Few-cycle solitons of an integrable generalization of the reduced Maxwell-Bloch equations

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(Received 1 May 2019; published 6 November 2019)

The physical generalization of the system of the reduced Maxwell-Bloch equations describing the nonlinear interaction of the laser pulses with the multilevel quantum medium is obtained without using the slowly varying envelope approximation. It is shown that this system is integrable in the frameworks of the inverse scattering transformation method. The soliton and breather solutions in the form of the unipolar and few-cycle pulses are constructed. Their distinctive features caused solely by the multilevel structure of the medium are discussed. It is revealed that the collision of the solitons can lead to an appearance of the large-amplitude short-living pulse, whose dynamics resembles that of rogue waves.

DOI: 10.1103/PhysRevA.100.053807

I. INTRODUCTION

An appearance in the 1960s of the powerful pulse lasers stimulated the experimental and theoretical investigations of the nonlinear interaction of the light pulses with matter. The resonant phenomenon of the self-induced transparency (SIT) was revealed this way [1]. A theoretical explanation of this effect was given on the basis of the notion of the optical soliton.

The duration τ_p of the SIT pulse is about 10^{-9} s, while its carrier frequency ω is 10^{15} s^{-1}. Therefore, such a pulse contains $N \sim \omega\tau_p \sim 10^6$ optical oscillations. The presence of the small parameter $\delta \sim 1/N \sim 10^{-6}$ gave an opportunity to apply the approximation of the slowly varying envelope and to obtain the system of the so-called SIT equations [1,2]. In the case of the sharp line resonance, the SIT equations are reduced to the famous sine-Gordon equation [2]. It was found under further research [3–5] that the sine-Gordon and SIT equations are integrable by the inverse scattering transformation method [6–9] and have multisoliton solutions corresponding to the discrete part of the scattering data [6]. These solitons known also as 2π pulses describe exactly the SIT phenomenon.

One of the tendencies of development of laser physics consists of shortening of the duration of the pulses generated in physical laboratories [10–13]. This leads to a gradual increase of the parameter δ. For the pulses with a duration of a few femtoseconds, this parameter can reach the values of the order of unit. In this case, the pulse contains about one period of optical oscillations. The term "few-cycle pulse" was strongly assigned to such pulses [10,13]. In that case, the concept of the envelope loses meaning, and the slowly varying envelope approximation is not yet applicable.

In 1973, an alternative approach to describe the SIT phenomenon, in which the slowly varying envelope approximation was not exploited, was offered [14]. It was suggested that the concentration of two-level is small. This allowed the derivatives on the coordinate and time in the wave equation for the pulse electric field from the second order to the first one to be reduced. Supplementing the equation obtained here by the system of the material equations on the density matrix elements of the two-level atom led to the system of the so-called reduced Maxwell-Bloch (RMB) equations.

The RMB equations contain no the envelopes of the electric field of the pulse and dipole moment of the atoms and are written for the full electric field and dipole moment as the whole. The system of the RMB equations is also integrable by the inverse scattering transformation method [14], and its soliton and breather solutions were studied in detail [9].

An obvious shortcoming of the system of the RMB equations is that its derivation is based on the model of medium consisting of the two-level atoms. This model rather adequately describes the propagation of resonant or quasiresonant solitons of the envelope, i.e., breathers with $\omega\tau_p \gg 1$ and $\omega \approx \omega_0$, where ω_0 is the frequency of quantum transition between the states of the two-level atom. However, in the case $\omega\tau_p \sim 1$, we have for the spectral width of the breather $\delta\omega \sim 1/\tau_p \sim \omega$. Owing to the large width of the spectrum of such a pulse, the large number of quantum transitions of the medium can be involved in the interaction with the pulse. Thus, the model of the two-level medium loses the adequacy and demands a replacement by a more realistic model.

Different models of the multilevel quantum medium were considered in Refs. [15–20] in particular. The approximation of the sudden perturbations [21,22] was applied in these cases. This approximation relies on the condition

$$\omega_{jk}\tau^* \ll 1, \tag{1}$$

where ω_{jk} is the frequency of quantum transition $j \leftrightarrow k$, and $\tau^* = \min\{\tau_p, \omega^{-1}\}$ is the minimum time scale of the pulse.

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It follows from condition (1) that the spectrum of the pulse overlaps all quantum transitions involved in the interaction. Therefore, the excitation of the medium is strong, i.e., the populations of the quantum levels are subject to considerable changes under the pulse propagation [17,18].

Another approximation used often when considering the multilevel quantum media is that of the optical transparency [17,18,23–27]. This approximation is based on the following condition:

$$\mu = (\omega_{jk}\tau^*)^{-1} \ll 1. \tag{2}$$

Contrary to condition (1), this one corresponds to rather weak excitation of the medium [18,25].

It was offered in Refs. [28,29] to approximate the multilevel medium by the model of the four-level atoms. The frequency of one allocated transition satisfied condition (1), while the frequencies of other allowed transitions to the remote quantum levels satisfied condition (2). The generalized sine-Gordon equation for the electric field of the pulse was obtained after applying the approximation of the sudden perturbations and optical transparency. This equation was shown in Ref. [29] to be integrable in the framework of the inverse scattering transformation method, and the scheme of construction of its multisoliton solutions was developed.

In order that the model used in Refs. [28,29] would have a wider physical reliability, it can be generalized in the case of any number of the remote quantum states. Besides, one can refuse from the approximation of the sudden perturbations and, as a consequence, from restrictions (1). This expands the domain of physical applicability of the model in the case, for example, of the electron-optical transitions corresponding to the visible range. On the other hand, it is possible this way to derive a generalization of the RMB system integrable by the inverse scattering transformation method and to investigate its soliton and breather solutions. This constitutes the essence of the present work.

The paper is organized as follows: In Sec. II, the model of the multilevel quantum medium with two allocated low-lying states is introduced. Then, the procedure of the exclusion of elements of the density matrix that correspond to the transitions to remote quantum levels is carried out by using condition (2). As result, we obtain the system of the equations containing the electric field of the pulse and the elements of the density matrix of the allocated transition only. We call these equations the generalized RMB (GRMB) system. In Sec. III, the integrability of the GRMB system in the framework of the inverse scattering transformation method is considered. The two-soliton and breather solutions are investigated in Sec. IV. These solutions are compared with the proper ones in the cases of the RMB and the generalized sine-Gordon equations. Also, particular attention is paid here to the collision of the solitons with opposite polarities, in which the large-amplitude short-living pulse having the dynamics similar to that of rogue waves can appear. Finally in Sec. V, the main results of the consideration are summarized, some concluding remarks are given, and future developments are discussed.

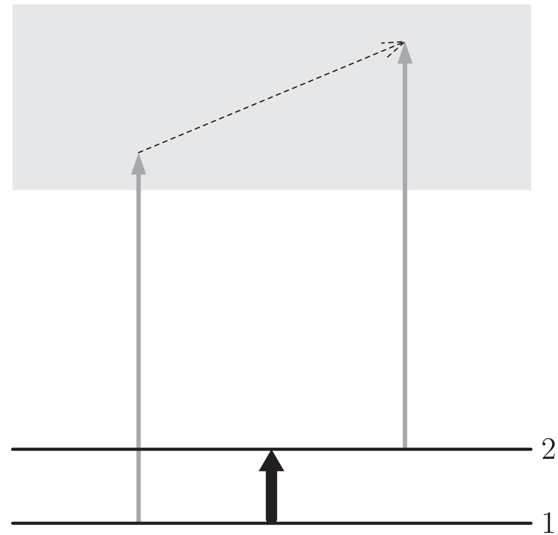

FIG. 1. The scheme of the allowed quantum transitions. Numbers 1 and 2 correspond to two lower levels. The gray rectangle designates a series of the quantum states removed up on the energy scale. The bold continuous arrow represents the transition changing the populations of the lower levels. The gray continuous arrows represent the transitions from the lower levels to the remote ones. The dotted arrow represents schematically the allowed transitions between the remote states.

II. GENELALIZED REDUCED MAXWELL-BLOCH SYSTEM

Let two quantum states (1 and 2) having the lowest values of the energy be allocated from a large number of the states (Fig. 1). Assume that these states possess opposite parities. Consequently, the electrodipole transition between them is allowed. Other quantum transitions allowed owing to the parity selection rule happen from state 1 or state 2 on the overlying ones and between overlying states.

Suppose that the overlying states are removed from states 1 and 2 so that inequalities

$$\omega_{j1},\ \omega_{k2} \gg \omega_{21}, \tag{3}$$

where $j = 3, 5, 7, \dots$ and $k = 4, 6, 8, \dots$, on the frequencies of allowed transitions take place. This situation is realized, for example, in the medium of the tunnel transitions between the minima of the deep two-pit potential. The ferroelectric materials [30,31], the metamaterials consisting of the quantum dots and quantum wells [32,33] can be considered as a medium having such properties.

We assume also that the frequencies of transitions $1 \leftrightarrow j$ and $2 \leftrightarrow k$ satisfy condition (2). In that case, they rather weakly interact with a light pulse, but have an impact on transition $1 \leftrightarrow 2$. The formal restrictions on quantum transition $1 \leftrightarrow 2$ are not imposed. This transition belongs to the far- or middle-infrared range in the examples given above. In turn, transitions $1 \leftrightarrow j$ and $2 \leftrightarrow k$ lie in the near-infrared range. The transitions $j \leftrightarrow k$ are weakened considerably here, and we neglect them [28,29].

The equations for elements of ρ_{ml} of the density matrix $\hat{\rho}$ of the multilevel atom in the representation of the eigenfunctions of the Hamiltonian of the free atom have the following form:

$$\frac{\partial \rho_{ml}}{\partial t} = -i\omega_{ml}\rho_{ml} + \frac{i}{\hbar}E[\hat{d}, \hat{\rho}]_{ml}, \tag{4}$$

where $\hbar$ is Planck's constant, E is the electric field of the pulse, $\hat{d}$ is the dipole moment operator, and the elements d_{jk} of matrix $\hat{d}$ are assumed to be real owing to its hermiticity, i.e., $d_{jk} = d_{kj}$.

Assuming that the pulse propagates along the z axis, we supplement the material equations (4) by the following wave equation on its electric field:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2}, \tag{5}$$

where

$$P = n\,\mathrm{Sp}\,(\hat{\rho}\hat{d}) = n\left\{d_{21}(\rho_{21} + \rho_{21}^*) + \sum_{q\neq 1,2}\left[d_{q1}(\rho_{q1} + \rho_{q1}^*) + d_{q2}(\rho_{q2} + \rho_{q2}^*) + \sum_{m\neq 1,2,q} d_{qm}\rho_{qm}\right]\right\} \tag{6}$$

(n is the concentration of the multilevel atoms).

Supposing that the concentration n is small (see below), we reduce the order of the derivatives in Eq. (5) as it was done in Refs. [14,18]. This gives us the following equation:

$$\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{2\pi}{c}\frac{\partial P}{\partial t}. \tag{7}$$

Let us exclude the density matrix elements different from ρ_{21}, ρ_{12}, ρ_{11}, and ρ_{22} from the system (4) in the first order with respect to the small parameter μ [see Eq. (2)]. For this, we rewrite Eqs. (4) in the following manner:

$$\frac{\partial \rho_{21}}{\partial t} = -i\omega_0\rho_{21} + i\frac{d_{21}}{\hbar}E(\rho_{11} - \rho_{22}) + i\frac{E}{\hbar}\sum_{m\neq 1,2}(d_{m2}\rho_{m1} - d_{m1}\rho_{m2}^*), \tag{8}$$

$$\frac{\partial \rho_{11}}{\partial t} = i\frac{d_{21}}{\hbar}E(\rho_{21} - \rho_{21}^*) + i\frac{E}{\hbar}\sum_{m\neq 1,2} d_{m1}(\rho_{m1} - \rho_{m1}^*), \tag{9}$$

$$\frac{\partial \rho_{q1}}{\partial t} = -i\omega_{q1}\rho_{q1} + i\frac{E}{\hbar}[d_{q1}(\rho_{11} - \rho_{qq}) + d_{q2}\rho_{21} - d_{21}\rho_{q2}] + i\frac{E}{\hbar}\sum_{m\neq 1,2,q}(d_{qm}\rho_{m1} - d_{m1}\rho_{qm}), \tag{10}$$

$$\frac{\partial \rho_{qr}}{\partial t} = -i\omega_{qr}\rho_{qr} + i\frac{E}{\hbar}(d_{q1}\rho_{r1}^* + d_{q2}\rho_{r2}^* - d_{r1}\rho_{q1} - d_{r2}\rho_{q2}) + i\frac{d_{qr}E}{\hbar}(\rho_{rr} - \rho_{qq}) + i\frac{E}{\hbar}\sum_{m\neq 1,2,q,r}(d_{qm}\rho_{mr} - d_{mr}\rho_{qm}^*), \tag{11}$$

where $\omega_0 \equiv \omega_{21}$ and $q,\ r = 3,\ 4,\ 5, \ldots$. The equations for elements ρ_{22} and ρ_{q2} are obtained under the replacement of indices $1 \leftrightarrow 2$ in Eqs. (9) and (10), respectively. The equation for elements ρ_{qq} are obtained from Eq. (11) in the case $r = q$.

We assume that the states 1 and 2 are populated only with probabilities w_1 and w_2, respectively, before the pulse impact.

Having put $\partial\rho_{q1}/\partial t = 0$ in Eq. (10) in the first-order approximation with respect to the small parameter μ, we find

$$\rho_{q1} = \frac{E}{\hbar\omega_{q1}}[d_{q1}(\rho_{11} - \rho_{qq}) + d_{q2}\rho_{21} - d_{21}\rho_{q2}] + \frac{E}{\hbar\omega_{q1}}\sum_{m\neq 1,2,q}(d_{qm}\rho_{m1} - d_{m1}\rho_{qm}^*). \tag{12}$$

It is seen from this relation and from Eq. (11) that the terms in the sum and the last term in the square brackets can be neglected in the first order with respect to μ. Thus, we have the following expressions for elements ρ_{q1} and ρ_{q2}:

$$\rho_{q1} = \frac{E}{\hbar\omega_{q1}}[d_{q1}(\rho_{11} - \rho_{qq}) + d_{q2}\rho_{21}],$$
$$\rho_{q2} = \frac{E}{\hbar\omega_{q2}}[d_{q2}(\rho_{22} - \rho_{qq}) + d_{q2}\rho_{21}^*].$$

We have $\omega_{q1} \approx \omega_{q2}$ owing to inequalities (3). Then, the approximations used here allow us to rewrite the last equalities in the following forms:

$$\rho_{q1} = \frac{E}{\hbar\omega_{q2}}[d_{q1}(\rho_{11} - \rho_{qq}) + d_{q2}\rho_{21}],$$
$$\rho_{q2} = \frac{E}{\hbar\omega_{q1}}[d_{q2}(\rho_{22} - \rho_{qq}) + d_{q2}\rho_{21}^*]. \tag{13}$$

For fixed value m, one of two matrix elements d_{m1} and d_{m2} is equal to zero according to the parity selection rules. Then $d_{m1}d_{m2} = 0$. Taking this into account, we obtain the following after substitution of relations (13) into the right-hand sides of Eqs. (8) and (9):

$$\frac{\partial \rho_{21}}{\partial t} = -\left[\omega_0 + \frac{E^2}{\hbar^2}\sum_{m\neq 1,2}\left(\frac{d_{m1}^2}{\omega_{m1}} - \frac{d_{m2}^2}{\omega_{m2}}\right)\right]\rho_{21} + i\frac{d_{21}}{\hbar}E(\rho_{11} - \rho_{22}), \tag{14}$$

$$\frac{\partial \rho_{11}}{\partial t} = -\frac{\partial \rho_{22}}{\partial t} = i\frac{d_{21}}{\hbar}E(\rho_{21} - \rho_{21}^*). \tag{15}$$

Thus, we have $\rho_{qr} = 0$ for $q,\ r > 2$ in the first-order approximation with respect to the small parameter μ. The pulse field causes a coherence in the system of transitions $q \leftrightarrow 1$ and $q \leftrightarrow 2$ ($q = 3,\ 4,\ 5, \ldots$), but does not populates the remote states: $\rho_{q1},\ \rho_{q2} \neq 0$ and $\rho_{qq} = 0$.

The quantum transitions between the states represented by the dotted arrow in Fig. 1 are considerably weakened and do not display themselves in any way. This allows us to neglect the last sum in the square brackets of expression (6) in the approximation considered. Then we have after the substitution of expressions (13) into relation (6):

$$P = n\left[d_{21}(\rho_{21} + \rho_{21}^*) + \frac{2E}{\hbar}\sum_{m\neq 1,2}\left(\frac{d_{m1}^2}{\omega_{m1}}\rho_{11} + \frac{d_{m2}^2}{\omega_{m2}}\rho_{22}\right)\right]. \tag{16}$$

Since states 1 and 2 remain populated only in the approximation accepted, we have $\rho_{11} + \rho_{22} = 1$. Then, introducing the Bloch's variables

$$U = \frac{\rho_{21} + \rho_{21}^*}{2}, \quad V = \frac{\rho_{21}^* - \rho_{21}}{2i}, \quad W = \frac{\rho_{22} - \rho_{11}}{2},$$

we obtain from Eqs. (14), (15), (7), and (16) the system of generalized RMB (GRMB) equations:

$$\frac{\partial U}{\partial \tau} = -(\omega_0 + \beta\Omega^2)V, \tag{17}$$

$$\frac{\partial V}{\partial \tau} = (\omega_0 + \beta\Omega^2)U + \Omega W, \tag{18}$$

$$\frac{\partial W}{\partial \tau} = -\Omega V, \tag{19}$$

$$\frac{\partial \Omega}{\partial z} = -\alpha\frac{\partial}{\partial \tau}(U - 2\beta\Omega W). \tag{20}$$

Here

$$\tau = t - z/v_0,$$

$$\frac{1}{v_0} = \frac{1}{c}\left[1 + \frac{2\pi n}{\hbar}\sum_{m\neq 1,2}\left(\frac{d_{m1}^2}{\omega_{m1}} + \frac{d_{m2}^2}{\omega_{m2}}\right)\right],$$

$$\alpha = \frac{8\pi d_{21}^2 n}{\hbar c}, \quad \beta = \frac{1}{4d_{21}^2}\sum_{m\neq 1,2}\left(\frac{d_{m1}^2}{\omega_{m1}} - \frac{d_{m2}^2}{\omega_{m2}}\right),$$

$$\Omega = \frac{2d_{21}E}{\hbar}.$$

From these formulas, we see that the condition of a small concentration of the atoms, at which the reduction of the wave equation (5) to Eq. (7) was performed, is written as

$$\frac{8\pi d^2 n}{\hbar\omega_0} \ll 1,$$

where $d^2 = \max(d_{21}^2, \{d_{m1}^2\}, \{d_{m2}^2\})$.

Parameter β considers here a digression from the two-level model. Taking $\beta = 0$ in Eqs. (17)–(20) gives us the well-known RMB system for the two-level case. The matrix elements d_{m1} in the sum of the parameter β definition are different from zero if m is odd. At the same time, $d_{m2} \neq 0$ if m is even.

It follows from Eqs. (17) and (18) that the presence of quantum transitions to overlying quantum states leads in the first order with respect to the small parameter μ to the dynamic Stark shift of frequency ω_0 of transition $1 \leftrightarrow 2$. Besides, as it is seen from Eq. (20), the transitions to overlying states cause a nonlinear contribution in the dynamic polarization response of the medium. At the same time, the changes of populations of the overlying quantum states are absent in this order of the approximation [see Eq. (19)]. All evolution of the population occurs between the allocated states 1 and 2.

It is important to note that the dipole moments d_{21}, d_{m1}, and d_{m2} ($m \geqslant 3$) of the allowed quantum transitions can be in various quantitative relations with each other (see the definition of the parameter β). Therefore, it is impossible to consider the terms in Eqs. (17), (18), and (20), which take into account a deviation from the model of the two-level medium, as small corrections to the RMB system. The GRMB system represents an independent full-fledged interest.

III. ZERO-CURVATURE REPRESENTATION

The GRMB system [Eqs. (17)–(20)] is integrable in the framework of the inverse scattering transformation method [34]. It admits the following zero-curvature condition:

$$\frac{\partial \hat{L}}{\partial z} - \frac{\partial \hat{A}}{\partial \tau} + [\hat{L}, \hat{A}] = 0,$$

where

$$\hat{L} = \begin{pmatrix} -i\sigma\Omega & \lambda\left[\frac{\omega_0+\beta\Omega^2}{2\omega_0} + i\varepsilon\Omega\right] \\ \lambda\left[\frac{\omega_0+\beta\Omega^2}{2\omega_0} - i\varepsilon\Omega\right] & i\sigma\Omega \end{pmatrix},$$

$$\hat{A} = \frac{\omega_0\alpha}{\lambda^2 + \omega_0^2}\begin{pmatrix} i(\omega_0\sigma U + \lambda^2\varepsilon V) & \lambda\left[\frac{W}{2} - i(\sigma V + \omega_0\varepsilon U)\right] \\ \lambda\left[\frac{W}{2} + i(\sigma V + \omega_0\varepsilon U)\right] & -i(\omega_0\sigma U + \lambda^2\varepsilon V) \end{pmatrix} + 2\alpha\beta W\hat{L},$$

with λ being the spectral parameter,

$$\sigma = \frac{\sqrt{1 + 4\beta\omega_0}}{2}, \quad \varepsilon = \sqrt{-\frac{\beta}{\omega_0}}. \tag{21}$$

Also, the GRMB system [Eqs. (17)–(20)] is connected through the change of variables $(T, Z, \Omega_0, U_0, V_0, W_0) \to (\tau, z, \Omega, U, V, W)$, where

$$d\tau = \left(1 + \sqrt{1 - \varepsilon^2\Omega_0^2}\right)dT + 2\varepsilon^2 W_0 dZ, \quad dz = \frac{dZ}{\omega_0\alpha},$$

$$\Omega(\tau, z) = \frac{\Omega_0(T, Z)}{1 + \sqrt{1 - \varepsilon^2\Omega_0^2(T, Z)}}, \quad W(\tau, z) = W_0(T, Z),$$

$$U(\tau, z) = V_0(T, Z), \quad V(\tau, z) = -U_0(T, Z), \tag{22}$$

with the system of the modified RMB equations

$$\frac{\partial U_0}{\partial T} = -2\omega_0\sqrt{1 - \varepsilon^2\Omega_0^2}\,V_0, \tag{23}$$

$$\frac{\partial V_0}{\partial T} = 2\omega_0\sqrt{1 - \varepsilon^2\Omega_0^2}\,U_0 + \Omega_0 W_0, \tag{24}$$

$$\frac{\partial W_0}{\partial T} = -\Omega_0 V_0, \tag{25}$$

$$\frac{\partial \Omega_0}{\partial Z} = -\frac{1}{\omega_0}\frac{\partial U_0}{\partial T}. \tag{26}$$

This system coincides with the RMB equations [14] in the case $\varepsilon = 0$ and is integrable by the inverse scattering transformation method also. The systems, which are equivalent to the modified RMB system, were considered in Refs. [35–38] under an investigation of the nonlinear dynamics of two-component electromagnetic and acoustic extremely short pulses in the two-level media.

IV. COLLISIONS OF SOLITONS AND BREATHER SOLUTIONS

The multisoliton solutions of the modified RMB equations (23)–(26) were studied well in Refs. [34,36,38]. These solutions and the change of variables (22) are used below to construct the multisoliton solutions of the GRMB system [Eqs. (17)–(20)]. Here, there exist three cases depending on the value of the parameter β. Looking ahead, we note that the solutions of the GRMB system are quite different from the ones of the modified RMB system.

A. Case $\beta < -\frac{1}{4\omega_0}$ ($i\sigma$ and ε are real)

In this case, we have the following for the effective frequency of quantum transition:

$$\omega_{\text{ef}} = \omega_0 + \beta\Omega^2 < \omega_0\left[1 - \left(\frac{\Omega}{2\omega_0}\right)^2\right]$$

[see Eqs. (17) and (18)].

The expression for the variable Ω_0 of the one-soliton solution of the modified RMB equations is written as follows:

$$\Omega_0 = \pm 2\sqrt{A}\,\frac{\sinh\theta}{\Delta_1}, \tag{27}$$

where

$$\Delta_1 = A\sinh^2\theta + \varepsilon^2, \quad A = \varepsilon^2\left(1 + \frac{\omega_0^2}{\nu^2}\right) - \frac{1}{4\nu^2},$$

$$\theta = 2\nu\left(T - \frac{W_0^{(0)}Z}{\nu^2 + \omega_0^2}\right) + \theta_0,$$

here $W_0^{(0)} = (w_2 - w_1)/2$, and ν and θ_0 are real constants. It is assumed here and below that the asymptotic values of variables U_0, V_0, and W_0 are equal to 0, 0, and $W_0^{(0)}$, respectively. Also, we put $\theta_0 = 0$ without loss of generality.

The profile of Ω_0 consists of two peaks with amplitudes equal to $1/|\varepsilon|$. The peaks have opposite polarities. The interval between them is determined by the value of the parameter ν.

The variable Ω of the one-soliton solution of the GRMB system [Eqs. (17)–(20)] is defined implicitly by the substitution of expression (27) into Eqs. (22). The square roots in Eqs. (22) change the branch in the points, where $|\Omega_0|$ takes its maximum value $1/|\varepsilon|$. Since the variable Ω_0 is equal to zero between the peaks, we see that the soliton solutions of the GRMB system are singular in the case considered.

The well-defined solution of the GRMB system can be obtained using the breather solution of the modified RMB equations. The expression for the variable Ω_0 of this solution has the form

$$\Omega_0 = \frac{1}{i\sigma}\frac{\partial}{\partial T}\ln\left|\frac{s_-}{s_+}\right|, \tag{28}$$

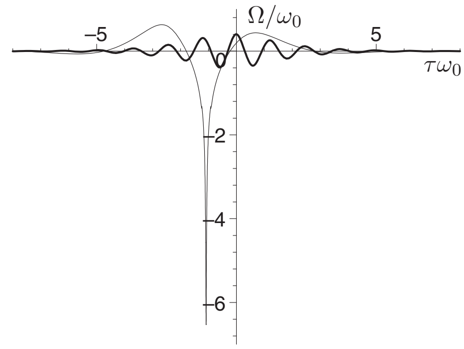

FIG. 2. Profiles of the variable Ω of breather solutions with parameters $\beta = -1/2\omega_0$, $W_0^{(0)} = -1/2$, $Z = 0$, $\nu_R = 0.7\omega_0$ and $\nu_I = \omega_0$ (normal line), and $\nu_I = 5\omega_0$ (bold line).

where

$$s_\pm = \nu_I[r_+\exp(-\theta_R) + r_-\exp(\theta_R)] \pm i\nu_R[r_+\exp(-i\theta_I) + r_-\exp(i\theta_I)],$$

$$r_\pm = \varepsilon(\nu_I - i\nu_R) \pm i\sigma,$$

$$\theta_R = 2\nu_R\left[T - \frac{W_0^{(0)}(\nu_R^2 + \nu_I^2 + \omega_0^2)Z}{\nu_R^4 + 2(\nu_I^2 + \omega_0^2)\nu_R^2 + (\nu_I^2 - \omega_0^2)^2}\right] + \theta_{R,0},$$

$$\theta_I = 2\nu_I\left[T + \frac{W_0^{(0)}(\nu_R^2 + \nu_I^2 - \omega_0^2)Z}{\nu_R^4 + 2(\nu_I^2 + \omega_0^2)\nu_R^2 + (\nu_I^2 - \omega_0^2)^2}\right] + \theta_{I,0},$$

where ν_R, ν_I, $\theta_{R,0}$, and $\theta_{I,0}$ are real constants. In what follows, we use the shifts of the independent variables to put $\theta_{R,0} = \theta_{I,0} = 0$.

The implicit definition of variable Ω of the breather solution of the GRMB system [Eqs. (17)–(20)] is obtained by the substitution of expression (28) into Eqs. (22). The profiles of the variable Ω of the breather solution for different values of the parameter ν_I determining mainly the carrier frequency are presented in Fig. 2. If the carrier frequency is high enough, then the breather solution of the GRMB system [Eqs. (17)–(20)] is similar to that of the RMB equations (see the plot with the bold line). If the carrier frequency tends from above to a finite limit depending on the parameter ν_R, then the oscillation with the amplitude exceeding ones of the nearest oscillations in a few times can appear. Here, the instantaneous value of the effective frequency of the quantum transition is negative: $\omega_{\text{ef}} < 0$. This means that the ground and excited states interchange their places in the location of such an oscillation of the breather.

B. Case $-\frac{1}{4\omega_0} < \beta < 0$ (σ and ε are real)

The expression for the variable Ω_0 of the one-soliton solution of the modified RMB system [Eqs. (23)–(26)] can be written in the form of Eq. (27) if $|\nu| \geqslant |\sigma/\varepsilon|$. The corresponding solution of the GRMB system is singular.

If $|\nu| < |\sigma/\varepsilon|$, then the variable Ω_0 of the one-soliton solution of the modified RMB system is defined as given:

$$\Omega_0 = \pm 2\sqrt{-A}\,\frac{\cosh\theta}{\Delta_2}, \tag{29}$$

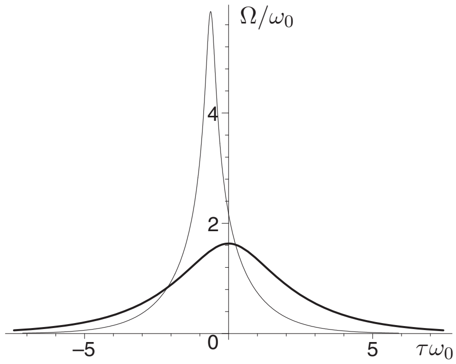

FIG. 3. Profiles of the variable Ω of one-soliton solutions with parameters $\beta = -1/8\omega_0$, $W_0^{(0)} = -1/2$, $Z = 0$, $\nu = 0.9\omega_0$ (normal line), and $\nu = 0.5\omega_0$ (bold line).

where

$$\Delta_2 = A\cosh^2\theta - \varepsilon^2.$$

It follows Eq. (29) that

$$\max|\Omega_0| = \begin{cases} \frac{2|\nu|}{|\sigma|}\sqrt{1-\nu^2\frac{\varepsilon^2}{\sigma^2}} & \text{for } |\nu| < \frac{|\sigma|}{\sqrt{2}|\varepsilon|}, \\ \frac{1}{|\varepsilon|} & \text{for } \frac{|\sigma|}{\sqrt{2}|\varepsilon|} \leqslant |\nu| < \frac{|\sigma|}{|\varepsilon|}. \end{cases} \tag{30}$$

In the second case, the profile of Ω_0 consists of two peaks having the same polarities and separated by the interval depending on the parameter ν.

The substitution of expression (29) into Eqs. (22) defines implicitly the variable Ω_0 of the one-soliton solution of the GRMB system [Eqs. (17)–(20)]. In this case, the first relation in Eqs. (22) gives

$$\tau = 2T + \frac{\varepsilon}{2\sigma}\ln\left[\frac{\sigma - \nu\varepsilon\tanh\theta}{\sigma + \nu\varepsilon\tanh\theta}\right] + \varepsilon^2 W_0^{(0)} Z.$$

It follows from this equality that the one-soliton solution of the GRMB system is steady state.

We have from Eqs. (21), (22), and (29) that

$$\max|\Omega| = \frac{|\nu|}{\sqrt{\sigma^2 - \nu^2\varepsilon^2}} \tag{31}$$

for the one-soliton solution of the GRMB system. It is assumed here that $|\nu| < |\sigma/\varepsilon|$. The single expression for the amplitude of the variable Ω [compare Eqs. (30) and (31)] is a result of the change of the sign by the square root in Eqs. (22) between the peaks of the variable Ω_0.

The profiles of the variable Ω of the one-soliton solutions of the GRMB system [Eqs. (17)–(20)] are presented in Fig. 3. It is seen that the amplitude of the soliton is not proportional to its inverse duration as it takes place in the case of the RMB system. Also, the amplitude of the variable Ω tends to infinity when $|\nu| \to |\sigma/\varepsilon|$ [see Eq. (31)].

The expression for the variable Ω_0 of the two-soliton solution of the modified RMB system [Eqs. (23)–(26)] can be written in the following form:

$$\Omega_0 = \frac{1}{\sigma}\frac{\partial}{\partial T}\left(\arctan\frac{\nu_+\sinh\theta_-}{\nu_-\cosh\theta_+} + \arctan\frac{\nu_+[\eta_-\sinh\theta_- - 2\nu_-\varepsilon\sigma\cosh\theta_-]}{\nu_-[\eta_+\cosh\theta_+ - 2\nu_+\varepsilon\sigma\sinh\theta_+]}\right), \tag{32}$$

where

$$\nu_\pm = \frac{\nu_1 \pm \nu_2}{2}, \quad \theta_\pm = \frac{\theta_1 \pm \theta_2}{2}, \quad \eta_\pm = \sigma^2 \pm \nu_1\nu_2\varepsilon^2,$$

$$\theta_{1,2} = 2\nu_{1,2}\left(T - \frac{W_0^{(0)}Z}{\nu_{1,2}^2 + \omega_0^2}\right) + \theta_{1,2}^{(0)} + ik_{1,2}\pi$$

($\nu_{1,2}$ and $\theta_{1,2}^{(0)}$ are the real constants, and parameters k_1 and k_2 are equal to 0 or 1). This solution describes the collision of the solitons considered above of the modified RMB equations. The shifts of the independent variables allow us to put $\theta_1^{(0)} = \theta_2^{(0)} = 0$ without loss of generality.

The implicit definition of the variable Ω of the two-soliton solution of the GRMB system is obtained by the substitution of expression (32) into Eqs. (22). We assume that $|\nu_{1,2}| < |\sigma/\varepsilon|$ for the two-soliton solution to be well-defined. Consider the collision of the solitons of the GRMB system [Eqs. (17)–(20)] in detail.

Let $(-1)^{k_1+k_2}\nu_1\nu_2 < 0$. In this case, the two-soliton solution describes an interaction of the solitons of the same polarities. The character of such an interaction is similar to that for the solitons of, e.g., the RMB system, the Korteweg-de Vries equation, or modified Korteweg-de Vries equations [39,40].

Now, let $(-1)^{k_1+k_2}\nu_1\nu_2 > 0$. The two-soliton solution of the GRMB system [Eqs. (17)–(20)] describes here an interaction of solitons with opposite polarities. When the amplitudes of the variable Ω of the solitons are much smaller than $1/|\varepsilon|$ [i.e., $|\nu_1|$, $|\nu_2| \ll |\sigma/\sqrt{2}\varepsilon|$; see Eq. (31)], this interaction is similar to that for the modified Korteweg-de Vries or RMB equations and leads to an appearance of the pulse of the variable Ω, whose amplitude is equal nearly to a sum of the amplitudes of the colliding solitons [39,40]. However, if at least one of the amplitudes of the variable Ω of the solitons is close enough to $1/|\varepsilon|$ (i.e., $|\nu_1| \approx |\sigma/\sqrt{2}\varepsilon|$ or/and $|\nu_2| \approx |\sigma/\sqrt{2}\varepsilon|$), then the collision of the well-defined solitons is accompanied by an appearance of the short-living pulse having extraordinarily large amplitude or even leads to the blow-up of the two-soliton solution.

Figure 4 shows the main phases of the collision of the solitons of the GRMB system [Eqs. (17)–(20)] in the case of opposite polarities. Here, the amplitude of the short-living pulse of the variable Ω that appears under the interaction of solitons exceeds significantly the sum of the amplitudes of the colliding solitons [Fig. 4(b)]. The dynamics of such short-living pulses resembles that of the rogue waves [41–44]. Note that $\omega_{\text{ef}} > 0$ in the locations of each of the solitons before their collision. An emergence of the short-living pulse with large amplitude is followed in its location by a drastic change of the state of the medium since $\omega_{\text{ef}} < 0$ there. Such character of the soliton interaction can be used for strengthening and shortening of the pulses.

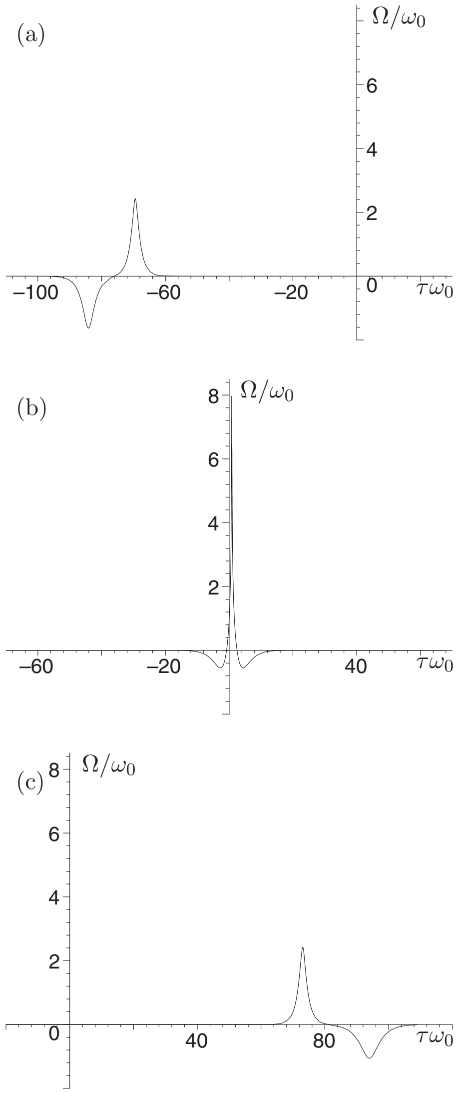

FIG. 4. Profiles of the variable Ω of two-soliton solutions with parameters $\beta = -1/8\omega_0$, $W_0^{(0)} = -1/2$, $k_{1,2} = 0$, $\nu_1 = 0.65\omega_0$, $\nu_2 = 0.5\omega_0$, $Z = 90\omega_0$ (a), $Z = -0.95\omega_0$ (b), and $Z = -90\omega_0$ (c).

The variable Ω_0 of the breather solution of the modified RMB system is defined as follows:

$$\Omega_0 = \frac{1}{\sigma}\frac{\partial}{\partial T}\left(\arctan\frac{\nu_R \sin\theta_I}{\nu_I \cosh\theta_R} + \arctan\frac{\nu_R[(\sigma^2 - |\nu|^2\varepsilon^2)\sin\theta_I - 2\nu_I\varepsilon\sigma\cos\theta_I]}{\nu_I[(\sigma^2 + |\nu|^2\varepsilon^2)\cosh\theta_R - 2\nu_R\varepsilon\sigma\sinh\theta_R]}\right), \tag{33}$$

where $|\nu|^2 = \nu_R^2 + \nu_I^2$. Substituting expression (33) into Eqs. (22), we obtain an implicit definition of the variable Ω of the breather solution of the GRMB system. The plot of the variable Ω of the breather solution is presented in Fig. 5. The oscillations having a sharp form can appear here as well as for the breathers of the previous case.

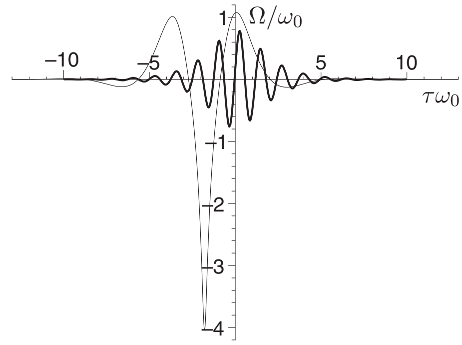

FIG. 5. Profiles of the variable Ω of breather solutions with parameters $\beta = -1/8\omega_0$, $W_0^{(0)} = -1/2$, $Z = 0$, $\nu_R = 0.7\omega_0$, $\nu_I = \omega_0$ (normal line), and $\nu_I = 5\omega_0$ (bold line).

C. Case $\beta > 0$ (σ and $i\varepsilon$ are real)

Here $\omega_{\rm ef} > 0$. This causes distinctive features in the structures of the solitons and breathers and also in the dynamics of their collision.

The variable Ω_0 of the one-soliton solution of the modified RMB system can be written in this case as

$$\Omega_0 = \frac{1}{\sigma}\frac{\partial}{\partial T}\arctan\frac{2\sigma^2\exp\theta}{\sigma^2[1-\exp(2\theta)] + \nu^2\tilde{\varepsilon}^2}, \tag{34}$$

where

$$\tilde{\varepsilon} = -i\varepsilon = \sqrt{\frac{\beta}{\omega_0}}.$$

The maximum value of $|\Omega_0|$ is equal to

$$\left|\frac{\nu}{\sigma}\right|\sqrt{\sigma^2 + \nu^2\tilde{\varepsilon}^2}.$$

The substitution of expression (34) into Eqs. (22) gives us an implicit definition of the variable Ω of the one-soliton solution of the GRMB system [Eqs. (17)–(20)]. Here, we have the following expression for the variable τ:

$$\tau = 2T + \frac{\tilde{\varepsilon}}{\sigma}\arctan\frac{\sigma^2[1+\exp(2\theta)] - \nu^2\tilde{\varepsilon}^2}{2\sigma\tilde{\varepsilon}\nu} - \tilde{\varepsilon}^2 W_0^{(0)} Z. \tag{35}$$

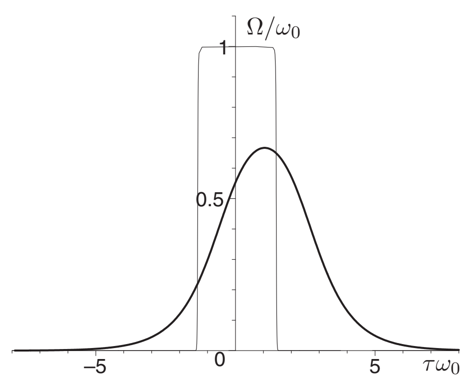

FIG. 6. Profiles of the variable Ω of one-soliton solutions with parameters $\beta = 1/\omega_0$, $W_0^{(0)} = -1/2$, $Z = 0$, $\nu = 100\omega_0$ (normal line), and $\nu = \omega_0$ (bold line).

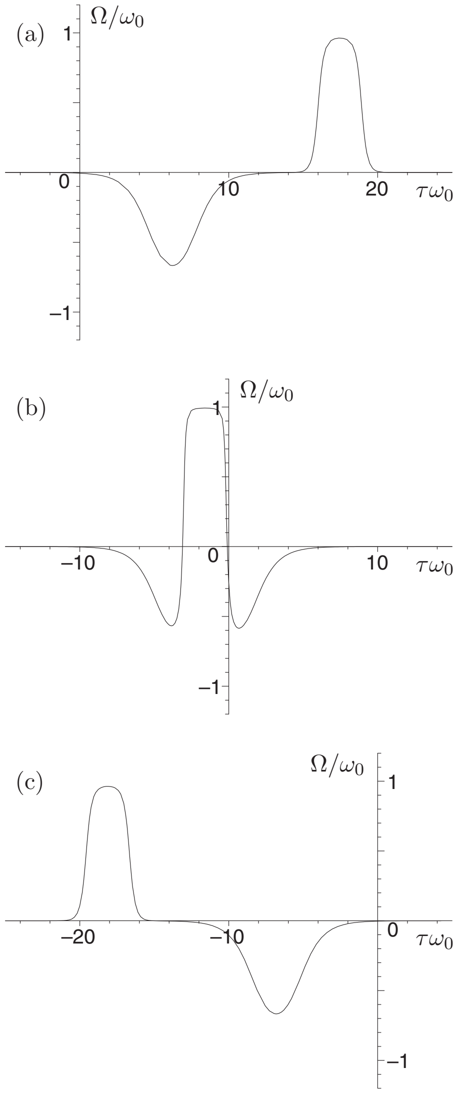

FIG. 7. Profiles of the variable Ω of two-soliton solutions with parameters $\beta = 1/\omega_0$, $W_0^{(0)} = -1/2$, $k_{1,2} = 0$, $\nu_1 = \omega_0$, $\nu_2 = 4\omega_0$, $Z = 20\omega_0$ (a), $Z = 0$ (b), and $Z = -20\omega_0$ (c).

It is seen from this relation that the one-soliton solution of the GRMB system is steady state in the case considered also. The profiles of the variable Ω for different values of the parameter ν are presented in Fig. 6.

It follows from Eq. (35) that the duration of the one-soliton solution tends in the limit $|\nu| \to \infty$ to the minimal value

$$\tau_{\min} = \frac{\pi|\tilde{\varepsilon}|}{|\sigma|},$$

while the amplitude of $|\Omega|$ tends in this limit to its maximum value $1/|\tilde{\varepsilon}|$. Due to these properties, the form of the one-soliton solution becomes "rectangular" if $|\nu|$ increases (see Fig. 6). It may be said that the one-soliton solution of the GRMB system [Eqs. (17)–(20)] has compact support in the limit $|\nu| \to \infty$. The solutions having compact support are known as compactons [45]. Similar compacton-like solutions have been found for different generalizations of the sine-Gordon equation [28,29,34].

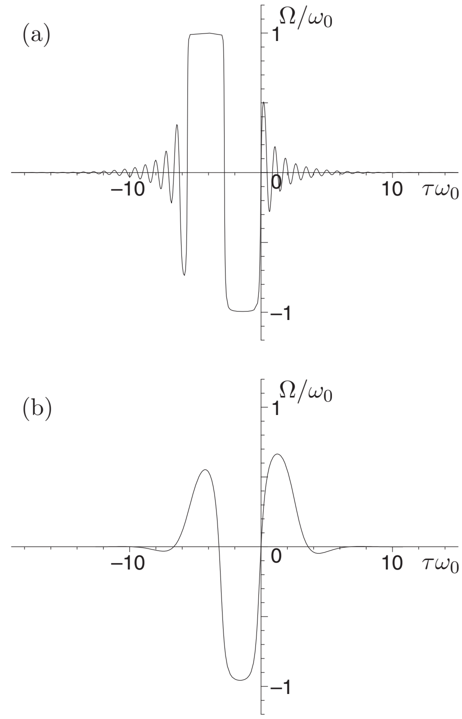

FIG. 8. Profiles of the variable Ω of breather solutions with parameters $\beta = 1/\omega_0$, $W_0^{(0)} = -1/2$, $Z = 0$, $\nu_R = 0.5\omega_0$, $\nu_I = 8\omega_0$ (a), and $\nu_R = \nu_I = \omega_0$ (b).

The variable Ω_0 of the two-soliton solution of the modified RMB system is written in the following manner:

$$\Omega_0 = \frac{1}{\sigma}\frac{\partial}{\partial T}\left(\arctan\frac{\sigma(\nu_1+\nu_2)\tilde{s}_+}{\tilde{r}_-} + \arctan\frac{\sigma(\nu_1+\nu_2)\tilde{s}_-}{\tilde{r}_+}\right), \tag{36}$$

where

$$\tilde{s}_\pm = \sigma[\exp(-\theta_1) - \exp(-\theta_2)] \pm \tilde{\varepsilon}(\nu_1 - \nu_2)\exp(-\theta_1 - \theta_2),$$
$$\tilde{r}_\pm = (\nu_1 - \nu_2)[\sigma^2 + (\sigma^2 - \tilde{\varepsilon}^2\nu_1\nu_2)\exp(-\theta_1 - \theta_2)]$$
$$\pm\sigma\tilde{\varepsilon}(\nu_1+\nu_2)[\nu_1\exp(-\theta_1) - \nu_2\exp(-\theta_2)].$$

Substitution of expression (36) into Eqs. (22) defines implicitly the variable Ω of the two-soliton solution of the GRMB system [Eqs. (17)–(20)]. The profiles of the variable Ω in the case of the collision of solitons having opposite polarities are presented in Fig. 7. The interaction of such solitons can lead to appearance of the "rectangular" pulse [Fig. 7(b)]. The duration and amplitude of this pulse tend to $\tau_{\min}$ and $1/|\tilde{\varepsilon}|$, respectively.

The variable Ω_0 of the breather solution of the modified RMB system is written in the case considered as

$$\Omega_0 = \frac{1}{\sigma}\frac{\partial}{\partial T}\left(\arctan\frac{\nu_R q_-}{\nu_I p_-} - \arctan\frac{\nu_R q_+}{\nu_I p_+}\right), \tag{37}$$

where

$$p_\pm = \sigma + \tilde{\varepsilon}\nu_I \pm 2\tilde{\varepsilon}\nu_R\sin(\theta_I)\exp(-\theta_R) + (\sigma - \tilde{\varepsilon}\nu_I)\exp(-2\theta_R),$$
$$q_\pm = p_\pm - \sigma[1 \mp 2\cos(\theta_I)\exp(-\theta_R) + \exp(-2\theta_R)].$$

Substituting expression (37) into Eqs. (22), we obtain implicit definition of the variable Ω of the breather solution of the GRMB system. The plot of the variable Ω of this solution is presented in Fig. 8. The form of two oscillations in the center of the breather becomes "rectangular" if $|\nu_I| > |\sigma/\tilde{\varepsilon}|$ and $\nu_R \to 0$ [see Fig. 8(a)]. The period of the "rectangular" oscillations is equal approximately to $\tau_{\min}$, while their amplitude tends to the maximum value $1/|\tilde{\varepsilon}|$.

V. CONCLUSION

In this study, the physical generalization of the RMB system was carried out by the refusal from the model of the two-level medium. It is important that this generalization is also integrable by the inverse scattering transformation method. The soliton and breather solutions of the GRMB system [Eqs. (17)–(20)] investigated here differ significantly from the corresponding solutions of the RMB system. We conclude that the existence in the case of the multilevel medium of the short-living pulses with extraordinarily large amplitude, which appear under the soliton collisions, and "rectangular" solitons or breathers is caused by the taking into account of the additional levels.

The properties of the solitons and breathers of the GRMB system in cases B and C are similar to those of the generalized sine-Gordon equation considered in Refs. [28,29]. This is a consequence of using models with remote quantum levels in both the cases. The solution in case A has no the counterparts among the solutions of the generalized sine-Gordon equation. This is a result of the application under its derivation of the approximation of sudden perturbations. This approximation implies that the frequency ω_0 is small enough for condition (1) to be valid. Then, the condition on parameter β corresponding to case A becomes unfeasible.

The profiles of the few-cycle breathers of the GRMB system with $\nu_I = \omega_0$ are represented in Figs. 2, 5, and 8(b). In this case, the spectrum of the breathers captures the quantum transition. An interaction of the pulses with the medium becomes resonant as a result. This allows us to remark on the SIT effect for the few-cycle pulses. It is extremely important to note in this regard that this effect is not considered in the limit $\nu_I \gg \omega_0$, in which the generalized sine-Gordon equation can be applied. On the other hand, the few-cycle resonant breathers of the GRMB system differ significantly from the ones of the RMB system. Indeed, we have breathers of the GRMB system with pointed ($\beta < 0$) and blunted ($\beta > 0$) oscillations. Such kinds of the few-cycle breathers are not observed in the case of the resonant effect of the SIT in the two-level medium.

It is of significant interest to consider the subsequent physical generalizations of the GRMB system and to investigate their integrability and the soliton solutions. For example, the effects of the permanent dipole moment, anisotropy, and inhomogeneous broadening of the quantum transitions can be taken into account.

ACKNOWLEDGMENT

This work was supported by the Russian Science Foundation (Project No. 17–11–01157).

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