Tripartite non-maximally-entangled mixed states as a resource for optimally controlled quantum teleportation fidelity

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Three-qubit mixed states are used as a channel for controlled quantum teleportation (CQT) of single-qubit states. The connection between different channel parameters to achieve maximum controlled teleportation fidelity is investigated. We show that for a given multipartite entanglement and mixedness a class of non-maximally-entangled mixed X states (X-NMEMS) achieves optimum controlled quantum teleportation fidelity; interestingly a class of maximally entangled mixed X states (X-MEMS) fails to do so. This demonstrates, for a given spectrum and mixedness, that X-MEMS are not sufficient to attain optimum controlled quantum teleportation fidelity, which is in contradiction with the traditional quantum teleportation of single qubits. In addition, we show that biseparable X-NMEMS, for a certain range of mixedness, are useful as a resource to attain high controlled quantum teleportation fidelity, which essentially lowers the requirements of quantum channels for CQT.

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I. INTRODUCTION

Quantum teleportation is the process of transferring quantum states across two parties separated by large distance without traversing the actual distance between them [1]. In the celebrated teleportation protocol, a single qubit's state is teleported between two parties, where the maximally entangled bipartite pure state shared by both parties acts as a quantum channel for the process. Teleportation fidelity determines the success of quantum teleportation; it is defined as the overlap of the state to be teleported and the output state at the receiver's end. It can be considered as an ascribed characteristic of the quantum channel used for the teleportation of an arbitrary quantum state. For a pure quantum channel, the existence of a monotonic relationship between entanglement and teleportation fidelity is well known [2–4]. In reality, quantum systems are open; interaction of the system with surroundings changes the properties of quantum states in general. Hence, the exploration of quantum states in noisy environments for implementing various quantum information processing protocols has attracted wide attention. In [2,3], it is shown that mixed quantum states can also be used as a channel to achieve imperfect teleportation. In the case of a mixed entangled teleportation channel, there exists no monotonic relationship between entanglement and teleportation fidelity, i.e., a higher value of entanglement of the quantum channel is not sufficient to achieve maximum fidelity [5]. The connection among different parameters of the quantum channel [2,3,6–8] should be known for the wise usage of channels for quantum teleportation under the effects of noise. For mixed quantum channels, both mixedness and entanglement contribute to the success of teleportation [5]. Manipulation of multipartite qubits [9–13] is an important task to scale up the quantum based technology efficiently. A multipartite variant of quantum teleportation has been proposed in [14], and it is known as controlled quantum teleportation (CQT). In CQT, an arbitrary single-qubit state is transferred from sender to receiver only with the permission of the controller. The authority power of the controller to decide the success or failure of teleportation for the tripartite CQT protocol shows its difference from the bipartite one. Recently, Barasinski et.al., experimentally implemented controlled quantum teleportation of single-qubit states on linear optical devices [15] and discussed the possibilities of controlled quantum teleportation by lowering the requirements of quantum channels. Conditioned and nonconditioned fidelity are two quantities that are measured with and without the permission of the controller, characterizing the CQT protocol. It is assumed that, in CQT, F_{CQT} (conditioned fidelity) should be always greater than the classical limit, whereas the value of F_{NC} (nonconditioned fidelity) [14,16,17] cannot exceed the classical limit $\frac{2}{3}$ ($F_{\mathrm{NC}} \leqslant \frac{2}{3}$). The classical limit of nonconditioned fidelity is calculated for the set of pure input states that are chosen according to the Haar measure [18]. The control power (CP), a quantity to define the authority of the controller in CQT, is estimated as the difference of conditioned and nonconditioned fidelity. As is known, for bipartite quantum states, purity of the quantum channel along with entanglement [5,19,20]

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plays a significant role in the implementation of the quantum teleportation process with maximum achievable fidelity. Different classes of states are considered as quantum channels for teleportation, among which a class of X states, having nonzero diagonal and antidiagonal elements, deserves special attention [21–23]. In the case of the bipartite qubit system, a given density matrix can be unitarily transformed to the X state with the same degree of entanglement and spectrum [24,25]. Thus quantum states in X structure form an important class of density matrices in general and are used as a representative class of states for quantum information processing. The use of tripartite quantum states as a channel for controlled quantum teleportation and the estimation of controlled teleportation fidelity for X states are shown in [26]. Both maximally and non-maximally-entangled pure Greenberger-Horne-Zeilinger (GHZ)-like states act as quantum channels for CQT. In [27] it is shown how genuine multipartite entanglement (GME) and CP affect the controlled quantum teleportation fidelity for a class of X states. Purity of tripartite quantum states is an important parameter that affects quantum correlations, and investigation of the efficacy of tripartite quantum states for CQT will not be conclusive without accounting for the purity of the quantum channel along with other channel parameters. We fill this gap by a detailed investigation on the performance of the mixed quantum channel for controlled quantum teleportation. We systematically investigate the roles played by various parameters, like purity, entanglement, and control power of tripartite qubit states, in achieving optimum controlled quantum teleportation fidelity (F_{CQT}). For this purpose, we consider different classes of multipartite X states and analyze their performance as CQT channels. First, we examine the faithfulness of a class of rank dependent maximally entangled mixed X states (X-MEMS), defined for a given spectrum of eigenvalues and linear entropy as a CQT resource. Since the performance of X-MEMS as a CQT channel is not optimum, a class of tripartite non-maximally-entangled mixed X states (X-NMEMS) is constructed and its teleportation fidelity is estimated. We show that our class of X-NMEMS outperforms X-MEMS as a quantum channel for CQT and rank-2 X-NMEMS give maximum achievable teleportation fidelity for a given entanglement and mixedness as shown in [27]. This clearly demonstrates that CQT protocol lowers the requirements of the quantum channel for the successful quantum teleportation of a single qubit's state. At a high value of mixedness, X-NMEMS become biseparable. Even with the biseparability condition, X-NMEMS are found to give high values for controlled quantum teleportation fidelity above the classical limit. This high value of fidelity of the biseparable quantum channel is a direct evidence that mixed tripartite quantum states can lower the requirements of the quantum channel for successful controlled teleportation. From our investigation on tripartite mixed quantum channels, we show that tripartite X-MEMS are not sufficient to achieve optimum CQT fidelity, whereas optimum controlled quantum teleportation fidelity is achieved using a class of X-NMEMS. Even though genuine multipartite entanglement of X-NMEMS vanishes for high values of mixedness, the process of controlled quantum teleportation of single-qubit states is enabled by the biseparability nature of X-NMEMS. These results, which lower the requirements of the quantum channel, are quite important for the experimental realization of controlled quantum teleportation in a noisy environment. The present paper is organized as follows. In Sec. II, we discuss the prerequisites for implementing the CQT protocol. Section III contains two subsections. The first subsection deals with the construction of tripartite qubit X-MEMS and its usefulness for controlled quantum teleportation. It is followed by the construction of a class of X-NMEMS, and its efficacy as a quantum channel for CQT is analyzed in the second subsection. Results and discussion in Sec. IV are followed by the concluding section (Sec. V).

II. PRELIMINARIES

Below, we define different parameters like GME, teleportation fidelity, control power, and linear entropy, which characterize the tripartite mixed entangled quantum channels for controlled quantum teleportation.

A. Genuine multipartite entanglement

The three-qubit symmetric mixed X states [28] are defined with diagonal elements denoted by $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \geqslant 1$ and antidiagonal elements given by $z_1, z_2, z_3, z_4, z_1^*, z_2^*, z_3^*, z_4^*$. The GME of a three-qubit X state is given as

$$C_{\text{GME}} = 2\max\{0, |z_j| - w_j\}, \tag{1}$$

where $\sum_i (a_i + b_i) = 1$ and $w_j = \sum_{k\neq j} \sqrt{a_k b_k}$. The positivity criterion of the X matrix is satisfied with the condition $|z_i| \leqslant \sqrt{a_i b_i}$. The tripartite X states are entangled for $0 < C_{\text{GME}} \leqslant 1$ and C_{GME} is zero for biseparable states [29].

B. Controlled quantum teleportation fidelity

Here, we describe the protocol of controlled quantum teleportation of a single qubit's state via the tripartite qubit channel. Consider that three parties, labeled as A, B, and C, share an entangled three-qubit quantum state ρ_{abc}, which acts as a channel connecting them to each other. Suppose party A wants to teleport an unknown state of qubit d to B with the consent of party C. At this moment controller C makes an orthogonal measurement on his qubit c, with ζ as the measurement outcome. This results in the projection of entangled channel ρ_{abc} onto the two-qubit state [17] ρ_{ab}^{ζ}:

$$\rho_{ab}^{\zeta} = \frac{\text{Tr}_c[1_2 \otimes 1_2 \otimes |\zeta\rangle\langle\zeta| U \rho_{abc} 1_2 \otimes 1_2 \otimes U^\dagger |\zeta\rangle\langle\zeta|]}{\langle\zeta|U\rho_c U^\dagger|\zeta\rangle}. \tag{2}$$

Here 1_2, a 2×2 identity matrix, acts on the qubit's state with observers A and B; U, a 2×2 unitary matrix, along with the projection operation acts on the qubit's state with observer C; and $\rho_c = \text{Tr}_{ab}[\rho_{abc}]$. Following this, party A makes a joint orthogonal measurement on qubits a and d and communicates the results to B, and appropriate unitary operations on qubit b complete the process of CQT. The controlled quantum teleportation fidelity $F_{\text{CQT}}(\rho)$ in this scenario is defined as

$$F_{\text{CQT}}(\rho) = \frac{2\max_U\left[\sum_{\zeta=0}^{1}\langle\zeta|U\rho_c U^\dagger|\zeta\rangle f\left(\rho_{ab}^{\zeta}\right)\right] + 1}{3}. \tag{3}$$

We have nonconditioned teleportation fidelity (without the controller's participation) given as

$$F_{\mathrm{NC}}(\rho) = \frac{2f(\rho_{ab}) + 1}{3}, \tag{4}$$

where $f(\rho)$ is the fully entangled fraction [30–32] and $\langle\zeta|U\rho U^{\dagger}|\zeta\rangle$ is the maximum probability of receiving outcome ζ. The fidelities derived in Eqs. (3) and (4) are estimated for general three-qubit mixed X states [26] as follows:

$$F_{\mathrm{CQT}}(\rho_X) = \max\{F^1_{\mathrm{CQT}}, F^2_{\mathrm{CQT}}, F^3_{\mathrm{CQT}}, F^4_{\mathrm{CQT}}\} \tag{5}$$

where

$$\begin{aligned} F^1_{\mathrm{CQT}} &= \frac{3 + |\Delta_1| + 4(|z_1| + |z_4|)}{6}, \\ F^2_{\mathrm{CQT}} &= \frac{3 + |\Delta_1| + 4(|z_2| + |z_3|)}{6}, \\ F^3_{\mathrm{CQT}} &= \frac{3 + \sqrt{\Delta_2^2 + 16(|z_1| + |z_4|)^2}}{6}, \\ F^4_{\mathrm{CQT}} &= \frac{3 + \sqrt{\Delta_2^2 + 16(|z_2| + |z_3|)^2}}{6}. \end{aligned} \tag{6}$$

Here $\Delta_1 = a_1 - a_2 - a_3 + a_4 + b_1 - b_2 - b_3 + b_4$, $\Delta_2 = a_1 - a_2 + a_3 - a_4 - b_1 + b_2 - b_3 + b_4$. The nonconditioned teleportation fidelity of the state ρ_X is

$$F_{\mathrm{NC}}(\rho_X) = \frac{3 + |\Delta_1|}{6}. \tag{7}$$

The influence of the control qubit in the CQT process is quantified by estimating CP and is defined as

$$\mathrm{CP}(\rho_X) = F_{\mathrm{CQT}}(\rho_X) - F_{\mathrm{NC}}(\rho_X). \tag{8}$$

The two conditions $F_{\mathrm{CQT}}(\rho) > \frac{2}{3}$ and $F_{\mathrm{NC}}(\rho) \leqslant \frac{2}{3}$ should be satisfied by tripartite quantum channels to ensure the active participation of the controller in the controlled quantum teleportation process. Mixedness of quantum states is an important parameter that influences fidelity of controlled quantum teleportation. We use linear entropy to estimate the mixedness of a state, which is defined for a multipartite qubit state ρ as

$$S_L(\rho) = \frac{2^N - 1}{2^N}[1 - \mathrm{Tr}(\rho^2)]. \tag{9}$$

Here N is the number of qubits and $\mathrm{Tr}(\rho^2)$ is the purity of the multipartite quantum state. Mixed states satisfy the condition $0 < S_L(\rho) \leqslant 1$ and $S_L(\rho) = 0$ for pure states.

III. MIXED X STATES: A RESOURCE FOR CONTROLLED QUANTUM TELEPORTATION

In this section, we investigate in detail the mixed three-qubit X states as a resource for controlled quantum teleportation. We show how purity and other quantum correlations of tripartite qubit states are connected to each other for their usage as a CQT channel. From the study of the bipartite mixed quantum channel as a resource for teleportation of single-qubit states, we infer the nontrivial dependence of teleportation fidelity on mixedness and entanglement of the quantum channel. In [5,20], one of the present authors has shown the existence of rank dependent bounds on mixedness and entanglement of quantum states for their usefulness for successful quantum teleportation. Among bipartite qubit quantum channels, a class of MEMS [33–36] gives maximum teleportation fidelity for a given mixedness and entanglement. This demonstrates its importance in investigating the efficacy of mixed entangled teleportation channels in higher-dimensional state space. We address this situation by considering tripartite mixed X quantum channel for CQT.

A. Tripartite maximally entangled mixed X states

The genuine maximally entangled mixed X states for N qubits are given in [37] for a given spectrum of eigenvalues. The class of three-qubit X-MEMS as a convex sum of maximally entangled pure GHZ and separable states is given as

$$\begin{aligned} \rho(X)_{\mathrm{MEMS}} = {} & p_1|\mathrm{GHZ}_1^+\rangle\langle\mathrm{GHZ}_1^+| + p_2|001\rangle\langle 001| \\ & + p_3|010\rangle\langle 010| + p_4|011\rangle\langle 011| \\ & + p_5|\mathrm{GHZ}_1^-\rangle\langle\mathrm{GHZ}_1^-| + p_6|100\rangle\langle 100| \\ & + p_7|101\rangle\langle 101| + p_8|110\rangle\langle 110|, \end{aligned} \tag{10}$$

where $p_1 \geqslant p_2 \geqslant p_3 \geqslant p_4 \geqslant p_5 \geqslant p_6 \geqslant p_7 \geqslant p_8 \geqslant 0$ are the eigenvalues of density matrix $\rho(X)_{\mathrm{MEMS}}$ and $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = 1$ satisfy the normalization condition of the density matrix. The maximally entangled three-qubit GHZ state basis is given as

$$\begin{aligned} |\mathrm{GHZ}_1^{\pm}\rangle &= \frac{1}{\sqrt{2}}[|000\rangle \pm |111\rangle], \\ |\mathrm{GHZ}_2^{\pm}\rangle &= \frac{1}{\sqrt{2}}[|001\rangle \pm |110\rangle], \\ |\mathrm{GHZ}_3^{\pm}\rangle &= \frac{1}{\sqrt{2}}[|010\rangle \pm |101\rangle], \\ |\mathrm{GHZ}_4^{\pm}\rangle &= \frac{1}{\sqrt{2}}[|011\rangle \pm |100\rangle]. \end{aligned} \tag{11}$$

It is shown that the given density matrix $\rho(X)_{\mathrm{MEMS}}$ possesses maximum value of GME for a given spectrum of eigenvalues $\{\Lambda\}$. We calculate the GME of $\rho(X)_{\mathrm{MEMS}}$, and it is given by

$$\begin{aligned} C^*[\rho(X)] = \max\{0, p_1 - p_5 - 2[\sqrt{p_2 p_8} \\ + \sqrt{p_3 p_7} + \sqrt{p_4 p_6}]\}. \end{aligned} \tag{12}$$

If GME of a given $\rho(X)$ is equal to $C^*[\rho(X)]$, then the state $\rho(X)$ belongs to the class of $\rho(X)_{\mathrm{MEMS}}$. The maximally entangled mixed three-qubit X states, defined with respect to the mixedness of quantum states [38], are given as

$$\rho(X) = \begin{pmatrix} f(\gamma) & 0 & 0 & 0 & 0 & 0 & 0 & \gamma \\ 0 & g(\gamma) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g(\gamma) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g(\gamma) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 & 0 & 0 & 0 & f(\gamma) \end{pmatrix} \tag{13}$$

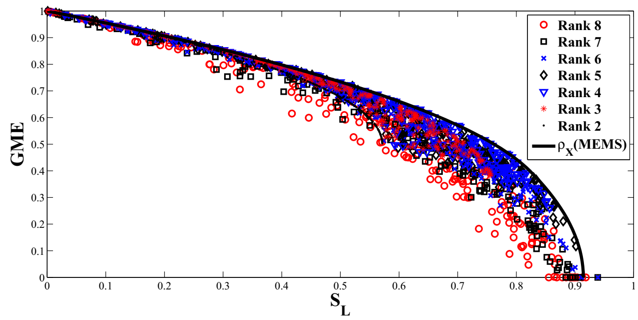

FIG. 1. Genuine multipartite entanglement of various rank X-MEMS is plotted as a function of linear entropy. Ranks of the states vary from 2 to 8. The entanglement of three-qubit X-MEMS [Eq. (13)] defined with respect to purity of the quantum states possesses maximum value for the full range of values of linear entropy.

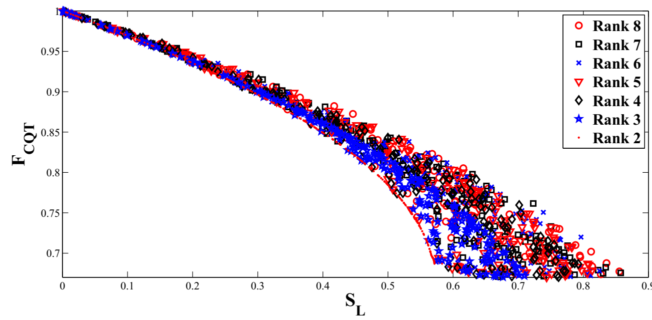

FIG. 2. Controlled teleportation fidelity F_{CQT} of X-MEMS is given as a function of linear entropy S_L. It is clear that higher rank MEMS give higher values of teleportation fidelity for a fixed linear entropy.

where

$$f(\gamma)=\begin{cases}1/5, & 0\leqslant\gamma\leqslant 1/5\\ \gamma, & 1/(5)<\gamma\leqslant 1/2\end{cases} \tag{14}$$

and

$$g(\gamma)=\begin{cases}1/5, & 0\leqslant\gamma\leqslant 1/5\\ (1-2\gamma)/3, & 1/5<\gamma\leqslant 1/2\end{cases}. \tag{15}$$

The tripartite X-MEMS, defined with respect to purity, are of rank 4 and 5. The GME of the above-defined maximally entangled mixed state is $\max[0,2|\gamma|]$. The GME of three-qubit X-MEMS of different ranks as a function of linear entropy is given in Fig. 1. From Fig. 1, it is clear that rank-4 and -5 X-MEMS possess the highest values of entanglement for a fixed linear entropy. The tripartite X-MEMS, defined with respect to purity, possess maximum achievable multipartite entanglement among all rank dependent X-MEMS. Here, we use this class of tripartite qubit X-MEMS as a channel for controlled quantum teleportation and show how the teleportation fidelity of different rank MEMS varies as a function of mixedness and other quantum correlations. The controlled quantum teleportation fidelity of X-MEMS is given as

$$F_{\mathrm{CQT}}(\mathrm{MEMS})=\tfrac{1}{6}[3+2(p_1-p_5)+|p_1-p_2-p_3+p_4+p_5+p_6-p_7-p_8|]. \tag{16}$$

The nonconditioned fidelity takes the value $\frac{1}{6}(3+|p_1-p_2-p_3+p_4+p_5+p_6-p_7-p_8|)$ and is always less than or equal to the classical limit of fidelity $\frac{2}{3}$. The CQT fidelity of X-MEMS as a function of linear entropy is given in Fig. 2. From Fig. 2, in which teleportation fidelity of X-MEMS of ranks varying from 2 to 8 is analyzed as a function of linear entropy, we infer that higher rank maximally entangled mixed states survive as a CQT channel for higher values of mixedness. In Fig. 3, we analyze the controlled teleportation fidelity of different rank X-MEMS as a function of genuine multipartite entanglement. It is seen that higher rank states possess higher values of teleportation fidelity for lower values of GME instead of maximally entangled mixed X states (with maximum GME) defined with respect to purity. This implies that there exists no monotonic relationship between entanglement and teleportation fidelity in the case of tripartite mixed channels. The control parameter is another quantity that captures the authority of the controller's qubit in the process of CQT. Control quantum teleportation fidelity as a function of control power for different rank X-MEMS is given in Fig. 4. The CQT fidelity of different rank X-MEMS under the authority of the controller qubit holds the bounds proposed in [27]. The boundaries for maximally entangled mixed X states of rank r ($2\leqslant r\leqslant 8$) are constructed by identifying the spectrum of eigenvalues as $p_1=\frac{1+(r-1)p}{r}$ and the rest of the $r-1$ eigenvalues equal to $\frac{1-p}{r}$. These boundary states act as an upper bound of corresponding rank dependent X-MEMS for the curves in which teleportation fidelity is analyzed as a function of linear entropy and multipartite entanglement. Since the CQT fidelity of X-MEMS is not optimum, we construct a class of non-maximally-entangled mixed states, X-NMEMS. The details of the investigation and its performance as a quantum channel for controlled quantum teleportation are discussed in the next section.

B. Tripartite non-maximally-entangled mixed X states

In this section, we construct a class of tripartite X-NMEMS; their performance as a CQT channel is investigated and is compared with that of X-MEMS. We show that our

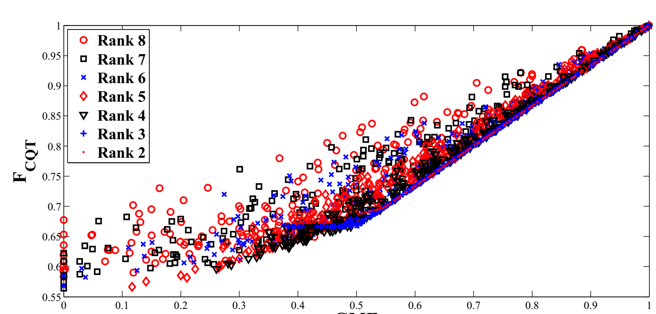

FIG. 3. Controlled quantum teleportation fidelity F_{CQT} of rank dependent tripartite X-MEMS is plotted as a function of genuine multipartite entanglement; it is shown that higher rank states possess higher values of teleportation fidelity for lower values of GME.

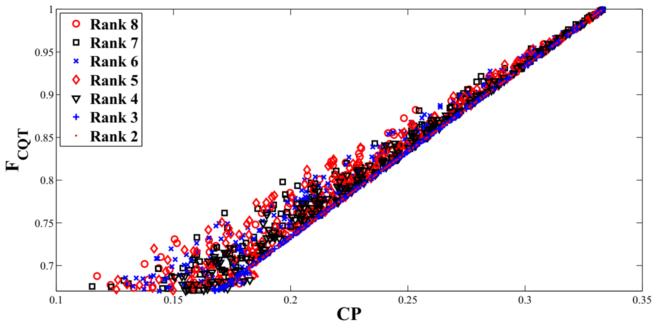

FIG. 4. X-MEMS of all ranks are generated and the controlled teleportation fidelity $F_{\rm CQT}$ of X-MEMS is plotted as a function of control power CP.

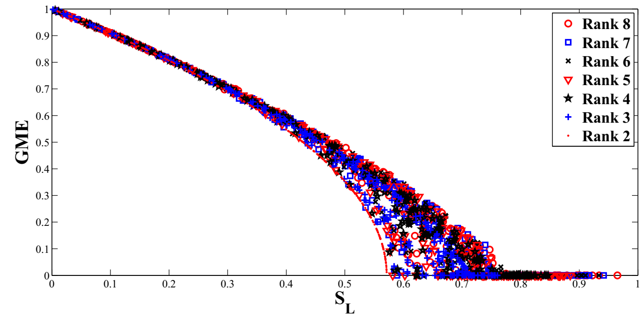

FIG. 5. Genuine multipartite entanglement of tripartite X-NMEMS of all ranks is plotted as a function of linear entropy S_L.

class of X-NMEMS is a potential candidate for controlled quantum teleportation of a qubit's state through a three-qubit quantum channel at a high value of mixedness and a low value of entanglement. The class of X-NMEMS, as a convex combination of maximally entangled GHZ states in Eq. (11), is given as

$$\begin{aligned}\rho(X)_{\rm NMEMS} = {} & p_1|{\rm GHZ}_1^+\rangle\langle+{\rm GHZ}_1^+| + p_2|{\rm GHZ}_4^+\rangle\langle{\rm GHZ}_4^+| \\ & + p_3|{\rm GHZ}_2^+\rangle\langle{\rm GHZ}_2^+| + p_4|{\rm GHZ}_3^+\rangle\langle{\rm GHZ}_3^+| \\ & + p_5|{\rm GHZ}_1^-\rangle\langle{\rm GHZ}_1^-| + p_6|{\rm GHZ}_4^-\rangle\langle{\rm GHZ}_4^-| \\ & + p_7|{\rm GHZ}_2^-\rangle\langle{\rm GHZ}_2^-| + p_8|{\rm GHZ}_3^-\rangle\langle{\rm GHZ}_3^-|. \end{aligned} \tag{17}$$

The eigenvalues $p_i^{\prime s}$ of non-maximally-entangled mixed X states satisfy the conditions of normalization and positivity discussed in Sec. III A. We investigate the details of the X-NMEMS quantum channel for CQT and show that tripartite mixed entangled states lower the requirements of the controlled quantum teleportation channel. The genuine multipartite entanglement of the non-maximally-entangled mixed state is estimated as

$$\begin{aligned}C[\rho(X)_{\rm NMEMS}] = {} & (p_1 - p_5) - (p_2 + p_3 \\ & + p_4 + p_6 + p_7 + p_8). \end{aligned} \tag{18}$$

The above-constructed tripartite X state does not fall in the class of X-MEMS, since the estimated GME of X states is not equal to $C^*\rho(X)$ in Eq. (12). The calculated controlled teleportation fidelity of X-NMEMS is given by

$$\begin{aligned}F_{\rm CQT}({\rm NMEMS}) = {} & \tfrac{1}{6}[3 + 3(p_1 + p_2) - (p_3 + p_4 \\ & + p_5 + p_6 + p_7 + p_8)]. \end{aligned} \tag{19}$$

The classical limit of teleportation fidelity of X-NMEMS is $F_{\rm NC} = \frac{1}{6}(3 + |p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8|)$. The genuine multipartite entanglement of different rank X-NMEMS is plotted as a function of linear entropy in Fig. 5. From Fig. 5, it is clear that the class of tripartite X-NMEMS possesses a lower value of GME as compared to that of X-MEMS for a defined spectrum of eigenvalues and linear entropy. Moreover, from Fig. 5, we infer that the entanglement of three-qubit X-NMEMS increases as rank increases, which is not the case for X-MEMS. We use this non-maximally-entangled mixed state as a CQT channel, and teleportation fidelity as a function of linear entropy is given in Fig. 6.

We analyze the performance of our X-NMEMS for the CQT process as a function of both entanglement and mixedness. It is clear from Eq. (19) that rank-2 X-NMEMS give maximum achievable controlled teleportation fidelity for all values of entanglement and mixedness. This is in contradiction with the case of the conventional quantum teleportation process. From Figs. 6 and 7, wherein controlled teleportation fidelity is analyzed as a function of mixedness and entanglement, respectively, it is seen that rank dependent X-NMEMS give maximum CQT fidelity in both cases as compared to X-MEMS. At the same time, as it is known from Fig. 5, even though GME of X-NMEMS is lower than that of X-MEMS for a given purity, its performance as a CQT channel is optimum. This indicates that maximum value of entanglement is not a necessary and sufficient condition to achieve optimum controlled quantum teleportation fidelity, which is not the case for bipartite quantum channels. At lowest rank (rank 2), X-NMEMS are the same as the states in [26] and they give maximum achievable controlled teleportation fidelity among all mixed states. The CQT fidelity of X-NMEMS is given as a function of control power in Fig. 8. X-NMEMS hold the lower and upper bounds defined for CQT fidelity, in terms of control power and multipartite entanglement. As we have discussed for maximally entangled mixed multi-

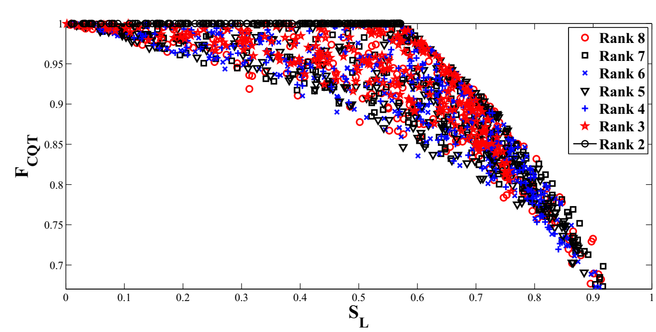

FIG. 6. Tripartite controlled quantum teleportation fidelity ($F_{\rm CQT}$) of X-NMEMS of various ranks is plotted for a given value of linear entropy (S_L). Rank-2 X-NMEMS give the maximum achievable CQT fidelity.

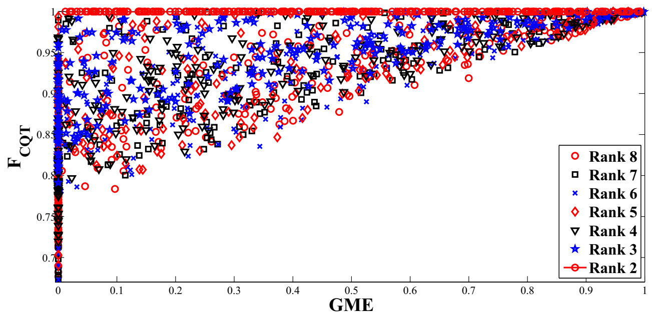

FIG. 7. Controlled quantum teleportation fidelity F_{CQT} of three-qubit X-NMEMS is given as a function of genuine multipartite entanglement, and it is shown that the maximum achievable CQT fidelity is attained for rank-2 X-NMEMS for the full range of values of GME.

partite states, the rank dependent boundary X-NMEMS are constructed by considering the eigenvalues, $p_1 = \frac{1+(r-1)p}{r}$, and the rest of the $r-1$ eigenvalues equal to $\frac{(1-p)}{r}$. In the case of X-NMEMS, rank dependent boundary states act as lower bounds of respective rank X-NMEMS for CQT fidelity, given as a function of multipartite entanglement and mixedness.

IV. RESULTS AND DISCUSSIONS

In this paper, we systematically investigated the efficacy of the tripartite mixed entangled state as a resource for controlled quantum teleportation. Mixedness and entanglement jointly decide the efficiency of mixed quantum channels for CQT. To investigate the interdependence of multipartite entanglement, mixedness, and control power of the quantum states on the success of controlled quantum teleportation in detail, we used a class of tripartite maximally entangled mixed X states as a channel for CQT. The rank dependent performance of X-MEMS as a CQT channel has been analyzed as a function of the aforementioned channel parameters and it is shown that the X-MEMS do not give optimum controlled quantum teleportation fidelity, as is true for the bipartite quantum states. The problem of controlled quantum teleportation via non-maximally-entangled pure states has already been studied. Here we extended this work to the usage of non-maximally-entangled mixed states as a CQT resource. We constructed a class of non-maximally-entangled mixed states and investigated its application as a controlled quantum teleportation channel. We showed that a class of X-NMEMS outperforms X-MEMS as a CQT channel, for a given mixedness and entanglement. This essentially proves that maximum multipartite entanglement is not sufficient for achieving optimum teleportation fidelity. From Fig. 5, it is known that for some values of linear entropy the genuine multipartite entanglement of tripartite X-NMEMS becomes zero. Zero GME implies that states are biseparable. From our investigation on tripartite X-NMEMS for CQT, it is evident that biseparable states are useful for CQT at a high degree of mixedness. This result is an important one for the experimental realization of CQT in a noisy environment. For example, consider the case of boundary X-NMEMS $[\rho_3(X)]$ of rank 3: the eigenvalues of $\rho_3(X)$ are $p_1 = \frac{1+2p}{3}$ and $p_2 = p_3 = \frac{1-p}{3}$. We calculated the channel parameters of $\rho_3(X)$ as $F_{CQT} = \frac{7+2p}{9}$, $F_{NC} = \frac{5+p}{9}$, $\text{GME} = \max\{0, \frac{4p-1}{3}\}$, and $S_L = \frac{16(1-p^2)}{21}$. Multipartite entanglement of $\rho_3(X)$ is zero for $0 \leqslant p \leqslant \frac{1}{4}$; i.e., for high values of mixedness $S_L \geqslant \frac{15}{21}$, rank-3 boundary X-NMEMS possess no genuine multipartite entanglement. However, the controlled teleportation fidelity of rank-3 X-NMEMS does not vanish above this value of mixedness, $F_{CQT} = \{\frac{7}{9}, \frac{5}{6}\}$ for $\frac{15}{21} \leqslant S_L \leqslant \frac{16}{21}$. Even for the biseparability nature of boundary X-NMEMS at high values of mixedness, the controlled quantum teleportation fidelity possesses a high value above the classical limit.

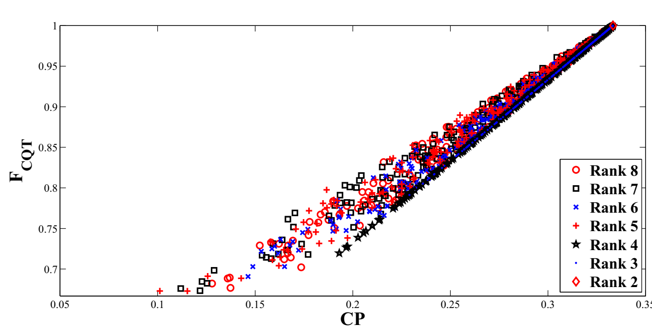

FIG. 8. Controlled quantum teleportation fidelity F_{CQT} of non-maximally-entangled mixed X states (X-NMEMS) is plotted as a function of control power CP.

V. CONCLUSIONS

Analysis of the performance of tripartite rank dependent X states as a resource for controlled quantum teleportation revealed many intriguing properties of multipartite systems that can be exploited for the efficient implementation of quantum information processing protocols. We showed that for a given multipartite entanglement and mixedness a class of non-maximally-entangled mixed X states achieves optimum controlled quantum teleportation fidelity. At the same time, investigation on X-MEMS as a resource for CQT proved that tripartite maximally entangled mixed states fail to attain optimum teleportation fidelity. From our investigation on X-NMEMS, we showed that the class of biseparable X-NMEMS can also be considered as a potential candidate for CQT, since it gives high controlled quantum teleportation fidelity for highly mixed cases. These results hold true for different measures of multipartite entanglement.

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