Trade-off relation among genuine three-qubit nonlocalities in four-qubit systems

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We study the trade-off relations satisfied by the genuine tripartite nonlocality in multipartite quantum systems. From the reduced three-qubit density matrices of the four-qubit generalized Greenberger-Horne-Zeilinger (GHZ) states and W states (4-qubit entangled state), we find that there exists a trade-off relation among the mean values of the Svetlichny operators associated with these reduced states. Namely, the genuine three-qubit nonlocalities are not independent. For four-qubit generalized GHZ states and W states, the summation of all their three-qubit maximal (squared) mean values of the Svetlichny operator has an upper bound. This bound is better than the one derived from the upper bounds of individual three-qubit mean values of the Svetlichny operator. Detailed examples are presented to illustrate the trade-off relation among the three-qubit nonlocalities.

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I. INTRODUCTION

Nonlocality is a fundamental feature of quantum mechanics [1,2]. It is also a key resource in information processing [3–6] and is related to various topics in quantum information theory such as the understanding of classical and quantum boundary [7,8], the entangling power of nonlocal unitary operations [9–11], and the efficient decomposition for realization in quantum circuits [12], unextendible product basis [13], and positive-partial-transpose entangled states [14].

Bell inequalities and nonlocality have been widely studied and are shown to be related to the monogamy trade-off obeyed by bipartite Bell correlations. It is believed that for general translation-invariant systems, two-qubit states should not violate the Bell inequality [15]. A nontrivial model is constructed to confirm that the Bell inequality can be violated in perfect translation-invariant systems with an even number of sites [16]. Monogamy relations between the violations of Bell's inequalities have been derived in Ref. [17]. Meanwhile, using the Bloch vectors, a trade-off relation has been derived, together with a complete classification of four-qudit quantum states [18].

In the multipartite case, nonlocality displays a much richer and more complex structure compared with the case of bipartite systems. This makes the study and the characterization of multipartite nonlocal correlations an interesting but challenging problem. It comes thus as no surprise that our understanding of nonlocality in the multipartite setting is much less advanced than in the bipartite case [19,20].

In Ref. [21], a complete characterization of entanglement of an entire class of mixed three-qubit states with the same symmetry as the Greenberger-Horne-Zeilinger state, known as GHZ-symmetric states, has been achieved. By analytical expressions of maximum violation value of most efficient Bell inequalities, one obtains the conditions of standard nonlocality and genuine nonlocality of this class of states. The relation between entanglement and nonlocality has been also discussed for this class of states. Interestingly, genuine entanglement of GHZ-symmetric states is necessary to reveal the standard nonlocality [22]. Nonlocal correlations are proposed in three-qubit generalized GHZ states and four-qubit generalized GHZ states [23]. Meanwhile, all multipartite pure states that are equivalent to the N-qubit W states (4-qubit entangled state) under stochastic local operation and classical communication (SLOCC) can be uniquely determined (among arbitrary states) from their bipartite marginals [24].

Two overlapping bipartite binary Bell inequalities cannot be simultaneously violated, which would contradict the usual no-signaling principle. It is known as the monogamy of Bell inequality violations. Generally Bell monogamy relations refer to trade-offs between simultaneous violations of multiple inequalities. The genuine multipartite nonlocality, as evidenced by a generalized Svetlichny inequality, does exhibit monogamy property [25]. There is a complementarity relation between dichotomic observables leading to the monogamy of Bell inequality violations [26].

To study the nonlocality of bipartite quantum states, one considers the Clauser-Horne-Shimony-Holt (CHSH) inequality [27]. For any two-qubit density matrix ρ, if there exist local hidden variable models to describe the system, the CHSH inequality says that

$$|\mathrm{Tr}(\rho B_{\mathrm{CHSH}})| \leqslant 2, \tag{1}$$

where B_{CHSH} is the CHSH operator

$$B_{\mathrm{CHSH}} = \vec{a}\cdot\vec{\sigma}\otimes(\vec{b}+\vec{b}')\cdot\vec{\sigma} + \vec{a}'\cdot\vec{\sigma}\otimes(\vec{b}-\vec{b}')\cdot\vec{\sigma},$$

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with $\vec{a}$, $\vec{a}'$, $\vec{b}$, and $\vec{b}'$ being the real three-dimensional unit vectors and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ being the Pauli matrices. Denote T the matrix with entries given by $t_{ij} = \mathrm{Tr}[\rho(\sigma_i \otimes \sigma_j]$. It has been shown that the maximal violation of the CHSH inequality (1) is given by [28,29]

$$\langle \mathrm{CHSH} \rangle_\rho = \max |\mathrm{Tr}(\rho B_{\mathrm{CHSH}})| = 2\sqrt{M(\rho)},$$

where $M(\rho) = \max_{j<k}\{\mu_j + \mu_k\}$, $j, k \in \{1, 2, 3\}$, μ_j, μ_k are the two largest eigenvalues of the real symmetric matrix $T^t T$ and t denotes the matrix transposition.

The distribution of nonlocality in multipartite systems based on the violation of Bell inequality has been investigated in Refs. [30,31]. For any three-qubit state $\rho_{ABC} \in \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C$, the maximal violation of CHSH inequality of pairwise bipartite states satisfies the following trade-off relation:

$$\langle \mathrm{CHSH} \rangle^2_{\rho_{AB}} + \langle \mathrm{CHSH} \rangle^2_{\rho_{AC}} + \langle \mathrm{CHSH} \rangle^2_{\rho_{BC}} \leqslant 12. \tag{2}$$

It implies that for a three-qubit system, it is impossible that all pairs of qubit states violate the CHSH inequality simultaneously.

For genuine tripartite nonlocality, consider three separated observers Alice, Bob, and Charlie, with their measurement settings x, y, z and outputs a, b, c, respectively. The correlations are said to be local if the joint probability distribution $p(abc|xyz)$ can be written as

$$p(abc|xyz) = \int d\lambda\, q(\lambda) p_\lambda(a|x) p_\lambda(b|y) p_\lambda(c|z), \tag{3}$$

where λ is the local random variable and $\int d\lambda q(\lambda) = 1$. A state is called genuine tripartite nonlocal if $p(abc|xyz)$ cannot be written as

$$\begin{aligned} p(abc|xyz) = &\int d\lambda q(\lambda) p_\lambda(ab|xy) p_\lambda(c|z) \\ &+ \int d\mu q(\mu) p_\mu(bc|yz) p_\mu(a|x) \\ &+ \int d\nu q(\nu) p_\nu(ac|xz) p_\nu(b|y), \end{aligned} \tag{4}$$

where $\int d\lambda\, q(\lambda) + \int d\mu\, q(\mu) + \int d\nu\, q(\nu) = 1$. A state satisfying (4) is said to admit the bi-LHV (local hidden variable) model. Svetlichny introduced an inequality to verify the genuine tripartite nonlocality. There are also two alternative definitions of n-way nonlocality and a series of Bell-type inequalities for the detection of three-way nonlocality [32]. Nevertheless, such n-way nonlocalities are strictly weaker than the Svetlichny's. The dynamics of the nonlocality measured by the violation of Svetlichny's Bell-type inequality has been investigated in the non-Markovian model [33].

To quantify the nonlocality of three-qubit states, in Ref. [34], a technique is developed to find the maximal violation of the Svetlichny inequality, and a tight upper bound is obtained. In this paper, we explicitly quantify the genuine tripartite nonlocality of the reduced states of four-qubit pure states. We first introduce the Svetlichny inequality whose violation is a signature of the genuine tripartite nonlocality. According to the maximal value of the Svetlichny operator, we show that there exists a trade-off relation among the mean values of the Svetlichny operators associated with the three-qubit reduced states of GHZ and W states. We present detailed examples to illustrate the trade-off relation among such genuine three-qubit nonlocalities. The rest of this paper is organized as follows. In Sec. II, we introduce the Svetlichny inequality. In Secs. III and IV, we investigate the trade-off for four-qubit symmetric pure states in the space spanned by Dicke states. Finally, we conclude in Sec. V.

II. SVETLICHNY INEQUALITY

We consider the nonlocality test scenario for three-qubit systems associated with Alice, Bob, and Chalie. Let the two measurement observables for Alice be $A = \vec{a} \cdot \vec{\sigma}$ and $A' = \vec{a}' \cdot \vec{\sigma}$, where $\vec{a}$ and $\vec{a}'$ are unit vectors in $\mathbb{R}^3$, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices. Each observable is an Hermitian operator with eigenvalues ± 1. Similarly, we have $B = \vec{b} \cdot \vec{\sigma}$ and $B' = \vec{b}' \cdot \vec{\sigma}$ for Bob, and $C = \vec{c} \cdot \vec{\sigma}$ and $C' = \vec{c}' \cdot \vec{\sigma}$ for Charlie. The Svetlichny operator corresponding to measurements A, A', B, B', C, and C' is defined by

$$\begin{aligned} S := &A((B + B')C + (B - B')C') \\ &+ A'((B - B')C - (B + B')C') \\ = &A(DC + D'C') + A'(D'C - DC'), \end{aligned} \tag{5}$$

where $D = B + B'$ and $D' = B - B'$.

If a three-qubit state ρ admits a bi-LHV model, then it satisfies the Svetlichny inequality [35],

$$\langle S(\rho) \rangle = \mathrm{Tr}(S\rho) \leqslant 4, \tag{6}$$

for all possible Svetlichny operators S. Conversely, a three-qubit state which violates this inequality for some S is genuine three-qubit nonlocal. To quantify the nonlocality of a three-qubit system, we need to compute the maximum of the so-called Svetlichny value,

$$S_{\max}(\rho) = \max \mathrm{Tr}(S\rho), \tag{7}$$

where the maximization is taken over all possible Svetlichny operators. Thus, $S_{\max}(\rho) > 4$ is a sufficient condition for ρ to be genuine three-qubit nonlocal. Moreover, the maximal Svetlichny value is $4\sqrt{2}$ when the Svetlichny inequality is maximally violated by, say, the GHZ state $(|000\rangle + |111\rangle)/\sqrt{2}$ [35,36]. It has been shown in Ref. [34] that for any three-qubit state ρ, the maximal value $S_{\max}$ related to the Svetlichny operator S satisfies

$$S_{\max}(\rho) \leqslant 4\lambda_1, \tag{8}$$

where λ_1 is the maximum singular value of the matrix $M = (m_{j,ik})$, with $m_{ijk} = \mathrm{Tr}\,(\rho(\sigma_i \otimes \sigma_j \otimes \sigma_k))$, $i, j, k = 1, 2, 3$.

III. TRADE-OFF RELATIONS WITH RESPECT TO FOUR-QUBIT SYMMETRIC STATES

Let $\vec{x} = (\sin\theta_x \cos\phi_x, \sin\theta_x \sin\phi_x, \cos\theta_x)$ for $x = a, a', b, b', c, c'$. Set $\vec{b} + \vec{b}' = 2\vec{d}\cos\omega$ and $\vec{b} - \vec{b}' = 2\vec{d}'\sin\omega$. If $\omega \neq \pi n/2$ for $n \in \mathbb{Z}$, $\vec{b} + \vec{b}'$ and $\vec{b} - \vec{b}'$ are mutually orthogonal. If $\omega = \pi n/2$ for $n \in \mathbb{Z}$, for example, $\omega = \pi/2$, then $\vec{d}' = \vec{b}$. We can still construct a $\vec{d}$ which is orthogonal to $\vec{d}'$ in this case. These two vectors $\vec{d}$ and $\vec{d}'$ satisfy

$$\vec{d} \cdot \vec{d}' = \cos\theta_d \cos\theta_{d'} + \sin\theta_d \sin\theta_{d'} \cos(\phi_d - \phi_{d'}) = 0, \tag{9}$$

that is, the maximum of $\cos^2\theta_d + \cos^2\theta_{d'}$ is 1, while the maximum of $\sin^2\theta_d + \sin^2\theta_{d'}$ is 2. Then, setting $D = \vec{d}\cdot\vec{\sigma}$ and $D' = \vec{d}'\cdot\vec{\sigma}$, we have

$$\begin{aligned}\langle S(\rho)\rangle &= 2|\cos\omega\langle ADC\rangle_\rho + \sin\omega\langle AD'C'\rangle_\rho \\ &\quad + \sin\omega\langle A'D'C\rangle_\rho - \cos\omega\langle A'DC'\rangle_\rho| \\ &\leqslant 2\big|\big(\langle ADC\rangle_\rho^2 + \langle AD'C'\rangle_\rho^2\big)^{1/2} \\ &\quad + \big(\langle A'D'C\rangle_\rho^2 + \langle A'DC'\rangle_\rho^2\big)^{1/2}\big|,\end{aligned} \tag{10}$$

where the following inequality has been taken into account,

$$x\cos\omega + y\sin\omega \leqslant (x^2+y^2)^{1/2}, \tag{11}$$

with the equality holding when $\tan\omega = \frac{y}{x}, x\cos\omega \geqslant 0, x \neq 0$; or $\sin\omega = \pm 1, y\sin\omega \geqslant 0, x = 0$. Equation (10) will be used in the following derivations.

Let us consider the four-qubit generalized Greenberger-Horne-Zeilinger (GGHZ) state $|\psi_{abcd}\rangle$ and the generalized maximal slice (MS) state $|\phi_{abcd}\rangle$:

$$\begin{aligned}|\psi_{abcd}\rangle &= \cos\theta|0000\rangle + \sin\theta|1111\rangle, \\ |\phi_{abcd}\rangle &= \tfrac{1}{\sqrt{2}}|0000\rangle + \tfrac{1}{\sqrt{2}}|111\rangle(\cos\theta|0\rangle + \sin\theta|1\rangle).\end{aligned} \tag{12}$$

Denote $\Psi_{abcd} = |\psi_{abcd}\rangle\langle\psi_{abcd}|$ and $\Phi_{abcd} = |\phi_{abcd}\rangle\langle\phi_{abcd}|$ as the corresponding density matrices.

Theorem 1. For four-qubit GGHZ state $\Psi_{abcd} = |\psi_{abcd}\rangle\langle\psi_{abcd}|$, the violation of the Svetlichny inequality on any three-qubit states satisfies the following relation:

$$\langle S(\Psi_{abc})\rangle + \langle S(\Psi_{abd})\rangle + \langle S(\Psi_{acd})\rangle + \langle S(\Psi_{bcd})\rangle \leqslant 16|\cos 2\theta|, \tag{13}$$

where $\Psi_{abc} = \Psi_{abd} = \Psi_{acd} = \Psi_{bcd} = \cos^2\theta|000\rangle\langle 000| + \sin^2\theta|111\rangle\langle 111|$ are the corresponding reduced three-qubit states. The equality holds in (13) when

$$|\cos\theta_a\cos\theta_c - \cos\theta_{a'}\cos\theta_{c'}| = 2, \omega = \theta_d = 0, \theta_{d'} = \pi/2.$$

See the proof in Appendix A.

When the equality holds in (13), namely, we have $S_{\max}(\Psi_{abc}) = S_{\max}(\Psi_{abd}) = S_{\max}(\Psi_{acd}) = S_{\max}(\Psi_{bcd}) = 4|\cos 2\theta| \leqslant 4$. It means that in this case all the reduced states of GGHZ state do not violate the Svetlichny inequality.

For the GGHZ state, the four reduced three-qubit states are the same. From (8), the maximal value of the Svetlichny operator is $4\max\{\cos^4\theta, \sin^4\theta\}$ for any one of such reduced three-qubit states. It is remarkable that the upper bound in (13) is always less or equal to the upper bound $16\max\{\cos^4\theta, \sin^4\theta\}$ derived from (8); see Fig. 1 for $\theta \in [0, \frac{\pi}{4}]$.

Generalizing Theorem 1 to general n-qubit case, we have for $n \geqslant 4$ the following:

Corollary 1. For n-qubit GGHZ state $|\Psi\rangle = \cos\theta|00\ldots0\rangle + \sin\theta|11\ldots1\rangle$, the violation of the Svetlichny inequality on any three-qubit states satisfies the following relation:

$$\sum_{1\leqslant I<J<K\leqslant n} \langle S(\Psi_{IJK})\rangle \leqslant 4\binom{n}{3}|\cos 2\theta|, \tag{14}$$

where $\Psi_{IJK} = \mathrm{Tr}_{\overline{IJK}}|\Psi\rangle\langle\Psi| = \cos^2\theta|000\rangle\langle 000|_{IJK} + \sin^2\theta|111\rangle\langle 111|_{IJK}$ are the corresponding reduced three-qubit states associated with qubits I, J, and K, and $\mathrm{Tr}_{\overline{IJK}}$ stands for the trace over the rest qubit systems.

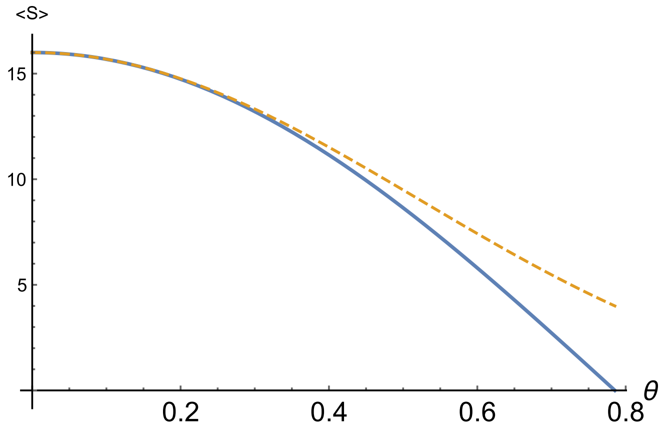

FIG. 1. For $\theta \in [0, \frac{\pi}{4}]$, the upper bound of the sum of violations of the Svetlichny inequality for four reduced three-qubit states is $16|\cos 2\theta|$. It is less or equal to $16\max\{\cos^4\theta, \sin^4\theta\} = 16\cos^4\theta$ derived from (8). The blue line is the bound from Theorem 1. The yellow dashed one comes from (8). When $\theta = 0$, two bounds are equal.

Theorem 2. For four-qubit generallized MS states Φ_{abcd}, the violation of the Svetlichny inequality on the reduced three-qubit density matrices satisfies the following relation:

$$\begin{aligned}&\langle S(\Phi_{abc})\rangle + \langle S(\Phi_{abd})\rangle + \langle S(\Phi_{acd})\rangle + \langle S(\Phi_{bcd})\rangle \\ &\quad\leqslant 4\sqrt{2}|\cos\theta| + 12|\cos^2\theta + \tfrac{1}{2}\sin 2\theta|,\end{aligned} \tag{15}$$

where

$$\begin{aligned}\Phi_{abc} &= \tfrac{1}{2}|000\rangle\langle 000| + \tfrac{1}{2}\cos\theta|000\rangle\langle 111| \\ &\quad + \tfrac{1}{2}\cos\theta|111\rangle\langle 000| + \tfrac{1}{2}|111\rangle\langle 111|, \\ \Phi_{abd} &= \Phi_{acd} = \Phi_{bcd} \\ &= \tfrac{1}{2}|000\rangle\langle 000| + \tfrac{1}{2}\cos^2\theta|110\rangle\langle 110| \\ &\quad + \tfrac{1}{2}\cos\theta\sin\theta|110\rangle\langle 111| + \tfrac{1}{2}\cos\theta\sin\theta|111\rangle\langle 110| \\ &\quad + \tfrac{1}{2}\sin^2\theta|111\rangle\langle 111|.\end{aligned} \tag{16}$$

See the proof in Appendix B.

Inequality (15) gives a trade off relation of among the three-qubit genuine nonlocalities in MS states. In fact, by using (8) for any three-qubit states of a MS state, one has

$$\langle S(\Phi_{abc})\rangle + \langle S(\Phi_{abd})\rangle + \langle S(\Phi_{acd})\rangle + \langle S(\Phi_{bcd})\rangle \leqslant 20\cos^2\theta. \tag{17}$$

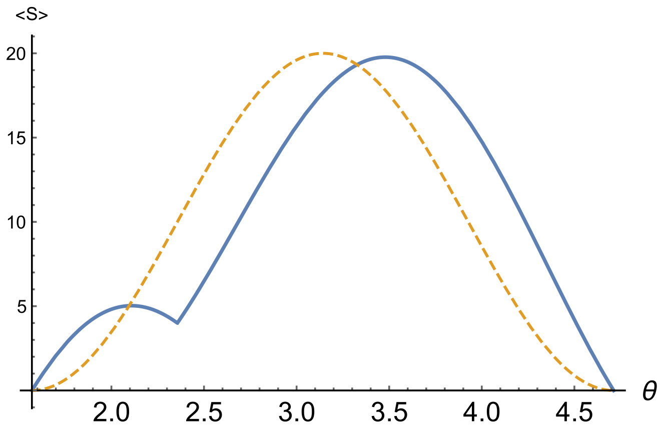

FIG. 2. The blue line is for the upper bound of (15), and the yellow dashed line is for the upper bound of (17) for $\theta \in (\pi/2, 3\pi/2)$.

Nevertheless, the upper bound of (17) is larger than the one of (15); see Fig. 2 for $\theta \in (\pi/2, 3\pi/2)$.

Now consider the n-qubit generalized MS states,

$$|\Psi_{12\ldots n}\rangle = \tfrac{1}{\sqrt{2}}|00\ldots 0\rangle + \tfrac{1}{\sqrt{2}}|11\ldots 1\rangle|\psi\rangle, \qquad (18)$$

where $|\psi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$. Let $\mathcal{I}$ denote a proper subset of $\{1, 2, \ldots, n\}$. We define the states with $n \notin \mathcal{I}$ as the class I ($\#\mathcal{I} = m < n$) and $n \in \mathcal{I}$ as the class II. Then, there are $[\binom{n}{m} - \binom{n}{m-1}]$ states in class I and $\binom{n}{m-1}$ states in class II:

$$\rho_{\mathcal{I}} = \begin{cases} \frac{1}{2}|0\ldots 00\rangle\langle 0\ldots 00| + \frac{1}{2}|1\ldots 11\rangle\langle 1\ldots 11| & n \notin \mathcal{I} \\ \frac{1}{2}|0\ldots 00\rangle\langle 0\ldots 00| + \frac{1}{2}|1\ldots 1\psi\rangle\langle 1\ldots 1\psi| & n \in \mathcal{I} \end{cases}. \qquad (19)$$

From Theorem 2, we have the following corollary:

Corollary 2. For n-qubit generallized MS states Φ_{abcd}, the violation of the Svetlichny inequality on the reduced three-qubit density matrices satisfies the following relation:

$$\sum_{1\leqslant I<J<K\leqslant n} \langle S(\Psi_{IJK})\rangle \leqslant 4\sqrt{2}\binom{n-1}{2}|\cos\theta| + 4\left[\binom{n}{3} - \binom{n-1}{2}\right]\left|\cos^2\theta + \frac{1}{2}\sin 2\theta\right|, \qquad (20)$$

where $\Psi_{IJK}=\mathrm{Tr}_{\overline{IJK}}|\Psi\rangle\langle\Psi|=\frac{1}{2}|000\rangle\langle 000|_{IJK}+\frac{1}{2}|111\rangle\langle 111|_{IJK}$ for Ψ_{IJK} belonging to class I and $\Psi_{IJK}=\mathrm{Tr}_{\overline{IJK}}|\Psi\rangle\langle\Psi| =\frac{1}{2}|000\rangle\langle 000|+\frac{1}{2}\cos^2\theta|110\rangle\langle 110| + \frac{1}{2}\cos\theta\sin\theta|110\rangle\langle 111| + \frac{1}{2}\cos\theta\sin\theta|111\rangle\langle 110| + \frac{1}{2}\sin^2\theta|111\rangle\langle 111|$ for Ψ_{IJK} belonging to class II.

IV. TRADE-OFF RELATIONS FOR THE W-CLASS STATES

For a four-qubit state,

$$|\varphi\rangle_{abcd} = \alpha|1000\rangle + \beta|0100\rangle + \gamma|0010\rangle + \delta|0001\rangle + \lambda|0000\rangle, \qquad (21)$$

with $\alpha, \beta, \gamma, \delta, \lambda$ are real numbers. It can generate four-qubit quantum states by unitary operators. We consider a trade-off relation between the reduced states of $|\psi_{abcd}\rangle$. Denote $G(x, y, u, v) = 2[(2x + 2y)^{\frac{1}{2}} + (2x + 8y + 8u^2v^2]^{\frac{1}{2}})$.

Theorem 3. For any four-qubit state $|\varphi\rangle_{abcd}$, the violation of Svetlichny operators on tripartite states satisfies the following relation:

$$\begin{aligned} &\langle S(\rho_{abc})\rangle + \langle S(\rho_{abd})\rangle + \langle S(\rho_{acd})\rangle + \langle S(\rho_{bcd})\rangle \\ &\quad \leqslant G(x_1, y_1, \beta, \gamma) + G(x_2, y_2, \beta, \delta) \\ &\quad\quad + G(x_3, y_3, \delta, \lambda) + G(x_4, y_4, \delta, \gamma), \end{aligned} \qquad (22)$$

where $\rho_{abc} = \mathrm{Tr}_d|\varphi\rangle\langle\varphi|_{abcd}$, $\rho_{abd} = \mathrm{Tr}_c|\varphi\rangle\langle\varphi|_{abcd}$, $\rho_{acd} = \mathrm{Tr}_b|\varphi\rangle\langle\varphi|_{abcd}$, $\rho_{bcd} = \mathrm{Tr}_a|\varphi\rangle\langle\varphi|_{abcd}$, and

$$\begin{aligned} x_1 &= (\alpha^2 + \beta^2 + \gamma^2 - \delta^2 - \lambda^2)^2, \\ y_1 &= \beta^2\gamma^2 + \alpha^2\lambda^2 + \tfrac{3}{2}\alpha^2\beta^2 + \gamma^2\lambda^2 + \tfrac{3}{2}\alpha^2\gamma^2 + \beta^2\lambda^2, \\ x_2 &= (\alpha^2 + \beta^2 - \gamma^2 + \delta^2 - \lambda^2)^2, \\ y_2 &= \beta^2\gamma^2 + \alpha^2\lambda^2 + \tfrac{3}{2}\alpha^2\beta^2 + \delta^2\lambda^2 + \tfrac{3}{2}\alpha^2\delta^2 + \delta^2\beta^2, \\ x_3 &= (\alpha^2 - \beta^2 + \gamma^2 + \delta^2 - \lambda^2)^2, \\ y_3 &= \tfrac{3}{2}\alpha^2\delta^2 + \alpha^2\lambda^2 + \tfrac{3}{2}\alpha^2\gamma^2 + \delta^2\lambda^2 + \delta^2\gamma^2 + \lambda^2\gamma^2, \\ x_4 &= (-\alpha^2 + \beta^2 + \gamma^2 + \delta^2 - \lambda^2)^2, \\ y_4 &= \tfrac{3}{2}\beta^2\gamma^2 + \beta^2\lambda^2 + \delta^2\gamma^2 + \delta^2\lambda^2 + \gamma^2\lambda^2 + \tfrac{3}{2}\delta^2\beta\gamma. \end{aligned}$$

See the proof in Appendix C.

The n-qubit Dicke state is an n-partite symmetric state defined as $|\mathfrak{D}(n, m)\rangle = \binom{n}{m}^{-1/2}\sum_{P\in\mathcal{P}} P(|0\rangle^{\otimes m} \otimes |1\rangle^{\otimes(n-m)})$, where $\mathcal{P}$ is the permutation group of n elements. The state $|\mathfrak{D}(4, 1)\rangle$ is the standard four-qubit W state. When $\lambda = 0$, the state (21) reduces to the four-qubit W-class state:

$$|\varphi\rangle_{W_{abcd}} = \alpha|1000\rangle + \beta|0100\rangle + \gamma|0010\rangle + \delta|0001\rangle. \qquad (23)$$

For the state (22) reduces to

$$\begin{aligned} &\langle S(W_{abc})\rangle^2 + \langle S(W_{abd})\rangle^2 + \langle S(W_{acd})\rangle^2 + \langle S(W_{bcd})\rangle^2 \\ &\quad \leqslant 64(1 + \alpha^2\gamma^2 + \beta^2\delta^2 + 2\alpha^2\beta^2 + 2\beta^2\gamma^2 + 2\gamma^2\delta^2), \end{aligned} \qquad (24)$$

where W_{abc}, W_{abd}, W_{acd}, and W_{bcd} denote the corresponding reduced states of $|\varphi\rangle\langle\varphi|_{W_{abcd}}$.

However, from (8) the violation of Svetlichny operators for tripartite states W_{abc}, W_{abd}, W_{acd}, and W_{bcd} satisfy the

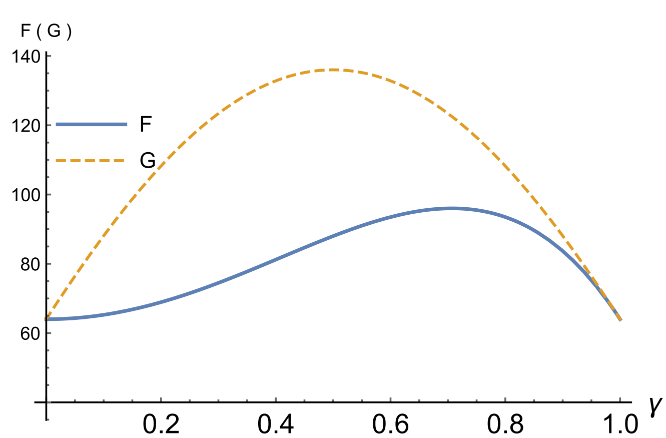

FIG. 3. In the range of $\gamma \in [0, 1]$, the bound F is smaller than G.

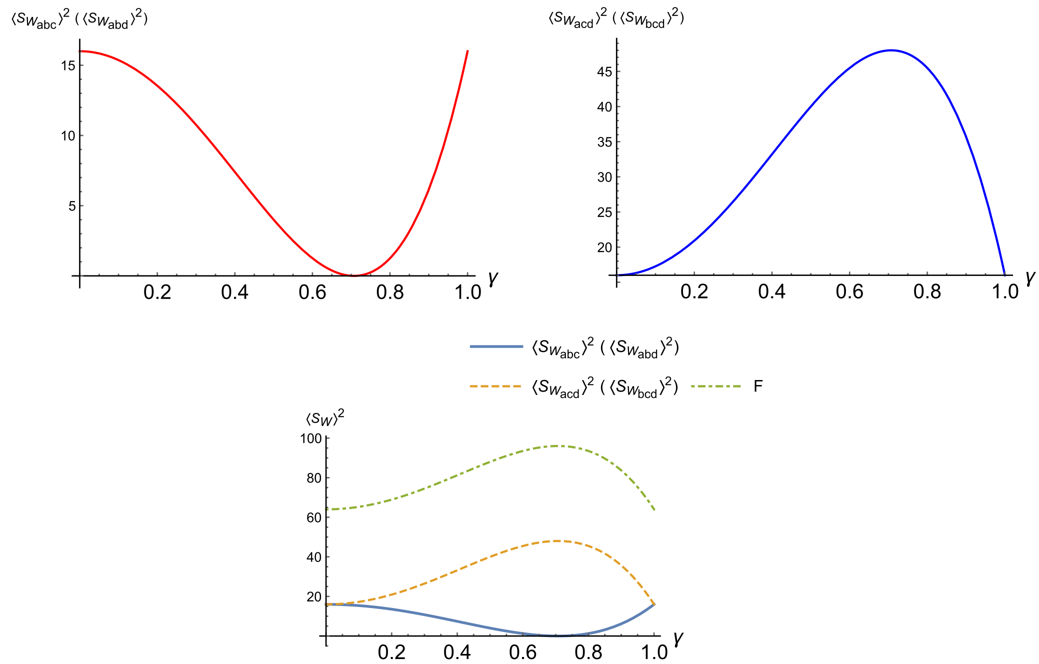

FIG. 4. Top left figure: violation of Svetlichny operators for states W_{abc} and W_{abd}. Top right figure: violation of Svetlichny operators for states W_{acd} and W_{bcd}. Bottom figure: trade-off relation among the nonlocallity of W-class states. For $\gamma \in [0, 1]$, the quantities $\langle S_{W_{abc}}\rangle^2$ $(= \langle S_{W_{abd}}\rangle^2)$ and $\langle S_{W_{acd}}\rangle^2 (= \langle S_{W_{bcd}}\rangle^2)$ vary in a way such that their summation kept to be bounded by F.

following relations:

$$\begin{aligned}
\langle S(W_{abc})\rangle &\leqslant 4\max\{\sqrt{4(\alpha\beta^2+\alpha\gamma^2)}, \sqrt{8\beta\gamma^2+(2\delta^2-1)^2}\},\\
\langle S(W_{abd})\rangle &\leqslant 4\max\{\sqrt{4(\alpha\beta^2+\alpha\delta^2)}, \sqrt{8\beta\delta^2+(2\gamma^2-1)^2}\},\\
\langle S(W_{acd})\rangle &\leqslant 4\max\{\sqrt{4(\alpha\gamma^2+\alpha\delta^2)}, \sqrt{8\gamma\delta^2+(2\beta^2-1)^2}\},\\
\langle S(W_{bcd})\rangle &\leqslant 4\max\{\sqrt{4(\beta\gamma^2+\beta\delta^2)}, \sqrt{8\gamma\delta^2+(2\alpha^2-1)^2}\}.
\end{aligned} \tag{25}$$

Accounting to the fact that for positive X and Y, $\max\{X, Y\} = \frac{|X-Y|+|X+Y|}{2}$, one has

$$\begin{aligned}
&\langle S(W_{abc})\rangle^2 + \langle S(W_{abd})\rangle^2 + \langle S(W_{acd})\rangle^2 + \langle S(W_{bcd})\rangle^2\\
&\leqslant 8[|4(\alpha\beta^2+\alpha\gamma^2) - 8\beta\gamma^2 - (2\delta^2-1)^2|\\
&\quad + |4(\alpha\beta^2+\alpha\delta^2) - 8\beta\delta^2 - (2\gamma^2-1)^2|\\
&\quad + |4(\beta\gamma^2+\beta\delta^2) - 8\gamma\delta^2 - (2\alpha^2-1)^2|\\
&\quad + |4(\alpha\gamma^2+\alpha\delta^2) - 8\gamma\delta^2 - (2\beta^2-1)^2|\\
&\quad + 8(\alpha\beta^2+\alpha\gamma^2+\alpha\delta^2+\tfrac{3}{2}\beta\gamma^2+\tfrac{3}{2}\beta\delta^2+2\gamma\delta^2)\\
&\quad + (2\alpha^2-1)^2 + (2\beta^2-1)^2 + (2\gamma^2-1)^2 + (2\delta^2-1)^2].
\end{aligned} \tag{26}$$

Denote F and G as the right sides of (24) and (26), respectively. Figure 3 shows that the value of F is always less than G in the range $\gamma \in [0, 1]$ for $\alpha = \beta = 0$ and $\delta^2 = 1 - \gamma^2$.

Equation (24) also gives a kind of trade-off relation among the quantum nonlocality of the reduced states. The maximum value $704/7$ of F is attained at $\{\alpha, \beta, \gamma, \delta\} = \{0, \sqrt{2/7}\sqrt{3/7}, \sqrt{2/7}\}$. Figure 4 shows the detailed trade-off relations among $\langle S_{W_{abc}}\rangle^2$, $\langle S_{W_{abd}}\rangle^2$, $\langle S_{W_{acd}}\rangle^2$, and $\langle S_{W_{bcd}}\rangle^2$. Here, for $\alpha = \beta = 0$ and $\delta^2 = 1 - \gamma^2$, we have $\langle S_{W_{abc}}\rangle^2 = \langle S_{W_{abd}}\rangle^2$ and $\langle S_{W_{acd}}\rangle^2 = \langle S_{W_{bcd}}\rangle^2$.

V. CONCLUSIONS

We have studied the trade-off relationship of genuine tripartite nonlocality in a multipartite system, and the corresponding tight upper bounds for GHZ-class states and W-class states have presented, showing that the genuine three-qubit nonlocalities are not independent in a four-qubit system. Meanwhile, we have identified that the reduced three-qubit states of a four-qubit GHZ state cannot violate the Svetchlity inequality. Our approach may be also used to investigate the trade-off relations of genuine nonlocalities satisfied by the reduced tripartite states of a more general multipartite system.

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APPENDIX A: PROOF OF THEOREM 1

By straightforward computation, we have

$$\langle ADC\rangle_{\Psi_{abc}} = \cos 2\theta \cos\theta_a \cos\theta_c \cos\theta_d, \tag{A1}$$

and similar expressions for $\langle AD'C'\rangle_{\Psi_{abc}}$, $\langle A'D'C\rangle_{\Psi_{abc}}$ and $\langle A'DC'\rangle_{\Psi_{abc}}$. From (10), we have

$$\begin{aligned}\langle S(\Psi_{abc})\rangle &= 2\big|\cos\omega\langle ADC\rangle_{\Psi_{abc}} + \sin\omega\langle AD'C'\rangle_{\Psi_{abc}} + \sin\omega\langle A'D'C\rangle_{\Psi_{abc}} - \cos\omega\langle A'DC'\rangle_{\Psi_{abc}}\big|\\ &\leqslant 2\big|\big(\langle ADC\rangle^2_{\Psi_{abc}} + \langle AD'C'\rangle^2_{\Psi_{abc}}\big)^{1/2} + \big(\langle A'D'C\rangle^2_{\Psi_{abc}} + \langle A'DC'\rangle^2_{\Psi_{abc}}\big)^{1/2}\big|\\ &= 2\big|\cos 2\theta\cos\theta_a(\cos^2\theta_c\cos^2\theta_d + \cos^2\theta_{c'}\cos^2\theta_{d'})^{\frac{1}{2}} + \cos 2\theta\cos\theta_{a'}(\cos^2\theta_c\cos^2\theta_{d'} + \cos^2\theta_{c'}\cos^2\theta_d)^{\frac{1}{2}}\big|.\end{aligned} \tag{A2}$$

Since the maximum of $\cos^2\theta_d + \cos^2\theta_{d'}$ is 1 [37], the above formula can be further reduced to

$$\langle S(\Psi_{abc})\rangle \leqslant 2|\cos 2\theta|(|\cos\theta_a| + |\cos\theta_{a'}|) \leqslant 4|\cos 2\theta|. \tag{A3}$$

Since $\langle S(\Psi_{abc})\rangle = \langle S(\Psi_{abd})\rangle = \langle S(\Psi_{acd})\rangle = \langle S(\Psi_{bcd})\rangle \leqslant 4|\cos 2\theta|$ for the state $\Psi_{abcd} = |\psi_{abcd}\rangle\langle\psi_{abcd}|$, one gets the inequality (13).

APPENDIX B: PROOF OF THEOREM 2

For the reduced state Φ_{abc}, one has the expectation value of the Svetlichny operator,

$$\langle ADC\rangle_{\Phi_{abc}} = \cos\theta\sin\theta_a\sin\theta_c\sin\theta_d\cos(\phi_a + \phi_c + \phi_d).$$

$\langle AD'C'\rangle_{\Phi_{abc}}$, $\langle A'D'C\rangle_{\Phi_{abc}}$, and $\langle A'DC'\rangle_{\Phi_{abc}}$ have similar expressions. Therefore, we have

$$\begin{aligned}\langle S(\Phi_{abc})\rangle &= 2\big|\cos\omega\langle ADC\rangle_{\Phi_{abc}} + \sin\omega\langle AD'C'\rangle_{\Phi_{abc}} + \sin\omega\langle A'D'C\rangle_{\Phi_{abc}} - \cos\omega\langle A'DC'\rangle_{\Phi_{abc}}\big|\\ &\leqslant 2\big|\big(\langle ADC\rangle^2_{\Phi_{abc}} + \langle AD'C'\rangle^2_{\Phi_{abc}}\big)^{1/2} + \big(\langle A'D'C\rangle^2_{\Phi_{abc}} + \langle A'DC'\rangle^2_{\Phi_{abc}}\big)^{1/2}\big|\\ &\leqslant 2|\{[\cos\theta\sin\theta_a\sin\theta_c\sin\theta_d\cos(\phi_a+\phi_c+\phi_d)]^2 + [\cos\theta\sin\theta_a\sin\theta_{c'}\sin\theta_{d'}\cos(\phi_{c'}+\phi_{d'}+\phi_a)]^2\}^{1/2}\\ &\quad + \{[\cos\theta\sin\theta_c\sin\theta_{a'}\sin\theta_{d'}\cos(\phi_{a'}+\phi_{d'}+\phi_c)]^2 + [\cos\theta\cos\omega\sin\theta_d\sin\theta_{a'}\sin\theta_{c'}\cos(\phi_{a'}+\phi_{c'}+\phi_d)]^2\}^{1/2}|\\ &\leqslant 2|(\cos^2\theta\sin^2\theta_d + \cos^2\theta\sin^2\theta_{d'})^{1/2} + (\cos^2\theta\sin^2\theta_{d'} + \cos^2\theta\sin^2\theta_d)^{1/2}|\\ &\leqslant 4|\cos\theta(\sin^2\theta_d + \sin^2\theta_{d'})^{1/2}|\\ &\leqslant 4\sqrt{2}|\cos\theta|.\end{aligned} \tag{B1}$$

When $\phi_i + \phi_j + \phi_k = 0$, where $i \in \{a, a'\}$, $j \in \{d, d'\}$, and $k \in \{c, c'\}$, one has $\langle S(\Phi_{abc})\rangle = 4\sqrt{2}|\cos\theta|$.

For the reduced state Φ_{abd}, we have

$$\langle ADC\rangle_{\Phi_{abd}} = \tfrac{1}{2}\cos\theta_a\cos\theta_d(\sin 2\theta\sin\theta_c\cos\phi_c + 2\cos^2\theta\cos\theta_c). \tag{B2}$$

The expressions for $\langle AD'C'\rangle_{\Phi_{abd}}$, $\langle A'D'C\rangle_{\Phi_{abd}}$, and $\langle A'DC'\rangle_{\Phi_{abd}}$ are similar. By direct computation, we obtain

$$\begin{aligned}\langle S(\Phi_{abd})\rangle &= 2\big|\cos\omega\langle ADC\rangle_{\Phi_{abd}} + \sin\omega\langle AD'C'\rangle_{\Phi_{abd}} + \sin\omega\langle A'D'C\rangle_{\Phi_{abd}} - \cos\omega\langle A'DC'\rangle_{\Phi_{abd}}\big|\\ &\leqslant 2\big|\big(\langle ADC\rangle^2_{\Phi_{abd}} + \langle AD'C'\rangle^2_{\Phi_{abd}}\big)^{1/2} + \big(\langle A'D'C\rangle^2_{\Phi_{abd}} + \langle A'DC'\rangle^2_{\Phi_{abd}}\big)^{1/2}\big|\\ &= 2\Big|\Big\{\Big[\tfrac{1}{2}\cos\theta_a\cos\theta_d(\sin 2\theta\sin\theta_c\cos\phi_c + 2\cos^2\theta\cos\theta_c)\Big]^2\\ &\quad + \Big[\tfrac{1}{2}\cos\theta_a\cos\theta_{d'}(\sin 2\theta\sin\theta_{c'}\cos\phi_{c'} + 2\cos^2\theta\cos\theta_{c'})\Big]^2\Big\}^{1/2}\\ &\quad + \Big\{\Big[\tfrac{1}{2}\cos\theta_{a'}\cos\theta_{d'}(\sin 2\theta\sin\theta_c\cos\phi_c + 2\cos^2\theta\cos\theta_c)\Big]^2\\ &\quad + \Big[\tfrac{1}{2}\cos\theta_d\cos\theta_{a'}(\sin 2\theta\sin\theta_{c'}\cos\phi_{c'} + 2\cos^2\theta\cos\theta_{c'})\Big]^2\Big\}^{1/2}\Big|\\ &\leqslant 4\Big|\Big(\tfrac{1}{4}\sin^2 2\theta + \sin 2\theta\cos^2\theta + \cos^4\theta\Big)^{1/2}\Big|\\ &\leqslant 4\Big|\cos^2\theta + \tfrac{1}{2}\sin 2\theta\Big|.\end{aligned} \tag{B3}$$

Taking into account that $\langle S(\Phi_{abd})\rangle = \langle S(\Phi_{acd})\rangle = \langle S(\Phi_{bcd})\rangle$, one proves the theorem.

APPENDIX C: PROOF OF THEOREM 3

For the reduced state ρ_{abc},

$$\begin{aligned}\rho_{abc} &= \mathrm{Tr}_d|\varphi\rangle\langle\varphi|_{abcd}\\ &= \alpha^2|100\rangle\langle 100| + \alpha\beta|100\rangle\langle 010| + \alpha\gamma|100\rangle\langle 001| + \alpha\lambda|100\rangle\langle 000| + \alpha\beta|010\rangle\langle 100| + \beta^2|010\rangle\langle 010|\\ &\quad + \beta\gamma|010\rangle\langle 001| + \beta\lambda|010\rangle\langle 000| + \alpha\gamma|001\rangle\langle 100| + \beta\gamma|001\rangle\langle 010| + \gamma^2|001\rangle\langle 001| + \gamma\lambda|001\rangle\langle 000|\\ &\quad + \delta^2|000\rangle\langle 000| + \alpha\lambda|000\rangle|100\rangle + \beta\lambda|000\rangle\langle 010| + \gamma\lambda|000\rangle\langle 001| + \lambda^2|000\rangle\langle 000|,\end{aligned} \tag{C1}$$

we can obtain

$$\begin{aligned}\langle ADC\rangle_{\rho_{abc}} &= -(\alpha^2+\beta^2+\gamma^2-\sigma^2-\lambda^2)\cos\theta_a\cos\theta_c\cos\theta_d + 2\beta\gamma\cos(\phi_c-\phi_d)\cos\theta_a\sin\theta_c\sin\theta_d\\ &\quad + 2\alpha\lambda\cos\phi_a\sin\theta_a\cos\theta_c\cos\theta_d + 2\alpha\beta\cos(\phi_a-\phi_d)\sin\theta_a\cos\theta_c\sin\theta_d\\ &\quad + 2\gamma\lambda\cos\phi_c\cos\theta_a\sin\theta_c\cos\theta_d + 2\alpha\gamma\cos(\phi_a-\phi_c)\sin\theta_a\sin\theta_c\cos\theta_d\\ &\quad + 2\beta\lambda\cos\phi_d\cos\theta_a\cos\theta_c\sin\theta_d.\end{aligned} \tag{C2}$$

Let

$$\begin{aligned} u_1 &= -(\alpha^2+\beta^2+\gamma^2-\sigma^2-\lambda^2), \quad v_1 = 0,\\ u_2 &= 2\beta\gamma\cos(\phi_c-\phi_d), \quad v_2 = 2\alpha\lambda\cos\phi_a,\\ u_3 &= 2\alpha\beta\cos(\phi_a-\phi_d), \quad v_3 = 2\gamma\lambda\cos\phi_c,\\ u_4 &= 2\alpha\gamma\cos(\phi_a-\phi_c), \quad v_4 = 2\beta\lambda\cos\phi_d,\end{aligned} \tag{C3}$$

and

$$\begin{aligned} x_1 &= \cos\theta_a\cos\theta_c\cos\theta_d, \quad y_1 = \sin\theta_a\sin\theta_c\sin\theta_d,\\ x_2 &= \cos\theta_a\sin\theta_c\sin\theta_d, \quad y_2 = \sin\theta_a\cos\theta_c\cos\theta_d,\\ x_3 &= \sin\theta_a\cos\theta_c\sin\theta_d, \quad y_3 = \cos\theta_a\sin\theta_c\cos\theta_d,\\ x_4 &= \sin\theta_a\sin\theta_c\cos\theta_d, \quad y_4 = \cos\theta_a\cos\theta_c\sin\theta_d.\end{aligned} \tag{C4}$$

One can verify that $\sum_{j=1}^4(x_j^2+y_j^2)=1$. Hence, we consider the following optimization:

$$\max\left(\sum_i u_i x_i + \sum_j v_j y_j\right) \text{ such that } \sum_{j=1}^4\left(x_j^2+y_j^2\right)=1. \tag{C5}$$

Using the Lagrange multiplier, we have the maximum $k=\sqrt{\sum_i(u_i^2+v_i^2)}$. It follows that the maximum is attained when each $\cos\phi=\pm1$.

Therefore, we have

$$\langle ADC\rangle_{\rho_{abc}} \leqslant \sqrt{(\alpha^2+\beta^2+\gamma^2-\sigma^2-\lambda^2)^2+4(\beta^2\gamma^2+\alpha^2\lambda^2+\alpha^2\beta^2+\gamma^2\lambda^2+\alpha^2\gamma^2+\beta^2\lambda^2)}.$$

Similarly, we have

$$\begin{aligned}\langle AD'C'\rangle_{\rho_{abc}} &\leqslant \sqrt{(\alpha^2+\beta^2+\gamma^2-\delta^2-\lambda^2)^2+4(\beta^2\gamma^2+\alpha^2\lambda^2+2\alpha^2\beta^2+\gamma^2\lambda^2+2\alpha^2\gamma^2+\beta^2\lambda^2)},\\ \langle A'D'C\rangle_{\rho_{abc}} &\leqslant \sqrt{(\alpha^2+\beta^2+\gamma^2-\delta^2-\lambda^2)^2+4(2\beta^2\gamma^2+\alpha^2\lambda^2+\alpha^2\beta^2+\gamma^2\lambda^2+2\alpha^2\gamma^2+\beta^2\lambda^2)},\\ \langle A'DC'\rangle_{\rho_{abc}} &\leqslant \sqrt{(\alpha^2+\beta^2+\gamma^2-\delta^2-\lambda^2)^2+4(2\beta^2\gamma^2+\alpha^2\lambda^2+2\alpha^2\beta^2+\gamma^2\lambda^2+\alpha^2\gamma^2+\beta^2\lambda^2)}.\end{aligned} \tag{C6}$$

Therefore, concerning the violation of the Svetlichny inequality with respect to the reduced state ρ_{abc}, we have

$$\begin{aligned}\langle S(\rho_{abc})\rangle &= 2\left|\cos\theta\langle ADC\rangle_{\rho_{abc}} + \sin\theta\langle AD'C'\rangle_{\rho_{abc}} + \sin\theta\langle A'D'C\rangle_{\rho_{abc}} - \cos\theta\langle A'DC'\rangle_{\rho_{abc}}\right|\\ &\leqslant 2\left|\left(\langle ADC\rangle_{\rho_{abc}}^2 + \langle AD'C'\rangle_{\rho_{abc}}^2\right)^{1/2} + \left(\langle A'D'C\rangle_{\rho_{abc}}^2 + \langle A'DC'\rangle_{\rho_{abc}}^2\right)^{1/2}\right|\\ &= 2\left[(2x_1+8y_1)^{\frac{1}{2}} + (2x_1+8y_1+8\beta^2\gamma^2)^{\frac{1}{2}}\right],\end{aligned} \tag{C7}$$

where $x_1=(\alpha^2+\beta^2+\gamma^2-\delta^2-\lambda^2)^2$, $y_1=\beta^2\gamma^2+\alpha^2\lambda^2+\frac{3}{2}\alpha^2\beta^2+\gamma^2\lambda^2+\frac{3}{2}\alpha^2\gamma^2+\beta^2\lambda^2$. Similarly, with respect to the reduced states ρ_{abd}, ρ_{acd} and ρ_{bcd}, we get

$$\langle S(\rho_{abd})\rangle \leqslant 2\left[(2x_2+8y_2)^{\frac{1}{2}} + (2x_2+8y_2+8\beta^2\delta^2)^{\frac{1}{2}}\right], \tag{C8}$$

where $x_2 = (\alpha^2 + \beta^2 - \gamma^2 + \delta^2 - \lambda^2)^2$, $y_2 = \beta^2\gamma^2 + \alpha^2\lambda^2 + \frac{3}{2}\alpha^2\beta^2 + \delta^2\lambda^2 + \frac{3}{2}\alpha^2\delta^2 + \delta^2\beta^2$.

$$\langle S(\rho_{acd})\rangle \leqslant 2\big[(2x_3 + 8y_3)^{\frac{1}{2}} + (2x_3 + 8y_3 + 8\delta^2\lambda^2)^{\frac{1}{2}}\big], \tag{C9}$$

where $x_3 = (\alpha^2 - \beta^2 + \gamma^2 + \delta^2 - \lambda^2)^2$, $y_3 = \frac{3}{2}\alpha^2\delta^2 + \alpha^2\lambda^2 + \frac{3}{2}\alpha^2\gamma^2 + \delta^2\lambda^2 + \delta^2\gamma^2 + \lambda^2\gamma^2$.

$$\langle S(\rho_{bcd})\rangle \leqslant 2\big[(2x_4 + 8y_4)^{\frac{1}{2}} + (2x_4 + 8y_4 + 8\delta^2\gamma^2)^{\frac{1}{2}}\big], \tag{C10}$$

where $x_4 = (-\alpha^2 + \beta^2 + \gamma^2 + \delta^2 - \lambda^2)^2$, $y_4 = \frac{3}{2}\beta^2\gamma^2 + \beta^2\lambda^2 + \delta^2\gamma^2 + \delta^2\lambda^2 + \gamma^2\lambda^2 + \frac{3}{2}\delta^2\beta\gamma$.

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