Distributing graph states over arbitrary quantum networks

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Multipartite entangled states are great resources for quantum networks. In this work, we study the distribution, or routing, of entangled states over fixed, but arbitrary, physical networks. Our simplified model represents each use of a quantum channel as the sharing of a Bell pair; local operations and classical communications are considered to be free. We introduce two protocols to distribute, respectively, Greenberger-Horne-Zeilinger (GHZ) states and arbitrary graph states over arbitrary quantum networks. The GHZ states' distribution protocol takes a single step and is optimal in terms of the number of Bell pairs used; the graph states' distribution protocol uses, at most, twice as many Bell pairs and steps as the optimal routing protocol for the worst-case scenario.

I. INTRODUCTION AND SETTING

Classical networks are actively present in many areas of day-to-day life. Whether on a global scale with worldwide communication or on a much smaller scale with multiprocessor devices distributing computation over several processors [1], these networks distribute, share, and use information. The study of quantum networks is a recent active field of quantum information with promising applications. These include secure communication, clock synchronization, exponential gains in communication complexity, distributed sensing to delegated computation in the cloud, distributed computation, and more. Distributing quantum states over all kinds of quantum networks is a necessary step to implement most of these applications and will consume quantum resources which may be difficult to replenish. It is thus necessary to find ways to distribute quantum states while minimizing the cost. Until recently, most of the work published about quantum networks and entanglement routing concerned point-to-point communications and bipartite setting [2–5], with a few recent exceptions [6–11].

As a simplified model, here we represent a quantum network by a graph, an example of which is depicted in Fig. 1. Nodes (dotted circles) represent physical locations in the network. Within these nodes, local computations (restricted to Clifford operations [12] in our protocols) are considered free. Quantum channels between nodes are represented as shared Bell pairs—pictured as solid vertices with edges between them. We consider classical communications to be free, and hence the Bell pairs can be considered as single uses of the quantum channel. Note that each node may contain several qubits, but each edge corresponds to a single Bell pair. These Bell pairs can be replenished at each step in our protocols, but only along the original edges of the network (representing the physical quantum channels). Our goal is to distribute entangled states across this network in a way that is most efficient in terms of the number of Bell pairs consumed and the number of steps taken.

The use of such a network to create a maximally entangled bipartite state between two distant nodes for point-to-point communications has been extensively studied in the literature [2–5]. This network can also be used to share multipartite entangled states, either by simultaneously sharing several entangled pairs between different sets of clients [13,14] or by sharing a genuinely multipartite state—a useful resource for quantum communication [15,16], computing [17–19], or metrology [20] protocols.

In this work, we will study the distribution of graph states—a large class containing many useful multipartite entangled states [15,21]—over networks of arbitrary topology. To simplify the study, we ignore the cost of classical communications and we also neglect the processing time of the local quantum processors and the cost of memory. We will also assume the distribution of Bell pairs to be perfect and to occur at perfectly synchronized times, and the node local computations to be similarly perfect. Note that operations between several qubits of the same node are considered here to be local; a wider range of graph-state transformations are thus available compared to other works tackling the manipulation of graph states [10,12].

The distribution of multipartite entangled states over quantum networks has also been studied in [6–11]. In [6], the authors investigate the creation of a graph state presenting the shape of the network in the presence of noise. References [7,8] present decomposition of graph states into various building blocks that can be purified and merged to construct graph states over a network. Reference [10] studies the possible transformations of an already shared graph state, with a single qubit per location. Reference [9] presents a modular architecture to fulfill graph-state creation requests. As in Ref. [11], here we outline the complete process of sharing entangled states from scratch. Recently, we noticed the publication of the independent work of Pirker and Dür [11], which includes a protocol very similar to ours. The modeling of the network in both works is different, as well as the optimized metrics. They describe a hierarchical network stack and use it to provide robustness against router or subnetwork failures, which we do not study. However, they do not have a cost metric, be it in entanglement use or time, and do not provide a statement of optimality as we do here.
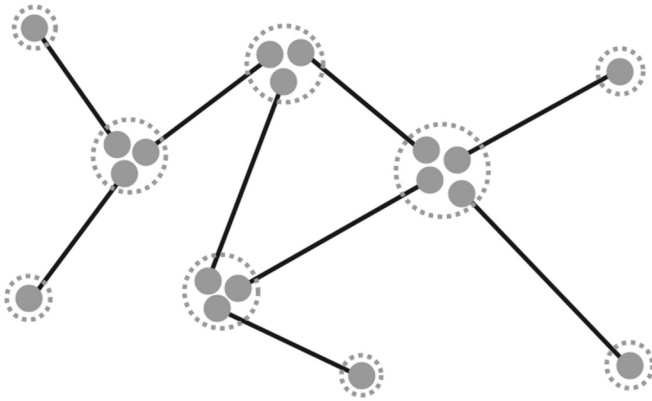
FIG. 1. A quantum network, where each vertex represents a qubit and each edge represents a Bell pair; each dotted circle represents a node of the network. Note that a node can hold several qubits.

The article is organized as follows. We begin in Sec. II by giving background on graph states and the graphical tools we use. Then, in Sec. III, we propose several protocols to distribute multipartite states over arbitrary quantum networks using only basic operations of graph states, starting in Sec. III A with a minimal protocol distributing Greenberger-Horne-Zeilinger (GHZ) states [22]. In Sec. III B, we then show a method to distribute arbitrary graph states by distributing several GHZ states over the network. Then, in Sec. III C, we analyze the optimality of these protocols. In Sec. IV, we briefly discuss the different imperfections overlooked in our idealized model and suggest ways to cope with them. We close in Sec. V with discussions.

II. GRAPH STATES AND GRAPHICAL TOOLS

The graph state $|G\rangle$ associated to the simple graph $G = (V, E)$, with vertices $a \in V$ and edges $a, b \in E$, is

$$|G\rangle := \prod_{(a,b)\in E} CZ_{a,b} |+\rangle^V , \quad \text{where} \quad |+\rangle^V := \bigotimes_{a\in V} |+\rangle^a \quad (1)$$

is the tensor product of all qubits of V in the state $|+\rangle^a := (|0\rangle^a + |1\rangle^a)/\sqrt{2}$, and $CZ_{a,b}$ is the controlled-Z operation between qubits a and b. That is, for a graph G, each vertex represents a qubit and each edge a CZ entangling operation. Note that the quantum network represented in Fig. 1 can be understood as a collection product of bipartite graph states, where each of the pairs represented is a Bell pair.

The work presented in this article is based on the graphical representation of operations and measurements. Indeed, several physical operations on graph states $|G\rangle$ can be represented as graph operations on G (up to local corrections, that we will neglect here and in the following). In particular, we will use three elementary graph operations as building blocks for our protocols [12,23]:

(i) Vertex deletion. This operation removes one vertex and all the associated edges from the graph. Physically, it is implemented by the Pauli measurement of the relevant qubit in the Z basis.

(ii) Local complementation on a vertex. This graph operation inverts the subgraph induced by the neighborhood N_a of the concerned vertex a—the set of vertices adjacent to a (see
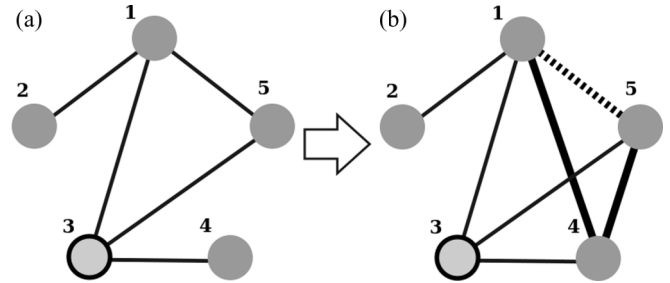

FIG. 2. Example of the application of local complementation applied on vertex 3. The associated physical operation is written as $U_3^\tau = e^{-i\frac{\pi}{4}X_3} \otimes e^{i\frac{\pi}{4}Z_1} \otimes e^{i\frac{\pi}{4}Z_4} \otimes e^{i\frac{\pi}{4}Z_5}$. The edge between 1 and 5 is removed and two edges are created between both the pairs (1,4) and (4,5).

Fig. 2). It is implemented by applying the relevant operation to the qubits of $a \cup N_a$, described by the quantum operator $U_a^\tau := e^{-i\frac{\pi}{4}X_a} \bigotimes_{b\in N_a} e^{i\frac{\pi}{4}Z_b}$ acting on $|G\rangle$.

(iii) Edge addition (deletion). By applying a controlled-Z operation between two qubits belonging to the same node, we create (delete) an edge between two nonadjacent (adjacent) vertices.

Another useful, if nonelementary operation is the measurement of a qubit in the Y basis, which corresponds graphically to a local complementation followed by the removal of the measured vertex. To see this, we note that the local complementation operations implement a basis change from Z to a Y on the concerned vertex.

As an example, Fig. 3 shows how entanglement swapping along a line of repeaters [2–4] can be depicted graphically with the above tools. The essential observation here is that a Bell measurement is equivalent to performing a CZ gate followed by two single-qubit Y measurements.

III. GRAPH-STATE DISTRIBUTION PROTOCOLS

We will now see protocols distributing graph states, starting with a minimal protocol to distribute GHZ states over an arbitrary quantum network.

A. GHZ-state distribution

GHZ states form a useful class of multipartite maximally entangled states. They are used in many multiparty
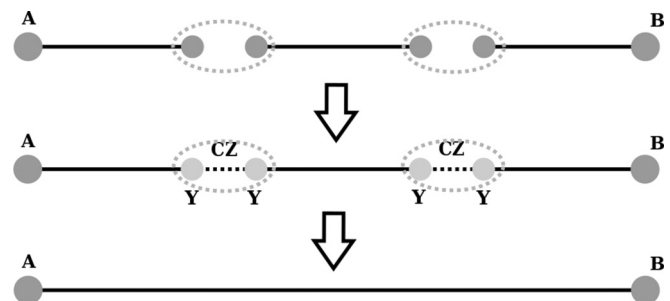

FIG. 3. Graphical representation of the quantum repeater protocol. Starting from a repeater line, we apply CZ and two measurements in the Y basis at each repeater to obtain a Bell pair between the end nodes.
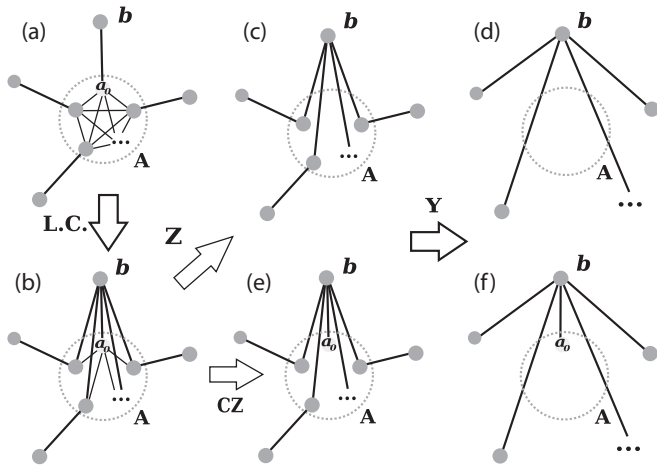
FIG. 4. The star expansion operation: (a) all qubits a_i, $i \geqslant 0$ of A are linked using CZ between all possible pairs; (b) local complementation is applied to the qubit a_0 linked to b; (c) if $A \notin W$, we remove this qubit and all edges within A by Z-measuring it; (e) else, when $A \in W$, we keep a_0 and apply CZ gates to remove all edges within A; (d),(f) finally, a Y measurement of all other qubits $a_i \in A$, $i > 0$ creates the desired star graph.

applications of quantum information such as quantum secret sharing [24] or quantum metrology [25]. Thus, establishing a protocol to distribute them is an important step toward the implementation of multipartite protocols over a quantum network. A N-GHZ state is written $|N\text{-GHZ}\rangle = (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$. It is locally equivalent to $(|0\rangle |+\rangle^{\otimes N-1} + |1\rangle |-\rangle^{\otimes N-1})/\sqrt{2}$, the star graph with an edge linking the first qubit—the center—to each of the others. The choice of which vertex is the center is arbitrary and can be changed by local operations such as two successive local complementations. We will distribute the star graph on the network using our graphical rules. Given an arbitrary set W of the network's nodes, we will now see how to distribute a star graph among all the nodes of W. The amount of Bell pairs consumed is minimal and we can distribute it in one time step.

This protocol relies on an operation we call *star expansion*, which acts on the qubit b of a graph state—b will be the center of a star in our case—and a node A of the network which contains a qubit $a_0 \in A \cap N_b$ in the neighborhood of b. Each of the other qubits $a_i \in A$, $i > 0$ of A constitutes a Bell pair with a qubit c_i in another node of the network. The star expansion operation, detailed in Fig. 4, uses the Bell pairs of the node A to add the edges (b, c_i) to the graph state, as well as the edge (b, a_0) iff $A \in W$.

The star expansion subprotocol defined above will help us to share the star graph state across the full set W. The first step is to find a minimal tree covering all the nodes of W—i.e., a subgraph connecting all the nodes in W with the minimum number of edges. Such a problem is *the Steiner tree problem*, well known in classical graph theory. Despite being NP-hard [26], the Steiner tree can be approximated in a polynomial time [27,28]. See Fig. 5 for an example Steiner tree defined on the network of Fig. 1.

Starting from any leaf ℓ of this tree, we can distribute the star graph by exploring the tree and applying star expansion
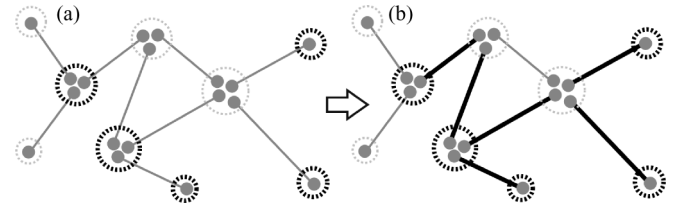

FIG. 5. Steiner tree example for a set of nodes W. The network vertices and edges are gray. The nodes of W are represented by a black-dotted circle. (a) The original network (gray vertices and edges); (b) the associated Steiner tree for the set W (black edges).

with the exploration's current node (a nonleaf neighbor of ℓ) as A, and ℓ as b. This process is depicted in Fig. 6.

The number of Bell pairs consumed in this process is equal to the number of edges in the tree, which by definition is (almost) the minimum possible number for the (approximate) Steiner tree. Since star expansion requires local operations to each node and the same Clifford operations at ℓ each time step, these commute and can all be done in one step, with a single step of correction afterwards [29].

B. Arbitrary graph-state distribution

We now show how to generalize the previous approach to distribute an arbitrary graph state over a set W of known nodes of the network. The procedure will be to distribute a specific resource graph state: the edge-decorated complete graph. From this graph state, nodes can construct any graph state by measuring each edge qubit in either the Z basis or the Y basis, as represented in Fig. 7. This graph is already used in [9] for a similar goal, with a protocol to distribute it in a different context. We present here a different approach to its distribution, adapted to our setting, as well as compute and optimize its cost in terms of resources.

The protocol to distribute the edge-decorated complete graph follows directly from multiple applications of the GHZ-state distribution. We consider the set W of the k participating
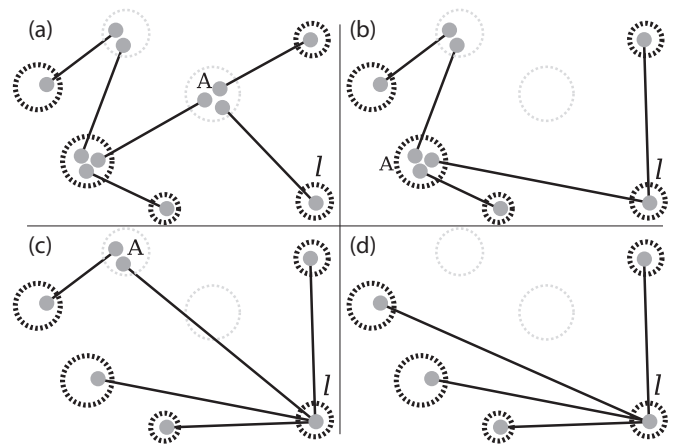

FIG. 6. Distribution of a star graph state. We distribute the star graph over the nodes of W, represented by black-dotted circles. To explore the tree, we arbitrarily choose a nonleaf neighbor of ℓ and apply star expansion on that node A (taking ℓ as Bob). We repeat this until we arrive at the desired star over W.
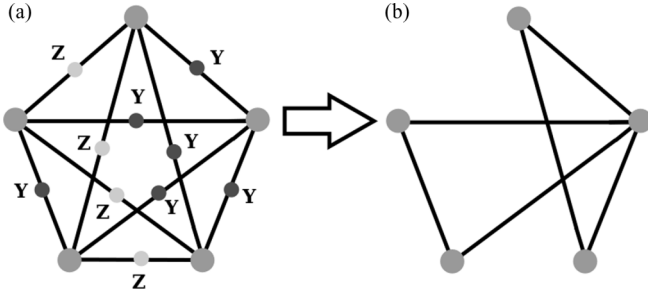
FIG. 7. The edge-decorated graph can be projected into any graph state by measuring its edge qubits. We can distribute it as a resource to generate an arbitrary graph state.

nodes. The first step is solving the *Steiner's tree problem* on the network for the k nodes. Then, we distribute a k-GHZ state starting from one arbitrary leaf ℓ_1. We delete vertices from the tree in order to have the covering tree for the set $W \setminus \{\ell_1\}$ and we distribute from a leaf ℓ_2 of this tree a $(k-1)$-GHZ state. This procedure iterates until the distribution of a final Bell pair between the two last nodes of W. As seen in Fig. 8, the resulting graph state is locally equivalent to the edge-decorated dotted graph.

Some optimizations are possible if the final graph state is a known quantity before the distribution. We call G the graph representing the graph state to distribute $|G\rangle$. We search to extract from G a star subgraph S_1 of maximum size. We distribute the GHZ state associated to S_1 using the GHZ-state distribution protocol. Then we iterate with $G \setminus S_1$ and so on, until $\bigcup S_i = G$.

C. Optimality

The presented protocol is strictly independent from the network topology and the wanted graph state. The consumption of this general protocol can be compared to the consumption of pathological expensive cases. The most expensive is the case of the line network where we pair each node with its opposite and distribute a Bell pair between them (see Fig. 9).
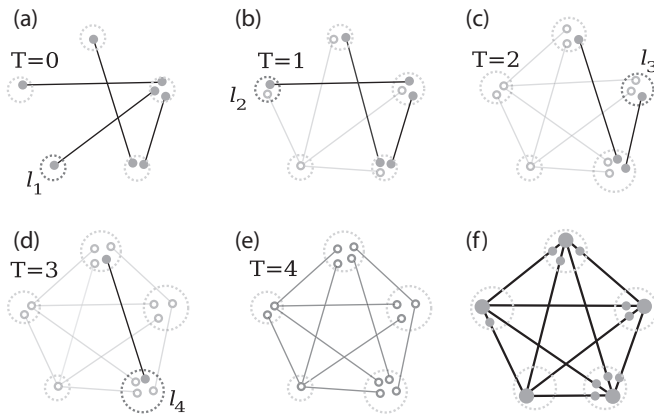

FIG. 8. Distribution of the edge-decorated graph of size 5 from a minimal tree (Steiner tree) starting with (a) at $T = 0$. At each step, a star graph is shared centered at ℓ_i (subsequently indicated in gray), then vertex i is ignored in the following steps. Finally, in (f), node local operations generate the desired edge-decorated graph state.
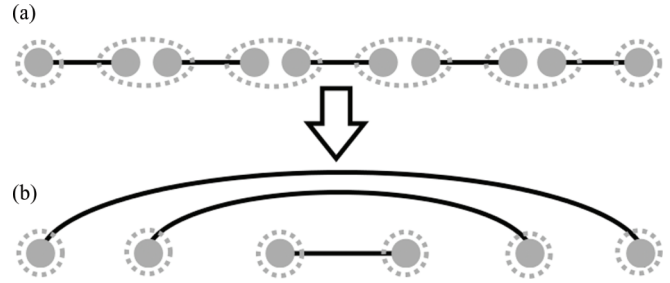

FIG. 9. Depiction of an expensive case. We want to entangle each qubit with the opposite one over a line network.

Our protocol gives a consumption of, at most, $\frac{N(N-1)}{2}$ Bell pairs in $N-1$ time steps. This upper bound is reached when all the network's nodes are part of the graph state. Both costs are equal up to a factor 2 to the cost of the pathological case (see Table I).

IV. IMPERFECTIONS

Up to this point, we have worked in a lossless and noiseless setting. Of course, such idealized setting is far from realistic. The full study of multipartite entanglement distribution in the presence of imperfections would depend on the details of its physical implementation, and is therefore beyond the scope of this article. However, several strategies can be discussed to cope with imperfections.

First, the dominant imperfection in a photonic network is expected to be losses. In a first approximation, they could be modeled by a dynamic network , i.e., a network with a graph changing at each time step, with each lost qubit erasing the corresponding edge. Since our GHZ-state distribution protocol is independent of the topology of the graph, as long as all nodes are connected, it should be quite robust to losses. Of course, if the relevant nodes are in subnetworks disconnected by the losses, no GHZ state can be distributed among them, but the use of quantum memories and the predistribution of "partial" graph states will probably allow one to bridge the gaps in the next round. The recent work of Khatri *et al.* [14] on entanglement percolation lets us hope that such techniques will have modest quantum memory requirements for large-scale quantum networks.

Another important imperfection is the noise itself, which can be introduced by the needed operations or is initially present in the distributed Bell pairs. A first step to reduce the operation-induced noise is to minimize the number of operations, a minimization we have not addressed here. However,

TABLE I. Creation cost on a network of size N, in terms of number of Bell pairs (EPR), and the total number of rounds (T).

		Protocol	Bound
N-GHZ	EPR	$N-1$	$N-1$
	T	1	1
Arbitrary graph state	EPR	$\leqslant \frac{N(N-1)}{2}$	$\lfloor \frac{N}{2} \rfloor^2$
	T	$\leqslant N-1$	$\lfloor \frac{N}{2} \rfloor$

since all operations here are Clifford, this kind of optimization is well known [29]. The noise can also be treated generally with direct purification of the final GHZ or graph states, which has been studied already [30,31]. More generally, many procedures to deal with noise in quantum information, including error correction [17,32] and secret sharing [33,34], are based on Clifford operations and often on graph states [21], and quantum fault-tolerant operations are easier within the Clifford group [35–37]. They are therefore likely to be compatible with our protocol.

One other idealization in our approach is our neglect of the cost of classical communications, which is already known to be a non-negligible overhead in (classical) network management. If the network is static and well known, the Clifford nature of operations allows us to essentially limit the communication to establishing the path and communicating the results of the measurements for the purpose of correction [29]. Classical communications in those cases should not be too onerous in practice, considering a quantum network will likely have much less traffic than the associated classical one. However, when losses are taken into account, the network graph itself becomes dynamic and updating every node about the state of the network will be costly [13]. We hope, however, that quantum secret-sharing techniques [33,34] can be used to mitigate this cost. We also assume that all nodes cooperate and know perfectly the actions of the others. In a more realistic setting, imperfect knowledge could lead to congestion issues on the network, and less altruistic nodes could exploit this.

V. CONCLUSION AND PERSPECTIVES

We have presented protocols for distributing GHZ and arbitrary graph states which work for networks of any topology. These protocols are close to optimal in terms of the number of steps T required and the number of Bell pairs consumed in the worst case.

Our model is naturally quite simplified, and there are many possibilities for trade-offs and improvements even within it. First, we note that the number of steps, T, does not necessarily represent time—for example, if nodes are allowed to share $N(N-1)$ Bell pairs, then everything can be done in one physical time step. One then has a potential trade-off between how parallel uses of the quantum channel can be, and the use of quantum memory. Indeed, in terms of memory, one may tweak the steps in our protocol so as to parallelize as much as possible—with a little thought, one can see that one only needs two qubits of memory per Bell-pair vertex per node at any one step. Furthermore, our optimality is for the worst-case topology and state—for a fixed graph state and topology, one may do much better (e.g., our protocol for the GHZ state). The advantage for our scheme is that it gives a method, and a bound, which works for all states and topology with the same efficiency.

Beyond this, there is much potential for developing more refined models for quantum networks, where one could, for example, consider memory, local processing or classical communication costs, as well as their mutual interactions. All of these choices can potentially change the optimal strategy and so must be made carefully. We leave this for future work.

[1] V. S. Sunderam, Concurr. Pract. Experience **2**, 315 (1990).

[2] E. Schoute, L. Mancinska, T. Islam, I. Kerenidis, and S. Wehner, arXiv:1610.05238.

[3] A. Acín, J. I. Cirac, and M. Lewenstein, Nat. Phys. **3**, 256 (2007).

[4] S. Perseguers, J. I. Cirac, A. Acín, M. Lewenstein, and J. Wehr, Phys. Rev. A **77**, 022308 (2008).

[5] S. Pirandola, arXiv:1601.00966.

[6] M. Epping, H. Kampermann, and D. Bruß, New J. Phys. **18**, 053036 (2016).

[7] M. Cuquet and J. Calsamiglia, Phys. Rev. A **86**, 042304 (2012).

[8] Y. Matsuzaki, S. C. Benjamin, and J. Fitzsimons, Phys. Rev. Lett. **104**, 050501 (2010).

[9] A. Pirker, J. Wallnöfer, and W. Dür, New J. Phys. **20**, 053054 (2018).

[10] F. Hahn, A. Pappa, and J. Eisert, npj Quantum Inf. **5**, 76 (2019).

[11] A. Pirker and W. Dür, New J. Phys. **21**, 033003 (2019).

[12] M. Van den Nest, J. Dehaene, and B. De Moor, Phys. Rev. A **69**, 022316 (2004).

[13] M. Pant, H. Krovi, D. Towsley, L. Tassiulas, L. Jiang, P. Basu, D. Englund, and S. Guha, npj Quantum Inf. **5**, 25 (2019).

[14] S. Khatri, C. T. Matyas, A. U. Siddiqui, and J. P. Dowling, Phys. Rev. Res. **1**, 023032 (2019).

[15] P. Agrawal and A. Pati, Phys. Rev. A **74**, 062320 (2006).

[16] S.-B. Zheng, Phys. Rev. A **74**, 054303 (2006).

[17] D. Schlingemann and R. F. Werner, Phys. Rev. A **65**, 012308 (2001).

[18] R. Raussendorf, D. E. Browne, and H. J. Briegel, Phys. Rev. A **68**, 022312 (2003).

[19] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, Nat. Phys. **5**, 19 (2009).

[20] I. L. Chuang, Phys. Rev. Lett. **85**, 2006 (2000).

[21] M. Hein, W. Dür, J. Eisert, R. Raussendorf, M. Van den Nest, and H. J. Briegel, in *Quantum Computers, Algorithms and Chaos*, Proceedings of the International School of Physics "Enrico Fermi", edited by G. C. D. L. Shepelyansky, P. Zoller, and G. Benenti (IOS Press, Amsterdam, and Società Italiana Di Fisica, Bologna, 2006), Vol. 162.

[22] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*,

edited by M. Kafatos (Kluwer Academic, Dordrecht, The Netherlands, 1989) pp. 69–72.

[23] D. E. Browne and H. J. Briegel, in *Lectures on Quantum Information*, edited by D. Bruß and G. Leuchs (Weinheim, Germany, 2007), Chap. 20.

[24] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A **59**, 1829 (1999).

[25] W. Dür, M. Skotiniotis, F. Fröwis, and B. Kraus, Phys. Rev. Lett. **112**, 080801 (2014).

[26] F. K. Hwang, D. S. Richards, and P. Winter, *The Steiner Tree Problem*, Annals of Discrete Mathematics, Vol. 53 (North Holland, Amsterdam, 1992).

[27] G. Robins and A. Zelikovsky, SIAM J. Discrete Math. **19**, 122 (2005).

[28] P. Berman, M. Karpinski, and A. Zelikovsky, in *Workshop on Algorithms and Data Structures* (Springer, New York, 2009), pp. 86–97.

[29] D. Browne, E. Kashefi, and S. Perdrix, in *Theory of Quantum Computation, Communication, and Cryptography*, Lecture Notes in Computer Science, Vol. 6519, edited by W. van Dam, V. M. Kendon, and S. Severini (Springer, Berlin, 2011), pp. 35–46 .

[30] C. Kruszynska, A. Miyake, H. J. Briegel, and W. Dür, Phys. Rev. A **74**, 052316 (2006).

[31] H. Aschauer, W. Dür, and H.-J. Briegel, Phys. Rev. A **71**, 012319 (2005).

[32] D. Gottesman, Ph.D. thesis, California Institute of Technology, 1997.

[33] R. Cleve, D. Gottesman, and H.-K. Lo, Phys. Rev. Lett. **83**, 648 (1999).

[34] D. Markham and B. C. Sanders, Phys. Rev. A **78**, 042309 (2008).

[35] P. W. Shor, in *Proceedings of 37th Conference on Foundations of Computer Science* (IEEE, Los Alamitos, California, 1996), pp. 56–65.

[36] J. Preskill, J. Proc. R. Soc. A **454**, 385 (1998).

[37] D. Gottesman, in *Group 22: Proceedings of the 12th International Colloquium on Group Theoretical Methods in Physics*, edited by S. Corney, R. Delbourgo, and P. Jarvis (International Press of Boston, Boston, 1999), pp. 32–43.