Frauchiger-Renner argument and quantum histories

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In this article, we reconstruct the Frauchiger and Renner argument, taking into account that the assertions of the argument are made at different times. To do this, we use a formalism of quantum histories, namely the Theory of Consistent Histories. We show that the supposedly contradictory conclusion of the argument requires computing probabilities in a family of histories that does not satisfy the consistency condition, i.e., an invalid family of histories for the theory.

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I. INTRODUCTION

In April 2016, Frauchiger and Renner published an article online in which they introduced a *Gedankenexperiment* that led them to conclude that "no single-world interpretation can be logically consistent" [1]. In a new version of the paper, the authors moderated their original claim, concluding "that quantum theory cannot be extrapolated to complex systems, at least not in a straightforward manner" [2].

Since its first online publication, the Frauchiger and Renner (FR) argument was extensively commented upon in the field of quantum foundations, since it was considered as a new no-go result for quantum mechanics whose strength relies on the fact that it is neutral regarding interpretation: on the basis of three seemingly reasonable assumptions that do not include interpretive premises, the argument leads to a contradiction. This fact has been conceived as pointing to a deep shortcoming of quantum mechanics itself, which contrasts with the extraordinary success of the theory.

In a previous article [3] a careful reconstruction of the FR argument has been offered, which shows that the contradiction resulting from the FR argument is inferred by making classical conjunctions between different and incompatible contexts, and, as a consequence, it is the result of a theoretically illegitimate inference. However, recently it has been suggested that the criticism does not take into account the fact that the inferences in the FR argument are all carefully timed, and that this fact would circumvent the objection based on the contextuality of quantum mechanics. The purpose of this article is to analyze such a defense of the FR argument.

If timing really matters in the FR argument, it seems natural to reconstruct the argument using a formalism of *quantum histories*, which allows us to define logical operations between quantum properties at different times. The idea of quantum histories was mainly motivated by this limitation of quantum mechanics. In 1984, Robert Griffiths presented the first version of his Theory of Consistent Histories [4]; some years later, he introduced some modifications to that original version [5,6]. Roland Omnès [7–11] also published a series of articles that contributed to the development of this theory. Simultaneously, Murray Gell-Mann and James Hartle developed a similar formalism [12–14]. The Theory of Consistent Histories extends the formalism of quantum mechanics. It introduces the notion of history, which generalizes the notion of event: an elemental history is defined as a sequence of events at different times, where an event is the occurrence of a property. But since it is not possible to assign probabilities to the set of all histories, it is necessary to select a subset of histories that satisfies additional conditions.

To analyze the defense of the FR argument on the basis of the fact that the assertions are made at different times, we will carefully reconstruct the argument in the framework of the Theory of Consistent Histories. This task will allow us to prove that the supposedly contradictory conclusion of the argument requires computing probabilities in a family of histories that is not consistent, i.e., an invalid family of histories for the theory.

II. THE FR ARGUMENT

The *Gedankenexperiment* proposed in Frauchiger and Renner's article is a sophisticated reformulation of Wigner's friend experiment [15]. In that original thought experiment, Wigner considers the superposition state of a particle in a closed laboratory where his friend is confined. When Wigner's friend measures the particle, the state collapses to one of its components. However, from the outside of the laboratory, Wigner still assigns a superposition state to the whole composite system: $Particle + Friend + Laboratory$.

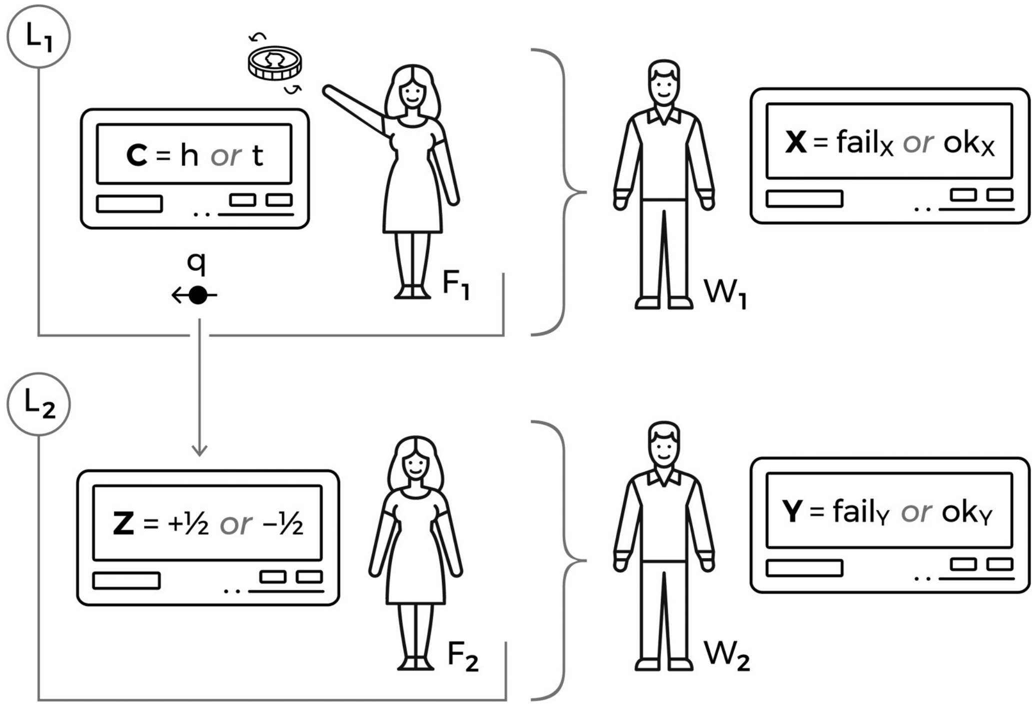

FIG. 1. Illustration of the *Gedankenexperiment*. Friend F_1 tosses a coin and measures its result. Depending on the outcome, she sends a qubit in a particular state. Then, friend F_2 measures the spin of the qubit in the z direction, obtaining $z = +\frac{1}{2}$ or $-\frac{1}{2}$. Finally, observers W_1 and W_2 measure the entire laboratories L_1 and L_2 obtaining outcomes fail_X or ok_X and fail_Y or ok_Y, respectively.

The FR argument relies on duplicating Wigner's setup (Fig. 1). Let us consider two friends F_1 and F_2 located in separate and isolated laboratories L_1 and L_2.[1] F_1 measures the observable C of a biased "quantum coin" in the state $|\phi\rangle = \frac{1}{\sqrt{3}}|h\rangle + \sqrt{\frac{2}{3}}|t\rangle$, where $|h\rangle$ and $|t\rangle$ are the eigenstates of C, and h and t are its respective eigenvalues. F_1 prepares a qubit in the state $|\downarrow\rangle$ if the outcome is h, or in the state $|\rightarrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$ if the outcome is t, and sends it to F_2. When F_2 receives the qubit, she measures its observable S_z. After these two measurements, the state of the whole system composed of the two laboratories is

$$|\Psi\rangle = \frac{1}{\sqrt{3}}|H\rangle|\Downarrow\rangle + \sqrt{\frac{2}{3}}|T\rangle|\Rightarrow\rangle, \tag{1}$$

where we have the following:

(i) $|H\rangle$ and $|T\rangle$, eigenstates of an observable A with eigenvalues H and T, are the states of the entire laboratory L_1 when the outcome of F_1's measurement is h and t, respectively.

(ii) $|\Uparrow\rangle$ and $|\Downarrow\rangle$, eigenstates of an observable B with eigenvalues $\Uparrow$ and $\Downarrow$, are the states of the entire laboratory L_2 when the outcome of F_2's measurement is $+1/2$ and $-1/2$, respectively.

(iii) $|\Rightarrow\rangle = \frac{1}{\sqrt{2}}|\Uparrow\rangle + \frac{1}{\sqrt{2}}|\Downarrow\rangle$.

The *Gedankenexperiment* continues by considering two "Wigner" observers, W_1 and W_2, located outside the laboratories, who will measure the observables X and Y of laboratories L_1 and L_2, respectively:

(i) X has the eigenvectors $|\mathrm{fail}_X\rangle$ and $|\mathrm{ok}_X\rangle$, such that

$$|\mathrm{fail}_X\rangle = \frac{1}{\sqrt{2}}|H\rangle + \frac{1}{\sqrt{2}}|T\rangle, \quad |\mathrm{ok}_X\rangle = \frac{1}{\sqrt{2}}|H\rangle - \frac{1}{\sqrt{2}}|T\rangle. \tag{2}$$

(ii) Y has the eigenvectors $|\mathrm{fail}_Y\rangle$ and $|\mathrm{ok}_Y\rangle$, such that

$$|\mathrm{fail}_Y\rangle = \frac{1}{\sqrt{2}}|\Downarrow\rangle + \frac{1}{\sqrt{2}}|\Uparrow\rangle, \quad |\mathrm{ok}_Y\rangle = \frac{1}{\sqrt{2}}|\Downarrow\rangle - \frac{1}{\sqrt{2}}|\Uparrow\rangle. \tag{3}$$

Before analyzing the consequences of the experiment, Frauchiger and Renner point out that their argument can be conceived as a no-go theorem [2] that proves that three "natural-sounding" assumptions, (Q), (C), and (S), cannot all be valid simultaneously:[2]

(Q) *Compliance with quantum theory*: Quantum mechanics is universally valid, that is, it applies to systems of any complexity, including observers. Moreover, an agent knows

[1] We modify the original terminology slightly for clarity.

[2] In their 2016 paper, Frauchiger and Renner implicitly consider (Q) and (C) as unavoidable: as a consequence, they claim that their argument shows that "no single-world interpretation can be logically consistent" and, therefore, "we are forced to give up the view that there is one single reality" [1]. In contrast, in their 2018 paper, they stress that "the theorem itself is neutral in the sense that it does not tell us which of these three assumptions is wrong" [2]; as a consequence, they admit the possibility of different theoretical and interpretive viewpoints regarding their result, and they include a table that shows which of the three assumptions each interpretation of quantum theory violates.

that a given proposition is true whenever the Born rule assigns probability 1 to it.

(C) *Self-consistency*: Different agents' predictions are not contradictory.

(S) *Single-world*: From the viewpoint of an agent who carries out a particular measurement, this measurement has one single outcome.

On the basis of the above considerations—experimental setup and assumptions—the FR argument proceeds as follows. First, in order to compute the probability that the measurements of X and Y yield the results $|\mathrm{ok}_X\rangle$ and $|\mathrm{ok}_Y\rangle$, respectively, the state described by Eq. (1) must be expressed as

$$|\Psi\rangle = \frac{1}{\sqrt{12}}|\mathrm{ok}_X\rangle|\mathrm{ok}_Y\rangle - \frac{1}{\sqrt{12}}|\mathrm{ok}_X\rangle|\mathrm{fail}_Y\rangle + \frac{1}{\sqrt{12}}|\mathrm{fail}_X\rangle|\mathrm{ok}_Y\rangle + \sqrt{\frac{3}{4}}|\mathrm{fail}_X\rangle|\mathrm{fail}_Y\rangle. \tag{4}$$

From this equation, it is clear that the probability of obtaining ok_X and ok_Y is $1/12$.

The second part of the argument consists in showing that the observers involved in the experiment can draw a conclusion different from the above one on the basis of the following reasoning. Let us consider the probability that F_2 obtains $-1/2$ in her S_z measurement and W_1 obtains $|\mathrm{ok}_X\rangle$ in her X measurement; in order to compute this probability, the state described by Eq. (1) must be expressed as

$$|\Psi\rangle = \sqrt{\frac{2}{3}}|\mathrm{fail}_X\rangle|\Downarrow\rangle + \frac{1}{\sqrt{6}}|\mathrm{fail}_X\rangle|\Uparrow\rangle - \frac{1}{\sqrt{6}}|\mathrm{ok}_X\rangle|\Uparrow\rangle. \tag{5}$$

From this equation, it can be inferred that such a probability is zero. Then, if W_1 obtains $|\mathrm{ok}_X\rangle$ in her X measurement on laboratory L_1, she can infer with certainty that the outcome of F_2's S_z measurement on the qubit was $+1/2$. In turn, if F_2 obtains $+1/2$ in her S_z measurement on the qubit, she can infer that the outcome of F_1's C measurement on the quantum coin was t, because otherwise F_1 would send F_2 the qubit in state $|\downarrow\rangle$, see Eq. (1). And if F_1 obtains t in her C measurement on the quantum coin, she can infer that the outcome of W_2's Y measurement on laboratory L_2 will be $|\mathrm{fail}_Y\rangle$, because the outcome t is perfectly correlated with the state $|\Rightarrow\rangle$ of laboratory L_2, and $|\Rightarrow\rangle = |\mathrm{fail}_Y\rangle$; see Eq. (3). Therefore, from a nested reasoning it can be concluded that, when W_1 gets $|\mathrm{ok}_X\rangle$, she can infer that W_2 certainly gets $|\mathrm{fail}_Y\rangle$. But this conclusion contradicts what was inferred from Eq. (4), that is, that there is a nonzero probability that W_1 gets $|\mathrm{ok}_X\rangle$ and W_2 gets $|\mathrm{ok}_Y\rangle$.

The reactions to the FR argument have been multiple and varied (see, for example, [16–22], just to mention some of them). However, since the argument from which the contradiction is obtained involves quantum properties at different times, it seems natural to consider a description of the *Gedankenexperiment* using the theory of quantum histories. This theory extends the formalism of quantum mechanics introducing the notion of quantum history: an elemental history is defined as a sequence of quantum properties at different times (see Sec. IV). As far as we know, there has not been a detailed reconstruction of the argument in terms of the Theory of Consistent Histories. In Sec. IV we will offer such a description and we will draw the conclusions that this formalism offers for this case.

Moreover, the vectors $|H\rangle$ and $|T\rangle$ of the previous discussion are states of the measurement instrument in laboratory L_1, while the vectors $|\Uparrow\rangle$ and $|\Downarrow\rangle$ are states of the measurement instrument in laboratory L_2. However, the states of the measurement instruments of observers W_1 and W_2 are not included. In the next section, we give a complete description of the process including the Hilbert spaces corresponding to all measurement instruments.

III. THE DIACHRONIC DEVELOPMENT OF THE ARGUMENT

Let us recall that in laboratory L_1 there is a quantum coin in the initial state

$$|\phi\rangle = \frac{1}{\sqrt{3}}|h\rangle + \sqrt{\frac{2}{3}}|t\rangle \in \mathcal{H}_C, \tag{6}$$

where $\mathcal{H}_C$ is the Hilbert space of the coin. The initial state of the rest of the laboratory L_1 (including observer F_1) is $|a_0\rangle \in \mathcal{H}_{F_1}$. Therefore, the Hilbert space of the entire laboratory L_1 is $\mathcal{H}_{L_1} = \mathcal{H}_C \otimes \mathcal{H}_{F_1}$. In turn, in the laboratory L_2 there is a qubit, which initially is in state $|q_0\rangle \in \mathcal{H}_q$, where $\mathcal{H}_q$ is the Hilbert space of the qubit. The initial state of the rest of laboratory L_2 (including observer F_2) is $|b_0\rangle \in \mathcal{H}_{F_2}$. Therefore, the Hilbert space of the entire laboratory L_2 is $\mathcal{H}_{L_2} = \mathcal{H}_q \otimes \mathcal{H}_{F_2}$.

Observer W_1 measures the observable X of the laboratory L_1 with an apparatus, which is initially in a state $|w_{10}\rangle \in \mathcal{H}_{W_1}$, where $\mathcal{H}_{W_1}$ is the Hilbert space of the apparatus. In turn, observer W_2 measures the observable Y of laboratory L_2 with an apparatus initially in a state $|w_{20}\rangle \in \mathcal{H}_{W_2}$, where $\mathcal{H}_{W_2}$ is the Hilbert space of the corresponding apparatus.

Summing up, the Hilbert space of the entire process is $\mathcal{H} = \mathcal{H}_{L_1} \otimes \mathcal{H}_{L_2} \otimes \mathcal{H}_{W_1} \otimes \mathcal{H}_{W_2}$, and the initial state at time t_0 is

$$|\Psi_0\rangle = |\phi\rangle \otimes |a_0\rangle \otimes |q_0\rangle \otimes |b_0\rangle \otimes |w_{10}\rangle \otimes |w_{20}\rangle \in \mathcal{H}. \tag{7}$$

In what follows, we describe the consecutive processes.

(i) Time interval (t_0, t_1): Observer F_1 measures the quantum coin. This process is represented by a unitary evolution U_{10} in the Hilbert space $\mathcal{H}_{L_1} = \mathcal{H}_C \otimes \mathcal{H}_{F_1}$, satisfying

$$\begin{aligned} U_{10}(|h\rangle \otimes |a_0\rangle) &= |h\rangle \otimes |a_h\rangle \equiv |H\rangle, \\ U_{10}(|t\rangle \otimes |a_0\rangle) &= |t\rangle \otimes |a_t\rangle \equiv |T\rangle. \end{aligned} \tag{8}$$

(ii) Time interval (t_1, t_2): Observer F_1 prepares the qubit. This process is represented by a unitary evolution U_{21} in the Hilbert space $\mathcal{H}_{F_1} \otimes \mathcal{H}_q$, satisfying

$$\begin{aligned} U_{21}(|a_h\rangle \otimes |q_0\rangle) &= |a_h\rangle \otimes |\downarrow\rangle, \\ U_{21}(|a_t\rangle \otimes |q_0\rangle) &= |a_t\rangle \otimes |\rightarrow\rangle. \end{aligned} \tag{9}$$

(iii) Time interval (t_2, t_3): Observer F_2 measures the qubit. This process is represented by a unitary evolution U_{32} in the Hilbert space $\mathcal{H}_{L_2} = \mathcal{H}_q \otimes \mathcal{H}_{F_2}$, satisfying

$$\begin{aligned} U_{32}(|\downarrow\rangle \otimes |b_0\rangle) &= |\downarrow\rangle \otimes |b_\downarrow\rangle \equiv |\Downarrow\rangle, \\ U_{32}(|\uparrow\rangle \otimes |b_0\rangle) &= |\uparrow\rangle \otimes |b_\uparrow\rangle \equiv |\Uparrow\rangle. \end{aligned} \tag{10}$$

(iv) Time interval (t_3, t_4): Observer W_1 measures laboratory L_1. This process is represented by a unitary evolution U_{43} in the Hilbert space $\mathcal{H}_{L_1} \otimes \mathcal{H}_{W_1}$, satisfying

$$\begin{aligned} U_{43}(|\mathrm{fail}_X\rangle \otimes |w_{10}\rangle) &= |\mathrm{fail}_X\rangle \otimes |w_{1\,\mathrm{fail}}\rangle, \\ U_{43}(|\mathrm{ok}_X\rangle \otimes |w_{10}\rangle) &= |\mathrm{ok}_X\rangle \otimes |w_{1\,\mathrm{ok}}\rangle. \end{aligned}$$

(v) Time interval (t_4, t_5): Observer W_2 measures laboratory L_2. This process is represented by a unitary evolution U_{54} in the Hilbert space $\mathcal{H}_{L_2} \otimes \mathcal{H}_{W_2}$, satisfying

$$U_{54}(|\text{ok}_Y\rangle \otimes |w_{20}\rangle) = |\text{ok}_Y\rangle \otimes |w_{2\,\text{ok}}\rangle,$$
$$U_{54}(|\text{fail}_Y\rangle \otimes |w_{20}\rangle) = |\text{fail}_Y\rangle \otimes |w_{2\,\text{fail}}\rangle.$$

Once the steps for the time evolution are established, the argument leading to the contradictory result, reviewed in Sec. II, should be written in terms of probabilities involving properties at different times. For example, in Sec. II the value $1/12$ was obtained for the probability for obtaining ok_X and ok_Y. Considering the description of the time evolution given in this section, we should write

$$\Pr(\{w_{2\,\text{ok}} \text{ at } t_5\} \wedge \{w_{1\,\text{ok}} \text{ at } t_4\}) = \tfrac{1}{12}, \tag{11}$$

where $\wedge$ represents the logical conjunction. This expression represents the probability for the measurement instrument of observer W_1 to indicate $w_{1\,\text{ok}}$ at time t_4 and for the measurement instrument of observer W_2 to indicate $w_{2\,\text{ok}}$ at the later time t_5.

The second part of the argument is based on the following conditional probabilities:

$$\Pr(\{b_\uparrow \text{ at } t_3\} \mid \{w_{1\,\text{ok}} \text{ at } t_4\}) = 1, \tag{12}$$

$$\Pr(\{a_t \text{ at } t_1\} \mid \{b_\uparrow \text{ at } t_3\}) = 1, \tag{13}$$

$$\Pr(\{w_{2\,\text{fail}} \text{ at } t_5\} \mid \{a_t \text{ at } t_1\}) = 1. \tag{14}$$

If the last three conditional probabilities could be considered simultaneously, then we could infer the following conditional probability:

$$\Pr(\{w_{2\,\text{fail}} \text{ at } t_5\} \mid \{w_{1\,\text{ok}} \text{ at } t_4\}) = 1. \tag{15}$$

Hence, $\Pr(\{w_{2\,\text{ok}} \text{ at } t_5\} \wedge \{w_{1\,\text{ok}} \text{ at } t_4\}) = 0$, which is in contradiction with Eq. (11).

Since the previous argument involves logical operations between quantum properties at different times, it seems natural to analyze it using a formalism of quantum histories. To search for the possibility of obtaining Eqs. (11), (12), (13), and (14) simultaneously, in the next section we will apply the Theory of Consistent Histories.

IV. THE FR ARGUMENT IN TERMS OF QUANTUM HISTORIES

In what follows, we present a brief summary of the Theory of Consistent Histories (TQH) [4–14]. In quantum mechanics, the properties of a system are represented by orthogonal projectors. Since an elementary history is a sequence of properties at consecutive times, the TQH represents each elementary history with a tensor product of orthogonal projectors. For example, a history of n times $\breve{\Pi} = \Pi_1 \otimes \cdots \otimes \Pi_n$ represents a sequence of properties $\Pi_1, \ldots, \Pi_n$ at times $t_1, \ldots, t_n$.

To define probabilities for quantum histories, it is necessary to define a family of histories. For this purpose, first we have to choose a context of properties at each time t_i, i.e., a set of projectors that sum the identity of $\mathcal{H}$ and that are mutually orthogonal:

$$\Pi_{k_i}\Pi_{k_i'} = \delta_{k_i k_i'}\,\Pi_{k_i}, \quad \textstyle\sum_{k_i}\Pi_{k_i} = I_{\mathcal{H}},$$
$$k_i, k_i' \in \sigma_i, \quad i = 1, \ldots, n,$$

where $I_{\mathcal{H}}$ is the identity of the Hilbert space $\mathcal{H}$, and each σ_i is an index set.

Second, we define the atomic histories $\breve{\Pi}_{k_1,\ldots,k_n}$, choosing one projector Π_{k_i} at each time t_i:

$$\breve{\Pi}_{k_1,\ldots,k_n} = \Pi_{k_1} \otimes \cdots \otimes \Pi_{k_n}, \quad (k_1, \ldots, k_n) \in \breve{\sigma},$$
$$\breve{\sigma} = \sigma_1 \times \cdots \times \sigma_n.$$

Then, we define the histories $\breve{\Pi}_\Lambda$ summing the histories $\breve{\Pi}_{k_1,\ldots,k_n}$ with $(k_1, \ldots, k_n) \in \Lambda \subseteq \breve{\sigma}$, i.e., $\breve{\Pi}_\Lambda = \sum_{(k_1,\ldots,k_n)\in\Lambda} \breve{\Pi}_{k_1,\ldots,k_n}$. These histories represent disjunctions of the histories $\breve{\Pi}_{k_1,\ldots,k_n}$. Finally, the family of histories is the set obtained by making arbitrary disjunctions between product histories.

If ρ_0 is the initial state at time t_0, the probability of a general history $\breve{\Pi}_\Lambda$ is defined in the following way:

$$\Pr_{\rho_0}(\breve{\Pi}_\Lambda) = \text{Tr}[C^\dagger(\breve{\Pi}_\Lambda)\rho_0 C(\breve{\Pi}_\Lambda)], \tag{16}$$

where we have introduced the chain operator $C(\breve{\Pi}_\Lambda) = \sum_{(k_1,\ldots,k_n)\in\Lambda} C(\breve{\Pi}_{k_1,\ldots,k_n})$, in which

$$C(\breve{\Pi}_{k_1,\ldots,k_n}) = U(t_0, t_1)\Pi_{k_1}U(t_1, t_2)\Pi_{k_2}\cdots U(t_{n-1}, t_n) \times \Pi_{k_n}U(t_n, t_0)$$

with $U(t_i, t_j) = e^{-iH(t_i - t_j)/\hbar}$.

In general, the probability definition given in Eq. (16) does not satisfy the axiom of additivity. Therefore, to have a well-defined probability, the atomic histories of a family of histories must satisfy an additional condition, called the *consistency condition*,

$$\text{Tr}[C^\dagger(\breve{\Pi}_{k_1,\ldots,k_n})\rho_0 C(\breve{\Pi}_{k_1',\ldots,k_n'})] = 0, \quad \forall\,(k_1, \ldots, k_n) \neq (k_1', \ldots, k_n'). \tag{17}$$

Intuitively, the consistency condition measures the amount of interference between pairs of histories. When $n = 1$, this condition is automatically satisfied, and the probability expression of Eq. (16) reduces to the Born rule. However, in the general case, the consistency condition is not trivial, and when it is satisfied the probability expression provides a generalization of the Born rule.

To describe the FR argument in terms of quantum histories, we first obtain the probability for the measurement instrument of the observer W_1 to indicate $w_{1\,\text{ok}}$ at time t_4 and for the measurement instrument of the observer W_2 to indicate $w_{2\,\text{ok}}$ at a later time t_5.

A suitable context of properties for time t_4 should include the properties $w_{1\,\text{ok}}$, $w_{1\,\text{fail}}$, and it has to be completed with the property $\neg(w_{1\,\text{ok}} \vee w_{1\,\text{fail}})$ (where $\vee$ is the disjunction and $\neg$ is the negation) in order to include all the degrees of freedom of the measurement instrument, for example the initial state $|w_{10}\rangle$ given in Eq. (7). These properties are represented by the following projectors:

$$\Pi_{w_{1\,\text{ok}}} = I_{L_1} \otimes I_{L_2} \otimes |w_{1\,\text{ok}}\rangle\langle w_{1\,\text{ok}}| \otimes I_{W_2},$$
$$\Pi_{w_{1\,\text{fail}}} = I_{L_1} \otimes I_{L_2} \otimes |w_{1\,\text{fail}}\rangle\langle w_{1\,\text{fail}}| \otimes I_{W_2},$$
$$\Pi_{\neg(w_{1\,\text{ok}} \vee w_{1\,\text{fail}})} = I_{\mathcal{H}} - \Pi_{w_{1\,\text{fail}}} - \Pi_{w_{1\,\text{ok}}}, \tag{18}$$

where each I_K is the identity of the corresponding Hilbert space $\mathcal{H}_K$. These three projectors provide a context of properties of the Hilbert space $\mathcal{H}$.

For time t_5, a suitable context of properties should include the properties of the measurement instrument of observer W_2, i.e., $w_{1\,\mathrm{ok}}$, $w_{1\,\mathrm{fail}}$, and it has to be completed with the property $\neg(w_{1\,\mathrm{ok}} \vee w_{1\,\mathrm{fail}})$. These properties are represented by the following projectors:

$$\Pi_{w_{2\,\mathrm{ok}}} = I_{L_1} \otimes I_{L_2} \otimes I_{W_1} \otimes |w_{2\,\mathrm{ok}}\rangle\langle w_{2\,\mathrm{ok}}|,$$
$$\Pi_{w_{2\,\mathrm{fail}}} = I_{L_1} \otimes I_{L_2} \otimes I_{W_1} \otimes |w_{2\,\mathrm{fail}}\rangle\langle w_{2\,\mathrm{fail}}|,$$
$$\Pi_{\neg(w_{2\,\mathrm{ok}} \vee w_{2\,\mathrm{fail}})} = I_{\mathcal{H}} - \Pi_{w_{2\,\mathrm{fail}}} - \Pi_{w_{2\,\mathrm{ok}}}. \tag{19}$$

These three projectors also provide a context of properties of the Hilbert space $\mathcal{H}$.

From the contexts of properties for times t_4 and t_5, we can generate a family of two-time histories, whose atomic histories are $\breve{\Pi}_{k_4,k_5} = \Pi_{k_4} \otimes \Pi_{k_5}$, with Π_{k_4} one of the projectors of Eqs. (18) and Π_{k_5} one of the projectors of Eqs. (19). It is easy to verify that the family generated by these atomic histories satisfies the consistency conditions given in Eq. (17). Therefore, Eq. (16) can be used to compute the probability of quantum history $\breve{\Pi}_{w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}} = \Pi_{w_{1\,\mathrm{ok}}} \otimes \Pi_{w_{2\,\mathrm{ok}}}$,

$$\Pr(\breve{\Pi}_{w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}}) = \tfrac{1}{12}. \tag{20}$$

This shows that using the Theory of Consistent Histories, and explicitly considering the measurement instruments as quantum systems, we obtain the same result given in Sec. III for the first part of the argument.

In the same way, different consistent families of two-time histories can be defined to express Eqs. (12), (13), and (14). However, if the three equations are going to be used together in the same argument, it is necessary to have a consistent family of four-time histories including the possible results of the instrument of the observer F_1 at time t_1, of the instrument of the observer F_2 at time t_3, of the instrument of observer W_1 at time t_4, and of the instrument of observer W_2 at time t_5.

For times t_4 and t_5, the contexts of properties given in Eqs. (18) and (19) are adequate. For time t_1, a suitable context of properties should include the properties a_h, a_t, and it has to be completed with the property $\neg(a_h \vee a_t)$. These properties are represented by the following projectors:

$$\Pi_{a_h} = I_C \otimes |a_h\rangle\langle a_h| \otimes I_{L_2} \otimes I_{W_1} \otimes I_{W_2},$$
$$\Pi_{a_t} = I_C \otimes |a_t\rangle\langle a_t| \otimes I_{L_2} \otimes I_{W_1} \otimes I_{W_2},$$
$$\Pi_{\neg(a_h \vee a_t)} = I_{\mathcal{H}} - \Pi_{a_h} - \Pi_{a_t}. \tag{21}$$

For time t_3, a suitable context of properties should include the properties of the measurement instrument of observer F_2, i.e., $b_\downarrow$, $b_\uparrow$, and it has to be completed with the property $\neg(b_\downarrow \vee b_\uparrow)$. These properties are represented by the following projectors:

$$\Pi_{b_\downarrow} = I_{L_1} \otimes I_q \otimes |b_\downarrow\rangle\langle b_\downarrow| \otimes I_{W_1} \otimes I_{W_2},$$
$$\Pi_{b_\uparrow} = I_{L_1} \otimes I_q \otimes |b_\uparrow\rangle\langle b_\uparrow| \otimes I_{W_1} \otimes I_{W_2},$$
$$\Pi_{\neg(b_\downarrow \vee b_\uparrow)} = I_{\mathcal{H}} - \Pi_{b_\downarrow} - \Pi_{b_\uparrow}. \tag{22}$$

From the contexts of properties for times t_1, t_3, t_4, and t_5 we can generate a family of four-time histories, whose atomic histories are

$$\breve{\Pi}_{k_1,k_3,k_4,k_5} = \Pi_{k_1} \otimes \Pi_{k_3} \otimes \Pi_{k_4} \otimes \Pi_{k_5}, \tag{23}$$

with Π_{k_1}, Π_{k_3}, Π_{k_4}, and Π_{k_5} projectors chosen from Eqs. (21), (22), (18), and (19), respectively.

The nonatomic histories that are involved in the FR argument are the following:

$$\breve{\Pi}_{1\,t} = \Pi_{a_t} \otimes I_{\mathcal{H}} \otimes I_{\mathcal{H}} \otimes I_{\mathcal{H}}, \tag{24}$$

$$\breve{\Pi}_{3\,\uparrow} = I_{\mathcal{H}} \otimes \Pi_{b_\uparrow} \otimes I_{\mathcal{H}} \otimes I_{\mathcal{H}}, \tag{25}$$

$$\breve{\Pi}_{4\,\mathrm{ok}} = I_{\mathcal{H}} \otimes I_{\mathcal{H}} \otimes \Pi_{w_{1\,\mathrm{ok}}} \otimes I_{\mathcal{H}}, \tag{26}$$

$$\breve{\Pi}_{5\,\mathrm{fail}} = I_{\mathcal{H}} \otimes I_{\mathcal{H}} \otimes I_{\mathcal{H}} \otimes \Pi_{w_{2\,\mathrm{fail}}}. \tag{27}$$

In terms of quantum histories, the FR argument can be formulated as follows:

First part:

$$\Pr(\breve{\Pi}_{5\,\mathrm{ok}} \wedge \breve{\Pi}_{4\,\mathrm{ok}}) = \tfrac{1}{12}. \tag{28}$$

Second part:

$$\Pr(\breve{\Pi}_{3\,\uparrow}|\breve{\Pi}_{4\,\mathrm{ok}}) = 1, \quad \Pr(\breve{\Pi}_{1\,t}|\breve{\Pi}_{3\,\uparrow}) = 1,$$
$$\text{and } \Pr(\breve{\Pi}_{5\,\mathrm{fail}}|\breve{\Pi}_{1\,t}) = 1. \tag{29}$$

This implies $\Pr(\breve{\Pi}_{5\,\mathrm{fail}}|\breve{\Pi}_{4\,\mathrm{ok}}) = 1$ and then $\Pr(\breve{\Pi}_{5\,\mathrm{ok}}|\breve{\Pi}_{4\,\mathrm{ok}}) = 0$. Therefore,

$$\Pr(\breve{\Pi}_{5\,\mathrm{ok}} \wedge \breve{\Pi}_{4\,\mathrm{ok}}) = 0. \tag{30}$$

The contradiction is obtained from Eqs. (28) and (30).

To infer Eq. (30) from Eqs. (29), the quantum histories must belong to a single consistent family of histories, generated by the atomic histories of Eqs. (23). But such a family of histories does not satisfy the consistency conditions given in Eq. (17).

To prove this statement, let us consider two atomic histories $\breve{\Pi}_{a_t,b_\uparrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}}$ and $\breve{\Pi}_{a_h,b_\downarrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}}$, representing different results for the four measurements,

$$\breve{\Pi}_{a_t,b_\uparrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}} = \Pi_{a_t} \otimes \Pi_{b_\uparrow} \otimes \Pi_{w_{1\,\mathrm{ok}}} \otimes \Pi_{w_{2\,\mathrm{ok}}}, \tag{31}$$

$$\breve{\Pi}_{a_h,b_\downarrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}} = \Pi_{a_h} \otimes \Pi_{b_\downarrow} \otimes \Pi_{w_{1\,\mathrm{ok}}} \otimes \Pi_{w_{2\,\mathrm{ok}}}, \tag{32}$$

and with the following chain operators:

$$C(\breve{\Pi}_{a_t,b_\uparrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}}) = U(t_0,t_1)\Pi_{a_t}U(t_1,t_3)\Pi_{b_\uparrow}U(t_3,t_4)$$
$$\times\ \Pi_{w_{1\,\mathrm{ok}}}U(t_4,t_5)\Pi_{w_{2\,\mathrm{ok}}}U(t_5,t_0), \tag{33}$$

$$C(\breve{\Pi}_{a_h,b_\downarrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}}) = U(t_0,t_1)\Pi_{a_h}U(t_1,t_3)\Pi_{b_\downarrow}U(t_3,t_4)$$
$$\times\ \Pi_{w_{1\,\mathrm{ok}}}U(t_4,t_5)\Pi_{w_{2\,\mathrm{ok}}}U(t_5,t_0). \tag{34}$$

Considering the unitary time evolution of the complete quantum system and the initial state defined in Eq. (7), we obtain

$$C^\dagger(\breve{\Pi}_{a_t,b_\uparrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}})|\Psi_0\rangle$$
$$= C^\dagger(\breve{\Pi}_{a_h,b_\downarrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}})|\Psi_0\rangle$$
$$= \frac{1}{\sqrt{12}}U(t_0,t_5)(|\mathrm{ok}_X\rangle \otimes |\mathrm{ok}_Y\rangle \otimes |w_{1\,\mathrm{ok}}\rangle \otimes |w_{2\,\mathrm{ok}}\rangle), \tag{35}$$

and therefore, according to Eq. (17), the consistency condition gives

$$\mathrm{Tr}[C^{\dagger}(\breve{\Pi}_{a_t,b_\uparrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}})|\Psi_0\rangle\langle\Psi_0|C(\breve{\Pi}_{a_h,b_\downarrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}})]$$
$$= \langle\Psi_0|C(\breve{\Pi}_{a_h,b_\downarrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}})C^{\dagger}(\breve{\Pi}_{a_t,b_\uparrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}})|\Psi_0\rangle$$
$$= \tfrac{1}{12} \neq 0. \tag{36}$$

This proves that the atomic histories $\breve{\Pi}_{a_t,b_\uparrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}}$ and $\breve{\Pi}_{a_h,b_\downarrow,w_{1\,\mathrm{ok}},w_{2\,\mathrm{ok}}}$ do not satisfy the consistency condition and, therefore, there is no family of consistent histories to describe the results of the four measurement instruments of the FR experiment. For this reason, the conclusion of the second part of the FR argument cannot be asserted.

Summing up, since the conclusion of the second part of the FR argument is based on an illegitimate inference, the supposed contradiction of the FR argument does not hold.

V. CONCLUSIONS

In a previous article [3], one of us argued that the contradiction resulting from the FR argument is inferred by making classical conjunctions between different and incompatible contexts, and, as a consequence, it is the result of a theoretically illegitimate inference. However, it has been suggested that the criticism does not take into account the fact that the inferences in the FR argument are all carefully timed, and this fact would circumvent the objection based on the contextuality of quantum mechanics.

If timing really matters in the FR argument, it seems natural to reconstruct it using a theory of quantum histories, a formalism that allows us to deal with quantum properties at different times. We applied the Theory of Consistent Histories, and we showed that the contradiction resulting from the FR argument is inferred by computing probabilities in a family of histories that is not consistent, i.e., an invalid family of histories for the theory.

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[1] D. Frauchiger and R. Renner, arXiv:1604.07422v1 (quant-ph).
[2] D. Frauchiger and R. Renner, Nat. Commun. **9**, 3711 (2018).
[3] S. Fortin and O. Lombardi, arXiv:1904.07412 (quant-ph).
[4] R. B. Griffiths, J. Stat. Phys. **36**, 219 (1984).
[5] R. B. Griffiths, *Consistent Quantum Theory* (Cambridge University Press, Cambridge, 2002).
[6] R. B. Griffiths, Stud. Hist. Philos. Mod. Phys. **44**, 93 (2013).
[7] R. Omnès, Phys. Lett. A **125**, 169 (1987).
[8] R. Omnès, J. Stat. Phys. **53**, 893 (1988).
[9] R. Omnès, J. Stat. Phys. **53**, 933 (1988).
[10] R. Omnès, *The Interpretation of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1994).
[11] R. Omnès, *Understanding Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1999).
[12] M. Gell-Mann and J. B. Hartle, in *Complexity, Entropy and the Physics of Information*, edited by W. Zurek (Addison-Wesley, Reading, MA, 1990), Vol. VIII.
[13] M. Gell-Mann J. B. Hartle, Phys. Rev. D **47**, 3345 (1993).
[14] J. B. Hartle, in *Quantum Cosmology and Baby Universes*, edited by S. Coleman, J. Hartle, and T. Piran (World Scientific, Singapore, 1991).
[15] E. Wigner, in *The Scientist Speculates*, edited by I. J. Good (Heinemann, London, 1961).
[16] A. Sudbery, Found. Phys. **47**, 658 (2017).
[17] J. Bub, Stud. Hist. Philos. Mod. Phys. (to be published), doi: 10.1016/j.shpsb.2018.03.002.
[18] C. Brukner, Entropy **20**, 350 (2018).
[19] R. Healey, Found. Phys. **48**, 1568 (2018).
[20] F. Laloë, arXiv:1802.06396 (quant-ph).
[21] D. Lazarovici and M. Hubert, Sci. Rep. **9**, 470 (2019).
[22] D. Dieks, in *Quantum Worlds: Perspectives on the Ontology of Quantum Mechanics*, edited by O. Lombardi, S. Fortin, C. López, and F. Holik (Cambridge University Press, Cambridge, 2019).