2]. \tag{46}$$

We should remark that this procedure to obtain reduced bases is useful only when the full system is divided into subsystems where a single mode promotes transitions between a few atomic levels. In addition, for large values of N_a, the reduction $\mathcal{B}^{\kappa}_{\sigma}[o_1, o_2]$ will give a good approximation to the exact quantum ground state because this state approaches better and better the symmetry-adapted variational case.

V. RESULTS FOR THE Ξ CONFIGURATION

As an example of the application of the reduced basis, we consider a three-level system in the Ξ configuration in resonance with two modes of the electromagnetic field (zero detuning).

A. Dimensions for different orders

In order to compare the dimension of the reduced basis (45) with the exact basis (23), we fix the dimensionless dipolar strength values at $x^{(1)}_{12} = x^{(2)}_{23} = 4$ and the error in the fidelity at $e_{\mathrm{rr}} = 10^{-10}$. For these equal values of $x^{(1)}_{12}$ and $x^{(2)}_{23}$, the values for $\overline{m}^{(1)}_{12}$ and $\overline{m}^{(2)}_{23}$ are equal. The bases will allow us to find the ground state as a function of the parameters in the region $[0,4] \times [0,4]$ as in Fig. 3(b).

By definition of the reduced bases, these satisfy

$$\begin{aligned} \mathcal{B}^{\kappa}_{\sigma}[o_1, o_2] &= \mathcal{B}^{\kappa}_{\sigma}[O_1, o_2], \quad \text{when } o_1 \geqslant O_1, \\ \mathcal{B}^{\kappa}_{\sigma}[o_1, o_2] &= \mathcal{B}^{\kappa}_{\sigma}[o_1, O_2], \quad \text{when } o_2 \geqslant O_2. \end{aligned}$$

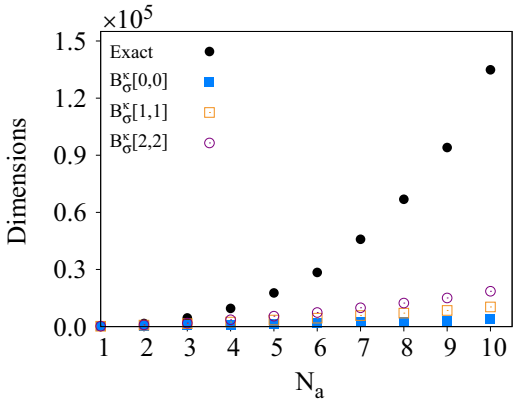

FIG. 5. The dimension of the Hilbert space (23) is shown as a function of the number of particles N_a, considering an error $e_{\mathrm{rr}} = 10^{-10}$ in the fidelity and maximum values of $x_{12} = x_{23} = 4$ for the Ξ configuration (solid circles) and even-even-parity $\sigma = ee$. This is compared to the corresponding dimensions of the reduced bases (45) with orders $o1 = o2 = 0$ (solid squares), $o1 = o2 = 1$ (empty squares), and $o1 = o2 = 2$ (empty circles).

In this sense, one may refer to any order of approximation independent of the number of particles N_a, which establishes the value O_1 in Eq. (37), and maximum number of photons ν_0 for the value of O_2 in Eq. (43).

The dimension of the reduced bases as function of the number of particles are shown in Fig. 5 for the even-even parity ($\sigma = ee$). Note that the savings are tremendous. In particular, for $N_a = 1$ and $N_a = 10$, we have

Basis	dimension for $N_a = 1$	dimension for $N_a = 10$
$\mathcal{B}^{\kappa}_{\sigma}$	397	133549
$\mathcal{B}^{\kappa}_{\sigma}[2,2]$	252	18452
$\mathcal{B}^{\kappa}_{\sigma}[1,1]$	216	10226
$\mathcal{B}^{\kappa}_{\sigma}[0,0]$	176	3754

In order to calculate the table above, we used

$$\overline{m}^{(1)}_{12} = \overline{m}^{(2)}_{23} = 22 \quad \text{for } N_a = 1,$$

and

$$\overline{m}^{(1)}_{12} = \overline{m}^{(2)}_{23} = 92 \quad \text{for } N_a = 10,$$

which imply that $(k_1, k_2) = (45, 22)$ and $(194, 92)$, respectively. Clearly, the reduced bases will be more important in calculations where the number of particles is large.

B. Comparison between energy surfaces

Previously, we have shown the exact ground energy surface for the case $N_a = 4$ in Fig. 3(b). For this case, we find that the dimensions of the reduced bases are $\dim(\mathcal{B}^{\kappa}_{\sigma}[0,0]) = 1020$, $\dim(\mathcal{B}^{\kappa}_{\sigma}[1,1]) = 2413$, and $\dim(\mathcal{B}^{\kappa}_{\sigma}[2,2]) = 3609$, in comparison with the dimension of the exact quantum basis $\dim(\mathcal{B}^{\kappa}_{\sigma}) = 9546$.

To compare the different energy surfaces, we define $E_{o1,o2}$ as the ground-state energy by using the reduced basis $\mathcal{B}^{\kappa}_{\sigma}[o_1, o_2]$ and calculate the percentual error with respect to

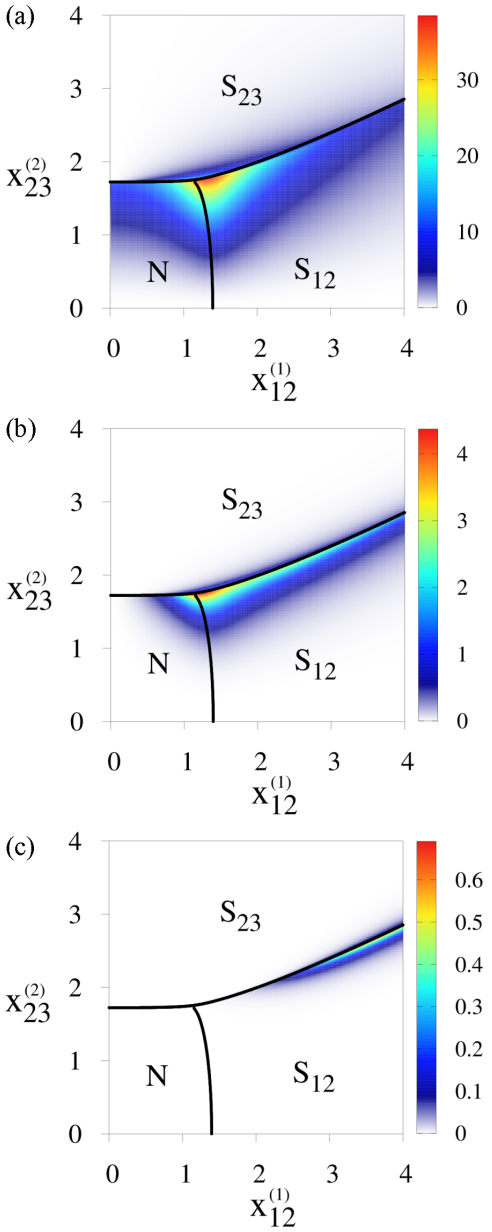

FIG. 6. Percentual error $\Delta_{o1,o2}$ in the quantum ground energy surface for the reductions: (a) $\mathcal{B}_\sigma^\kappa[0,0]$, (b) $\mathcal{B}_\sigma^\kappa[1,1]$, and (c) $\mathcal{B}_\sigma^\kappa[2,2]$. Note that the plots are given at different scales. The parameters are discussed in the text.

the exact quantum ground energy $\mathcal{E}_g$ as

$$\Delta_{o1,o2} = \left| \frac{(\mathcal{E}_g - E_{o1,o2})}{\mathcal{E}_g} \right| \times 100\%. \tag{47}$$

We define $\Delta_{o1,o2} = 0$ when $\mathcal{E}_g = 0$ since this value is obtained at points on the axes, and one may see easily that any basis reduction provides the same results as the exact basis when $\mathcal{E}_g = 0$.

We choose the parameters for the system indicated in Sec. III C for $N_a = 4$ atoms in the Ξ configuration and compare the energy surfaces in Fig. 6 for the reductions (a) $\mathcal{B}_\sigma^\kappa[0,0]$, (b) $\mathcal{B}_\sigma^\kappa[1,1]$, and (c) $\mathcal{B}_\sigma^\kappa[2,2]$.

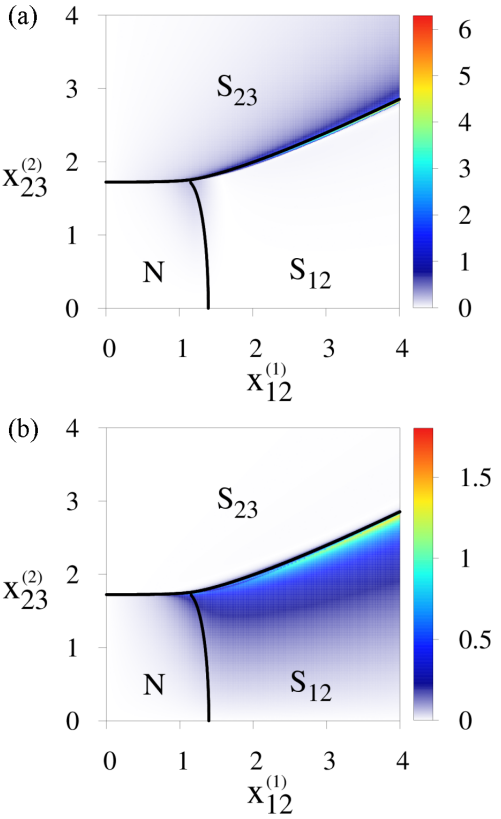

FIG. 7. Absolute error in the fluctuation of the number of photons $\Delta(\sigma_v)$ for the quantum ground state in comparison with the ground state in the reduction $\mathcal{B}_\sigma^\kappa[0,0]$, (a) for photon Ω_1 and (b) for photon Ω_2. Note that the plots are given at different scales.

Each energy surface divides itself into three regions, denoted by N, S_{12}, and S_{23}. N is associated with the normal sector where the ground state is dominated by the state with four particles in the lowest-energy level and zero photons. The S_{ij} are superradiant regions in which the ground state is dominated by the two-level subsystem $\omega_i \rightleftharpoons \omega_j$ where only one type of photon is present. The dark lines in the figure represent the separatrices which separate these phases, and the percentual error $\Delta_{o1,o2}$ is color mapped to the scale beside each plot.

In all cases, one may observe that the maximum percentual error lies around the separatrix where discontinuous transitions occur, whereas, away from the separatrix, it tends rapidly to zero. The reader should note the differences in scale given in each plot: While we have a maximum error around 30% for the $\mathcal{B}_\sigma^\kappa[0,0]$ basis, this reduces to approximately 0.4% for the basis $\mathcal{B}_\sigma^\kappa[2,2]$. In other words, we can obtain excellent agreement with the exact value for the ground-state energy when using our reduced bases.

We should also note that the error appears larger in the superradiant region S_{12} than in the region S_{23}. This responds to the fact that the value of the coupling constant in the subsystem $\omega_2 \rightleftharpoons \omega_3$ is already larger than that of the subsystem $\omega_1 \rightleftharpoons \omega_2$ when the corresponding separatrix with the normal region is crossed.

C. Photon number fluctuations

In the previous subsection, we saw that the value of the ground-state energy found with the reduced bases is in very good agreement with the exact quantum calculation. Similar results are obtained for any expectation value of both number of photons or atomic populations. Here, we show the absolute error in the fluctuations in the number of photons,

$$\Delta(\sigma_\nu) := |\sigma_g(\nu) - \sigma_{o1,o2}(\nu)|, \tag{48}$$

a quantity that is not well approximated by variational methods.

In Fig. 7, the absolute error in the fluctuation of the number of photons is shown for the calculations with the reduced basis $\mathcal{B}_\sigma^\kappa[0,0]$ for photons Ω_1 in Fig. 7(a) and for photons Ω_2 in Fig. 7(b). The normal N and superradiant regions S_{ij} in the phase diagram are also shown. Once again, the maximum error lies around the separatrix, and the difference in scale for each plot should be noted. The error will tend to zero as the order of the reduced basis increases (in fact, for the bases $\mathcal{B}_\sigma^\kappa[1,1]$ and $\mathcal{B}_\sigma^\kappa[2,2]$ plotted errors would be difficult to see). This shows that, in addition to the expectation values, one has an excellent agreement for their fluctuations and, hence, the ground state obtained with a reduced basis provides the same statistical properties than the exact calculation.

We should also stress that as the number of particles increases, the exact quantum ground state tends to the one obtained with the reduced basis $\mathcal{B}_\sigma^\kappa[0,0]$.

VI. CONCLUDING REMARKS

In this paper, we built a sequence of ever-approximating bases for the infinite-dimensional Hilbert space $\mathcal{H}$ of matter interacting with radiation. The ground state (which is the one under study) obtained in these truncated spaces differs from the exact ground state by no more than a certain allowed error $e_{\rm rr}$ as measured by the fidelity F between the two states. We have shown examples for both $e_{\rm rr} = 10^{-10}$ and $e_{\rm rr} = 10^{-15}$. The reduced bases provide solutions with the same statistical properties as those of the exact solution and are especially useful when the number of particles is large. In fact, at different orders of the approximation, one may study the physical properties of the system for any number of particles as was exemplified even for a single particle in Ref. [22].

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APPENDIX: DIMENSIONS OF THE RWA BASES

For the three-level atoms interacting dipolarly with two modes of electromagnetic field, the degeneracy of states with fixed values $\kappa = \{k_1, k_2\}$ of the operators $\boldsymbol{K}_1, \boldsymbol{K}_2$, respectively (see Table I) is given by the dimension of the subspace $\mathcal{B}_{\rm RWA}^{(\kappa)}$ Eq. (19). To find the dimension as a function of the parameters, we proceed as follows: For a fixed set of values $\{N_a, k_1, k_2\}$, the number of elements of the basis $\mathcal{B}_{\rm RWA}^{(\kappa)}$ is calculated by means of equations of the form

$$a_1 N_a^2 + a_2 N_a + b_1 k_1^2 + b_2 k_1 + c_1 k_2^2 + c_2 k_2 + d. \tag{A1}$$

Note that an equation of second order in the variables N_a, k_1, and k_2 is proposed because the states are the direct product of five Fock states, similar to a problem of five harmonic oscillators, whose degeneracy is given by a second-order equation.

By comparing expression (A1) with the dimension found from the basis, given (seven sets of) values for N_a, k_1, and k_2, the coefficients a_j, b_j, c_j, and d are determined and provide us with an analytical expression for the dimension. The region of the validity of the expression is also obtained.

After this procedure is finished, the expression for the dimension of the basis for each atomic configuration and any value of $\{N_a, k_1, k_2\}$ is obtained.

For the Λ configuration, one gets

$$\dim\left(\mathcal{B}_{\rm RWA}^{(\kappa)}\right) = \begin{cases} g(N_a + k_1 - k_2 + 1), & N_a < k_2 \wedge k_1 \leqslant k_2, \\ g(k_1 + 1), & k_2 \leqslant N_a \wedge k_1 \leqslant k_2, \\ g(k_2 + 1), & k_2 \leqslant N_a \wedge k_2 < k_1, \\ g(N_a + 1), & \text{other case}, \end{cases} \tag{A2}$$

where we defined

$$g(x) := \frac{(x)(x+1)}{2}, \tag{A3}$$

and the operators $\boldsymbol{K}_1, \boldsymbol{K}_2$ are given in Table I. Given the values of $k_1 = 0, 1, 2, \ldots$, the values of k_2 are limited by $k_2 = 0, 1, \ldots, k_1 + N_a$; in all other cases the subspace is empty.

For the Ξ configuration, one finds

$$\dim\left(\mathcal{B}_{\rm RWA}^{(\kappa)}\right) = \begin{cases} g(k_1 - k_2 + 1), & k_1 - k_2 \leqslant N_a \wedge k_1 \leqslant 2k_2, \\ g(k_1 + 1) - g(k_1 - k_2) - 2g(k_2), & k_1 - k_2 \leqslant N_a \wedge 2k_2 < k_1, \\ g(N_a + 1) - g(N_a - k_2), & N_a < k_1 - k_2 \wedge k_2 < N_a, \\ g(N_a + 1), & \text{other case}. \end{cases} \tag{A4}$$

In a similar way to the Λ configuration, the values satisfy $k_1 = 0, 1, 2, \ldots$ and $k_2 = 0, 1, \ldots, k_1$.

For the V configuration,

$$\dim\left(\mathcal{B}^{(\kappa)}_{\mathrm{RWA}}\right) = \begin{cases} (k_2+1)(k_1-k_2+1), & k_1 \leqslant N_a, \\ h(k_2, N_a), & (2N_a \leqslant k_1 \ \wedge \ k_2 \leqslant N_a) \\ & \vee\,(N_a < k_1 \ \wedge \ k_1 < 2N_a \ \wedge \ N_a < k_1 - k_2), \\ h(k_1-k_2, N_a), & (N_a < k_1 \ \wedge \ k_1 < 2N_a \ \wedge \ N_a < k_2) \\ & \vee\,(2N_a \leqslant k_1 \ \wedge \ k_1 - k_2 \leqslant N_a), \\ (k_1-k_2)k_2 + h(N_a, k_1) - g(k_1), & N_a < k_1 \ \wedge \ k_1 < 2N_a \ \wedge \ k_2 \leqslant N_a \ \wedge \ k_1 - k_2 \leqslant N_a, \\ g(N_a+1), & \text{other case}, \end{cases} \tag{A5}$$

with $k_1 = 0, 1, 2, \ldots$, and $k_2 = 0,\ 1,\ \ldots,\ k_1$ and where we defined

$$h(x, y) := (x+1)(y+1) - g(x) \tag{A6}$$

to simplify the notation.

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