

Enhanced excitation of a driven bistable system induced by spectrum degeneracyEvgeny V. Anikin¹, Natalya S. Maslova,² Nikolay A. Gippius,¹ and Igor M. Sokolov³¹*Skolkovo Institute of Science and Technology, 121205 Moscow, Russia*²*Chair of Quantum Electronics and Quantum Technology Centrum, Department of Physics, Lomonosov Moscow State University, 119991, Moscow, Russia*³*Institut für Physik and IRIS Adlershof, Humboldt Universität zu Berlin, Newtonstraße 15, 12489 Berlin, Germany*

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The nonequilibrium statistics and kinetics of a simple bistable system (resonantly driven nonlinear oscillator coupled to reservoir) have been investigated by means of master equation for the density matrix and quasiclassical Fokker–Planck equation in quasienergy space. We found that the system’s statistical and kinetic properties drastically change when the quasienergy states become nearly degenerate and the occupation of the most excited state is strongly enhanced. It has been revealed that, in nearly degenerate case, a critical quasienergy parameter emerges. Below the critical quasienergy value the eigenstates are superpositions of the quasiclassical states from different phase-space regions, while above this value the eigenstates correspond to only one particular region of the phase space. We have also generalized Keldysh theory for ionization of atoms in the electromagnetic field for bistable systems. It has been demonstrated that Keldysh parameter in the bistability region is large when pumping intensity is smaller than the critical value. It has been shown by direct calculations that multiphoton transition amplitude coincides with the tunneling amplitude. So, multiphoton transitions and tunneling between the regions of the phase space are just the same effects. We also demonstrated that, for bistable systems, the Keldysh parameter logarithmically depends on the external field amplitude.

DOI: [10.1103/PhysRevA.100.043842](https://doi.org/10.1103/PhysRevA.100.043842)**I. INTRODUCTION**

Nowadays bistability is one of the most pronounced phenomena in modern optics and electronics. It has broad applications in all-optical logic and memory performance. So controllable changing of the occupation of different stable states and the control of transition rates between them are among the most important problems. The solution of these problems relies on one’s knowledge of optimal perturbation which transfers the system from one stable state to another. Another problem is how to control the structure of stable states by changing the system parameters such as external field frequency and intensity.

In the pioneering works on bistability [1–3], the systems of two-level atoms in a laser cavity under external pumping were considered. It was shown that the intensity of light transmitted through the cavity varies discontinuously under changing of external pumping and exhibits a hysteresis cycle. In later works, much effort was put into studying the effects of quantum and thermal fluctuations and noise on the properties of these bistable systems [4,5]. Also the spectrum of scattered light was analyzed [6,7], as well as instabilities and chaotic behavior [8]. Bistability has been widely studied in different experimental setups: cold clouds of atoms inside the optical cavity [9,10], exciton-polariton modes in microcavities with external pumping [11], fiber ring cavities [12], and mesoscopic Josephson junction array resonators [13] in the external field, etc.

The results of many of these experiments can be understood by investigating bistable single-mode systems with Kerr-like nonlinearity. The resonance response of a bistable

single-mode system with a Kerr-like nonlinearity to the external field is described by a model of a driven nonlinear oscillator interacting with dissipative environment [14]. This model also describes an atomic system with several energy levels coupled to the cavity mode after adiabatically excluding atomic variables [15–17].

Moreover, the model of a driven nonlinear oscillator is a minimal model of a bistable driven system out of equilibrium. In the quasiclassical limit, its statistics and kinetics can be described by a Fokker–Planck equation (FPE) [18]. By means of a FPE, it is possible to find the stationary distributions, relaxation rates, and the occupations of two classical stable states of the oscillator. In the classical case, the noise-induced transitions between the stable states can also be described in terms of probability distributions of the first passage time through the saddle point. Moreover, for small damping constant, the ratio between the occupation probabilities of the stable states obtained from Fokker–Planck equation is close to the ratio of the mean first passage times through the saddle point. The first passage times can be calculated by direct numerical simulation of the stochastic equation of motion for slowly varying amplitudes, as in Refs. [19,20]. This happens because the first passage times are exponentially large compared with the inverse damping constant describing the first (fast) stage of relaxation in each region of the phase space. After crossing the saddle point from one of the stable states, the system undergoes the fast relaxation towards another stable state.

Previously, the nonequilibrium statistics and relaxation kinetics at different temperatures were studied numerically for the quantum oscillator by using the rate equation [21,22]. Also for the quantum oscillator the exact Glauber–Sudarshan

function of the steady state was obtained [14] in the case of zero bath temperature.

In the described model of the bistable driven system, it has been shown [18] that there exist different regions of the classical phase space with degenerate energies. Among the quantum effects, the effects of tunneling transitions between these regions are of particular interest.

For proper treatment of tunneling effects, a fully quantum-mechanical analysis is necessary. In the quasiclassical limit, the kinetic equation of Fokker-Planck type requires the additional tunneling term which leads to a nonzero flow of the probability distribution. As mentioned in Refs. [23,24], tunneling increases the population of the higher-amplitude state, which is also a squeezed state. It also increases the relaxation rate to the stationary distribution. From the quasiclassical point of view, tunneling can lead to hybridization of quasiclassical states from different regions of the classical phase plane. It can be shown to be very strong in the case of integer or half-integer detuning-nonlinearity ratio, when the quasiclassical states from the different regions of the phase space become degenerate. This also corresponds to the multiphoton resonance between the real energy levels of the driven system. The hybridization of the quasienergy states from different regions of the phase space in the case of multiphoton resonance can strongly change the kinetics of the considered system. With hybridization, the kinetics drastically differ from the nondegenerate case investigated previously [23] and are not studied yet.

In the case of strong hybridization between the states from different regions of the phase plane, the transition rate between them can be explained by a generalization of Keldysh theory for ionization of atoms in an electromagnetic field [25] for bistable systems. The Keldysh theory explains the interplay between tunneling and multiphoton ionization. However, for bistable systems the correspondence between multiphoton transitions and tunneling effects is not clear. To understand it, one should define the Keldysh parameter γ_K as the ratio of “tunneling time” to the period of motion along the classical trajectory. It will be demonstrated that, when the field intensity is much smaller than the critical value defining the range of bistability, the Keldysh parameter is large, $\gamma_K \gg 1$, and tunneling probability is just the same as the probability of multiphoton excitation. It will be also shown that γ_K logarithmically depends on the strength of the external field f while in the case of ionization of atoms it is proportional to f^{-1} .

II. THE SIMPLE MODEL OF BISTABLE DRIVEN SYSTEM

The Hamiltonian of a driven resonant mode with Kerr-like nonlinearity in the rotating-frame approximation reads

$$\hat{H} = -\Delta \hat{a}^\dagger \hat{a} + \frac{\alpha}{2} (\hat{a}^\dagger \hat{a})^2 + f(\hat{a} + \hat{a}^\dagger). \quad (1)$$

Here Δ is the detuning between the driving force and the resonant oscillator frequency, α is the blueshift due to nonlinearity, and f is proportional to the amplitude of the driving force.

In the classical limit, one should replace the operators \hat{a} , \hat{a}^\dagger in Eq. (1) with classical field amplitudes a , a^* to obtain the classical Hamiltonian. The classical phase portrait of the system is shown in Fig. 1: the classical trajectories in the a

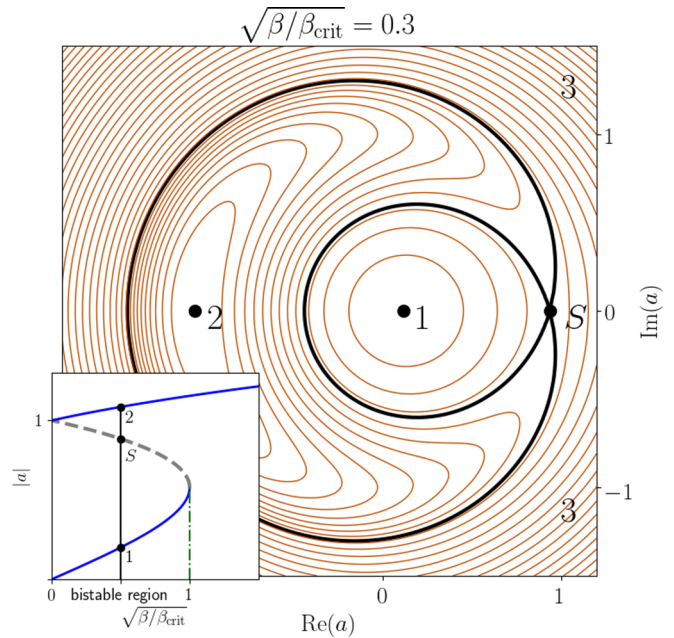


FIG. 1. The phase portrait of the nonlinear oscillator with Hamiltonian (1) where the operators \hat{a} , \hat{a}^\dagger are replaced by c numbers. The parameters are $\Delta = \alpha = 1$, $\sqrt{\beta/\beta_{\text{crit}}} = 0.3$. The stationary points 1, 2, and S are shown by black dots. The separatrix is denoted by a thick black line. It divides the phase plane into regions which are shown by numbers 1, 2, 3. In the inset, the dependence of amplitude stationary values on β is shown for $\Delta = \alpha = 1$. The blue solid (gray dashed) curves correspond to the stable states (unstable states).

plane are given by the contour lines of the classical Hamiltonian [see surface plot of the Hamiltonian function $H(a, a^*)$ in Fig. 2(b)]. Each classical trajectory corresponds to a certain quasienergy value ϵ . The only dimensionless parameter governing the system classical dynamics is $\alpha f^2/\Delta^3 \equiv \beta$, which can be treated as the rephasing rate of the nonlinear driven oscillator [26]. This parameter can also be identified with the Dicke cooperation parameter determining the typical rate of the intensity growth of a superradiance pulse. Bistability appears when $0 < \beta < \beta_{\text{crit}} \equiv 4/27$; see inset of Fig. 1.

For the quantum Hamiltonian, there exists another dimensionless parameter $m \equiv 2\Delta/\alpha$. The quasiclassical limit is acquired at large m .

A key feature of the driven nonlinear oscillator is the presence of two stable stationary states, which means bistability. Another important feature is the presence of a self-intersecting trajectory called separatrix, which divides the phase plane into regions 1, 2, and 3 and passes through the unstable stationary point S . The regions 1 and 2 contain the stationary states with smaller and larger amplitude, respectively. The quasienergies of the states 1, 2, S are denoted by ϵ_1 , ϵ_2 and ϵ_{sep} , and they always obey the inequality $\epsilon_2 < \epsilon_{\text{sep}} < \epsilon_1$. In further discussion, we also use the dimensionless quasienergy defined as $E = \alpha\epsilon/\Delta^2$, $E_1 = \alpha\epsilon_1/\Delta^2$, $E_2 = \alpha\epsilon_2/\Delta^2$, $E_{\text{sep}} = \alpha\epsilon_{\text{sep}}/\Delta^2$.

The states of the quantum Hamiltonian in the limit of large numbers of excitation quanta correspond to a discrete set of classical trajectories on the phase portrait. The corresponding

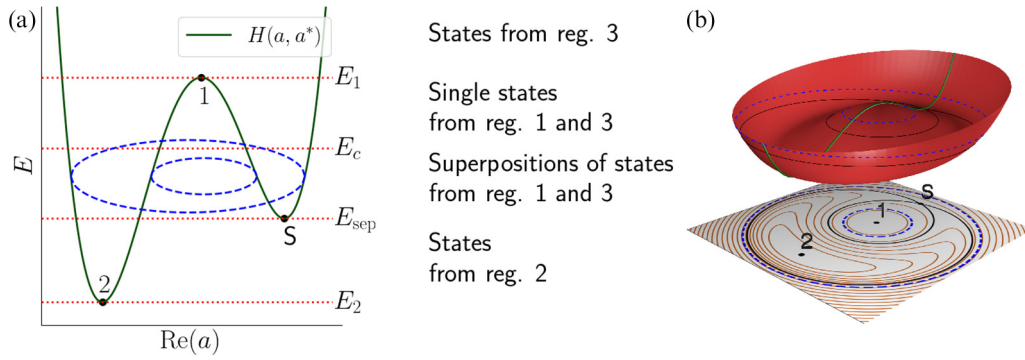


FIG. 2. (a) The quasienergy ranges containing quasienergy states with different structure are shown. The dark green thick line schematically denotes the Hamiltonian function $H(a, a^*)$ at $\text{Im } a = 0$. Its extrema correspond to the stationary states 1, 2, and S. Also a quantum state which is a superposition of two quasiclassical states is schematically shown by a blue dashed line. (b) A surface plot of the Hamiltonian function $H(a, a^*)$ is shown together with the classical phase portrait of the system. The dark green line corresponds to the intersection of the surface with the vertical plane $\text{Im } a = 0$. The blue dashed lines are the intersection of the surface with a horizontal plane. They correspond to a pair of classical trajectories with the same quasienergy from regions 1 and 3.

energies can be obtained from the Bohr–Sommerfeld quantization rule.

In the considered system, various sources of noise are possible including different parametric sources or interaction with dissipative environment. We investigate the nonequilibrium statistics of the system in the presence of fluctuations caused by interaction with the environment, which can be described by the following interaction Hamiltonian:

$$H_{\text{int}} = \hat{\xi}^\dagger \hat{a} + \hat{\xi} \hat{a}^\dagger, \quad (2)$$

where $\hat{\xi}$, $\hat{\xi}^\dagger$ are the operators of random force with correlation functions defined as

$$\begin{aligned} \langle \hat{\xi}(t) \hat{\xi}^\dagger(t') \rangle &= \gamma(N+1)\delta(t-t'), \\ \langle \hat{\xi}^\dagger(t) \hat{\xi}(t') \rangle &= \gamma N \delta(t-t'), \end{aligned} \quad (3)$$

where N is the number of noise quanta and γ is the damping caused by the interaction with the environment. The kinetics of the quantum bistable driven oscillator in the limit of weak coupling with the environment can be treated in the diagonal approximation for the density matrix. In this approximation, one obtains the rate equation dealing with probabilities P_n of occupation of the n th quasienergy state:

$$\begin{aligned} \frac{dP_n}{dt} &= \sum_{n'} w_{mn'} P_{n'} - w_{n'n} P_n, \\ w_{mn'} &= \gamma[(N+1)|\langle n|\hat{a}|n'\rangle|^2 + N|\langle n'|\hat{a}|n\rangle|^2]. \end{aligned} \quad (4)$$

If each quasienergy state can be uniquely attributed to one of the regions of the phase space, in the limit of large number of excitation quanta the rate equation transforms to the classical Fokker–Planck equation in the quasienergy space:

$$\begin{aligned} \frac{\partial P_i(E)}{\partial t} &= \frac{1}{T_i(E)} \frac{\partial J_i(E)}{\partial E}, \\ J_i(E) &= \vartheta K_i(E) P_i(E) + Q D_i(E) \frac{\partial P_i}{\partial E}, \end{aligned} \quad (5)$$

where $T_i(E)$ is the period of motion along the classical trajectory, the coefficients $K_i(E)$ and $D_i(E)$ are the drift and diffusion coefficients in quasienergy space. The coefficients $T_i(E)$, $K_i(E)$, and $D_i(E)$ are defined in Appendix A. The dimensionless parameters ϑ , Q are defined as $\vartheta = \gamma/\Delta$

and $Q = \vartheta \alpha(N+1/2)/\Delta$. The probability densities $P_i(E)$ are the continuous limits of P_n , where the probabilities P_n are considered as functions of dimensionless quasienergy $E \equiv \alpha \epsilon_n / \Delta^2$, and i denotes the region of the phase space according to Fig. 1. The function $P_2(E)$ is defined for $E_2 < E < E_{\text{sep}}$, $P_1(E)$ for $E_{\text{sep}} < E < E_1$, and $P_3(E)$ for $E > E_{\text{sep}}$.

However, as the quasienergy states in regions 1 and 3 can have same energies (see Fig. 2), it is possible that the true quasienergy states are superpositions of the quasiclassical states from regions 1 and 3 due to quantum tunneling. So the classical Fokker–Planck equation should be generalized to take this effect into account. It will be shown that hybridization of quasiclassical states from regions 1 and 3 strongly modifies the nonequilibrium statistics and kinetics of the system.

III. THE STRUCTURE OF QUASIENERGY STATES

In this section, we consider the model quasiclassically, although the results obtained here are valid beyond the quasiclassical approximation. Within the quasiclassical approach, the eigenstates of the quantum Hamiltonian correspond to a discrete set of trajectories which are obtained by using the Bohr–Sommerfeld rule:

$$\frac{1}{2\pi} \oint p dq = 2\pi n. \quad (6)$$

The variables q , p are the canonical coordinate and momentum defined by $a = \frac{q+ip}{\sqrt{2}}$ and $a^* = \frac{q-ip}{\sqrt{2}}$. For the states from region 2, the value of quasienergy uniquely defines the classical trajectory. However, if a state has quasienergy $\epsilon_{\text{sep}} < \epsilon < \epsilon_1$, it can lie either in region 1 or region 3. Therefore, if the quasienergies of some states obtained from the Bohr–Sommerfeld rule are close enough, the true eigenstates of the quantum Hamiltonian can be superpositions of the quasiclassical states due to quantum tunneling.

Let the set of quasiclassical states from regions 1 and 3 be $|n_1\rangle$, $|n_3\rangle$ with corresponding quasienergies ϵ_{n_1} , ϵ_{n_3} . If for some n , n' the quasienergies are almost equal, $\epsilon_{n_1} = \epsilon_{n'_3}$, the quasiclassical states $|n_1\rangle$ and $|n'_3\rangle$ form superpositions. Formally, for each n , n' it is possible to find the parameters

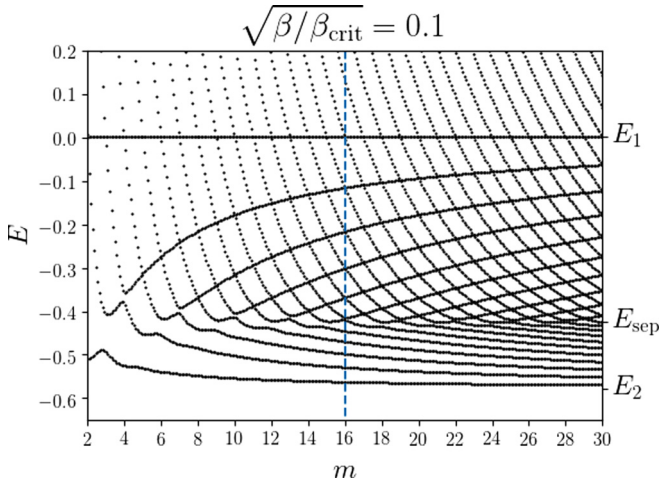


FIG. 3. The dependency of the quantum driven nonlinear oscillator quasienergy levels on m . Different behavior of levels above and below E_{sep} is evident: for $E_{\text{sep}} < E < E_1$, there are two families of almost-intersecting (anticrossing) lines which correspond to the quasiclassical states from regions 1 and 3. All anticrossings occur at integer values of m : the blue dashed line corresponds to $m = 16$. For $E_2 < E < E_{\text{sep}}$, there is only one family of lines which corresponds to the quasiclassical states from region 2.

β , m such that $\epsilon_{n_1} = \epsilon_{n'_3}$. In principle, these values of β and m could be different for different pairs n, n' . However, a special feature of the Hamiltonian (1) is that the degeneracy of the quasiclassical states occurs exactly at integer values of m for all β in the region of bistability. Therefore, at each integer value of m all quasiclassical states from regions 1 and 3 can be grouped in pairs so that within each pair the values of quasienergy are equal. When m is close to an integer, the states still can be grouped in pairs with close values of quasienergies. This is clearly seen from the exact diagonalization of the Hamiltonian (1). In Fig. 3, the eigenvalues of Hamiltonian (1) are shown as functions of m at constant β . It can be seen that, for $E_{\text{sep}} < E < E_1$, there exist two families of lines which correspond to the states from regions 1 and 3. The anticrossings of these lines indicate the degeneracy of the quasiclassical states. It is evident that all these anticrossings occur exactly at integer values of m .

To put the statements of the previous paragraph on the theoretical ground, let us first consider the case $f = 0$. At $f = 0$, the Hamiltonian commutes with the number of excitation quanta operator $a^\dagger a$, and eigenstates have the form $|n\rangle$ with corresponding quasienergies

$$\begin{aligned} \epsilon_n^{(0)} &= -\Delta n + \frac{\alpha}{2} n^2 = \frac{2\Delta^2}{\alpha} \frac{n}{m} \left(\frac{n}{m} - 1 \right), \\ m &\equiv \frac{2\Delta}{\alpha}. \end{aligned} \quad (7)$$

At integer m , it is clear that all quasienergy levels split into pairs with same energy, as $\epsilon_n^{(0)} = \epsilon_{m-n}^{(0)}$.

Let us proceed to the case $f > 0$. Let us consider the pair of states $|n\rangle$ and $|m-n\rangle$, $n < m/2$ which are degenerate at $f = 0$. From the quasiclassical point of view, these states corresponds to circular trajectories on the phase plane with different radii. From the Bohr–Sommerfeld rule, it follows

that the values of adiabatic invariant $\frac{1}{2\pi} \oint p dq$ for these states are $-n$ and $m-n$. Then let us switch on the external field adiabatically. After that, the quasienergies of the states change, but the values of the adiabatic invariant remain the same. The corresponding trajectories lie in regions 1 and 3, respectively, unless f is so large that they merge into a single trajectory from region 2. Therefore the quasienergies of the resulting states are $\epsilon_1(-n, \beta)$ and $\epsilon_3(m-n, \beta)$, where the energies are understood as functions of the adiabatic invariant.

For the considered model, it is possible to prove the identity

$$\epsilon_1(-n, f) = \epsilon_3(m-n, f), \quad (8)$$

or, equivalently,

$$n_3(\epsilon, f) - n_1(\epsilon, f) = m. \quad (9)$$

This happens because $n_1(E, f)$ and $n_3(E, f)$ have an analytic expression through the same elliptic integral with different contours (see Appendix A). With the same reasoning, it is also easy to verify that the classical periods of the trajectories from regions 1 and 3 with equal quasienergies are the same [27]. Thus, in the quasiclassical limit the energies of the states with numbers n and $m-n$ obtained from Bohr–Sommerfeld rule remain the same even at finite f . This supports the statement that the degeneracy of the quasiclassical states from regions 1 and 3 happens simultaneously for all states at integer values of m .

To find the eigenstates in the case when the quasiclassical states from regions 1 and 3 are degenerate, it is necessary to find the tunneling amplitude, which can also be obtained quasiclassically. For that, one should consider the motion of the system in the classically forbidden area. The quasiclassical tunneling exponent equals half of the action of a classical closed trajectory in imaginary time. The resulting amplitude is

$$\omega_R(E) \sim \frac{1}{T(E)} e^{-S_{\text{tunn}}(E)}, \quad (10)$$

$$S_{\text{tunn}}(E) = \frac{1}{2} \oint p_{\text{im}} dq = \frac{m}{2} \int_{s_1}^{s_2} \text{acosh} \left\{ \frac{E + \frac{s^2}{2} - \frac{s^4}{8}}{\sqrt{2\beta}s} \right\} ds, \quad (11)$$

where p_{im} is defined from the equation $H(q, ip_{\text{im}}) = \epsilon$. When $\beta \ll \beta_{\text{crit}}$,

$$S_{\text{tunn}}(E) = \frac{m}{2} \ln \frac{1}{\beta} \sqrt{1 + 2E} + O(1). \quad (12)$$

For nearly degenerate states, when $E_n \approx E_{m-n}$, the tunneling exponent takes the form

$$S_{\text{tunn}}(E_n) = \frac{m}{2} \ln \frac{1}{\beta} \left(1 - \frac{2n}{m} \right), \quad (13)$$

which is especially convenient for comparison with the multiphoton transition amplitude obtained by perturbation theory.

The eigenstates with account for tunneling between the nearly degenerate eigenstates can be found from an effective two-level Hamiltonian for two near-degenerate quasiclassical states. It has the form

$$H_n = \begin{pmatrix} \epsilon_{n_1} & -\omega_R^n \\ -\omega_R^n & \epsilon_{n_3} \end{pmatrix}, \quad (14)$$

where ϵ_{n_1} and ϵ_{n_3} are the quasienergies of the quasiclassical states $|n_1\rangle$, $|n_3\rangle$, and ω_R^n is the tunneling amplitude. When m slightly deviates from an integer, $m = m_0 + \delta m$, the difference of quasiclassical energies is

$$\delta\epsilon_n = \epsilon_{n_1} - \epsilon_{n_3} = -\frac{2\pi\delta m}{T(E_n)}. \quad (15)$$

The eigenstates of the effective Hamiltonian are

$$\begin{aligned} |n_+\rangle &= c_n|n_1\rangle + s_n|n_3\rangle, \\ |n_-\rangle &= -s_n|n_1\rangle + c_n|n_3\rangle. \end{aligned} \quad (16)$$

where the coefficients c_n , s_n are defined by the ratio between ω_R and $\delta\epsilon_n$:

$$\begin{aligned} c_n &\equiv \cos\theta_n, \quad s_n \equiv \sin\theta_n, \\ \tan 2\theta_n &= \frac{2\omega_R}{\delta\epsilon_n} = \frac{1}{\pi\delta m} e^{-S_{\text{tunn}}(E_n)}. \end{aligned} \quad (17)$$

Because of exponential dependence of ω_R on E , the coefficients c_n and s_n have a step-like dependence on n . When $\omega_R(E_n) \ll \delta\epsilon_n$, $s_n \approx 0$, $c_n \approx 1$, and the eigenstates correspond to distinct trajectories in the regions 1 and 3 of the phase portrait. On the contrary, when $\omega_R(E_n) \gg \delta\epsilon_n$, $s_n \approx c_n \approx 1/\sqrt{2}$, and the states are very close to symmetric and antisymmetric superpositions of trajectories. These alternatives are separated by the critical value of quasienergy E_c for which $\omega_R(E_c) \approx \delta\epsilon(E_c)$. Combining this with Eq. (15), one gets the equation for E_c :

$$\frac{1}{\pi\delta m} e^{-S_{\text{tunn}}(E_c)} = 1. \quad (18)$$

The resulting structure of quasienergy states is schematically depicted in Fig. 2. Below E_{sep} , there are only states from classical region 2. Between E_{sep} and E_c , the states from regions 1 and 3 form superpositions and, above E_c , the states from regions 1 and 3 do not hybridize.

IV. THE SYMMETRY OF THE HAMILTONIAN

The quasiclassical arguments of the previous section are valid only in the leading order in the quasiclassicity parameter m . In particular, the quasienergy values of the system can be expanded in asymptotic series in $1/m$, and the Bohr–Sommerfeld rule gives only the leading term of these series. Therefore additional arguments are needed to explain the simultaneous anticrossings of the quasienergy levels at integer m .

The rigorous proof can be given by using the perturbation theory in f for the quantum Hamiltonian (1). At $f = 0$, the Hamiltonian commutes with $\hat{n} = a^\dagger a$, so the quasienergies of the Hamiltonian (1) are given by Eq. (7). At small f , one can use the perturbation theory to find the corrections to the energy of the state $|n\rangle$. For example, the second-order correction is

$$\delta\epsilon_n^{(2)} = \frac{f^2}{\alpha} \frac{(m+1)}{(m-2n)^2 - 1}. \quad (19)$$

The perturbative correction $\epsilon_n^{(2)}$ is symmetric with respect to replacement $n \rightarrow m - n$. This is in agreement with the quasiclassical arguments of Sec. III: it was shown that the

changes of quasiclassical quasienergy due to adiabatic change of f are the same for states $|n\rangle$ and $|m - n\rangle$. This statement is not only valid in the quasiclassical framework but also holds exactly in the second order of perturbation theory. Moreover, we obtained a rigorous proof that the same holds for higher orders k of perturbation theory, $\epsilon_n^{(k)} = \epsilon_{m-n}^{(k)}$ up to order $m - 2n$. The complete proof is given in Appendix B. For integer m one should utilize the degenerate perturbation theory which also takes into account multiphoton transitions between the states $|n\rangle$ and $|m - n\rangle$. These transitions occur only at order $m - 2n$, and up to this order, the nondegenerate perturbation theory remains valid. Therefore, the quasienergy splitting between the states $|n\rangle$ and $|m - n\rangle$ occurs only at the order $m - 2n$.

V. THE GENERALIZATION OF KELDYSH THEORY FOR IONIZATION OF ATOMS IN ELECTROMAGNETIC FIELD

Splitting between the quasienergy states and the transitions between the regions of the phase space can be treated not only as tunneling between the regions of the phase space but also in $(m - 2n)$ th order of perturbation theory in f , as mentioned in Sec. IV. Both approaches lead to just the same effects. Such behavior can be understood in the frame of Keldysh theory for ionization of atoms in an electromagnetic field generalized for bistable systems with a discrete spectrum. Transition amplitude depends on Keldysh parameter $\gamma_K \equiv T_{\text{im}}(E)/T(E)$ where T_{im} is the tunneling time, which is defined as $T_{\text{im}} \equiv (\alpha/\Delta^2)\partial_E S_{\text{tunn}}$, where $S_{\text{tunn}}(E)$ is the tunneling action defined by Eq. (12). The tunneling time has the meaning of the half period of motion along the closed trajectory in the imaginary time. From the expression (13), we obtain that $T_{\text{im}}(E) \sim \Delta^{-1} \ln \frac{1}{\beta}$. The period in the real time is always $\sim \Delta^{-1}$ unless the quasienergy is close to E_{sep} . So, $\gamma_K \sim \ln \frac{1}{\beta} \gg 1$ for $\beta \ll \beta_{\text{crit}}$. In this limit, the tunneling amplitude coincides with $(m - 2n)$ -order perturbation theory multiphoton transition amplitude. Indeed, as was shown in Ref. [23], the multiphoton transition amplitude reads

$$\omega_R^{n,m-n} = \frac{V_{n,n+1} \cdots V_{m-n-1,m-n}}{(\epsilon_n^{(0)} - \epsilon_{n-1}^{(0)}) \cdots (\epsilon_n^{(0)} - \epsilon_{m-n-1}^{(0)})} \propto \Delta \beta^{\frac{m}{2}-n}, \quad (20)$$

where $\hat{V} = f(\hat{a} + \hat{a}^\dagger)$. This coincides with the expression $\omega_R(E) \propto \Delta e^{-S_{\text{tunn}}(E)}$ where $S_{\text{tunn}}(E)$ is defined by Eq. (13). According to both formulas, $\omega_R \sim \Delta \beta^{\frac{m}{2}-n}$. Moreover, not only the power-law dependence on β coincides but also the numerical coefficient which is accurately derived in Appendix C. Therefore, tunneling and multiphoton transitions are just the same effects. Also let us note that, for bistable driven systems, the Keldysh parameter γ_K logarithmically depends on the external field amplitude f whereas in the case of ionization of atoms by strong electromagnetic field, $\gamma_K \sim f^{-1}$.

VI. THE EFFECT OF DEGENERACY ON KINETICS

The fact that the eigenstates of the quantum driven nonlinear oscillator can be superpositions of quasiclassical states from regions 1 and 3 has a strong effect on kinetics described by the rate equation. When each state can be

uniquely attributed to a single region of the phase space, in the limit of large number of excitation quanta the rate equation transforms to the classical Fokker–Planck equation in quasienergy representation. However, this is not the case when m is close to an integer. The actual eigenstates are the superpositions of the quasiclassical states from regions 1 and 3, which breaks the crucial assumption under which the Fokker–Planck equation is derived.

However, even when m is close to integer, there exists a quasiclassical limit of the rate equation corresponding to large m which has also the form of the classical Fokker–Planck equation in the quasienergy space. It is obtained under the assumption that the occupations P_n^+ and P_n^- of states $|n_+\rangle$ and $|n_-\rangle$ [see Eq. (16)] still depend smoothly on n . The rate equation in terms of P_n^s , $s = \pm$, takes the form

$$\frac{\partial P_n^s}{\partial t} = \sum_{n'} w_{nn'}^{ss'} P_{n'}^{s'} - w_{n'n}^{s's} P_n^s, \quad (21)$$

where the transition probabilities $w_{nn'}^{ss'}$ between the states $|n_s\rangle$, $|n'_s\rangle$ are defined by the general formula (4), and the states $|n_s\rangle$ are defined by Eq. (16). In Eq. (21), it is possible to perform a gradient expansion of P_n^\pm . To perform such a procedure, it is convenient to rewrite Eq. (21) in a slightly different form:

$$\begin{aligned} \frac{\partial P_n^+}{\partial t} &= \sum_{n'} W_{nn'}^+ P_{n'}^+ - W_{n'n}^+ P_n^+ + w_{nn'}^{+-} P_{n'}^- - w_{n'n}^{-+} P_n^-, \\ \frac{\partial P_n^-}{\partial t} &= \sum_{n'} W_{nn'}^- P_{n'}^- - W_{n'n}^- P_n^- + w_{nn'}^{-+} P_{n'}^+ - w_{n'n}^{+-} P_n^+, \end{aligned} \quad (22)$$

with the newly defined coefficients $W_{nn'}^+ = w_{nn'}^{++} + w_{n'n}^{-+}$ and $W_{nn'}^- = w_{nn'}^{--} + w_{n'n}^{+-}$. They are expressed via matrix elements between the states from regions 1 and 3 as

$$\begin{aligned} W_{nn'}^+ &= (N+1)(c_{n'}^2 |a_{n_1 n'_1}|^2 + s_{n'}^2 |a_{n_3 n'_3}|^2) + N(c_{n'}^2 |a_{n'_1 n_1}|^2 + s_{n'}^2 |a_{n'_3 n_3}|^2), \\ W_{nn'}^- &= (N+1)(s_{n'}^2 |a_{n_1 n'_1}|^2 + c_{n'}^2 |a_{n_3 n'_3}|^2) + N(s_{n'}^2 |a_{n'_1 n_1}|^2 + c_{n'}^2 |a_{n'_3 n_3}|^2), \\ w_{nn'}^{+-} &= (N+1)(c_n^2 s_{n'}^2 |a_{n_1 n'_1}|^2 + s_n^2 c_{n'}^2 |a_{n_3 n'_3}|^2 - 2c_n c_{n'} s_n s_{n'} \operatorname{Re} a_{n_1 n'_1} a_{n_3 n'_3}^*) \\ &\quad + N(c_n^2 s_{n'}^2 |a_{n'_1 n_1}|^2 + s_n^2 c_{n'}^2 |a_{n'_3 n_3}|^2 - 2c_n c_{n'} s_n s_{n'} \operatorname{Re} a_{n'_1 n_1}^* a_{n_3 n'_3}), \\ w_{nn'}^{-+} &= (N+1)(s_n^2 c_{n'}^2 |a_{n_1 n'_1}|^2 + c_n^2 s_{n'}^2 |a_{n_3 n'_3}|^2 - 2s_n s_{n'} c_n c_{n'} \operatorname{Re} a_{n_1 n'_1} a_{n_3 n'_3}^*) \\ &\quad + N(s_n^2 c_{n'}^2 |a_{n'_1 n_1}|^2 + c_n^2 s_{n'}^2 |a_{n'_3 n_3}|^2 - 2s_n s_{n'} c_n c_{n'} \operatorname{Re} a_{n'_1 n_1}^* a_{n_3 n'_3}). \end{aligned} \quad (23)$$

Using the rate equations in the form (22), it is easy to understand the structure of the quasiclassical limit of Eqs. (22). Following the derivation of the Fokker–Planck equation which was described in detail in Ref. [23], it is clear that, after gradient expansion, the terms $\sum_{n'} W_{nn'}^\pm P_{n'}^\pm - W_{n'n}^\pm P_n^\pm$ transform to the expressions $\frac{1}{T} \frac{\partial}{\partial E} [\vartheta K^\pm P^\pm + Q D^\pm \frac{\partial P^\pm}{\partial E}]$ where $K^\pm(E)$ and $D^\pm(E)$ describe the drift and the diffusion correspondingly in quasienergy space. The terms $\pm(w_{nn'}^{+-} P_{n'}^- - w_{n'n}^{-+} P_n^+)$ describe tunneling transitions between the states $|n_+\rangle$ and $|n_-\rangle$. Thus, the whole system of equations takes the form

$$\frac{\partial P^\pm}{\partial t} = \frac{1}{T} \frac{\partial J_\pm}{\partial E} + \Lambda_{\text{tunn}}^\pm, \quad (24)$$

where

$$J_\pm = \vartheta K^\pm + Q D^\pm \frac{\partial P^\pm}{\partial E}, \quad (25)$$

and $\Lambda_{\text{tunn}}^{+(-)}$ are the terms responsible for tunneling:

$$\begin{aligned} \Lambda_{\text{tunn}}^{+(-)}(E) &\equiv \int_{E_{\text{sep}}}^{E_1} dE' T(E') [w_{EE'}^{+(-+)} P^{-(+)}(E') \\ &\quad - w_{E'E}^{-+(+-)} P^{+(-)}(E)]. \end{aligned} \quad (26)$$

The system (24) is exactly the Fokker–Planck equation with the tunneling term from Ref. [23]. The features of Eqs. (24) can be understood from the structure of the eigenstates which was described in Sec. III; see Eq. (16) and Fig. 2.

As was mentioned in Sec. III, the hybridization of the quasiclassical states from different regions of the phase portrait is strong for quasienergies below E_c and it is very small for quasienergies above E_c . This defines the behavior of the coefficients $K^\pm(E)$, $D^\pm(E)$, and terms $\Lambda_{\text{tunn}}^\pm(E)$ in Eq. (24), which is different for $E < E_c$ and $E > E_c$. For $E > E_c$ there is almost no hybridization of states from regions 1 and 3. Therefore, the drift and diffusion coefficients of the Fokker–Planck equation $K^\pm = K_{1,3}$, $D^\pm = D_{1,3}$ and the tunneling transition rate is small: $\Lambda_{\text{tunn}}^\pm T \ll 1$. On the contrary, for $E < E_c$ the actual eigenstates are symmetric and antisymmetric superpositions of the quasiclassical states from different regions. Using the expressions (23) and keeping in mind that, in the considered case $s_n \approx c_n \approx 1/\sqrt{2}$, it is obvious that the drift and diffusion coefficients for $|n_\pm\rangle$ states are $K^+ \approx K^- \approx \frac{1}{2}(K_1 + K_3)$, $D^+ \approx D^- \approx \frac{1}{2}(D_1 + D_3)$, and the tunneling rates are large and almost equal: $\Lambda_{\text{tunn}}^+ T \approx \Lambda_{\text{tunn}}^- T \sim 1$.

From the considerations of the previous paragraph, it is easy to obtain the stationary distribution. As for $E < E_c$, the drift and diffusion coefficients and tunneling transition rates are almost equal for states $|n_\pm\rangle$ and the tunneling transition rate is large; the stationary probability densities P^+ and P^- are almost equal, too, $P^+ \approx P^- \approx \bar{P}$. Thus, the stationary probability density \bar{P} is obtained from the condition of zero flow $J^\pm(E)$ and is given by the expression

$$\bar{P}(E) = \exp \left\{ -\frac{\vartheta}{Q} \int_{E_{\text{sep}}}^E \frac{K_1 + K_3}{D_1 + D_3} dE' \right\}, \quad E_{\text{sep}} < E < E_c. \quad (27)$$

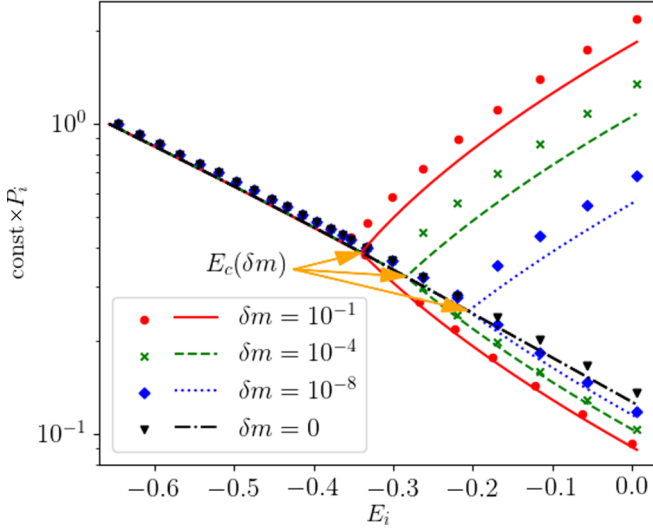


FIG. 4. The distribution functions at $\sqrt{\beta/\beta_{\text{crit}}} = 0.2$, $N_{\text{th}} = 4$, $m = 30 + \delta m$ for different small δm . The exact quantum distributions are denoted by red circles, green crosses, blue diamonds, and black triangles for $\delta m = 10^{-1}$, 10^{-4} , 10^{-8} , 0. The quasiclassical approximations of distribution functions given by Eqs. (27) and (29) are shown with red solid, green dashed, blue dotted, and black dash-dotted lines for different values of δm , correspondingly. The orange arrows indicate the position of critical quasienergy $E_c(\delta m)$.

For $E_c < E < E_1$, the tunneling term in Eq. (24) is small, and the stationary distributions $P^+(P^-)$ can be obtained by perturbation theory in $\Lambda_{\text{tunn}}^{\pm}$ as was done in Ref. [23].

Let us define

$$P^{\pm}(E) = P_0^{\pm}(E) + \delta P^{\pm}(E), \quad (28)$$

where $P_0^{+(-)}(E)$ are stationary distributions without the tunneling term and $\delta P^{+(-)}$ are the first-order corrections caused by the tunneling terms. Then,

$$P_0^+(E) = \bar{P}(E_c) \exp \left\{ -\frac{\vartheta}{Q} \int_{E_c}^E \frac{K_1}{D_1} dE' \right\}, \quad E_c < E < E_1,$$

$$P_0^-(E) = \bar{P}(E_c) \exp \left\{ -\frac{\vartheta}{Q} \int_{E_c}^E \frac{K_3}{D_3} dE' \right\}, \quad E_c < E < \infty. \quad (29)$$

The coefficients in Eqs. (29) are defined by the continuity condition at $E = E_c$. The tunneling corrections have the form

$$\delta P^+ = -P_0^+ \int_{E_{\text{sep}}}^E \frac{dE'}{Q D_1(E') P_0^+} \int_{E_1}^{E'} \Lambda_{\text{tunn}}^+(E'') T(E'') dE'',$$

$$\delta P^- = P_0^- \int_{E_{\text{sep}}}^E \frac{dE'}{Q D_3(E') P_0^-} \int_{E'}^{\infty} \Lambda_{\text{tunn}}^-(E'') T(E'') dE'', \quad (30)$$

where $\Lambda_{\text{tunn}}^{\pm}$ are expressed by Eqs. (26) with $P_0^{\pm}(E)$.

In Fig. 4, the distribution functions obtained from analytical formulas (29) and (30) are compared with those obtained by numerical solution of Eqs. (21) and (22) for different values of δm . It is evident that the critical quasienergy E_c shifts towards E_{sep} with increasing δm , according to Eq. (18). Also it can be seen that, for exactly degenerate quasienergy levels $\delta m = 0$, the system remains close to the stable state 2, which

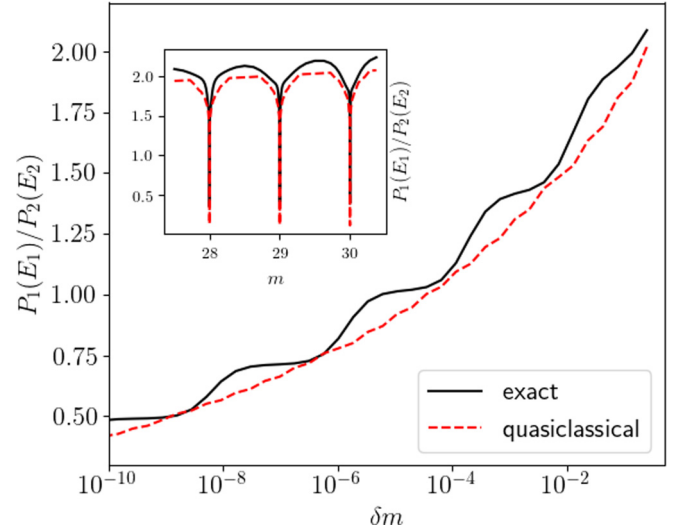


FIG. 5. The ratio of probability densities in the stationary states 1 and 2 at $\sqrt{\beta/\beta_{\text{crit}}} = 0.2$, $N_{\text{th}} = 4$, $m = 30 + \delta m$. The exact value obtained from transition matrix diagonalization for the quantum oscillator is compared with analytical formula (31). On the inset, the dependence of probabilities ratio $P_2(E_2)/P_1(E_1)$ on m is shown on a linear scale for the same parameters. At integer values of m , there are exponentially narrow dips.

is squeezed [23]. In this case, the states corresponding to the regions 1 and 3 are equally occupied and the occupation probabilities are exponentially small.

The ratio between the probability densities in two stable stationary states is

$$\frac{Q}{\vartheta} \ln \frac{P_2(E_2)}{P_1(E_1)} = \int_{E_2}^{E_{\text{sep}}} \frac{K_2(E')}{D_2(E')} dE' + \int_{E_{\text{sep}}}^{E_c} \frac{K_1(E') + K_3(E')}{D_1(E') + D_3(E')} dE' + \int_{E_c}^{E_1} \frac{K_1(E')}{D_1(E')} dE' + \ln \left[1 + \frac{\delta P^+}{P_0^+} \right]. \quad (31)$$

The tunneling correction for $E > E_c$ does not lead to any qualitative effects because it is of order α/Δ compared with P_0^{\pm} .

In Fig. 5, the analytical formula (31) is compared with the numerical result for the dependence of $P_1(E_1)/P_2(E_2)$ on δm . The analytical formula fits the numerical result quite well. However, the discreteness of the quasienergy levels manifests itself in smooth steps in the dependence $P_1(E_1)/P_2(E_2)$ on δm which are not reproduced by Eq. (31). These steps can be explained by the fact that the crossover energy ϵ_c can take only discrete values. Thus, for

$$e^{-S_{\text{tunn}}(E_n)} < \delta m < e^{-S_{\text{tunn}}(E_{n+1})}, \quad (32)$$

the effective position of E_c remains the same. When $\delta m \approx e^{-S_{\text{tunn}}(E_{n+1})}$, the value of E_c abruptly changes from E_n to E_{n+1} . This explains the presence of steps in Fig. 5. The width of the steps on the logarithmic scale is $S_{\text{tunn}}(E_n) - S_{\text{tunn}}(E_{n+1}) \approx 2\pi T_{\text{im}}(E_n)/T(E_n)$. In the latter expression, we recognize the previously defined Keldysh parameter γ_K . For n much smaller than m and $\beta \ll \beta_{\text{crit}}$, $\gamma_K \sim \ln \frac{1}{\beta}$.

In the inset in Fig. 5, the dependence of probabilities ratio $P_1(E_1)/P_2(E_2)$ on m is shown on a linear scale for

$\sqrt{\beta/\beta_{\text{crit}}} = 0.2$. For the nondegenerate case, when m is far from an integer, the state with smaller amplitude and quasienergy E_1 is the most probable. However, when m becomes close to an integer, the occupation of the state with quasienergy E_1 abruptly drops, and the value at the minima is exponentially small for large m (see the dips on the inset of Fig. 5). The width of the dips on the linear scale is also exponentially small.

VII. CONCLUSIONS

We considered the nonequilibrium statistics and kinetics of the model of resonantly driven quantum nonlinear oscillator interacting with a dissipative environment. We found that the nonequilibrium statistics and kinetics are strongly modified when quasienergy states are nearly degenerate which occurs at integer or half-integer detuning-nonlinearity ratio. In particular, the occupation of the classical stable state with smaller amplitude is strongly reduced. Thus, in the case of exactly degenerate quasienergy levels, the system always occupies the state with higher amplitude, which is squeezed.

The coefficients of the Fokker–Planck equation, which describes the kinetics in the quasiclassical limit are very sensitive to the structure of eigenstates of the system’s Hamiltonian. We found that, in the case of integer or half-integer detuning-nonlinearity ratio, which corresponds to the exact multiphoton resonance between the genuine energy levels of the unperturbed nonlinear oscillator, the quasienergy states from different regions of the phase space simultaneously hybridize and form symmetric and antisymmetric superpositions. This fact can be proven by applying a special symmetry transformation to the Hamiltonian. Also, we revealed that, when the quasienergy levels of the system are nearly degenerate, an important critical quasienergy parameter ϵ_c emerges. Below ϵ_c , all quasienergy states are superpositions of the quasiclassical states from regions 1 and 3, and above ϵ_c , the quasienergy states correspond to either region 1 or 3. We found out that the coefficients of the Fokker–Planck equation which describes the quasiclassical kinetics of the oscillator in the almost-degenerate case have different behavior above and below ϵ_c . In particular, the tunneling term is large below ϵ_c and exponentially small above ϵ_c . Also, the drift and diffusion coefficients are not affected by tunneling above ϵ_c whereas below ϵ_c they are strongly modified. The distribution functions and the ratio between occupations of the classical stable states calculated analytically fit well the numerical results.

We generalized Keldysh theory for ionization of atoms in electromagnetic field for bistable systems. It was demonstrated that Keldysh parameter defined as the ratio of tunneling time to the quasiclassical period of motion along the phase trajectory is large in the bistability region for external field intensity smaller than the critical value. So the multiphoton transition and tunneling between different regions of the phase space can be treated as the same effects. This fact was proved by direct calculation of transition amplitude using both tunneling and perturbation theory approach. Also we revealed that the Keldysh parameter for the considered system depends logarithmically on the amplitude f of the external field. In contrast, in the case of multiphoton ionization of atoms, the

Keldysh parameter is inversely proportional to the amplitude of the external field.

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APPENDIX A: THE COEFFICIENTS OF THE CLASSICAL FOKKER–PLANCK EQUATION

The coefficients of the classical FPE are defined as line integrals along the classical trajectories of the nonlinear oscillator:

$$\begin{aligned} K_i(E) &= \frac{i}{2} \oint da da^* - a^* da, \\ D_i(E) &= \frac{i}{2} \oint \frac{\partial H}{\partial a} da - \frac{\partial H}{\partial a^*} da^*, \\ T_i(E) &= \int da^* da \delta(E - H(a^*, a)). \end{aligned} \quad (\text{A1})$$

The classical trajectory is a contour line of the classical Hamiltonian

$$H_{\text{cl}} = -\Delta|a|^2 + \frac{\alpha}{2}|a|^2 + f(a + a^*), \quad (\text{A2})$$

where $a = \frac{q+ip}{\sqrt{2}}$, and q, p form a pair of canonically conjugate variables. The index i denotes the region of the phase space according to Fig. 1. The coefficient $K_i(E)$ is proportional to the adiabatic invariant of the trajectory defined as

$$n_i(E, f) = \frac{1}{2\pi} K_i(E) = \frac{1}{2\pi} \oint pdq. \quad (\text{A3})$$

The expressions (A1) and (6) can be rewritten as two-dimensional integrals in the p, q plane with a Dirac δ function, as in expression for T_i . Then, it is convenient to use variables q, t instead of q, p where $t = q^2 + p^2$. Then the coefficients are transformed to one-dimensional integrals by t . Now let us focus on the expression for adiabatic invariant:

$$n_i(E, f) = \frac{m}{2} \oint_{C_i} \frac{dt}{4\pi} \frac{3t^2/16 - t/4 + E/2}{\sqrt{2f^2t - (E + \frac{t}{2} - \frac{t^2}{8})^2}}. \quad (\text{A4})$$

Using this expression, we will prove the identity of Eq. (9). The contour of integration in Eq. (A4) depends on the region of the phase space in which the trajectory lies. In the range of energies corresponding to region 2 of the phase space, the polynomial has only two real roots, and the contour of integration encloses them. In the range of energies corresponding to region 1, there are four real roots: $t_1 < t_2 < t_3 < t_4$ (see Fig. 6). The range $t_1 < t < t_2$ ($t_3 < t < t_4$) corresponds to the trajectories from region 1 (3). Thus, the contour of integration for n_1 (n_3) encloses t_1 and t_2 (t_3 and t_4).

By deformation of the contour, it is easy to show that $n_3(\epsilon, f) - n_1(\epsilon, f)$ is expressed through the residue of integrand (A4) on infinity. Expanding the integrand of Eq. (A4)

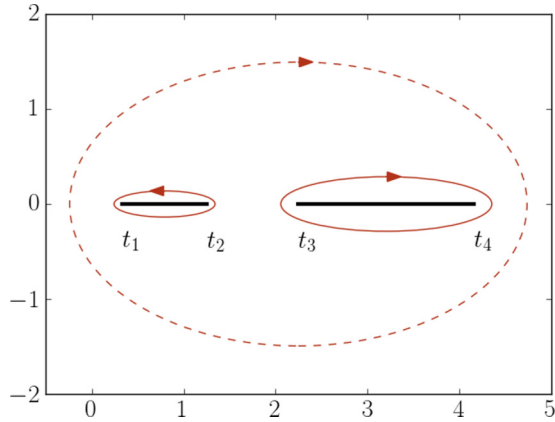


FIG. 6. The contours of integration in Eq. (A4) corresponding to $n_1(\epsilon, f)$ (left contour) and $n_3(\epsilon, f)$ (right contour) are shown on the complex plane of t . The difference $n_3 - n_1$ is expressed as an integral over the outer contour (dashed line).

in t^{-1} , one gets the desired identity:

$$n_3(\epsilon, f) - n_1(\epsilon, f) = m = \frac{2\Delta}{\alpha}. \quad (\text{A5})$$

APPENDIX B: THE PROOF OF THE SYMMETRY OF THE PERTURBATIVE CORRECTIONS

Here we prove that the perturbation theory corrections in f to the quasienergies $\epsilon_n^{(0)}$ of the eigenstate with n excitation quanta of the model (1) at $f = 0$ are symmetric with respect to replacement $n \rightarrow m - n$. This fact was mentioned in Refs. [22] and [27], but the authors did not give any proof of this fact.

For several low-order corrections, this can be verified by straightforward calculation, as for the second-order correction (19):

$$\epsilon_n^{(2)} = \frac{f^2}{\alpha} \frac{(m+1)}{(m-2n)^2 - 1}. \quad (\text{B1})$$

However, it is necessary to clarify what the expression $\epsilon_{m-n}^{(k)}$ means for noninteger m because the corrections $\epsilon_n^{(k)}$ are defined only for integer n , which has the meaning of the number of excitation quanta. Thus, for noninteger m , the identity $\epsilon_n^{(k)} = \epsilon_{m-n}^{(k)}$ holds only for formal expressions. Up to now, we did not give any meaning to $\epsilon_\nu^{(k)}$ for noninteger ν except as an analytic continuation of perturbation theory formulas.

However, it is in fact possible to give direct meaning to ϵ_ν and $\epsilon_\nu^{(k)}$ for noninteger ν . For that, we should formally assume that the Hamiltonian (1) acts on the space of all possible real “numbers of excitation quanta” ν with operators a, a^\dagger defined as follows:

$$\langle \nu | a | \nu + 1 \rangle = \langle \nu + 1 | a^\dagger | \nu \rangle = \sqrt{\nu}. \quad (\text{B2})$$

Then, for $f = 0$ each state $|\nu\rangle$ is an eigenstate with energy $\epsilon_\nu^{(0)} = \nu(m - \nu)$. For $f \neq 0$, they become coupled with $|\nu \pm 1\rangle, |\nu \pm 2\rangle, \dots, |\nu \pm k\rangle, \dots$. If none of the states $|\nu \pm k\rangle$ are degenerate with ν (equivalently, $2\nu - m$ is noninteger), the amplitudes of $|\nu \pm k\rangle$ remain small at small f , and it is possible to define continuous f -dependent energy $\epsilon_\nu(f)$. This

is the energy of the eigenstate which evolves from $|\nu\rangle$ after adiabatic switching of the perturbation. For integer $2\nu - m$, the energy $\epsilon_\nu(f)$ cannot be defined that way because of degeneracy between $|\nu\rangle$ and $|m - \nu\rangle$.

The series of perturbation theory for ϵ_ν in the cases of integer and noninteger ν are completely identical because of definition (B2). Thus, the claim that $\epsilon_n^{(k)} = \epsilon_{m-n}^{(k)}$ follows from the even more general statement $\epsilon_\nu = \epsilon_{m-\nu}$.

We prove the identity $\epsilon_\nu = \epsilon_{m-\nu}$ in several steps. First, it is obvious from the previous considerations that $\epsilon_\nu(f)$ is an eigenvalue of the operator

$$\mathcal{H}_\nu = \frac{\alpha}{2} \sum_{\sigma-\nu \in \mathbb{Z}} \sigma(\sigma - m) |\sigma\rangle \langle \sigma| + f \sqrt{\sigma} (|\sigma - 1\rangle \langle \sigma| + |\sigma\rangle \langle \sigma - 1|), \quad (\text{B3})$$

which corresponds to the state $|\nu\rangle$. Analogously, $\epsilon_{m-\nu}(f)$ arises from the operator $\mathcal{H}_{m-\nu}$. For convenience in the later discussion, we change the numeration of basis vectors in $\mathcal{H}_{m-\nu}$ so that $|\sigma\rangle$ becomes $|m - \sigma\rangle$. After such relabeling,

$$\mathcal{H}_{m-\nu} = \frac{\alpha}{2} \sum_{\sigma-\nu \in \mathbb{Z}} \sigma(\sigma - m) |\sigma\rangle \langle \sigma| + f \sum \sqrt{m - \sigma} (|\sigma + 1\rangle \langle \sigma| + |\sigma\rangle \langle \sigma + 1|). \quad (\text{B4})$$

Both \mathcal{H}_ν and $\mathcal{H}_{m-\nu}$ act on a single space with a set of basis vectors $|\sigma\rangle$ with such σ that $\sigma - \nu$ is integer. We should emphasize that \mathcal{H}_ν and $\mathcal{H}_{m-\nu}$ are substantially different and could not be transformed to each other by any permutation of eigenvectors.

However, there exists a nontrivial linear operator T which transforms \mathcal{H}_ν to $\mathcal{H}_{m-\nu}$:

$$\mathcal{H}_\nu = T \mathcal{H}_{m-\nu} T^{-1}. \quad (\text{B5})$$

It has the form

$$T = U T U'^{-1}, \quad (\text{B6})$$

where

$$U = \sum_{\sigma} \sqrt{\Gamma(\sigma + 1)} |\sigma\rangle \langle \sigma|, \\ U' = \sum_{\sigma} \sqrt{\Gamma(m - \sigma)} |\sigma\rangle \langle \sigma|, \\ T = \exp \frac{2f}{\alpha} \sum |\sigma\rangle \langle \sigma + 1|. \quad (\text{B7})$$

The identities (B5)–(B7) are checked by direct calculation.

The existence of the operator T is possible only because of special form of $\epsilon_\nu = \frac{\alpha}{2} \nu(m - \nu)$. For any other dependence of ϵ_ν on ν , no analogous operator can be found this way. So the symmetry property expressed by T is a special feature of Kerr-like nonlinearity.

The equivalence of Hamiltonians \mathcal{H}_ν and $\mathcal{H}_{m-\nu}$ proves that the energies $\epsilon_\nu(f)$ and $\epsilon_{m-\nu}(f)$ are equal when $2\nu - m$ is not integer. However, we are interested in the case of integer m and integer numbers of excitation quanta. For this case, one should utilize degenerate perturbation theory to find the energies. Nevertheless, the corrections to energies of $|n\rangle$ and $|m - n\rangle$ obtained by degenerate perturbation theory are just the same as in nondegenerate perturbation theory up to the

order $f^{|m-2n|}$. This happens because the leading contribution to composite matrix element (multiphoton Rabi frequency) between $|n\rangle$ and $|m-n\rangle$ is a product of $|m-2n|$ matrix elements of the perturbation \hat{V} [see Eqs. (20) and (C1)]. For a series of nondegenerate perturbation theory the identity for k th-order corrections $\epsilon_n^{(k)} = \epsilon_{m-n}^{(k)}$ holds even for integer m and n if $k < 2|m-2n|$. For $k \geq 2|m-2n|$, the corrections of nondegenerate perturbation theory do not make sense because of a zero in the denominator, which is a manifestation of degeneracy. This means that the degeneracy of $|n\rangle$ and $|m-n\rangle$ is lifted only in the order $|m-2n|$, and the energy splitting happens only due to multiphoton Rabi oscillations. The leading term at small f is $\Delta\epsilon_{n,m-n} = 2\omega_R^{n,m-n}$.

APPENDIX C: THE IDENTITY OF TUNNELING SPLITTING AND MULTIPHOTON TRANSITION AMPLITUDE

The multiphoton Rabi splitting between the quasienergy states $|n\rangle$ and $|m-n\rangle$ is given by

$$\begin{aligned} \omega_R^{n,m-n} &= \frac{V_{n,n+1} \cdots V_{m-n-1,m-n}}{(\epsilon_n^{(0)} - \epsilon_{n-1}^{(0)}) \cdots (\epsilon_n^{(0)} - \epsilon_{m-n-1}^{(0)})} \\ &= \alpha \left(\frac{2f}{\alpha} \right)^{m-2n} \frac{1}{(m-2n-1)!^2} \sqrt{\frac{(m-n)!}{n!}}. \end{aligned} \quad (\text{C1})$$

At large values of n , m , it is possible to approximate the factorials by using the Stirling formula. One then gets the

following expression for $\omega_R^{n,m-n}$:

$$\begin{aligned} \ln \frac{\omega_R^{n,m-n}}{\Delta} &= \frac{m}{2} \left[(1-r) \ln \frac{1}{\beta} + (1-r)(2 \ln 2 - 3) \right. \\ &\quad \left. + 4(1-r) \ln(1-r) \right. \\ &\quad \left. - \frac{1}{2}([2-r] \ln(2-r) - r \ln r) \right], \end{aligned} \quad (\text{C2})$$

where $r = 2n/m$. In this form, it is easy to compare it with tunneling splitting given by Eqs. (10) and (11). At small β , the tunneling action (11) can be approximated as

$$\begin{aligned} S_{\text{tunn}} &= \frac{m}{2} \left[\sqrt{1+2E} \ln \frac{2}{\beta} + \int_{x_1}^{x_2} \ln \frac{E + \frac{x^2}{2} - \frac{x^4}{8}}{x} dx \right], \\ x_{1,2} &= \sqrt{2 \mp 2\sqrt{1+2E}}. \end{aligned} \quad (\text{C3})$$

For small external force the quasienergy E is related with number of excitation quanta by the formula $E = \frac{2n}{m}(\frac{n}{m} - 1) = r^2/2 - r$. Evaluating the integral in Eq. (C3) and substituting the expression for E via r , it is easy to obtain that, with the current approximations, $S_{\text{tunn}} = \ln(\omega_R^{n,m-n}/\Delta)$. Thus, for $\beta \ll \beta_{\text{crit}}$ perturbation theory is consistent with the tunneling approach.

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