Probability of radiation of twisted photons in an inhomogeneous isotropic dispersive medium

O. V. Bogdanov,[1,2,*] P. O. Kazinski,[1,†] and G. Yu. Lazarenko[1,‡]
[1]Physics Faculty, Tomsk State University, Tomsk 634050, Russia
[2]Division for Mathematics and Computer Sciences, Tomsk Polytechnic University, Tomsk 634050, Russia

(Received 11 July 2019; published 25 October 2019)

The general formula for probability to record a twisted photon produced by a charged particle moving in an inhomogeneous isotropic dispersive medium is derived. The explicit formulas for probability to record a twisted photon are obtained for the radiation of a charged particle traversing a dielectric plate or an ideally conducting foil. It is shown that, in the case when the charged particle moves along the detector axis, all the radiated twisted photons possess a zero projection of the total angular momentum and the probability of their radiation is independent of the photon helicity. The radiation produced by helically microbunched beams of charged particles is also considered. The fulfillment of the strong addition rule for the projection of the total angular momentum of radiated twisted photons is demonstrated. Thus the helical beams allow one to generate coherent transitions and Vavilov-Cherenkov radiation with large projections of the total angular momentum. The radiation produced by charged particles in a helical medium is studied. Typical examples of such a medium are metallic spirals and cholesteric liquid crystals. It is shown that the radiation of a charged particle moving along the helical axis of such a medium is a pure source of twisted photons.

DOI: 10.1103/PhysRevA.100.043836

I. INTRODUCTION

The use of media with nontrivial permittivity is the most common method to generate twisted photons [1–14]. However, as a rule, the medium is employed only as a converter of plane-wave photons to twisted ones. We investigate in the present paper a direct means for production of twisted photons by charged particles moving in an inhomogeneous dispersive medium. As far as a homogeneous medium is concerned, the theory of radiation of twisted photons is known in this case (see the description of the Vavilov-Cherenkov process in [15,16]). However, the detectors of twisted photons [17–22] or the objects that should be irradiated by them [23–26] are usually positioned outside the medium. When the twisted photons escape from the medium, the form of their peculiar phase front can be destructed. Therefore, to describe properly the production of twisted photons by charged particles moving in inhomogeneous media, the corresponding theory has to be constructed.

The classical and quantum theories of radiation by particles propagating in dispersive media are well elaborated (see, e.g., [27–50]). So we adapt these theories for the description of radiation of twisted photons. Recall that the twisted photons are the states of the electromagnetic field that possess a definite energy, unique projections of the momentum and of the total angular momentum onto the detector axis, and a well-defined helicity. First of all, we develop QED in an inhomogeneous dispersive medium with twisted photons and derive the general formula for the probability to record a twisted photon created by a classical current. The range of applicability of such a theory is of course the same as for the corresponding theory of radiation of plane-wave photons. Then, employing this formalism, we investigate several representative examples [38,40,45,48,51,52] to demonstrate the main features of twisted photon radiation by charged particles in a medium. Our primary goal is to describe the distribution of radiation of twisted photons by charged particles over the projections m of the total angular momentum onto some axis (the detector axis), to establish its general properties, and to find situations when this distribution has a desired form, for example, when it is concentrated at a certain value of m. Notice that this objective cannot be achieved in the plane-wave basis of photon mode functions. Namely, we consider the radiation of twisted photons by a charged particle moving uniformly along a straight line intersecting a dielectric plate or a metal foil. With the aid of the results of [53,54], we investigate the coherence of such a radiation created by particle beams and the conditions when transition and Vavilov-Cherenkov (VC) radiations can be used as a pure bright source of twisted photons. In particular, we show that the helically microbunched beams [55–64] with a certain helix pitch and transverse size produce coherent twisted transition and VC radiations with large projection of the total angular momentum. Note that coherent transition and VC radiations of plane-wave photons were discussed in many works (see, e.g., [44,65–70]). As for helically microbunched beams, they were used to generate twisted photons by undulators [56,57,59–61] and by hitting metal foils [55,58]. The peculiarities of twisted photon radiation produced by particle bunches of different profiles can also be employed for the diagnostics of the particle beam structure [55,58,71].

Another pure source of twisted photons that we study in the present paper is a helical medium. Typical examples of such

*bov@tpu.ru
†kpo@phys.tsu.ru
‡laz@phys.tsu.ru

a medium are metallic helical ribbons [13], cholesteric liquid crystals [7,9,72], and helically arranged dielectrics. Usually, the helical media are used to convert ordinary plane-wave photons to twisted ones. We however consider the direct process of radiation by charged particles moving in such a medium. It turns out that a charged particle moving uniformly along the helical axis of such a medium is a pure source of twisted photons. Its radiation obeys the selection rules that are pertinent to ideal helical undulators [53,73–83] or to scattering on helical targets [82]. This fact has a simple explanation in terms of transition scattering of the permittivity wave on the charged particle. This source can be employed to generate twisted photons with energies up to the x-ray spectral range. Of course, there are other pure sources of twisted photons such as undulators and laser waves where the twisted photons are produced directly by charged particles [56,57,59–61,73–81,83–86]. The hard twisted photons with energies of the order of hundreds of MeV can be generated by inverse Compton scattering of low-energy twisted photons [87,88] or in channeling [89,90].

The paper is organized as follows. In Sec. II we develop the QED in an inhomogeneous dispersive medium. In Sec. III we find the probability to record a twisted photon by a detector in a vacuum and obtain the general formula for the radiation probability of twisted photons by classical currents. In Sec. IV we obtain the general formula for the probability to record a twisted photon created by a charged particle moving in a homogeneous medium and reproduce the known results for VC radiation of twisted photons in such media. In Sec. V we apply the general theory to particular examples. In Secs. V A and V B we consider the radiation of twisted photons by a charged particle or a beam of them that traverses a dielectric plate. In Sec. V C transition radiation by a charged particle hitting a metal foil is investigated. The radiation produced by charged particles in helical media is studied in Sec. V D. In Sec. VI we summarize the results. The evaluation of the incoherent and coherent interference factors [53,54] for Gaussian helically microbunched beams is given in the Appendix.

We work in a system of units such that $\hbar = c = 1$ and $e^2 = 4\pi\alpha$, where $\alpha \approx 1/137$ is the fine-structure constant. The notation from [81] is vastly employed.

II. QUANTUM ELECTROMAGNETIC FIELD IN A MEDIUM

For the reader's convenience and concordance of notation, in this section we develop the QED in an inhomogeneous transparent isotropic dispersive medium. As regards the homogeneous medium, such a construction is well known (see, e.g., [30–37,39,40,42,43,45,47,50]). We suppose that the spatial dispersion is negligible, the magnetic permeability $\mu = 1$, and the permittivity $\varepsilon(k_0, \mathbf{x}) > 0$. In particular, $\operatorname{Im}\varepsilon(k_0, \mathbf{x}) = 0$, where $k_0 \in \mathbb{R}$ is found from solution of the Maxwell equations (1), i.e., it is "on shell." The absence of absorption is a necessary requirement for quantum field theory to be unitary. The generalization to the case of a medium with small absorption will be given below. The procedure developed below is analogous to the one used in [81] for the description of radiation of twisted photons by classical currents in a vacuum.

The free Maxwell equations in a medium have the form (see, e.g., [36,42,43])

$$\big[k_0^2\varepsilon(k_0, \mathbf{x}) - \hat{h}^2\big]A_i(k_0, \mathbf{x}) = 0, \quad \partial_i[\varepsilon(k_0, \mathbf{x})A_i(k_0, \mathbf{x})] = 0, \tag{1}$$

where $A_i(k_0, \mathbf{x})$ is the Fourier transform of the vector potential and we have introduced the Maxwell Hamiltonian (the curl operator)

$$\hat{h}_{ij} := \varepsilon_{ikj}\partial_k. \tag{2}$$

The second condition in (1) is the generalization of the Coulomb gauge such that the Fourier transform of the electric field strength $E_i(k_0, \mathbf{x}) = -ik_0A_i(k_0, \mathbf{x})$. If the solution to the first equation in (1) is found, then it satisfies the second equation in (1) provided $k_0 \neq 0$. Further, we assume that the electromagnetic field obeys boundary conditions such that $k_0 > 0$. In addition, we suppose that $\varepsilon(k_0, \mathbf{x}) \neq 1$ in the region M of a finite volume. The rest of the space possessing an infinite volume will be denoted by Ω. Since we assume $\operatorname{Im}\varepsilon(k_0, \mathbf{x}) = 0$ on shell then, for these values of k_0 [42],

$$\varepsilon(k_0, \mathbf{x}) = \varepsilon(-k_0, \mathbf{x}). \tag{3}$$

Introducing the operator

$$\hat{\tilde{h}}^2(k_0) := \varepsilon^{-1/2}\hat{h}^2\varepsilon^{-1/2}, \tag{4}$$

we cast Eqs. (1) into an explicitly self-adjoint form

$$\big(k_0^2 - \hat{\tilde{h}}^2\big)\tilde{A}_i(k_0, \mathbf{x}) = 0, \quad \partial_i[\varepsilon^{1/2}(k_0, \mathbf{x})\tilde{A}_i(k_0, \mathbf{x})] = 0, \tag{5}$$

where $\tilde{A}_i(k_0, \mathbf{x}) = \varepsilon^{1/2}(k_0, \mathbf{x})A_i(k_0, \mathbf{x})$. The operators entering Eqs. (5) remain unchanged under the replacement $k_0 \to -k_0$.

It is not difficult to find the general solution of Eqs. (5) for real k_0 such that $\operatorname{Im}\varepsilon(k_0, \mathbf{x}) = 0$. Let us pose the eigenvalue problem

$$\hat{\tilde{h}}^2(k_0)\tilde{\psi}_\alpha(k_0) = \chi_\alpha^2(k_0)\tilde{\psi}_\alpha(k_0), \quad \chi_\alpha(k_0) \geqslant 0, \tag{6}$$

where α marks the eigenvalues $\chi_\alpha^2(k_0)$. The operator on the left-hand side of (6) is self-adjoint in the Hilbert space of complex vectors $\tilde{\psi}_i(k_0; \mathbf{x})$ subject to the condition

$$\partial_i[\varepsilon^{1/2}(k_0)\tilde{\psi}_i(k_0)] = 0, \tag{7}$$

with the standard scalar product

$$\langle\tilde{\varphi}, \tilde{\psi}\rangle = \int d\mathbf{x}\, \tilde{\varphi}_i^*(\mathbf{x})\tilde{\psi}_i(\mathbf{x}). \tag{8}$$

The eigenfunctions (6) constitute a complete orthonormal set. The completeness relation reads

$$\begin{aligned}\sum_\alpha \tilde{\psi}_{\alpha i}(k_0; \mathbf{x})\tilde{\psi}^*_{\alpha j}(k_0; \mathbf{y}) &= \delta^{\perp}_{ij}(k_0; \mathbf{x}, \mathbf{y}) \\ &= \{\delta_{ij} - \varepsilon^{1/2}(k_0, \mathbf{x})\partial_i\Delta_\varepsilon^{-1}\partial_j\varepsilon^{1/2}(k_0, \mathbf{y})\}\delta(\mathbf{x} - \mathbf{y}),\end{aligned} \tag{9}$$

where Δ_ε^{-1} is the inverse to the operator $\Delta_\varepsilon := \partial_i\varepsilon(k_0, \mathbf{x})\partial_i$. Taking the complex conjugate of (6), we see that the complete set contains the function $\tilde{\psi}^*_\alpha$ along with $\tilde{\psi}_\alpha$ corresponding to the same eigenvalue χ^2_α but with a different quantum number $\alpha'(\alpha)$. Alternatively,

$$\tilde{\psi}^*_\alpha(k_0)=\tilde{\psi}_{\alpha'(\alpha)}(k_0),$$
$$\chi_\alpha(k_0)=\chi_{\alpha'(\alpha)}(k_0),\quad \alpha'(\alpha'(\alpha))=\alpha, \tag{10}$$

and $\alpha'(\alpha)=\alpha$ for real-valued $\tilde{\psi}_\alpha(k_0)$. In particular, it follows from this property that the left-hand side of (9) is real.

If the permittivity is discontinuous on some closed hypersurface Σ, then the standard boundary conditions for the electromagnetic field strength on Σ become

$$[\psi_\tau]=0,\quad [(\hat{h}\psi)_\tau]=0, \tag{11}$$

where the square brackets denote a discontinuity jump of the corresponding function on the hypersurface Σ and the index τ means that only components tangent to Σ of the complex vector ψ should be taken. It is implied in (11) that the surface charge and the current density are absent on Σ. The conditions (11) entail in particular that [36]

$$[\hat{h}\psi]=0,\quad [\varepsilon\psi_n]=0, \tag{12}$$

where n denotes the vector component normal to Σ. The boundary conditions (11) are consistent with Eqs. (5) in the sense that they make the operator $\hat{\tilde{h}}^2$ self-adjoint in the space of divergence-free complex vector fields obeying (11). The corresponding boundary terms (the singular current) for the fields defined in the region bounded by the surface Σ have the form (118). The fields defined from the outside of Σ result in the same singular current but with the opposite sign ($n^i\to -n^i$). The fulfillment of boundary conditions (11) leads to cancellation of the singular currents. Note that the boundary conditions on the surface of an ideal conductor look like

$$\psi_\tau=0, \tag{13}$$

and $\psi=0$ inside the conductor. The operator $\hat{\tilde{h}}^2$ is self-adjoint on account of these boundary conditions.

Assuming that the operator (4) does not have zero eigenvalues, the complete set (6) allows one to write the quantum field satisfying (5):

$$\hat{\tilde{A}}_i(t,\mathbf{x})=\sum_\alpha\big[\hat{a}_\alpha f_\alpha^{1/2}\tilde{\psi}_{\alpha i}(k_{0\alpha};\mathbf{x})e^{-ik_{0\alpha}t}$$
$$+\hat{a}^\dagger_\alpha f_\alpha^{1/2}\tilde{\psi}^*_{\alpha i}(k_{0\alpha};\mathbf{x})e^{ik_{0\alpha}t}\big]. \tag{14}$$

Here

$$[\hat{a}_\alpha,\hat{a}_\beta]=[\hat{a}^\dagger_\alpha,\hat{a}^\dagger_\beta]=0,\quad [\hat{a}_\alpha,\hat{a}^\dagger_\beta]=\delta_{\alpha\beta}, \tag{15}$$

the on-shell condition takes the form

$$k^2_{0\alpha}=\chi^2_\alpha(k_{0\alpha}),\quad k_{0\alpha}>0, \tag{16}$$

and the normalization coefficients $f_\alpha>0$ are found from the requirement that the residue of the propagator of the field (14) is equal to unity. The quantum field $\hat{A}_i(x)$ is obtained from (14) in an obvious manner

$$\hat{A}_i(t,\mathbf{x})=\sum_\alpha\big[\hat{a}_\alpha f_\alpha^{1/2}\psi_{\alpha i}(k_{0\alpha};\mathbf{x})e^{-ik_{0\alpha}t}$$
$$+\hat{a}^\dagger_\alpha f_\alpha^{1/2}\psi^*_{\alpha i}(k_{0\alpha};\mathbf{x})e^{ik_{0\alpha}t}\big], \tag{17}$$

where $\psi_{\alpha i}(k_{0\alpha};\mathbf{x})=\varepsilon^{-1/2}(k_{0\alpha};\mathbf{x})\tilde{\psi}_{\alpha i}(k_{0\alpha};\mathbf{x})$.

The commutator Green's function is written as

$$\tilde{G}_{ij}(t,\mathbf{x};t',\mathbf{y}):$$
$$=[\hat{\tilde{A}}_i(t,\mathbf{x}),\hat{\tilde{A}}_j(t',\mathbf{y})]$$
$$=\sum_\alpha f_\alpha[\tilde{\psi}_{\alpha i}(k_{0\alpha};\mathbf{x})\tilde{\psi}^*_{\alpha j}(k_{0\alpha};\mathbf{y})e^{-ik_{0\alpha}(t-t')}-\text{c.c.}]. \tag{18}$$

On the other hand, the retarded Green's function for the operator (5) is

$$G^-_{ij}(t,\mathbf{x};t',\mathbf{y})=\int\frac{dk_0}{2\pi}\sum_\alpha\tilde{\psi}_{\alpha i}(k_0;\mathbf{x})\frac{e^{-ik_0(t-t')}}{k_0^2-\chi^2_\alpha(k_0)}\tilde{\psi}^*_{\alpha j}(k_0;\mathbf{y}), \tag{19}$$

where the integration contour over k_0 runs a little bit higher than the real axis. Taking into account that

$$G^-_{ij}(t,\mathbf{x};t',\mathbf{y})=-i\theta(t-t')\tilde{G}_{ij}(t,\mathbf{x};t',\mathbf{y}) \tag{20}$$

and evaluating (19) for $t>t'$ by residues, we obtain [40,42,45,91–93]

$$f_\alpha^{-1}=\big[k_0^2-\chi^2_\alpha(k_0)\big]'_{k_0=k_{0\alpha}}=2k_{0\alpha}[1-\chi'_\alpha(k_{0\alpha})], \tag{21}$$

where $k_{0\alpha}$ is the solution of (16). For the quantum theory to be unitary, the right-hand side of (21) must be positive for $k_{0\alpha}>0$. Note that this condition is fulfilled for a homogeneous isotropic medium due to the inequalities that hold for the permittivity $\varepsilon(k_0)$ and its derivative (see Sec. 84 in [42] and Sec. IV below).

We define

$$\bar{\delta}_{ij}(\mathbf{x},\mathbf{y}):=\sum_\alpha\tilde{\psi}_{\alpha i}(k_{0\alpha};\mathbf{x})\tilde{\psi}^*_{\alpha j}(k_{0\alpha};\mathbf{y}). \tag{22}$$

As long as $k_{0\alpha}=k_{0\alpha'(\alpha)}$ and the relations (10) hold, we have

$$\bar{\delta}^*_{ij}(\mathbf{x},\mathbf{y})=\bar{\delta}_{ij}(\mathbf{x},\mathbf{y}). \tag{23}$$

The operator $\bar{\delta}$ is the identity on the space of solutions to the Maxwell equations (5) in the sense that

$$\int d\mathbf{y}\,\bar{\delta}_{ij}(\mathbf{x},\mathbf{y})\hat{\tilde{A}}_j(t,\mathbf{y})=\hat{\tilde{A}}_i(t,\mathbf{x}). \tag{24}$$

Furthermore, it follows from (10) that

$$\tilde{G}_{ij}(t,\mathbf{x};t,\mathbf{y})=[\hat{\tilde{A}}_i(t,\mathbf{x}),\hat{\tilde{A}}_j(t,\mathbf{y})]=0. \tag{25}$$

By analogy with quantum field theory in a vacuum, one can introduce the canonical momentum

$$\hat{\tilde{\pi}}_i(t,\mathbf{x}):=-\frac{i}{2}\sum_\alpha\big[\hat{a}_\alpha f_\alpha^{-1/2}\tilde{\psi}_{\alpha i}(k_{0\alpha};\mathbf{x})e^{-ik_{0\alpha}t}$$
$$-\hat{a}^\dagger_\alpha f_\alpha^{-1/2}\tilde{\psi}^*_{\alpha i}(k_{0\alpha};\mathbf{x})e^{ik_{0\alpha}t}\big]. \tag{26}$$

Then

$$[\hat{\tilde{A}}_i(t,\mathbf{x}),\hat{\tilde{\pi}}_j(t,\mathbf{y})]=i\bar{\delta}_{ij}(\mathbf{x},\mathbf{y}),\quad [\hat{\tilde{\pi}}_i(t,\mathbf{x}),\hat{\tilde{\pi}}_j(t,\mathbf{y})]=0. \tag{27}$$

The last property follows from the relations (10). The evolution operator of the electromagnetic field in the presence of the external conserved classical current is given by (see, e.g., [94])

$$\hat{U}_{t_2,t_1} = \hat{U}^0_{t_2,0}\hat{S}_{t_2,t_1}\hat{U}^0_{0,t_1},$$
$$\hat{S}_{t_2,t_1} = T\exp\left[-i\int_{t_1}^{t_2} dx\,\hat{A}_i(x)j^i(x) - i\int_{t_1}^{t_2} dt\,V_{\text{Coul}}\right], \quad (28)$$

where $\hat{U}^0_{t_2,t_1}$ is the evolution operator of a free electromagnetic field (17) and V_{Coul} is the energy of Coulomb self-interaction of the current. In the case at hand, this energy gives only the contribution to the common phase of transition amplitudes, and we do not take it into account anymore. It is noteworthy, however, that this phase is divergent in the infrared limit, if the system possesses an uncompensated charge [94,95].

III. RECORDING PHOTONS IN A STATIONARY STATE

Having constructed quantum field theory in a medium, we ought to define what we mean under the twisted photon and its recording by the detector. We follow the postulate of quantum theory that a detector allows one to measure the projection of quantum state to a given state, i.e., the probability to find a quantum system in a given state. This given state is determined by the structure of the detector. The concrete form of the detector will not be relevant for us. It is only needed that the detector projects the quantum state to the state close to a vacuum twisted photon. Notice that the detectors were elaborated that allow one to decompose the electromagnetic field in terms of twisted photons even at a single-photon level [18–21].

Suppose that the detector of twisted photons is located in the region Ω, which has an infinite volume, and $\varepsilon(k_0,\mathbf{x}) = 1$ in it. Let $\tilde{\psi}_\alpha(k_{0\alpha};\mathbf{x})$ be a complete set of solutions to the Maxwell equations described in the preceding section. Using these mode functions, we construct the wave packets φ_β such that (i) $\varphi_\beta \approx \varphi_{0\beta}$ in the vicinity of the detector of twisted photons, where $\varphi_{0\beta}$ are the modes corresponding to twisted photons in a vacuum [see the explicit expression in [81,87,88,96–99] and (55)], and (ii) φ_β are sufficiently narrow with respect to the energy quantum number, i.e., they are composed of the mode functions $\tilde{\psi}_\alpha(k_{0\alpha};\mathbf{x})$ with small dispersion of energy $\Delta k_{0\alpha}$. It is the photon state φ_β which is assumed to be recorded by the detector. These conditions imply in particular that the detector is positioned in the wave zone and the region where $\varphi_\beta \approx \varphi_{0\beta}$ is sufficiently large. As follows from the uncertainty relation,

$$\Delta k_{0\alpha} \sim 2\pi/L_{\text{vac}}, \quad (29)$$

where L_{vac} is a typical size of this region.

In the vicinity of the detector, under the above conditions, we can write

$$\hat{A}_i(t,\mathbf{x}) \approx \int d\mathbf{y}\sum_\beta \varphi_{\beta i}(\mathbf{x})\varphi^*_{\beta j}(\mathbf{y})\hat{A}_j(t,\mathbf{y})$$
$$= \sum_{\alpha,\beta}\left[\varphi_{\beta i}(\mathbf{x})f_\alpha^{1/2}\hat{a}_\alpha\langle\varphi_\beta,\psi_\alpha(k_{0\alpha})\rangle e^{-ik_{0\alpha}t} + \text{H.c.}\right]. \quad (30)$$

Assuming that

$$\Delta k_{0\beta}T \ll 1, \quad (31)$$

we obtain

$$\hat{A}_i(t,\mathbf{x}) \approx \sum_{\alpha,\beta}\left[\varphi_{\beta i}(\mathbf{x})e^{-ik_{0\beta}t}f_\alpha^{1/2}\hat{a}_\alpha\langle\varphi_\beta,\psi_\alpha(k_{0\alpha})\rangle + \text{H.c.}\right]. \quad (32)$$

The parameter T is the observation period. In fact, in order to describe the radiation correctly, we need the fulfillment of a weaker condition in (30) and (32),

$$\Delta k_{0\beta}t_f \ll 1, \quad (33)$$

where t_f is the radiation formation time. The fulfillment of this last condition can always be achieved by using the detector of twisted photons with sufficiently narrow bandwidth. We introduce the notation

$$\hat{b}_\beta := \sqrt{2k_{0\beta}}\sum_\alpha f_\alpha^{1/2}\hat{a}_\alpha\langle\varphi_\beta,\psi_\alpha(k_{0\alpha})\rangle. \quad (34)$$

Then, in the vicinity of the detector, the quantum field becomes

$$\hat{A}_i(t,\mathbf{x}) = \sum_\beta\left[\varphi_{\beta i}(\mathbf{x})\frac{e^{-ik_{0\beta}t}}{\sqrt{2k_{0\beta}}}\hat{b}_\beta + \text{H.c.}\right], \quad (35)$$

which coincides with the decomposition of the free quantum electromagnetic field in terms of twisted photons.

In the first Born approximation with respect to the external current, the transition amplitude of the process

$$0 \to \gamma \quad (36)$$

reads

$$S(\beta;0) = \langle 0|\hat{b}_\beta\hat{U}_{T/2,-T/2}|0\rangle \approx -ie^{-iTE_{\text{vac}}}\sqrt{2k_{0\beta}}\sum_\alpha\int_{-T/2}^{T/2}dy$$
$$\times\, f_\alpha\langle\varphi_\beta,\psi_\alpha(k_{0\alpha})\rangle\psi^*_{\alpha i}(\mathbf{y})j^i(y)e^{-ik_{0\alpha}(T/2-y^0)}, \quad (37)$$

where it is assumed that $\hat{H}_0|0\rangle = E_{\text{vac}}|0\rangle$. Let us introduce the positive-frequency Green's function

$$G^{(+)}_{ij}(x,y) := -i\langle 0|\hat{A}_i(x)\hat{A}_j(y)|0\rangle$$
$$= -i\sum_\alpha f_\alpha e^{-ik_{0\alpha}(x^0-y^0)}\psi_{\alpha i}(k_{0\alpha};\mathbf{x})\psi^*_{\alpha j}(k_{0\alpha};\mathbf{y}). \quad (38)$$

Then, employing condition (ii), we can write

$$S(\beta;0) = e^{-iTE_{\text{vac}}}\sqrt{2k_{0\beta}}\int d\mathbf{x}\int_{-T/2}^{T/2}dy\,\varphi^*_{\beta i}(\mathbf{x})G^{(+)}_{ij}$$
$$\times\,(T/2,\mathbf{x};y^0,\mathbf{y})j^j(y). \quad (39)$$

Consequently, in the first Born approximation, the probability to record one photon in the process (36) is written as

$$dP(\beta) = 2k_{0\beta}\left|\int d\mathbf{x}\int_{-T/2}^{T/2}dy\,\varphi^*_{\beta i}(\mathbf{x})G^{(+)}_{ij}\right.$$
$$\left.\times\,(T/2,\mathbf{x};y^0,\mathbf{y})j^j(y)\right|^2 d\beta, \quad (40)$$

where $d\beta$ is the measure in the space of quantum numbers of twisted photons in a vacuum [see formula (14) of [81]]. Since

$$G^{(+)}_{ij}(x,y) = G_{ij}(x,y), \quad x^0 > y^0, \quad (41)$$

where $G_{ij}(x,y)$ is the Feynman propagator, the positive-frequency Green's function can be replaced by the Feynman propagator in (40).

The formula (40) can be simplified, if one strengthens condition (i) and demands that $\varphi_\beta \approx \varphi_{0\beta}$ everywhere except from a bounded neighborhood of the region M. Then, as long as Ω is unbounded, the contribution of the bounded neighborhood of the region M can be neglected in the scalar product entering (37) and φ_β can be replaced by $\varphi_{0\beta}$. As a result,

$$dP(\beta)=2k_{0\beta}\bigg|\int d\mathbf{x}\int_{-\infty}^{\infty} dy\,\varphi^*_{0\beta i}(\mathbf{x})G^{(+)}_{ij}(0,\mathbf{x};y^0,\mathbf{y})j^j(y)\bigg|^2 d\beta, \tag{42}$$

where we have taken the limit $T\to\infty$. The concrete choice of the instant of time $x^0=0$ in the argument of the positive-frequency Green's function in (42) is irrelevant as the different choices of x^0 just result in a common phase factor which disappears in evaluating the modulus. Performing the Fourier transforms

$$G^{(+)}_{ij}(k_0;\mathbf{x},\mathbf{y}) = -2\pi i\sum_\alpha \delta(k_0-k_{0\alpha})f_\alpha\psi_{\alpha i}(k_0;\mathbf{x})\psi^*_{\alpha j}(k_0;\mathbf{y}),$$
$$j^i(k_0;\mathbf{x}) = \int_{-\infty}^{\infty} dt\, e^{ik_0t}j^i(t,\mathbf{x}), \tag{43}$$

we obtain

$$dP(\beta) = 2k_{0\beta}\bigg|\int d\mathbf{x}\,d\mathbf{y}\int_0^\infty \frac{dk_0}{2\pi}\varphi^*_{0\beta i}(\mathbf{x})G^{(+)}_{ij} \times (k_0;\mathbf{x},\mathbf{y})j^j(k_0;\mathbf{y})\bigg|^2 d\beta. \tag{44}$$

For $k_0>0$, the formula (43) can be written as

$$G^{(+)}_{ij}(k_0;\mathbf{x},\mathbf{y}) = G_{ij}(k_0+i0;\mathbf{x},\mathbf{y}) - G^*_{ji}(k_0+i0;\mathbf{y},\mathbf{x}), \tag{45}$$

where $G_{ij}(k_0)$ is the resolvent of the Maxwell operator (1), i.e.,

$$\big[k_0^2\varepsilon(k_0,\mathbf{x})-\hat{h}^2\big]\hat{G}(k_0)=\hat{1},\quad k_0\in\mathbb{C}. \tag{46}$$

As was discussed in Sec. 28 of [36] and Sec. 75 of [100], this resolvent is analytic for $\operatorname{Im}k_0>0$ and

$$G_{ij}(k_0;\mathbf{x},\mathbf{y}) = G_{ji}(k_0;\mathbf{y},\mathbf{x}) = G^*_{ij}(-k_0^*;\mathbf{x},\mathbf{y}). \tag{47}$$

The formula (45) can be generalized to the case of a finite temperature β_T^{-1}. In this case (see Secs. 76 and 86 in [100]),

$$G^{(+)}_{ij}(k_0;\mathbf{x},\mathbf{y}) = (1-e^{-\beta_Tk_0})^{-1}[G_{ij}(k_0+i0;\mathbf{x},\mathbf{y}) - G^*_{ji}(k_0+i0;\mathbf{y},\mathbf{x})]. \tag{48}$$

The temperature-dependent factor is $1+n_{\mathrm{BE}}(\beta_Tk_0)$, where the last term (the Bose-Einstein distribution) is responsible for stimulated radiation. The temperature effects are relevant only when

$$\beta_Tk_0\lesssim 1,\quad 1\ \mathrm{K}\approx 8.6\times10^{-5}\ \mathrm{eV}, \tag{49}$$

where k_0 is the energy of a radiated photon. Thus, the probability to record a twisted photon (44) reads

$$dP(\beta) = 2k_{0\beta}\bigg|\int d\mathbf{x}\,d\mathbf{y}\int_{-\infty}^\infty \frac{dk_0}{2\pi}\frac{\operatorname{sgn}(k_0)}{1-e^{-\beta_T|k_0|}}\varphi^*_{0\beta i}(\mathbf{x}) \times G_{ij}(k_0+i0;\mathbf{x},\mathbf{y})j^j(|k_0|;\mathbf{y})\bigg|^2 d\beta. \tag{50}$$

Of course, the background blackbody radiation is neglected in this formula. Such a representation allows one to use the formula (50) in the case of a weakly absorbing medium (cf. [101]). The Green's function for the Maxwell equations (1) can be found perturbatively by treating $k_0^2[\varepsilon(k_0,\mathbf{x})-1]$ as a perturbation.

Since the model we consider is exactly solvable (see, e.g., [81,96,102,103]), we can find the average number of created twisted photons and the probability of the inclusive process

$$0\to\gamma+X. \tag{51}$$

The procedure is completely the same as that given in [81], Sec. 3. The probabilities (40) and (44) coincide with the average number of twisted photons and, with good accuracy, are equal to the probability of inclusive process (51) (see the details in [81]). If the probability to record a twisted photon with quantum numbers $\beta\in D$ is needed, then

$$w_{\mathrm{incl}}(\beta\in D;0) = 1-\exp\bigg[-\int_D dP(\beta)\bigg]. \tag{52}$$

For the radiation of a point charged particle, formulas (44) and (50) look like

$$dP(\beta) = 2k_{0\beta}e^2\bigg|\int_{-\infty}^\infty d\tau\int d\mathbf{x}\int_0^\infty\frac{dk_0}{2\pi}\varphi^*_{0\beta i}(\mathbf{x}) \times G^{(+)}_{ij}(k_0;\mathbf{x},\mathbf{x}(\tau))\dot{x}^j(\tau)e^{ik_0x^0(\tau)}\bigg|^2 d\beta$$
$$= 2k_{0\beta}e^2\bigg|\int_{-\infty}^\infty d\tau\int d\mathbf{x}\int_{-\infty}^\infty\frac{dk_0}{2\pi}\frac{\operatorname{sgn}(k_0)}{1-e^{-\beta_T|k_0|}}\varphi^*_{0\beta i}(\mathbf{x}) \times G_{ij}(k_0+i0;\mathbf{x},\mathbf{x}(\tau))\dot{x}^j(\tau)e^{i|k_0|x^0(\tau)}\bigg|^2 d\beta, \tag{53}$$

where e is the particle charge and $x^\mu(\tau)$ specifies the particle world line. Recall that we work in the system of units where $e^2=4\pi\alpha$.

IV. RADIATION OF TWISTED PHOTONS IN A HOMOGENEOUS MEDIUM

Let us consider separately the generation of twisted photons in the case when the dispersive medium is homogeneous $\varepsilon=\varepsilon(k_0)$, $\varepsilon(k_0)>0$, and the detector of twisted photons is located in the medium or the twisted photons escape the medium, preserving its phase front structure and are recorded by the detector out of the medium. In that case, the procedure developed above can be considerably simplified.

We denote by ψ_α the orthonormal set of eigenfunctions of the Maxwell Hamiltonian

$$\hat{h}\psi_\alpha = s\tilde{\chi}_\alpha\psi_\alpha,\quad \tilde{\chi}_\alpha>0,\ s=\pm1, \tag{54}$$

taken in the form of twisted photons (see for details [81])

$$
\begin{aligned}
\psi_3(m,k_3,k_\perp) &= \frac{1}{\sqrt{RL_z}} j_m(k_\perp x_+, k_\perp x_-) e^{ik_3x_3},\\
\psi_\pm(s,m,k_3,k_\perp) &= \frac{ik_\perp}{sk_0 \pm k_3}\psi_3(m\pm 1,k_3,k_\perp),\\
\psi(s,m,k_3,k_\perp) &= \frac{1}{2}[\psi_-(s,m,k_3,k_\perp)\mathbf{e}_+\\
&\quad + \psi_+(s,m,k_3,k_\perp)\mathbf{e}_-]\\
&\quad + \psi_3(m,k_3,k_\perp)\mathbf{e}_3, \qquad (55)
\end{aligned}
$$

where s is the photon helicity, m is the projection of the total angular momentum onto the axis 3,

$$k_0 := \sqrt{k_3^2 + k_\perp^2}, \qquad (56)$$

and R and L_z characterize the normalization volume. The function

$$j_m(p,q) := (p/q)^{m/2} J_m(p^{1/2}q^{1/2}) \qquad (57)$$

is an entire function of p and q for integer m. The helicity s is an eigenvalue of the helicity operator

$$S_{ij} := J_{lij}k_l/|\mathbf{k}|, \qquad (58)$$

where the total angular momentum operator

$$J_{lij} := \varepsilon_{lmn}x_m k_n \delta_{ij} - i\varepsilon_{lij}, \quad k_n := -i\partial_n. \qquad (59)$$

The index l in J_{lij} marks the components of the angular momentum operator. The basis vectors $\mathbf{e}_\pm := \mathbf{e}_1 \pm i\mathbf{e}_2$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a right-handed orthonormal triple. The set of functions (55) is complete in the space of divergence-free square-integrable complex vector fields. The Maxwell equations (1) entail the on-shell condition

$$k_{0\alpha}^2 \varepsilon(k_{0\alpha}) = \tilde{\chi}_\alpha^2, \quad k_{0\alpha} > 0. \qquad (60)$$

The free quantum electromagnetic field has the form (17) with the normalization coefficients

$$f_\alpha^{-1} = \left[k_0^2 - \tilde{\chi}_\alpha^2/\varepsilon(k_0)\right]'_{k_0=k_{0\alpha}} = \left[k_0^2\varepsilon(k_0)\right]'_{k_0=k_{0\alpha}} = \frac{d\tilde{\chi}_\alpha^2}{dk_{0\alpha}}. \qquad (61)$$

In this last equality, it is assumed that $\tilde{\chi}_\alpha$ is expressed though $k_{0\alpha}$ by means of the on-shell condition (60). As long as (see [42], Sec. 84)

$$\frac{d}{dk_0}\left[k_0^2\varepsilon(k_0)\right] > \frac{d}{dk_0}k_0^2 > 0 \qquad (62)$$

for $k_0 > 0$, then $f_\alpha > 0$ for $k_{0\alpha} > 0$.

Repeating the considerations presented in the previous sections (see also [81]), we find that the probability to record a twisted photon produced by point charged particles is given by

$$
\begin{aligned}
dP(s,m,k_3,k_\perp) &= \frac{k_\perp^3 f_\alpha}{(2\tilde{\chi}_\alpha)^2}\frac{dk_3dk_\perp}{2\pi^2}\bigg|\sum_l e_l\int d\tau_l\\
&\quad\times e^{-ik_{0\alpha}x_l^0(\tau_l)+ik_3x_{l3}(\tau_l)}\\
&\quad\times\bigg\{\frac{1}{2}[\dot{x}_{l+}(\tau_l)a_-(s,m,k_3,k_\perp;\mathbf{x}_l(\tau_l))\\
&\quad+\dot{x}_{l-}(\tau_l)a_+(s,m,k_3,k_\perp;\mathbf{x}_l(\tau_l))]\\
&\quad+\dot{x}_{l3}(\tau_l)a_3(m,k_\perp;\mathbf{x}_l(\tau_l))\bigg\}\bigg|^2, \qquad (63)
\end{aligned}
$$

where the notation borrowed from [81] has been used, l numbers the particles with charges e_l, and

$$\tilde{\chi}_\alpha = \sqrt{k_3^2 + k_\perp^2}. \qquad (64)$$

In the formula (63), it is convenient to pass from the variable k_3 to k_0, $k_0 > 0$. Then

$$f_\alpha dk_3 = dk_0/2k_3, \quad k_3 = \sqrt{k_0^2\varepsilon(k_0) - k_\perp^2}, \qquad (65)$$

and

$$
\begin{aligned}
dP(s,m,k_0,k_\perp) &= \frac{n_\perp^3}{n_3}\frac{dk_0dk_\perp}{16\pi^2}\bigg|\sum_l e_l\int d\tau_l e^{-ik_0x_l^0(\tau_l)+ik_3x_{l3}(\tau_l)}\\
&\quad\times\bigg\{\frac{1}{2}[\dot{x}_{l+}(\tau_l)a_-(s,m,k_3,k_\perp;\mathbf{x}_l(\tau_l))\\
&\quad+\dot{x}_{l-}(\tau_l)a_+(s,m,k_3,k_\perp;\mathbf{x}_l(\tau_l))]\\
&\quad+\dot{x}_{l3}(\tau_l)a_3(m,k_\perp;\mathbf{x}_l(\tau_l))\bigg\}\bigg|^2. \qquad (66)
\end{aligned}
$$

In this section, it is convenient to use the notation

$$n_3 := \frac{k_3}{k_0\varepsilon^{1/2}(k_0)} = \left(1 - \frac{k_\perp^2}{k_0^2\varepsilon(k_0)}\right)^{1/2}, \quad n_\perp := \frac{k_\perp}{k_0\varepsilon^{1/2}(k_0)}. \qquad (67)$$

As can be seen, the effect of a medium is reduced in this case to the replacement $k_0 \to k_0\varepsilon^{1/2}(k_0)$ in the expression for the spatial part of the mode functions defining the probability of twisted photon radiation by a classical current in a vacuum [81].

Vavilov-Cherenkov radiation

As an immediate application of the general formula (66), we consider the generation of twisted photons by means of the VC radiation by a charged particle moving strictly along the detector axis. In that case,

$$x_\pm = 0, \quad x_3 = \beta_3 x^0, \qquad (68)$$

where $\beta_3 = \text{const} > 0$ is the particle velocity in the laboratory frame. The expression under the modulus sign in (66) is equal to

$$2\pi e\beta_3\delta(k_0 - k_3(k_0,k_\perp)\beta_3)\delta_{m0}. \qquad (69)$$

Squaring this expression, we obtain the probability of radiation of a twisted photon per unit time

$$
\begin{aligned}
&dP(s,m,k_0,k_\perp)/T\\
&\quad= 2\pi e^2\delta(k_0 - k_3(k_0,k_\perp)\beta_3)\delta_{m0}\beta_3^2\frac{n_\perp^3}{n_3}\frac{dk_0dk_\perp}{16\pi^2}. \qquad (70)
\end{aligned}
$$

We see that the VC radiation consists of the twisted photons with $m = 0$ [15,16]. This property is a consequence of the general statement [82] that the current density $j^i(t,\mathbf{x})$ invariant with respect to rotations around the detector axis for any t

produces the twisted photons only with $m=0$. The δ function entering (70) defines the Cherenkov cone

$$k_\perp = k_0\sqrt{\varepsilon(k_0)-\beta_3^{-2}},\quad n_\perp=\sqrt{1-\varepsilon^{-1}(k_0)\beta_3^{-2}},\\ n_3=\varepsilon^{-1/2}(k_0)\beta_3^{-1}. \tag{71}$$

Recall that the speed of light in a medium is $\varepsilon^{-1/2}(k_0)$. Integrating (70) over $k_\perp$, we have

$$dP(s,m,k_0)/T\\ =\frac{\alpha}{2}\delta_{m0}\beta_3\big(1-\varepsilon^{-1}(k_0)\beta_3^{-2}\big)\theta\big(1-\varepsilon^{-1}(k_0)\beta_3^{-2}\big)dk_0. \tag{72}$$

This expression does not depend on s.

Using (72) and the general formulas obtained in [53,54], it is not difficult to find the probability per unit time of twisted photon radiation in the case when the charged particle moves with constant velocity along the line parallel to the detector axis,

$$dP(s,m,k_0)/T=\frac{\alpha}{2}J_m^2\big(k_\perp(k_0)|x_+|\big)\beta_3\big(1-\varepsilon^{-1}(k_0)\beta_3^{-2}\big)\\ \times\ \theta\big(1-\varepsilon^{-1}(k_0)\beta_3^{-2}\big)dk_0, \tag{73}$$

where $|x_+|$ is the distance from the detector axis to the particle trajectory. Note that the dependence of the expression (73) on m is the same as for the edge radiation for a charged particle moving along the detector axis [82]. As for the radiation by a bunch of N identical particles moving along parallel trajectories, we obtain

$$dP_\rho(s,m,k_0)/T\\ =\frac{\alpha}{2}[Nf_m(k_\perp(k_0)\sigma_\perp)+N(N-1)|\varphi_m(k_0/\beta_3,k_\perp(k_0)\sigma_\perp)|^2]\\ \times\ \beta_3\big(1-\varepsilon^{-1}(k_0)\beta_3^{-2}\big)\theta\big(1-\varepsilon^{-1}(k_0)\beta_3^{-2}\big)dk_0, \tag{74}$$

where $\sigma_\perp$ is the transverse size of the particle bunch, $f_m(x)$ is the incoherent interference factor, and φ_m is the corresponding coherent interference factor (see the notation and explicit expressions in [53,54]).

V. EXAMPLES

A. Dielectric plate

Let us consider the radiation of twisted photons produced by a charged particle passing through a homogeneous isotropic dielectric plate with the width L. We suppose that the plate is positioned at $z\in[-L,0]$, and $\varepsilon(k_0)>0$ does not depend on the choice of a point in the plate. Out of the plate, $\varepsilon(k_0)=1$. The detector of twisted photons is located at $z>0$. The charged particle moves along the detector axis with constant velocity (68). The general procedure expounded in Secs. II and III cannot be directly applied to this case as the medium occupies an infinite volume. Nevertheless, this procedure is readily generalized to such a configuration. Having suitably defined the modes of twisted photons, we will find the expression for the probability of excitation of these modes by constructing the complete set of solutions of the Maxwell equations (5) with the boundary conditions (11). Note that the transition radiation of plane-wave photons produced by twisted electrons was investigated in [104,105]. In this section we consider the transition radiation of twisted photons generated by usual plane-wave charged particles or by helical beams of them.

The wave functions of twisted photons in a vacuum have the form (55). The wave functions of twisted photons with energy k_0 in a homogeneous isotropic medium $z\in[-L,0]$ are written as

$$\psi'_3(m',k'_3,k'_\perp)=\psi_3(m',k'_3,k'_\perp),\\ \psi'_\pm(s',m',k'_3,k'_\perp)=\frac{ik'_\perp}{s'\varepsilon^{1/2}(k_0)k_0\pm k'_3}\psi'_3(m'\pm1,k'_3,k'_\perp), \tag{75}$$

where

$$\varepsilon^{1/2}(k_0)k_0=\sqrt{k_3'^2+k_\perp'^2}, \tag{76}$$

and we have used the notation in (55). Note that

$$\hat h\psi'(s',m',k'_3,k'_\perp)=s'\varepsilon^{1/2}(k_0)k_0\psi'(s',m',k'_3,k'_\perp). \tag{77}$$

It follows from the first boundary condition in (11) at $z=0$ that

$$k_\perp=k'_\perp,\quad m=m'. \tag{78}$$

For given k_0 and k_3, the momentum of photon in a medium, k'_3, is found from Eq. (76) and can take two values differing by the sign. For $z>0$, the mode function of a twisted photon reads

$$a\psi(s,m,k_3,k_\perp). \tag{79}$$

For $z\in[-L,0]$, using the boundary conditions (11), we obtain the wave function

$$a[b_+\psi'(1,m,k'_3,k_\perp)+c_+\psi'(1,m,-k'_3,k_\perp)\\ +\ b_-\psi'(-1,m,k'_3,k_\perp)+c_-\psi'(-1,m,-k'_3,k_\perp)], \tag{80}$$

where the coefficients of the linear combination are

$$b_\pm=\frac{\varepsilon^{1/2}\pm s}{4\varepsilon k'_3}(\pm sk'_3+\varepsilon^{1/2}k_3),\\ c_\pm=\frac{\varepsilon^{1/2}\pm s}{4\varepsilon k'_3}(\pm sk'_3-\varepsilon^{1/2}k_3). \tag{81}$$

The constant a is found from the normalization condition for the mode functions.

For $z<-L$, the mode function is the linear combination

$$a[a_+\psi(1,m,k_3,k_\perp)+d_+\psi(1,m,-k_3,k_\perp)\\ +\ a_-\psi(-1,m,k_3,k_\perp)+d_-\psi(-1,m,-k_3,k_\perp)]. \tag{82}$$

Then the boundary conditions at $z=-L$ lead to

$$a_\pm=\frac{2(1\pm s)\varepsilon k_3k'_3\cos(k'_3L)-i\big[\varepsilon^2k_3^2+k_3'^2\pm s\varepsilon\big(k_3^2+k_3'^2\big)\big]\sin(k'_3L)}{4\varepsilon k_3k'_3}e^{ik_3L},\quad d_\pm=-i\frac{\varepsilon^2k_3^2-k_3'^2\pm s\varepsilon\big(k_3^2-k_3'^2\big)}{4\varepsilon k_3k'_3}\sin(k'_3L)e^{-ik_3L}, \tag{83}$$

where s is the helicity of the mode function (79). The coefficients (83) obey the unitarity relation

$$1+|d_+|^2+|d_-|^2=|a_+|^2+|a_-|^2 \tag{84}$$

for real k_3. The dielectric plate is ideally transparent for the modes with real k_3 and $k'_3L=\pi n$, $n\in\mathbb{Z}$.

Let $\Phi(s,m,k_\perp,k_3)$ be the mode function taking the values (79), (80), and (82) on the corresponding intervals of the variable z without the common factor a and with the same values of the quantum numbers s, m, $k_\perp$, and k_0. The various linear combinations of the modes $\Phi(s,m,k_\perp,k_3)$ with the same energy give the complete set of solutions to the Maxwell equations (5) with the boundary conditions (11). There are bound states among these mode functions. They are exponentially damped out of the dielectric plate and correspond to the case of complete internal reflection of the electromagnetic wave in the dielectric. This occurs when

$$k_3'^2=\varepsilon(k_0)k_0^2-k_\perp^2\geqslant 0,\quad k_3^2=k_0^2-k_\perp^2<0 \tag{85}$$

and is only possible for $\varepsilon>1$. Setting $k_3=i|k_3|$, we find that the linear combination

$$\alpha_+\Phi(1,m,k_\perp,k_3)+\alpha_-\Phi(-1,m,k_\perp,k_3) \tag{86}$$

is exponentially damped out of the dielectric plate if one of the following two conditions is satisfied:

$$\cot(k'_3L)=\frac{1}{2}\left(\frac{k'_3}{|k_3|}-\frac{|k_3|}{k'_3}\right),$$
$$\cot(k'_3L)=\frac{1}{2}\left(\frac{k'_3}{\varepsilon|k_3|}-\frac{\varepsilon|k_3|}{k'_3}\right). \tag{87}$$

These equations define the multivalued function $k_\perp=k_\perp(k_0,L)$. Such states are irrelevant for further investigation of the radiation of twisted photons. They are exponentially suppressed in the domain where the detector is positioned and give a negligible contribution to (42). Note that the bound states (86) are the eigenfunctions of the operator of the total angular momentum.

It is useful to numerate the mode functions of twisted photons recorded by the detector ($k_3>0$) by the quantum numbers $\alpha:=(s,m,k_3,k_\perp)$ and set

$$k'_3=\sqrt{\varepsilon(k_0)k_0^2-k_\perp^2},\quad k_0=\sqrt{k_\perp^2+k_3^2}. \tag{88}$$

Then

$$f_\alpha^{-1}=2k_{0\alpha}=2\sqrt{k_3^2+k_\perp^2}>0. \tag{89}$$

Since, in the region $z>0$, where the detector is positioned, the wave functions $\Phi(s,m,k_\perp,k_3)$ coincide with the wave functions of twisted photons in a vacuum, we will call these wave functions mode functions of twisted photons. For $L_z\to\infty$ and $R\to\infty$, the normalization condition for the mode function $a\Phi(s,m,k_\perp,k_3)$ is written as

$$\begin{aligned}&\frac{|a|^2}{2}(1+|a_+|^2+|a_-|^2+|d_+|^2+|d_-|^2)\\&\quad=|a|^2(|a_+|^2+|a_-|^2)=1.\end{aligned} \tag{90}$$

The factor $\frac{1}{2}$ in this expression is due to the fact that the dielectric plate divides all the space into two parts. The wave function of a photon in the plate gives a negligible contribution to the normalization condition when $L_z\to\infty$ and $R\to\infty$. Substituting the explicit expressions (83), we deduce

$$\begin{aligned}|a|^{-2}&=|a_+|^2+|a_-|^2\\&=\left|1+\frac{1}{8}\left[(\varepsilon^2+1)\left(\frac{k_3^2}{k_3'^2}+\frac{k_3'^2}{\varepsilon^2k_3^2}\right)-4\right]\sin^2(k'_3L)\right|.\end{aligned} \tag{91}$$

As a result, in accordance with the general formulas obtained in the previous sections, the probability to record a twisted photon produced by the particles with charges e_l looks like

$$\begin{aligned}dP(s,m,k_\perp,k_3)=|a|^2\bigg|\sum_l e_l\int_{-\infty}^{\infty}d\tau\, e^{-ik_0x_l^0(\tau_l)}\Big\{&\dot{x}_{3l}(\tau_l)\Phi_3(s,m,k_\perp,k_3;\mathbf{x}_l(\tau_l))\\&+\frac{1}{2}[\dot{x}_{+l}(\tau_l)\Phi_-(s,m,k_\perp,k_3;\mathbf{x}_l(\tau_l))+\dot{x}_{-l}(\tau_l)\Phi_+(s,m,k_\perp,k_3;\mathbf{x}_l(\tau_l))]\Big\}\bigg|^2\left(\frac{k_\perp}{2k_0}\right)^3\frac{dk_3dk_\perp}{2\pi^2},\end{aligned} \tag{92}$$

where $x_l^\mu(\tau)$ are the particle world lines.

Considering the particles with the trajectories (68), the expression under the modulus sign in (92) at $m=0$ is given by

$$\begin{aligned}A:=\frac{i|\beta_3|}{k_0}\Bigg[&\frac{\left[\cos(k'_3L)-\frac{i}{2}\left(\frac{\varepsilon k_3}{k'_3}+\frac{k'_3}{\varepsilon k_3}\right)\sin(k'_3L)\right]e^{ik_0L/\beta_3}-1}{1-n_3\beta_3}+\frac{-\frac{i}{2}\left(\frac{\varepsilon k_3}{k'_3}-\frac{k'_3}{\varepsilon k_3}\right)\sin(k'_3L)e^{ik_0L/\beta_3}}{1+n_3\beta_3}\\&+\frac{1+\frac{\varepsilon k_3}{k'_3}}{2\varepsilon(1-n'_3\beta_3)}(1-e^{ik_0L(1-n'_3\beta_3)/\beta_3})+\frac{1-\frac{\varepsilon k_3}{k'_3}}{2\varepsilon(1+n'_3\beta_3)}(1-e^{ik_0L(1+n'_3\beta_3)/\beta_3})\Bigg],\end{aligned} \tag{93}$$

where $n'_3:=k'_3/k_0$. The probability to record a twisted photon produced by the particle with charge e takes the form

$$dP(s,m,k_\perp,k_3)=e^2|a|^2|A|^2\delta_{0,m}\left(\frac{k_\perp}{2k_0}\right)^3\frac{dk_3dk_\perp}{2\pi^2}. \tag{94}$$

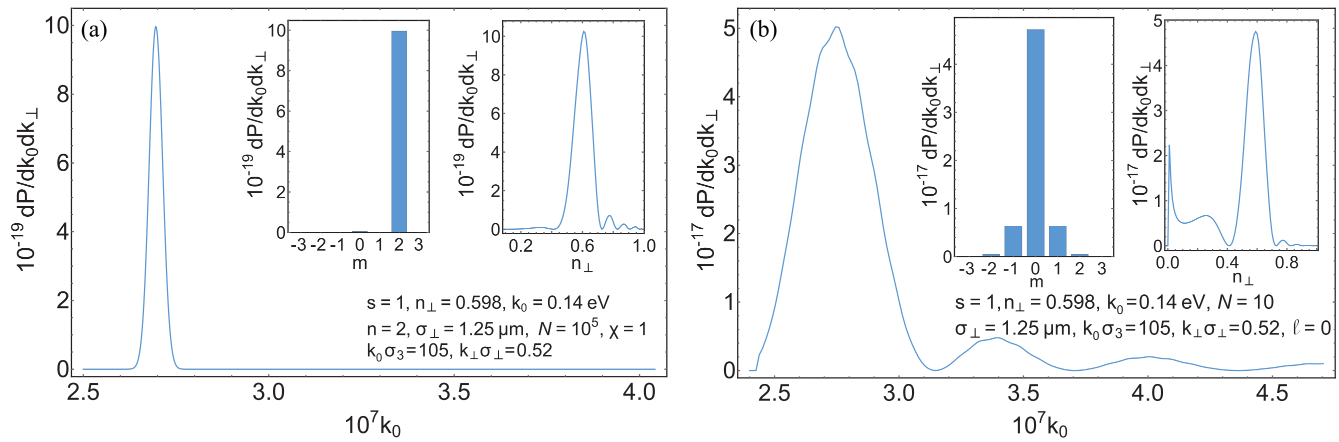

FIG. 1. The VC radiation of twisted photons produced by beams of charged particles in the LiF plate of thickness $L = 100$ μm. The number of particles in the beam is $N = 10^5$ and the Lorentz factor of particles is $\gamma = 235$. The beam is supposed to have a Gaussian profile (see the Appendix) with longitudinal dimension $\sigma_3 = 150$ μm (duration 0.5 ps) and the transverse size $\sigma_\perp = 1.25$ μm. The other parameters of the beam are the same as in Fig. 1 in [54]. The observation photon energy $k_0 = 0.14$ eV, which corresponds to a wavelength of 9 μm. The photon energy and momentum are measured at electron rest energies of 0.511 MeV. It is assumed that the particles move along the detector axis towards the detector. The peak at $n_\perp := k_\perp/k_0 = 0.598$ corresponds to the VC radiation and is found from the standard expression $n_\perp \equiv \sin\theta_{\text{VC}} = \sqrt{\varepsilon'(k_0) - \beta^{-2}}$, where $\varepsilon'(k_0)$ is the real part of the permittivity. The experimental data for the permittivity $\varepsilon(k_0)$ of LiF are taken from [106]. The projection of the total angular momentum per photon is denoted by ℓ. (a) The VC radiation produced by the helically microbunched beam of particles at the second coherent harmonic $k_0 = 2\pi\chi n\beta_3/\delta$, where $n = 2$ and δ is the helix pitch [54]. The fulfillment of the strong addition rule is evident since the one-particle radiation is concentrated at $m = 0$. (b) The VC radiation produced by the uniform Gaussian beam of particles. The coherent contribution to radiation is negligible. As long as $k_\perp\sigma_\perp < 1$, the radiation is concentrated near $m = 0$ as in the one-particle case [53,54]. The peak at small $n_\perp \sim 1/\gamma$ is the transition radiation.

This probability is concentrated at $m = 0$ and does not depend on the photon helicity. The first two terms in (93) describe transition radiation, and the terms on the second line in (93) correspond to VC radiation [45,51]. For

$$k_0 L|1 \mp n_3'\beta_3|/|\beta_3| \lesssim \pi/5, \tag{95}$$

the latter contributions possess a sharp maximum of the order

$$e^2L^2|a|^2 \frac{\left|1 \pm \frac{\varepsilon k_3}{k_3'}\right|^2}{4|\varepsilon|^2} \delta_{0,m} \left(\frac{k_\perp}{2k_0}\right)^3 \frac{dk_3 dk_\perp}{2\pi^2}, \tag{96}$$

where the sign $\pm$ agrees with the choice of the sign in (95). The contribution of transition radiation to (94) reaches a maximum at

$$n_\perp\gamma \approx \sqrt{3}, \tag{97}$$

provided $k_0 L/\gamma^2 \ll 1$. Hereinafter $n_\perp = k_\perp/k_0$. Notice that this notation does not agree with formula (63). For $\beta_3 > 0$, the formula (94) describes the radiation of a charged particle moving towards the detector, while for $\beta_3 < 0$ the particle moves from the detector.

The fact that probability (94) is concentrated at $m = 0$ is a consequence of the general rule [82]. This property holds for any current density symmetric under the rotations around the detector axis. If instead of one charged particle the radiation of N identical charged particles moving along parallel trajectories is considered, then, as was shown in [53,54], the probability to record a twisted photon becomes

$$dP_\rho(s, m, k_\perp, k_3) = [N f_m + N(N-1)|\varphi_m|^2] dP_1(s, 0, k_\perp, k_3), \tag{98}$$

where $dP_1(s, 0, k_\perp, k_3)$ is the probability of radiation of a twisted photon by one charged particle, i.e., in our case (94). Of course, it is assumed in (98) that the particle beam falls normally onto the dielectric plate. The functions f_m and φ_m are the incoherent and coherent interference factors, respectively. The explicit expressions for f_m and φ_m are given in [53,54]. In particular, the probability distribution over m for the coherent radiation of twisted photons by a helically microbunched beam of charged particles moving along parallel trajectories is shifted with respect to the one-particle probability distribution by the signed number n of coherent harmonic (the strong addition rule). This allows one to generate the twisted photons with large m by means of coherent transition and VC radiations (see Figs. 1–4). In particular, the projection of the total angular momentum per photon ℓ_ρ of the radiation at the nth coherent harmonic is given by [54]

$$\ell_\rho = \ell_1 + n\frac{(N-1)|\varphi_n|^2}{1 + (N-1)|\varphi_n|^2}, \tag{99}$$

where ℓ_1 is the projection of the total angular momentum per photon for the radiation produced by one charged particle. In our case, $\ell_1 = 0$. Notice that, in the paraxial regime $n_\perp \ll 1$, even the twisted photons with $m = 0$ and definite helicity s possess a nontrivial phase front corresponding to the orbital angular momentum $l = m - s = -s$ [16]. The probability to record a twisted photon radiated by one particle moving along a straight line parallel to the detector axis at a distance $|x_+|$ is obtained from (98) by the substitution

$$f_m = J_m^2(k_\perp|x_+|), \quad N = 1. \tag{100}$$

The radiation of twisted photons by the helically microbunched particle beam traversing the dielectric plate at a large angle to its normal is described in Fig. 5.

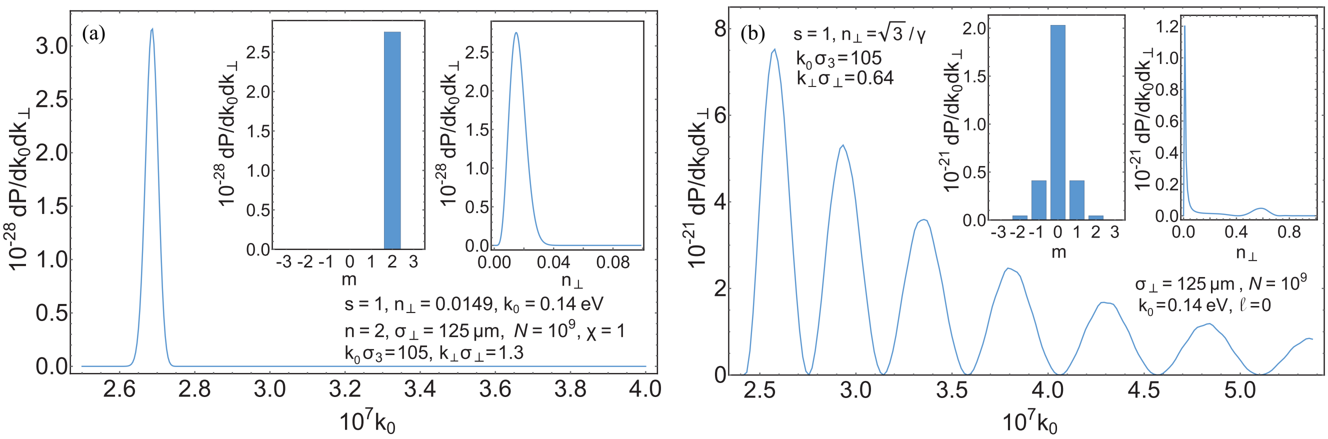

FIG. 2. Transition radiation from charged particles traversing the LiF dielectric plate. The particles move along the detector axis towards the detector. The parameters of the plate, the profile of the beam, and the Lorentz factor of particles are the same as in Fig. 1. (a) Transition radiation of the helically microbunched beam of particles. The fulfillment of the strong addition rule on the second coherent harmonic of radiation is clearly seen. (b) Transition radiation of the uniform Gaussian beam of particles. The contribution of coherent radiation is strongly suppressed. The small hump at $n_\perp \approx 0.6$ is the VC radiation.

B. Thick dielectric plate

Now we consider the case when the thickness of the dielectric plate is so large that one may suppose that the half space $z < 0$ is filled by the dielectric, i.e., $L \to \infty$. The mode functions $\Phi(s, m, k_\perp, k_3)$ are given by formulas (79) and (80) without the common factor a. The states exponentially damped in the region $z > 0$ correspond to $\Phi(s, m, k_\perp, k_3)$ with the quantum numbers k_0 and $k_\perp$ satisfying (85). These states are not recorded by the detector located sufficiently far from the medium in the domain $z > 0$.

We will be interested in the states with $k_3 > 0$ undamped in the region $z > 0$. It is convenient to number them by $\alpha := (s, m, k_3, k_\perp)$. Then the normalization coefficient takes the form (89). For $R \to \infty$ and $L_z \to \infty$, the normalization of the wave functions leads to the relation

$$\frac{|a|^2}{2}[1 + \varepsilon(|b_+|^2 + |b_-|^2 + |c_+|^2 + |c_-|^2)] = 1, \qquad (101)$$

where the factor $\frac{1}{2}$ comes from the fact that the whole space is split into two parts by the interface Σ and ε in front of the parenthesis results from the integration measure in the scalar product (8). Substituting the explicit expressions (81), we find the normalization factor

$$|a|^{-2} = \frac{1}{2}\left|1 + (\varepsilon + 1)\frac{\varepsilon k_3^2 + k_3'^2}{4\varepsilon k_3'^2}\right|. \qquad (102)$$

Inasmuch as the mode functions $\Phi(s, m, k_\perp, k_3)$ coincide with the vacuum twisted photons for $z > 0$, we will call them twisted photons. The probability of radiation of twisted photons by charged particles is described by the formula (92).

As for one charged particle moving along the trajectory (68), the probability to record a twisted photon produced by it

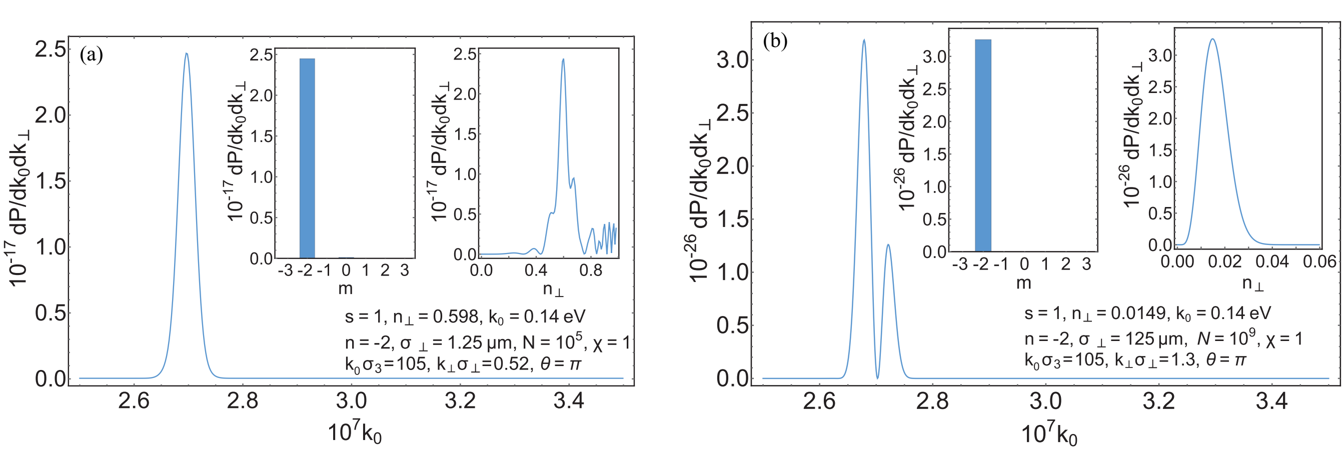

FIG. 3. Same as in Figs. 1 and 2, but the charged particles move from the detector, i.e., $\theta = \pi$. (a) The VC radiation of a helically microbunched beam of particles. In accordance with the strong addition rule, since the beam of particles moves from the detector, the distribution over m is shifted to the opposite direction in comparison with Fig. 1. (b) Transition radiation of a helically microbunched beam of particles.

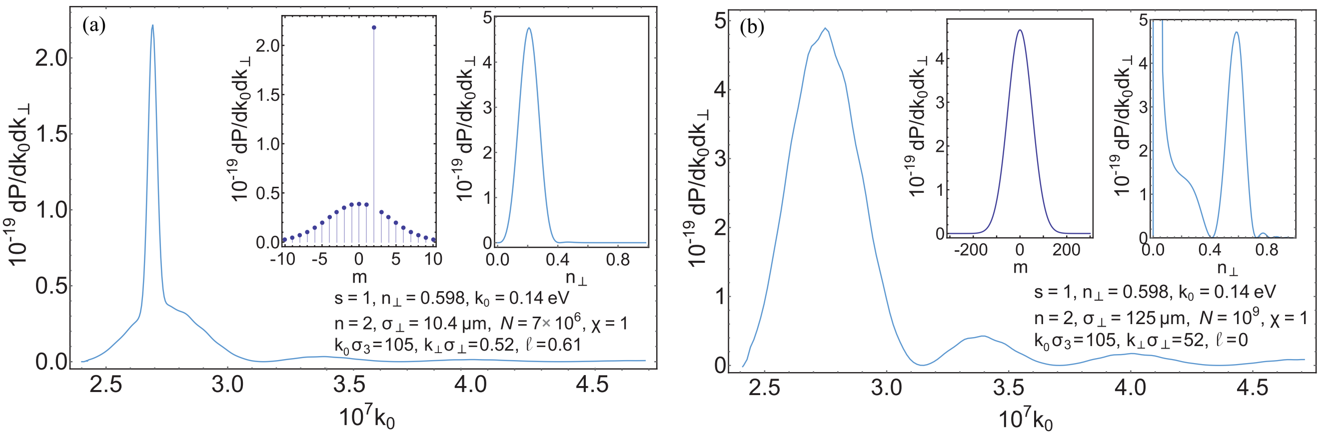

FIG. 4. Suppression of the coherent contribution to VC radiation in increasing the transverse size of the helically microbunched particle beam (cf. Fig. 1). The numbers of particles in the beams are such that the densities of particles are equal. The other parameters are the same as in Fig. 1. The large peaks at small $n_\perp$ correspond to transition radiation. Plots against k_0 and $n_\perp$ are given for (a) $m = 2$ and (b) $m = 0$.

becomes

$$dP(s,m,k_\perp,k_3) = e^2|a|^2\frac{\beta_3^2}{k_0^2}\left|\frac{\frac{1}{2\varepsilon}\left(1+\frac{\varepsilon k_3}{k_3'}\right)}{1-n_3'\beta_3}+\frac{\frac{1}{2\varepsilon}\left(1-\frac{\varepsilon k_3}{k_3'}\right)}{1+n_3'\beta_3} - \frac{1}{1-n_3\beta_3}\right|^2\delta_{0,m}\left(\frac{k_\perp}{2k_0}\right)^3\frac{dk_3dk_\perp}{2\pi^2}, \tag{103}$$

where e is the particle charge and $n_3 := k_3/k_0$. The last notation disagrees with formula (63). The singularity of this expression when the Cherenkov condition is fulfilled is resolved by taking into account the medium absorption, i.e., one has to suppose that $\varepsilon(k_0) \in \mathbb{C}$ and k_3' is found from (88). The last term under the modulus sign describes the contribution of transition radiation. It reaches a maximum when the condition (97) holds. Due to the symmetry of the problem, the radiation probability is concentrated at $m = 0$. Moreover, it does not depend on the photon helicity.

The probability to radiate a twisted photon by a bunch of particles moving along parallel trajectories is given by (98). The radiation of twisted photons produced by helically microbunched beams of particles obeys the strong addition rule and the sum rule (99) is satisfied. As for radiation created by one particle moving along a straight line parallel to the detector axis, one just needs to substitute (100) in (98).

It is not difficult to find the probability to record a twisted photon produced by a charged particle moving along the trajectory

$$x_\pm = \beta_\pm x^0, \quad x_3 = \beta_3 x^0, \tag{104}$$

where $\beta_\pm$ and β_3 are the constant projections of the particle velocity in the laboratory frame. Let us introduce the notation

$$\kappa(n_3,\beta_3) := \sqrt{(1-n_3\beta_3)^2 - n_\perp^2\beta_\perp^2},$$
$$q(n_3,\beta_3) := \frac{n_\perp\beta_\perp}{1-n_3\beta_3+\kappa(n_3,\beta_3)} = \frac{1-n_3\beta_3-\kappa(n_3,\beta_3)}{n_\perp\beta_\perp}, \tag{105}$$

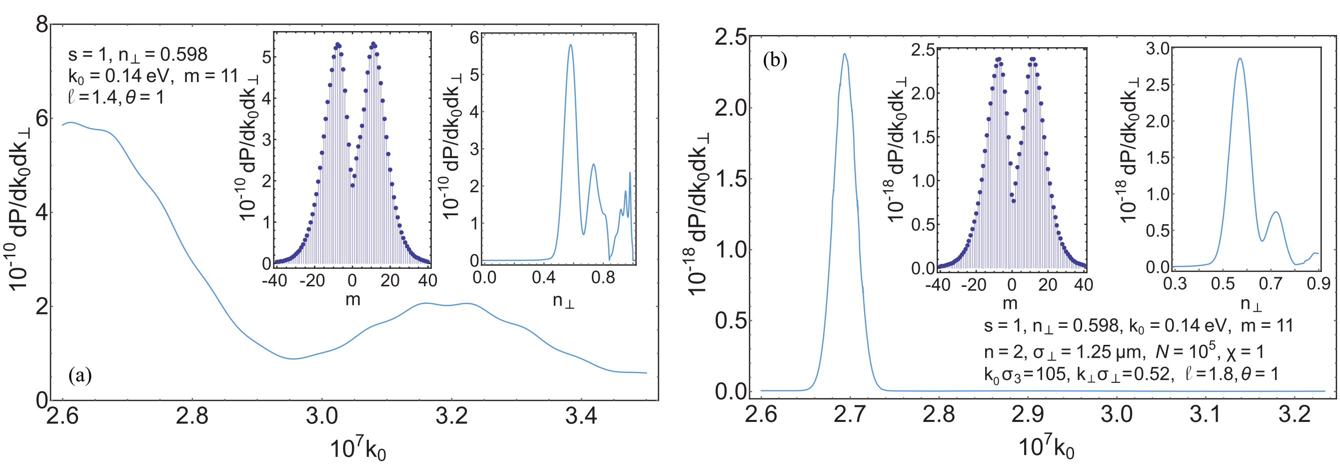

FIG. 5. Radiation of twisted photons by charged particles traversing the LiF dielectric plate at the angle $\theta = 1$. The angle is counted from the normal to the plate. The other parameters are the same as in Fig. 1. (a) Radiation of twisted photons produced by one charged particle. The plots against k_0 and $n_\perp$ are given at $m = 11$. (b) Radiation of twisted photons produced by a helically microbunched particle beam at the second coherent harmonic (A23). The plots against k_0 and $n_\perp$ are given at $m = 11$. Of course, the strong addition rule does not hold, but the harmonic (A23) is clearly seen.

and $\delta := \arg\beta_+$. Then the expression under the modulus sign in (92) reads

$$A = I(\beta_3) + I'(n'_3) + I'(-n'_3), \tag{106}$$

where $I(\beta_3)$ is the amplitude of the edge radiation [82]

$$\begin{aligned} I(\beta_3) &:= \operatorname{sgn}(\beta_3)\frac{i^{-1-m}}{k_\perp n_\perp}\left(\frac{\beta_3 - n_3}{\kappa(n_3,\beta_3)} - s\operatorname{sgn}(m)\right)\\ &\quad\times e^{im\delta}q^{|m|}(n_3,\beta_3) \quad \text{for } |m|>0,\\ I(\beta_3) &:= \operatorname{sgn}(\beta_3)\frac{i^{-1}}{k_\perp n_\perp}\left(\frac{\beta_3 - n_3}{\kappa(n_3,\beta_3)} + n_3\right) \quad \text{for } |m| = 0, \end{aligned} \tag{107}$$

while the other two terms give the amplitude of the transition radiation

$$\begin{aligned} I'(n'_3) &:= -\operatorname{sgn}(\beta_3)\frac{i^{-1-m}}{2k_\perp n_\perp}\left[\frac{\varepsilon\beta_3 - n'_3}{\kappa(n'_3,\beta_3)}\left(\frac{n_3}{n'_3} + \frac{1}{\varepsilon}\right)\right.\\ &\quad\left. - s\operatorname{sgn}(m)\left(\frac{n_3}{n'_3} + 1\right)\right]e^{im\delta}q^{|m|}(n'_3,\beta_3) \quad \text{for } |m|>0,\\ I'(n'_3) &:= -\operatorname{sgn}(\beta_3)\frac{i^{-1}}{2k_\perp n_\perp}\left(\frac{n_3}{n'_3} + \frac{1}{\varepsilon}\right)\left[\frac{\varepsilon\beta_3 - n'_3}{\kappa(n'_3,\beta_3)} + n'_3\right]\\ &\quad \text{for } |m| = 0. \end{aligned} \tag{108}$$

The probability of twisted photon radiation is written as

$$dP(s,m,k_\perp,k_3) = e^2|a|^2|A|^2\left(\frac{k_\perp}{2k_0}\right)^3\frac{dk_3dk_\perp}{2\pi^2}, \tag{109}$$

with $|a|^2$ presented in (102).

It was shown in [82] that the probability of radiation of twisted photons for an arbitrary QED process in a vacuum possesses the reflection symmetry,

$$dP(s,m,k_\perp,k_3) = dP(-s,-m,k_\perp,k_3), \tag{110}$$

in the infrared regime. It is clear from (106)–(109) that, in the case we consider, this symmetry also holds. Moreover, this symmetry takes place for the processes involving an arbitrary number of charged particles and evolving near the boundary of the thick dielectric plate. Namely, the probability to record a twisted photon produced by an arbitrary number of charged particles moving along the trajectories (104) with different velocities $\boldsymbol{\beta}$ possesses the reflection symmetry (110) provided the medium is transparent, i.e., $\varepsilon > 1$, and

$$(1 - n'_3|\beta_3|)^2 \geqslant n_\perp^2\beta_\perp^2, \quad n'_3 = \sqrt{\varepsilon - n_\perp^2}, \tag{111}$$

for all the particles participating in the process. This property also holds when the particles transmute one into another, are created, or cease to exist at the origin. The proof of the symmetry property (110) is completely analogous to that given in [82].

C. Conducting plate

In order to describe the radiation of twisted photons by charged particles moving near an ideally conducting plane $z = 0$, it is necessary to construct the corresponding mode functions in the domain $z > 0$ satisfying the boundary condition (13). The linear combination

$$\begin{aligned} \Phi(s,m,k_\perp,k_3) &= \psi(s,m,k_\perp,k_3) + b_+\psi(1,m,k_\perp,-k_3)\\ &\quad + b_-\psi(-1,m,k_\perp,-k_3), \quad k_3>0, \end{aligned} \tag{112}$$

satisfies (13) when

$$b_\pm = \frac{1\mp s}{2}. \tag{113}$$

The normalization of the wave function $a\Phi(s,m,k_\perp,k_3)$ results in

$$\frac{|a|^2}{2}(1 + |b_+|^2 + |b_-|^2) = |a|^2 = 1. \tag{114}$$

We will call these mode functions twisted photons since the wave traveling to the detector coincides with the vacuum twisted photon.

The probability to record a twisted photon produced by a charged particle moving along the trajectory (68) is given by (92). The expression under the modulus sign in (92) becomes

$$\begin{aligned} A :&= -\frac{i|\beta_3|}{k_0}\left[\frac{1}{1-n_3\beta_3} + \frac{1}{1+n_3\beta_3}\right]\delta_{0,m}\\ &= -\frac{i|\beta_3|}{k_0}\frac{2}{1-(n_3\beta_3)^2}\delta_{0,m}, \end{aligned} \tag{115}$$

where $n_3 := k_3/k_0$. This expression has a clear physical interpretation (see, e.g., [40,45]): The two terms in (115) correspond to the contributions of the edge radiation produced by the charge and its image with the opposite charge in the ideal conductor (the mirror). The probability to record the twisted photon is

$$dP(s,m,k_\perp,k_3) = \frac{4e^2\beta_3^2\delta_{0,m}}{k_0^2[1-(n_3\beta_3)^2]^2}\left(\frac{k_\perp}{2k_0}\right)^3\frac{dk_3dk_\perp}{2\pi^2}. \tag{116}$$

It is concentrated at $m = 0$ and does not depend on the photon helicity. Just as in the case of a dielectric, the probability of radiation of twisted photons by a bunch of charged particles moving along parallel trajectories normal to the surface of the ideal conductor has the form (98). The use of helically microbunched beams of particles allows one to shift the coherent radiation probability distribution over m by the signed number of coherent harmonics [54,55] (see Fig. 6). The sum rule (99) with $\ell_1 = 0$ is fulfilled.

If the charged particle moves at an angle to the detector axis along the trajectory (104), then the expression under the modulus sign in (92) is reduced to

$$\begin{aligned} A &= I(\beta_3) - I(-\beta_3)\\ &= \operatorname{sgn}(\beta_3)\frac{i^{-1-m}e^{im\delta}}{k_\perp n_\perp}\left[\left(\frac{\beta_3-n_3}{\kappa(n_3,\beta_3)} - s\operatorname{sgn}(m)\right)\right.\\ &\quad\left.\times q^{|m|}(n_3,\beta_3) - (\beta_3\leftrightarrow-\beta_3)\right]. \end{aligned} \tag{117}$$

The first term comes from the edge radiation of the charge and the second one is from its image in the mirror. Then the probability to record a twisted photon is given by (109) with the normalization factor $|a|^2 = 1$. The reflection symmetry

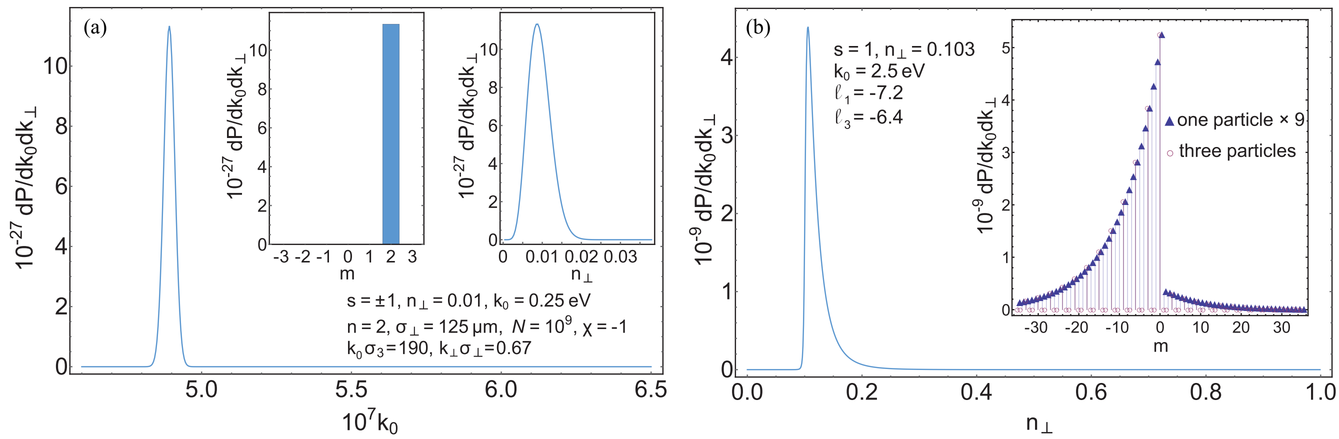

FIG. 6. Transition radiation produced by charged particles striking a conducting plate. (a) Transition radiation of a helically microbunched beam of particles falling normally onto the conducting plate. The parameters of the particle beam are the same as in Fig. 2. The strong addition rule is evidently satisfied. The radiation probability is independent of the photon helicity s. In the paraxial regime $n_\perp \ll 1$, which obviously holds in our case, the orbital angular momentum can be introduced as $l = m - s$. Thus the radiation summed over the photon helicities is an equiprobable mixture of twisted photons with $l = \{1, 3\}$. The twisted photons with a definite value of the orbital angular momentum l can be extracted from this radiation with the help of a circular polarizer selecting $s = 1$ or $s = -1$. (b) Transition radiation of three charged particles falling onto the conducting plate at the angle $\theta = 1/10$ with the Lorentz factor $\gamma = 235$. The trajectories of these particles are obtained from one another by rotation around the detector axis by an angle of $2\pi/3$. The fulfillment of the selection rules discussed in [82] is shown. The maximum of the projection of the total angular momentum per photon is reached at $n_\perp \approx K(1 + 1/\sqrt{2}K)/\gamma \approx 0.103$, where $K := \gamma\beta \sin\theta$.

(110) holds for the radiation created by an arbitrary number of charged particles moving along the trajectories (104) with different velocities $\boldsymbol{\beta}$.

It was pointed out in [82] that the edge radiation can be used as a superradiant source of soft twisted photons. As we see, the use of a conducting mirror allows one to simplify the configuration of particle beams generating twisted photons in the spectral range where the conductor can be regarded as ideal. For example, if one focuses the three beams of charged particles at one point on the mirror such that the angles of incidence are equal and the angles between the beams are the same, then the probability distribution over m of recorded twisted photons is asymmetric for a given s (see Fig. 6), obeys the selection rule $m = 3k$, $k \in \mathbb{Z}$, and the reflection symmetry (110). The absolute value of projection of the total angular momentum per photon is of order $K/2\sqrt{2}$ [82], where $K := \gamma\beta_\perp$ and $\beta_\perp = |\beta_+|$.

As for an ideal conductor with the surface of a general form, the solution of the problem (1) and (13) can be reduced to the solution of the Maxwell equations (1) with a singular source. The nontrivial boundary conditions on the closed hypersurface Σ are standardly replaced by the singular source

$$j^i[\psi;\mathbf{x}) = \int_\Sigma d\sigma\sqrt{h}\{n^{[i}\,\mathrm{pr}^{j]k}\,\partial_j\delta(\mathbf{x}-\mathbf{x}(\sigma))\psi_k + \delta(\mathbf{x}-\mathbf{x}(\sigma))\delta^{ik}\partial_{[k}\psi_{j]}n^j\}, \tag{118}$$

where $x^i(\sigma)$, $\sigma = \{\sigma^a\}$, $a = \overline{1,2}$, is the embedding map of the closed hypersurface Σ into $\mathbb{R}^3$, n^i is the unit normal to Σ directed into the conductor,

$$\mathrm{pr}^{ij} := h^{ab}\partial_a x^i\partial_b x^j, \tag{119}$$

with h^{ab} the inverse to the induced metric $h_{ab} = \partial_a x^i\partial_b x^j\delta_{ij}$, and $h = \det h_{ab}$. Taking into account the boundary conditions (13), we obtain

$$j_i[\psi;\mathbf{x}) = \int_\Sigma d\sigma\sqrt{h}n^j\delta(\mathbf{x}-\mathbf{x}(\sigma))\partial_{[i}\psi_{j]}. \tag{120}$$

From the physical point of view, this current describes the currents induced by the external electromagnetic field on the ideal conductor. The corresponding vacuum Maxwell equations become

$$\big(k_0^2 - \hat{h}^2\big)\psi_i(k_0,\mathbf{x}) - j_i[\psi;\mathbf{x}) = 0. \tag{121}$$

The resolvent (the Green's function) for this equation can be found perturbatively regarding the contribution $j_i[\psi]$ as a perturbation.

D. Helical medium

A promising pure source of twisted photons with large projection of the total angular momentum m is the radiation produced by charged particles in helical media. Let the permittivity $\varepsilon(k_0,\mathbf{x})$ be invariant under the transformations

$$z \to z + \varphi_r/q_0 = z + \frac{\lambda_0}{r}, \quad \varphi \to \varphi + \varphi_r, \tag{122}$$

where $q_0 = 2\pi/\lambda_0$, φ is the azimuth angle of the cylindrical system of coordinates with the reference axis z, and $\varphi_r = 2\pi/r$, $r = \overline{1,\infty}$, is a fixed rotation angle. We will call the medium possessing such a symmetry a helical medium. We denote by V_r the unitary operator acting in the space of solutions to the Maxwell equations and realizing the symmetry transformation (122). It follows from (122) that $\varepsilon(k_0,\mathbf{x})$ is a periodic function of z with the period $|\lambda_0|$. Of course, such a situation is never realized. However, if the number of periods $N \gtrsim 10$, then, in describing the radiation generated by charged particles, the edge effects can be neglected (discussed below).

In accordance with the standard theorems (see, e.g., [107]), the complete set of solutions of the Maxwell equations (1) or (5) can be found in the form of the eigenfunctions of the symmetry operator

$$\hat{V}_r\psi = e^{i\xi}\psi. \tag{123}$$

We will assume that the complex vector field ψ is periodic with respect to the variable z with the period $N|\lambda_0|$ for sufficiently large N. Developing the mode functions as a Fourier series

$$\psi_3(\rho,\varphi,z) = \sum_{k,n} a_{kn}(\rho)e^{ik\varphi}e^{in(q_0/N)z}, \tag{124}$$

where ρ is the distance from the z axis, and substituting into (123), we obtain

$$\xi = \frac{2\pi}{Nr}\varkappa, \quad \varkappa\in\mathbb{Z}, \tag{125}$$

and

$$\psi_3(\rho,\varphi,z) = \sum_{k,n}\tilde{a}_{k,N(rn-k)+\varkappa}(\rho)e^{ik(\varphi-q_0z)}e^{i(\varkappa/N+rn)q_0z}. \tag{126}$$

Introducing

$$k_3 = \frac{q_0}{N}\varkappa \tag{127}$$

and shifting $k\to k+m$ and $\varkappa\to\varkappa+mN$, where $m\in\mathbb{Z}$, we have

$$\psi_3(m,k_3;\rho,\varphi,z) = \sum_{k,n}\bar{a}_{k+m}(k_3+q_0(rn-k);\rho) \times e^{ik(\varphi-q_0z)}e^{im\varphi}e^{i(k_3+q_0rn)z}. \tag{128}$$

As a result,

$$\psi_3(m,k_3;\rho,\varphi,z) = f_3(m,k_3;\rho,z,\varphi-q_0z)e^{ik_3z}e^{im\varphi}, \tag{129}$$

where

$$f_3(m,k_3;\rho,z,\varphi) = f_3\left(m,k_3;\rho,z+\frac{\lambda_0}{r},\varphi\right) = f_3(m,k_3;\rho,z,\varphi+2\pi). \tag{130}$$

The same procedure applied to the transverse components of the mode functions leads to

$$\psi_\pm(m,k_3;\rho,\varphi,z) = f_\pm(m,k_3;\rho,z,\varphi-q_0z)e^{ik_3z}e^{i(m\pm1)\varphi}, \tag{131}$$

where the functions $f_\pm$ obey the periodicity conditions (130).

Further, we suppose that the permittivity $\varepsilon(k_0,\mathbf{x})$ tends sufficiently fast to unity as $\rho\to+\infty$. The probability to record a twisted photon by the detector is determined only by the scattering states, which tend to the solutions of the free Maxwell equations as $\rho\to+\infty$. For these states, we have

$$f_3(m,k_3;\rho,z,\varphi)\to f_3(m,k_3;\rho), \quad f_\pm(m,k_3;\rho,z,\varphi)\to f_\pm(m,k_3;\rho), \tag{132}$$

as $\rho\to+\infty$, i.e., the dependence of functions $f_{3,\pm}$ on the last two arguments disappears in this limit. It is this asymptotics that conditions the presence of factors

$$e^{ik_3z}e^{im\varphi}, \quad e^{ik_3z}e^{i(m\pm1)\varphi} \tag{133}$$

in ψ_3 and $\psi_\pm$, respectively.

If, additionally, the permittivity $\varepsilon(k_0,\mathbf{x})$ is symmetric under the rotations

$$\varphi\to\varphi+\varphi_q, \tag{134}$$

where q is some natural number, then the expansion (128) turns into

$$\psi_3(m,k_3;\rho,\varphi,z) = \sum_{k,n} b_{qk+m}(k_3+q_0(rn-qk);\rho) \times e^{iqk(\varphi-q_0z)}e^{im\varphi}e^{i(k_3+q_0rn)z}, \tag{135}$$

the formula (129) is left intact, and the periodicity property (130) becomes

$$f_3(m,k_3;\rho,z,\varphi) = f_3\left(m,k_3;\rho,z+\frac{\lambda_0}{r},\varphi\right) = f_3(m,k_3;\rho,z,\varphi+2\pi/q). \tag{136}$$

The analogous formulas are valid for the transverse components $\psi_\pm$.

Now we can use the general formula (53). As long as the integral entering this formula is saturated in the region Ω where the asymptotics (132) holds, the amplitude of radiation of a twisted photon with the momentum k_3 and the projection of the total angular momentum m by a charged particle moving along the trajectory (68) is proportional to

$$\int_\infty^\infty dt\, e^{-ik_0t}\psi_3(m,k_3;0,\varphi,\beta_3t). \tag{137}$$

Substituting the expansion (135) into this integral and bearing in mind that the dependence of the mode functions on the azimuth angle φ must disappear at $\rho=0$, we conclude that the integral (137) is proportional to

$$\delta(k_0-[k_3+q_0(m+rn)]\beta_3), \quad m=-qk. \tag{138}$$

To put it differently, the radiation of twisted photons is concentrated at the harmonics

$$k_0 = |q_0|\frac{|\beta_3|\bar{n}}{1-\beta_3n_3} = \frac{2\pi}{|\lambda_0|}\frac{|\beta_3|\bar{n}}{1-\beta_3n_3}, \tag{139}$$

where $\bar{n}=\overline{1,\infty}$ is the harmonic number, and the selection rule

$$m = \mathrm{sgn}(\lambda_0\beta_3)\bar{n}+rn = qk \tag{140}$$

holds, where n and k are some integer numbers. The intensity of radiation at these harmonics is determined by the form of the permittivity $\varepsilon(k_0,\mathbf{x})$.

The case of the permittivity invariant with respect to the continuous transformations

$$z\to z+\psi/q_0, \quad \varphi\to\varphi+\psi, \tag{141}$$

where $\psi\in\mathbb{R}$, is formally obtained from (122) in the limit $r\to\infty$. Then the selection rule (139) is left unchanged and (140) is replaced by

$$m = \mathrm{sgn}(\lambda_0\beta_3)\bar{n} = qk. \tag{142}$$

As we see, the radiation at the $\bar{n}$th harmonic is a pure source of twisted photons with a definite value of the projection of the total angular momentum m.

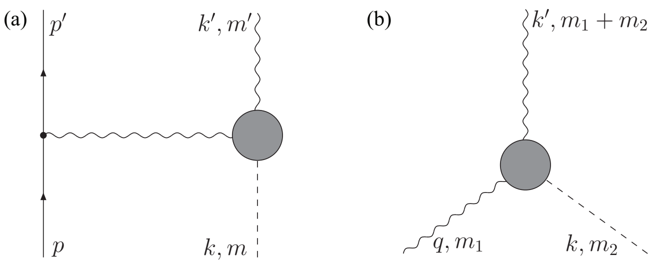

FIG. 7. (a) Feynman diagram for the leading contribution to transition scattering of the permittivity wave on the charged particle. The permittivity wave is represented by the dashed line and the large circle denotes the interaction vertex of the permittivity with the electromagnetic field. This vertex is the first nontrivial term of the expansion of the photon propagator in a medium in powers of susceptibility. (b) First-order diagram describing the conversion of a plane-wave photon to a twisted one. For a normal incidence of the plane-wave photon onto the medium interface, the projection of the total angular momentum $m_1 = s_1$, where s_1 is the photon helicity. This process was employed in [9,13,72] for generation of twisted photons.

The above derivation is completely applicable to the case when the permittivity is not isotropic and the permittivity tensor has the form

$$\varepsilon_{ij}(k_0, \mathbf{x}) = \varepsilon_\perp(k_0, \mathbf{x})\delta_{ij} + [\varepsilon_\parallel(k_0, \mathbf{x}) - \varepsilon_\perp(k_0, \mathbf{x})]n_i n_j, \tag{143}$$

where the functions $\varepsilon_\perp(k_0, \mathbf{x})$ and $\varepsilon_\parallel(k_0, \mathbf{x})$ are invariant under (141) and

$$n_i = (\cos(q_0 z), \sin(q_0 z), 0). \tag{144}$$

The permittivity tensor (143) is inherent to cholesteric liquid crystals (see, e.g., [108]). Therefore, the radiation produced by a charged particle moving along a helical axis of the cholesteric is a pure source of twisted photons obeying the selection rules (139) and (142) with $q = 1$ and $|\lambda_0|$ being equal to a half of the chiral pitch. Note that the cholesteric liquid crystals were used in [7,9,72] to convert an ordinary laser wave to a twisted one. In fact, we consider the process depicted by the diagram in Fig. 7(a), while the conversion of plane-wave photons is described by the diagram in Fig. 7(b). The permittivity of the form (143) is the simplest model for permittivity of cholesteric liquid crystals which describes their electromagnetic properties rather well. The selection rules (139) and (142) are valid in the photon energy domain where (143) holds, i.e., up to the extreme ultraviolet spectral range.

Another realization of a medium possessing symmetry (141) is represented by the conductor in the form of a helix with q branches. The bunch of charged particles moves along the axis of this helix. The conductors of such a form were used in [13] to transform ordinary plane-wave photons to twisted ones. The medium with permittivity symmetric with respect to (122) can also be made by arranging properly the dielectrics with different ε. One more means to construct a helical medium with symmetry (141) is to deform the medium in a twisted manner by using, for example, the helical sound waves. The wave packet of twisted phonons, which is sufficiently wide in m and obeys (123) with good accuracy, represents a helical sound wave and leads to an appropriate variation of permittivity. As a rule, the phonon velocity can be ignored in comparison with the velocity of a relativistic particle and so the formulas derived above are applicable to this case without any modifications. Thereby the twisted photons are generated by transition scattering [40,45] of helical waves of permittivity on charged particles (see Fig. 7). The same situation happens in a plasma perturbed by the helical wave of phonons or photons [62–64,109]. However, in order to describe the radiation of twisted photons quantitatively, one needs to take a spatial dispersion into account in this case.

The selection rules (139) and (140) coincide exactly with the selection rules found in [82] for the scattering of particles on helical targets. It is not surprising as (139) and (140) can be deduced from the following considerations (a similar derivation of selection rules for the radiation produced by charged particles in a layered medium can be found, e.g., in [38,44]). Let the charged particle move in a medium along the z axis with constant velocity, the z axis coinciding with the detector axis. We assume that the medium permittivity is invariant under the transformations (122) and (134). The Coulomb field of the charged particle induces the current in the medium. In virtue of the symmetry of the problem, the density of the total current j^i, which is the sum of the current density of the charged particle and the density of the induced current, is symmetric with respect to the transformations

$$z \to z + \frac{\lambda_0}{2\pi}\varphi_r, \quad \varphi \to \varphi + \varphi_r, \quad x^0 \to x^0 + \frac{\lambda_0}{2\pi\beta_3}\varphi_r. \tag{145}$$

The components of the current density transform as

$$j_3 \to j_3, \quad j_\pm \to j_\pm e^{\pm i\varphi_r}. \tag{146}$$

The mode functions of the vacuum twisted photons (55) change accordingly

$$\begin{aligned} \psi_3(m, k_3, k_\perp) &\to \psi_3(m, k_3, k_\perp) e^{-i[k_0(\lambda_0/2\pi\beta_3) - k_3(\lambda_0/2\pi) - m]\varphi_r}, \\ \psi_\pm(s, m, k_3, k_\perp) &\to \psi_\pm(s, m, k_3, k_\perp) \\ &\quad \times e^{-i[k_0(\lambda_0/2\pi\beta_3) - k_3(\lambda_0/2\pi) - m]\varphi_r} e^{\pm i\varphi_r}. \end{aligned} \tag{147}$$

These relations allow us to rewrite the probability to record a twisted photon produced by the current j^i in such a way that the fulfillment of the selection rules (139) and (140) becomes evident.

Indeed, the probability to record a twisted photon is given by the general formula

$$\begin{aligned} dP(s, m, k_3, k_\perp) &= \bigg| \int d^4x\, e^{-ik_0x^0 + ik_3x_3} \bigg[\frac{1}{2} a_+(s, m, k_3, k_\perp; \mathbf{x}) j_-(x) \\ &\quad + \frac{1}{2} a_-(s, m, k_3, k_\perp; \mathbf{x}) j_+(x) + a_3(m, k_3, k_\perp; \mathbf{x}) j_3(x) \bigg] \bigg|^2 \\ &\quad \times \left(\frac{k_\perp}{2k_0}\right)^3 \frac{dk_3 dk_\perp}{2\pi^2}, \end{aligned} \tag{148}$$

where the renormalized mode functions a_3 and $a_\pm$ have been introduced [see (35) of [81]]. Then we consider the integration over N periods in (148) and neglect the edge effects. Parting the integral over x^0 as

$$\int_0^{N|\lambda_0|} dx^0 \cdots = \sum_{n=1}^{Nr} \int_{|\lambda_0|(n-1)/r|\beta_3|}^{|\lambda_0|n/r|\beta_3|} dx^0 \cdots =: \sum_{n=1}^{Nr} A_n \quad (149)$$

and employing the symmetry properties (146) and (147), we find that

$$A_n = e^{-i[k_0(\lambda_0/2\pi\beta_3)-k_3(\lambda_0/2\pi)-m]\,\mathrm{sgn}(\lambda_0\beta_3)\varphi_r(n-1)} A_1, \quad (150)$$

where

$$A_1 = \int_0^{|\lambda_0|/r|\beta_3|} dx^0 \int d\mathbf{x}\, e^{-ik_0x^0+ik_3x_3} \bigg[\frac{1}{2}a_+(s,m,k_3,k_\perp;\mathbf{x})j_-(x) + \frac{1}{2}a_-(s,m,k_3,k_\perp;\mathbf{x})j_+(x) + a_3(m,k_3,k_\perp;\mathbf{x})j_3(x)\bigg]. \quad (151)$$

As a result, the radiation probability (148) is written as

$$dP(s,m,k_3,k_\perp) = \sum_{n=1}^{Nr} |e^{-i(k_0/q_0\beta_3-k_3/q_0-m)\,\mathrm{sgn}(\lambda_0\beta_3)\varphi_r(n-1)}|^2 \times dP_1(s,m,k_3,k_\perp), \quad (152)$$

where

$$dP_1(s,m,k_3,k_\perp) := |A_1|^2 \left(\frac{k_\perp}{2k_0}\right)^3 \frac{dk_3 dk_\perp}{2\pi^2}. \quad (153)$$

The interference factor

$$I(m,k_3,k_\perp) = \sum_{n=1}^{Nr} |e^{-i(k_0/q_0\beta_3-k_3/q_0-m)\,\mathrm{sgn}(\lambda_0\beta_3)\varphi_r(n-1)}|^2 = \frac{\sin^2(\pi N\delta)}{\sin^2(\pi\delta/r)}, \quad \delta := m - \frac{k_0\lambda_0}{2\pi}(\beta_3^{-1} - n_3), \quad (154)$$

has the same form as in [82]. It possesses the sharp maxima at $\delta = nr$, where $n \in \mathbb{Z}$. Taking into account that permittivity is invariant under rotations (134), the condition $\delta = nr$ leads to the selection rules (139) and (140). At the maxima, the interference factor is equal to

$$I_{\max} = N^2 r^2, \quad (155)$$

whence it is evident that, for sufficiently large N, the edge effects give a negligible contribution to radiation at harmonics (139).

The selection rules (139) and (140) can also be deduced with the aid of the conservation laws of the momentum and total angular momentum applied to the process of transition scattering of the wave of permittivity on the charged particle moving uniformly along a straight line [45]. Let us consider a stationary permittivity tensor invariant with respect to (122). Then the Fourier modes of the wave corresponding to such a permittivity tensor obey the relations

$$k_0 = 0, \quad (m + k_3/q_0)\varphi_r = 2\pi n \Rightarrow$$
$$k_3 = q_0(nr - m), \quad n \in \mathbb{Z}, \quad (156)$$

where m is the projection of the total angular momentum onto the twisted photon detector axis. Hence, for the diagram depicted in Fig. 7, which describes the leading-order contribution to the twisted photon radiation by a charged particle moving along the detector axis, we have

$$M\gamma\beta_3 + k_3 = M\gamma'\beta_3' + k_3', \quad M\gamma = M\gamma' + k_0', \quad m = m', \quad (157)$$

where M is the charged particle mass, γ is its Lorentz factor, and the prime marks the quantities after scattering. The last equality in (157) expresses conservation of projection of the total angular momentum and we suppose that the charged particle does not change its projection of the total angular momentum. Assuming that $k_0'/M\gamma \ll 1$, we have, from (157),

$$k_3 = k_3' - k_0'/\beta_3. \quad (158)$$

Therefore,

$$k_0' = q_0\beta_3 \frac{m - nr}{1 - \beta_3 n_3'}, \quad n_3' := k_3'/k_0'. \quad (159)$$

Defining

$$\bar{n} := \mathrm{sgn}(\lambda_0\beta_3)(m - nr), \quad (160)$$

we arrive at the selection rules (139) and (140) with $q = 1$. The generalization to the case $q \in \mathbb{N}$ is obvious.

The above selection rules were obtained for radiation of twisted photons by one particle or by a sufficiently narrow uniform beam of them in a helical medium. The use of sufficiently long helically microbunched beams allows one to shift the radiation probability distribution over m in accordance with the strong addition rule [54].

VI. CONCLUSION

Let us sum up the results. In Secs. II and III we developed a general quantum theory of radiation of twisted photons by charged particles propagating in an inhomogeneous dispersive medium. The production of twisted photons in homogeneous dispersive media was already studied in [15,16], but the inhomogeneity of a medium was not taken into account. As we have already discussed in the Introduction, one usually needs a source of twisted photons in a vacuum and not in the medium. Therefore, it is relevant to consider the production of twisted photons in inhomogeneous media since the twisted photons can be destroyed by the inhomogeneities of permittivity.

In Sec. III we derived the general formula for the probability to record a twisted photon produced by a classical current in an inhomogeneous dispersive medium. In the case of a homogeneous medium investigated in Sec. IV, we reproduced the known results. In Sec. V we applied the general formula to the description of radiation of twisted photons in several particular configurations.

In Secs. V A and V B we deduced the explicit expressions for the probability to record a twisted photon produced by a charged particle moving with constant velocity and crossing a dielectric plate. The axis of the detector of twisted photons was assumed to be normal to the surface of the dielectric plate. As expected, we found that, in the case when a charged particle moves along the detector axis, all the radiated twisted photons possess a zero projection of the total angular momentum and the probability to record a twisted photon does not depend on the photon helicity. Moreover, we proved that the probability to record a twisted photon produced by an arbitrary number of charged particles moving uniformly along straight lines intersecting at one point of the surface of a thick transparent dielectric plate possesses the reflection symmetry [82]. Such a radiation describes infrared asymptotics of radiation of twisted photons in an arbitrary QED process near a dielectric plate. Of course, for the asymptotics to take place, it is necessary that the wavelength of radiated photons will be much smaller than the typical sizes of the dielectric plate.

In Sec. V C we considered the radiation of twisted photons by a charged particle falling onto an ideally conducting plate. In fact, the radiation in this case is the edge radiation completely described in terms of twisted photons in [82]. As in the case of the edge radiation, the probability to record a twisted photon obeys the reflection symmetry.

In Sec. V D we investigated the radiation produced by charged particles moving along the axis of a twisted photon detector in a helical medium. A typical example of such a medium is a cholesteric liquid crystal (see, e.g., [108]). Using different approaches, we proved that, in this case, the radiation of twisted photons obeys the same selection rules as were found in [82] for the scattering of charged particles on helical targets. We provided a simple explanation of this fact in terms of transition scattering on a helical permittivity wave. These specific properties of helical media can be employed for elaboration of the active medium for coherent amplification of stimulated radiation of twisted photons.

Employing the general formulas [53,54] for the radiation of twisted photons by particle beams, we also described the radiation produced by uniform Gaussian and helically microbunched beams of relativistic charged particles. In particular, we showed explicitly the fulfillment of the strong addition rule [54–61] for the radiation of twisted photons by helical beams of particles falling normally onto a medium surface. Such beams can be used to generate VC and transition radiations with large projections of the total angular momentum.

ACKNOWLEDGMENTS

The work was supported by the Ministry of Science and Higher Education of the Russian Federation, Project No. 3.9594.2017/8.9. The numerical simulations of radiation of twisted photons by beams of particles were performed on TPU Supercomputing Cluster within the framework of Tomsk Polytechnic University Competitiveness Enhancement Program.

APPENDIX: INTERFERENCE FACTORS FOR GAUSSIAN PARTICLE BEAMS

The general formulas for the incoherent, f_{mn}, and coherent, φ_m, interference factors read [54]

$$f_{mn}=\int d\mathbf{b}\,\rho(\mathbf{b})j^*_m(k_\perp\delta_+(\mathbf{b}),k_\perp\delta_-(\mathbf{b}))j_n(k_\perp\delta_+(\mathbf{b}),k_\perp\delta_-(\mathbf{b})),$$
$$\varphi_m=\int d\mathbf{b}\,\rho(\mathbf{b})e^{ik_0\delta^0(\mathbf{b})-ik_3\delta^3(\mathbf{b})}j^*_m(k_\perp\delta_+(\mathbf{b}),k_\perp\delta_-(\mathbf{b})). \quad \text{(A1)}$$

We particularize these formulas to the case when the normal $\boldsymbol{\xi}=(0,0,1)$ to the vacuum-medium interface at the points where the beam of particles enters and exits the medium is parallel to the axis of the twisted photon detector. The particle beam moves initially with the velocity

$$\boldsymbol{\beta}=\beta(\sin\theta,0,\cos\theta). \quad \text{(A2)}$$

The one-particle probability distribution has the form (see Sec. 3.5 of [54])

$$\rho(\mathbf{b}')=\frac{e^{-b_3'^2/2\sigma_3^2}}{\sqrt{2\pi}\sigma_3}\frac{e^{-b'_+b'_-/2\sigma_\perp^2}}{2\pi\sigma_\perp^2}\times{\sum_{k=0}^{\infty}}'\left[\alpha_k\left(\frac{b'_+}{\sigma_\perp}\right)^k e^{2\pi i\chi kb'_3/\delta}+\alpha_{-k}\left(\frac{b'_-}{\sigma_\perp}\right)^k e^{-2\pi i\chi kb'_3/\delta}\right], \quad \text{(A3)}$$

where $\alpha^*_k=\alpha_{-k}$ are given in (74) of [54], the prime to the sum sign says that the term with $k=0$ should be multiplied by 1/2, and

$$b'_1=\cos\theta b_1-\sin\theta b_3,\quad b'_2=b_2,$$
$$b'_3=\sin\theta b_1+\cos\theta b_3. \quad \text{(A4)}$$

Using the general formula (12) of [54], we find

$$\delta_1=\frac{b'_1}{\cos\theta},\quad \delta_2=b'_2,\quad \delta_3=0,\quad \delta^0=\frac{\tan\theta b'_1-b'_3}{\beta}. \quad \text{(A5)}$$

The probability distribution (A3) describes a helically microbunched particle beam with the helix pitch δ and the handedness χ. The case $\delta\to\infty$ corresponds to a uniform Gaussian beam of particles.

It is convenient to change the integration variables in (A1) and pass from $\mathbf{b}$ to $\mathbf{b}'$. The Jacobian of this change equals unity. Having performed such a change of variables, we will not write the primes at $\mathbf{b}'$ to simplify the notation. Let us start with the incoherent interference factor. First, we write

$$j^*_m j_n=i^{m-n}\int_{-\pi}^{\pi}\frac{d\psi_1 d\psi_2}{(2\pi)^2}e^{im\psi_1-in\psi_2}e^{ik_\perp b_2(\sin\psi_1-\sin\psi_2)}\times e^{ik_\perp b_1(\cos\psi_1-\cos\psi_2)/\cos\theta}. \quad \text{(A6)}$$

Due to periodicity of the integrand, the change of variables

$$\frac{\psi_1-\psi_2}{2}\to\psi_1,\quad \frac{\psi_1+\psi_2}{2}\to\psi_2 \quad \text{(A7)}$$

results in

$$\int_{-\pi}^{\pi}\frac{d\psi_1 d\psi_2}{(2\pi)^2}\to\int_{-\pi}^{\pi}\frac{d\psi_1 d\psi_2}{(2\pi)^2}. \quad \text{(A8)}$$

Hence,

$$j^*_m j_n=i^{m-n}\int_{-\pi}^{\pi}\frac{d\psi_1 d\psi_2}{(2\pi)^2}e^{i(n-m)\psi_2+i(n+m)\psi_1}e^{-ik_\perp\sin\psi_1(b_+z^*+b_-z)}, \quad \text{(A9)}$$

where

$$z:=\frac{\sin\psi_2}{\cos\theta}-i\cos\psi_2. \quad \text{(A10)}$$

The integrals over $b_\pm$ in (A1) become Gaussian and can be evaluated with the aid of the relations

$$\int \frac{db_+db_-}{2i} b_+^m e^{-b_+b_-/2+\beta_+b_-+\beta_-b_+} = 2\pi(2\beta_+)^m e^{2\beta_+\beta_-},$$
$$\int \frac{db_+db_-}{2i} b_-^m e^{-b_+b_-/2+\beta_+b_-+\beta_-b_+} = 2\pi(2\beta_-)^m e^{2\beta_+\beta_-}, \tag{A11}$$

where $db_+db_- = 2idb_1db_2$. In addition,

$$\int \frac{db_3}{\sqrt{2\pi}\sigma_3} e^{-b_3^2/2\sigma_3^2 \pm 2\pi i\chi kb_3/\delta} = e^{-2\pi^2k^2\sigma_3^2/\delta^2}. \tag{A12}$$

Using these formulas, we find that the term at α_k is given by the integral

$$i^{m-n}e^{-2\pi^2k^2\sigma_3^2/\delta^2}\int_{-\pi}^{\pi}\frac{d\psi_1d\psi_2}{(2\pi)^2}e^{i(n-m)\psi_2+i(n+m)\psi_1}$$
$$\times(-2ik_\perp\sigma_\perp\sin\psi_1 z)^k e^{-2k_\perp^2\sigma_\perp^2\sin^2\psi_1|z|^2}. \tag{A13}$$

The term at α_{-k} becomes

$$i^{m-n}e^{-2\pi^2k^2\sigma_3^2/\delta^2}\int_{-\pi}^{\pi}\frac{d\psi_1d\psi_2}{(2\pi)^2}e^{i(n-m)\psi_2+i(n+m)\psi_1}$$
$$\times(2ik_\perp\sigma_\perp\sin\psi_1 z^*)^k e^{-2k_\perp^2\sigma_\perp^2\sin^2\psi_1|z|^2}. \tag{A14}$$

One integration in this double integral can be performed. The integral over ψ_1 is reduced to

$$\int_{-\pi}^{\pi}\frac{d\psi}{2\pi}e^{i(n+m)\psi}\sin^k\psi\, e^{-a^2\sin^2\psi} = \left(\frac{i}{2}\right)^k$$
$$\times\begin{cases}\sum_{s=0}^k\frac{(-1)^sk!}{s!(k-s)!}e^{-a^2/2}I_{s+(n+m-k)/2}\left(\frac{a^2}{2}\right), & \frac{n+m-k}{2}\in\mathbb{Z}\\ 0, & \frac{n+m-k}{2}\notin\mathbb{Z}.\end{cases} \tag{A15}$$

As a result, we arrive at

$$f_{mn} = i^{m-n}\int_{-\pi}^{\pi}\frac{d\psi}{2\pi}e^{i(n-m)\psi}{\sum_{k=0}^{\infty}}'(k_\perp\sigma_\perp)^k$$
$$\times e^{-2\pi^2k^2\sigma_3^2/\delta^2}[\alpha_kz^k+\alpha_{-k}z^{*k}]$$
$$\times\sum_{s=0}^k\frac{(-1)^sk!}{s!(k-s)!}e^{-k_\perp^2\sigma_\perp^2|z|^2}I_{s+(n+m-k)/2}(k_\perp^2\sigma_\perp^2|z|^2), \tag{A16}$$

where $I_k(x)$ is the modified Bessel function of the first kind and ψ_2 in the definition of z should be replaced by ψ. The prime to the sum sign reminds us that the term at $k=0$ should be taken with the factor $1/2$ and the terms such that $n+m-k$ is an odd number must be omitted.

Note that when

$$\pi\sigma_3/\delta \gtrsim 1, \tag{A17}$$

the terms with $k\neq 0$ are strongly suppressed, i.e., the incoherent radiation is the same as for a round particle beam [53] and

$$f_{mn}\approx i^{m-n}\int_{-\pi}^{\pi}\frac{d\psi}{2\pi}e^{i(n-m)\psi}$$
$$\times\begin{cases}e^{-k_\perp^2\sigma_\perp^2|z|^2}I_{(n+m)/2}(k_\perp^2\sigma_\perp^2|z|^2), & \frac{n+m}{2}\in\mathbb{Z}\\ 0, & \frac{n+m}{2}\notin\mathbb{Z}.\end{cases} \tag{A18}$$

For $\theta\ll 1$ or $\pi-\theta\ll 1$, the incoherent interference factor (A16) reduces to the corresponding factor found in [54] in the case of a forward radiation.

Now we turn to the coherent interference factor. Upon substitution

$$e^{ik_0\delta^0(\mathbf{b})-ik_3\delta^3(\mathbf{b})}j_m^*(k_\perp\delta_+(\mathbf{b}),k_\perp\delta_-(\mathbf{b}))$$
$$= i^m\int_{-\pi}^{\pi}\frac{d\psi}{2\pi}e^{-im\psi}e^{ik_0(\tan\theta b_1-b_3)/\beta}e^{-ik_\perp(b_2\sin\psi+b_1\cos\psi/\cos\theta)}$$
$$= e^{-ik_0b_3/\beta}\int_{-\pi}^{\pi}\frac{d\psi}{2\pi}e^{-im\psi}e^{ik_\perp(b_+\tilde{z}^*+b_-\tilde{z})/2}, \tag{A19}$$

where

$$\tilde{z} := \frac{\tan\theta}{\beta n_\perp}+\frac{\sin\psi}{\cos\theta}-i\cos\psi, \tag{A20}$$

the integrals over $\mathbf{b}$ in (A1) become Gaussian. Then we have

$$\int\frac{db_3}{\sqrt{2\pi}\sigma_3}e^{-b_3^2/2\sigma_3^2\pm 2\pi i\chi kb_3/\delta-ik_0b_3/\beta} = e^{-\sigma_3^2(k_0/\beta\mp 2\pi\chi k/\delta)^2/2}. \tag{A21}$$

The integrals over $b_\pm$ can be performed by the use of the formulas (A11). After a little algebra, we arrive at

$$\varphi_m = \int_{-\pi}^{\pi}\frac{d\psi}{2\pi}e^{-im\psi}e^{-k_\perp^2\sigma_\perp^2|\tilde{z}|^2/2}\left({\sum_{k=0}^{\infty}}'ik_\perp\sigma_\perp\right)^k$$
$$\times\left[\alpha_k\tilde{z}^ke^{-\sigma_3^2(k_0/\beta-2\pi\chi k/\delta)^2/2}\right.$$
$$\left.+\alpha_{-k}\tilde{z}^{*k}e^{-\sigma_3^2(k_0/\beta+2\pi\chi k/\delta)^2/2}\right], \tag{A22}$$

where the prime to the sum sign says that the term with $k=0$ should be taken with the factor $1/2$. The integral over ψ can be represented as the series of products of the modified Bessel functions of the first kind, but this representation does not facilitate the evaluation of (A22). Therefore, we do not write it here.

Note that when the condition (A17) is satisfied, the coherent interference factor and the probability of coherent radiation are concentrated near the coherent harmonics

$$k_0 = 2\pi\chi n\beta/\delta,\quad \chi n>0,\ n\in\mathbb{Z}. \tag{A23}$$

The radiation probability at these harmonics is proportional to $|\alpha_n|^2$. However, in general, the strong addition rule [54] is not fulfilled. Only when

$$\Delta\theta\ll 1,\quad \frac{|\tan\theta|}{\beta n_\perp}\ll 1,\quad k_\perp^2\sigma_\perp^2\frac{|\tan\theta|}{\beta n_\perp}\ll 1, \tag{A24}$$

where $\Delta\theta=\theta$ or $\Delta\theta=\pi-\theta$, does the strong addition rule hold. This case of course corresponds to the forward radiation.

[1] B. A. Knyazev and V. G. Serbo, Beams of photons with nonzero projections of orbital angular momenta: New results, Phys. Usp. **61**, 449 (2018).

[2] M. J. Padgett, Orbital angular momentum 25 years on, Opt. Express **25**, 11267 (2017).

[3] H. Rubinsztein-Dunlop *et al.*, Roadmap on structured light, J. Opt. **19**, 013001 (2017).

[4] *The Angular Momentum of Light*, edited by D. L. Andrews and M. Babiker (Cambridge University Press, New York, 2013).

[5] *Twisted Photons*, edited by J. P. Torres and L. Torner (Wiley-VCH, Weinheim, 2011).

[6] *Structured Light and its Applications*, edited by D. L. Andrews (Academic, Amsterdam, 2008).

[7] D. Voloschenko and O. D. Lavrentovich, Optical vortices generated by dislocations in a cholesteric liquid crystal, Opt. Lett. **25**, 317 (2000).

[8] L. Marrucci, C. Manzo, and D. Paparo, Optical Spin-to-Orbital Angular Momentum Conversion in Inhomogeneous Anisotropic Media, Phys. Rev. Lett. **96**, 163905 (2006).

[9] R. Barboza, U. Bortolozzo, G. Assanto, E. Vidal-Henriquez, M. G. Clerc, and S. Residori, Vortex Induction via Anisotropy Stabilized Light-Matter Interaction, Phys. Rev. Lett. **109**, 143901 (2012).

[10] C. Loussert, U. Delabre, and E. Brasselet, Manipulating the Orbital Angular Momentum of Light at the Micron Scale With Nematic Disclinations in a Liquid Crystal Film, Phys. Rev. Lett. **111**, 037802 (2013).

[11] R. Barboza, U. Bortolozzo, G. Assanto, E. Vidal-Henriquez, M. G. Clerc, and S. Residori, Harnessing Optical Vortex Lattices in Nematic Liquid Crystals, Phys. Rev. Lett. **111**, 093902 (2013).

[12] L. K. Migara, C.-M. Lee, K. Kwak, H. Lee, and J.-K. Song, Tunable optical vortex arrays using spontaneous periodic pattern formation in nematic liquid crystal cells, Curr. Appl. Phys. **18**, 819 (2018).

[13] Y. Liu, Q. Guo, H. Liu, C. Liu, K. Song, B. Yang, Q. Hou, X. Zhao, S. Zhang, and M. Navarro-Cía, Circular-Polarization-Selective Transmission Induced by Spin-Orbit Coupling in a Helical Tape Waveguide, Phys. Rev. Appl. **9**, 054033 (2018).

[14] N. Kravets, N. Podoliak, M. Kaczmarek, and E. Brasselet, Self-induced liquid crystal q-plate by photoelectric interface activation, Appl. Phys. Lett. **114**, 061101 (2019).

[15] I. Kaminer, M. Mutzafi, A. Levy, G. Harari, H. H. Sheinfux, S. Skirlo, J. Nemirovsky, J. D. Joannopoulos, M. Segev, and M. Soljačić, Quantum Čerenkov Radiation: Spectral Cutoffs and the Role of Spin and Orbital Angular Momentum, Phys. Rev. X **6**, 011006 (2016).

[16] I. P. Ivanov, V. G. Serbo, and V. A. Zaytsev, Quantum calculation of the Vavilov-Cherenkov radiation by twisted electrons, Phys. Rev. A **93**, 053825 (2016).

[17] J. Leach, M. J. Padgett, S. M. Barnett, S. Franke-Arnold, and J. Courtial, Measuring the Orbital Angular Momentum of a Single Photon, Phys. Rev. Lett. **88**, 257901 (2002).

[18] G. C. G. Berkhout, M. P. J. Lavery, J. Courtial, M. W. Beijersbergen, and M. J. Padgett, Efficient Sorting of Orbital Angular Momentum States of Light, Phys. Rev. Lett. **105**, 153601 (2010).

[19] T. Su, R. P. Scott, S. S. Djordjevic, N. K. Fontaine, D. J. Geisler, X. Cai, and S. J. B. Yoo, Demonstration of free space coherent optical communication using integrated silicon photonic orbital angular momentum devices, Opt. Express **20**, 9396 (2012).

[20] M. P. J. Lavery, J. Courtial, and M. J. Padgett, in *The Angular Momentum of Light*, edited by D. L. Andrews and M. Babiker (Cambridge University Press, New York, 2013).

[21] G. Ruffato, M. Girardi, M. Massari, E. Mafakheri, B. Sephton, P. Capaldo, A. Forbes, and F. Romanato, A compact diffractive sorter for high-resolution demultiplexing of orbital angular momentum beams, Sci. Rep. **8**, 10248 (2018).

[22] B. Paroli, M. Siano, L. Teruzzi, and M. A. C. Potenza, Single-shot measurement of phase and topological properties of orbital angular momentum radiation through asymmetric lateral coherence, Phys. Rev. Accel. Beams **22**, 032901 (2019).

[23] O. Matula, A. G. Hayrapetyan, V. G. Serbo, A. Surzhykov, and S. Fritzsche, Atomic ionization of hydrogen-like ions by twisted photons: Angular distribution of emitted electrons, J. Phys. B **46**, 205002 (2013).

[24] A. A. Peshkov, S. Fritzsche, and A. Surzhykov, Ionization of H_2^+ molecular ions by twisted Bessel light, Phys. Rev. A **92**, 043415 (2015).

[25] A. Afanasev, V. G. Serbo, and M. Solyanik, Radiative capture of cold neutrons by protons and deuteron photodisintegration with twisted beams, J. Phys. G **45**, 055102 (2018).

[26] M. Solyanik-Gorgone, A. Afanasev, C. E. Carlson, C. T. Schmiegelow, and F. Schmidt-Kaler, Excitation of E1-forbidden atomic transitions with electric, magnetic, or mixed multipolarity in light fields carrying orbital and spin angular momentum [Invited], J. Opt. Soc. Am. B **36**, 565 (2019).

[27] P. A. Cherenkov, Visible light from clear liquids under the action of γ-radiation, Dokl. Akad. Nauk SSSR **2**, 451 (1934).

[28] S. I. Vavilov, On possible causes of dark blue γ-radiation in liquids, Dokl. Akad. Nauk SSSR **2**, 457 (1934).

[29] I. E. Tamm and I. M. Frank, Coherent radiation of a fast electron in a medium, Dokl. Akad. Nauk SSSR **14**, 107 (1937).

[30] V. L. Ginzburg, On quantum theory of radiation of an electron uniformly moving in a medium, Zh. Eksp. Teor. Fiz. **10**, 589 (1940).

[31] A. A. Sokolov, Quantum theory of radiation of elementary particles, Dokl. Akad. Nauk SSSR **28**, 415 (1940).

[32] A. A. Sokolov and I. M. Loskutov, Polarization of Cerenkov radiation, Zh. Eksp. Teor. Fiz. **32**, 630 (1957) [Sov. Phys. JETP **5**, 523 (1957)].

[33] M. I. Riazanov, Phenomenological study of the effect of non-conducting medium in quantum electrodynamics, Zh. Eksp. Teor. Fiz. **32**, 1244 (1957) [Sov. Phys. JETP **5**, 1013 (1957)].

[34] A. A. Sokolov and I. M. Loskutov, Cerenkov radiation of longitudinally polarized electrons, Zh. Eksp. Teor. Fiz. **34**, 1022 (1958) [Sov. Phys. JETP **7**, 706 (1958)].

[35] M. I. Riazanov, Radiative corrections to Compton scattering taking into account polarization of the surrounding medium, Zh. Eksp. Teor. Fiz. **34**, 1258 (1958) [Sov. Phys. JETP **7**, 869 (1958)].

[36] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Prentice Hall, Englewood Cliffs, 1963).

[37] A. I. Alekseev and Y. P. Nikitin, Quantization of the electromagnetic field in a dispersive medium, Zh. Eksp. Teor. Fiz. **50**, 915 (1966) [Sov. Phys. JETP **23**, 608 (1966)].

[38] M. L. Ter-Mikaelian, *High-Energy Electromagnetic Processes in Condensed Media* (Wiley Interscience, New York, 1972).

[39] J. Schwinger and W.-Y. Tsai, Classical and quantum theory of synergic synchrotron-Čerenkov radiation, Ann. Phys. (NY) **96**, 303 (1976).

[40] V. L. Ginzburg, *Theoretical Physics and Astrophysics* (Pergamon, London, 1979).

[41] I. M. Frank, Vavilov-Cherenkov radiation for electric and magnetic multipoles, Sov. Phys. Usp. **27**, 772 (1984).

[42] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1984).

[43] M. I. Ryazanov, *Electrodynamics of a Condensed Medium* (Nauka, Moscow, 1984).

[44] V. A. Bazylev and N. K. Zhevago, *Radiation from Fast Particles in a Medium and External Fields* (Nauka, Moscow, 1987).

[45] V. L. Ginzburg and V. N. Tsytovich, *Transition Radiation and Transition Scattering* (Hilger, Bristol, 1990).

[46] R. J. Glauber and M. Lewenstein, Quantum optics of dielectric media, Phys. Rev. A **43**, 467 (1991).

[47] B. D. Orisa, Cerenkov radiation in anisotropic media by the methods of quantum electrodynamics, Int. J. Theor. Phys. **34**, 545 (1995).

[48] G. N. Afanasyev, *Vavilov-Cherenkov and Synchrotron Radiation: Foundations and Applications* (Kluwer Academic, Dordrecht, 2005).

[49] I. B. Khriplovich, Čerenkov radiation of a spinning particle, Zh. Eksp. Teor. Fiz. **135**, 51 (2009) [J. Exp. Theor. Phys. **108**, 43 (2009)].

[50] K. Tuchin, Chiral Cherenkov and chiral transition radiation in anisotropic matter, Phys. Rev. D **98**, 114026 (2018).

[51] V. E. Pafomov, Radiation of a charged particle in the presence of interfaces, Proc. P. N. Lebedev Phys. Inst. **44**, 28 (1971).

[52] V. G. Baryshevsky and I. M. Frank, Light emission by an oscillator moving through a refractive plate, Yad. Fiz. **36**, 1442 (1982) [in Russian].

[53] O. V. Bogdanov and P. O. Kazinski, Probability of radiation of twisted photons by axially symmetric bunches of particles, Eur. Phys. J. Plus (to be published).

[54] O. V. Bogdanov, P. O. Kazinski, and G. Y. Lazarenko, Probability of radiation of twisted photons by cold relativistic particle bunches, arXiv:1905.07688.

[55] E. Hemsing and J. B. Rosenzweig, Coherent transition radiation from a helically microbunched electron beam, J. Appl. Phys. **105**, 093101 (2009).

[56] E. Hemsing, A. Marinelli, and J. B. Rosenzweig, Generating Optical Orbital Angular Momentum in a High-Gain Free-Electron Laser at the First Harmonic, Phys. Rev. Lett. **106**, 164803 (2011).

[57] E. Hemsing and A. Marinelli, Echo-Enabled X-Ray Vortex Generation, Phys. Rev. Lett. **109**, 224801 (2012).

[58] E. Hemsing, A. Knyazik, F. O'Shea, A. Marinelli, P. Musumeci, O. Williams, S. Tochitsky, and J. B. Rosenzweig, Experimental observation of helical microbunching of a relativistic electron beam, Appl. Phys. Lett. **100**, 091110 (2012).

[59] E. Hemsing, A. Knyazik, M. Dunning, D. Xiang, A. Marinelli, C. Hast, and J. B. Rosenzweig, Coherent optical vortices from relativistic electron beams, Nat. Phys. **9**, 549 (2013).

[60] P. R. Ribič, D. Gauthier, and G. De Ninno, Generation of Coherent Extreme-Ultraviolet Radiation Carrying Orbital Angular Momentum, Phys. Rev. Lett. **112**, 203602 (2014).

[61] E. Hemsing, G. Stupakov, D. Xiang, and A. Zholents, Beam by design: Laser manipulation of electrons in modern accelerators, Rev. Mod. Phys. **86**, 897 (2014).

[62] C. Liu *et al.*, Generation of gamma-ray beam with orbital angular momentum in the QED regime, Phys. Plasmas **23**, 093120 (2016).

[63] X.-L. Zhu, T.-P. Yu, M. Chen, S.-M. Weng, and Z.-M. Sheng, Generation of GeV positron and γ-photon beams with controllable angular momentum by intense lasers, New J. Phys. **20**, 083013 (2018).

[64] L. B. Ju, C. T. Zhou, K. Jiang, T. W. Huang, H. Zhang, T. X. Cai, J. M. Cao, B. Qiao, and S. C. Ruan, Manipulating the topological structure of ultrarelativistic electron beams using Laguerre-Gaussian laser pulse, New J. Phys. **20**, 063004 (2018).

[65] V. M. Arutyunyan and S. G. Oganesyan, The stimulated Cherenkov effect, Phys. Usp. **37**, 1005 (1994).

[66] M. V. Kuzelev and A. A. Rukhadze, Spontaneous and stimulated emission induced by an electron, electron bunch, and electron beam in a plasma, Phys. Usp. **51**, 989 (2008).

[67] W.-X. Liu, W.-H. Huang, Y.-C. Du, L.-X. Yan, D. Wu, and C.-X. Tang, Terahertz coherent transition radiation based on an ultrashort electron bunching beam, Chin. Phys. B **20**, 074102 (2011).

[68] L. A. Gevorgian, K. A. Ispirian, and A. H. Shamamian, Crystalline undulator radiation of microbunched beams taking into account the medium polarization, Nucl. Instrum. Methods Phys. Res. Sect. B **309**, 63 (2013).

[69] T. Yu. Alekhina, A. V. Tyukhtin, and S. N. Galyamin, Radiation of a bunch in a waveguide with a semibounded anisotropic dielectric, Phys. Rev. Accel. Beams **22**, 041301 (2019).

[70] A. Curcio, M. Bergamaschi, R. Corsini, D. Gamba, W. Farabolini, T. Lefevre, S. Mazzoni, V. Dolci, M. Petrarca, and S. Lupi, Beam-based sub-THz source at the CERN linac electron accelerator for research facility, Phys. Rev. Accel. Beams **22**, 020402 (2019).

[71] H. Larocque, F. Bouchard, V. Grillo, A. Sit, S. Frabboni, R. E. Dunin-Borkowski, M. J. Padgett, R. W. Boyd, and E. Karimi, Nondestructive Measurement of Orbital Angular Momentum for an Electron Beam, Phys. Rev. Lett. **117**, 154801 (2016).

[72] R. Barboza, U. Bortolozzo, M. G. Clerc, S. Residori, and E. Vidal-Henriquez, Optical vortex induction via light-matter interaction in liquid-crystal media, Adv. Opt. Photon. **7**, 635 (2015).

[73] S. Sasaki and I. McNulty, Proposal for Generating Brilliant X-Ray Beams Carrying Orbital Angular Momentum, Phys. Rev. Lett. **100**, 124801 (2008).

[74] J. Bahrdt, K. Holldack, P. Kuske, R. Müller, M. Scheer, and P. Schmid, First Observation of Photons Carrying Orbital Angular Momentum in Undulator Radiation, Phys. Rev. Lett. **111**, 034801 (2013).

[75] E. Hemsing, M. Dunning, C. Hast, T. Raubenheimer, and D. Xiang, First Characterization of Coherent Optical Vortices from Harmonic Undulator Radiation, Phys. Rev. Lett. **113**, 134803 (2014).

[76] P. R. Ribič *et al.*, Extreme-Ultraviolet Vortices from a Free-Electron Laser, Phys. Rev. X **7**, 031036 (2017).

[77] M. Katoh, M. Fujimoto, H. Kawaguchi, K. Tsuchiya, K. Ohmi, T. Kaneyasu, Y. Taira, M. Hosaka, A. Mochihashi, and Y. Takashima, Angular Momentum of Twisted Radiation from

an Electron in Spiral Motion, Phys. Rev. Lett. **118**, 094801 (2017).
[78] M. Katoh *et al.*, Helical phase structure of radiation from an electron in circular motion, Sci. Rep. **7**, 6130 (2017).
[79] Y. Taira, T. Hayakawa, and M. Katoh, Gamma-ray vortices from nonlinear inverse Thomson scattering of circularly polarized light, Sci. Rep. **7**, 5018 (2017).
[80] Y. Taira and M. Katoh, Generation of optical vortices by nonlinear inverse Thomson scattering at arbitrary angle interactions, Astrophys. J. **860**, 45 (2018).
[81] O. V. Bogdanov, P. O. Kazinski, and G. Y. Lazarenko, Probability of radiation of twisted photons by classical currents, Phys. Rev. A **97**, 033837 (2018).
[82] O. V. Bogdanov, P. O. Kazinski, and G. Y. Lazarenko, Probability of radiation of twisted photons in the infrared domain, Ann. Phys. (NY) **406**, 114 (2019).
[83] O. V. Bogdanov, P. O. Kazinski, and G. Y. Lazarenko, Semiclassical probability of radiation of twisted photons in the ultrarelativistic limit, Phys. Rev. D **99**, 116016 (2019).
[84] A. Afanasev and A. Mikhailichenko, On generation of photons carrying orbital angular momentum in the helical undulator, arXiv:1109.1603.
[85] V. A. Bordovitsyn, O. A. Konstantinova, and E. A. Nemchenko, Angular momentum of synchrotron radiation, Russ. Phys. J. **55**, 44 (2012).
[86] V. Epp and U. Guselnikova, Angular momentum of radiation from a charge in circular and spiral motion, Phys. Lett. A **383**, 2668 (2019).
[87] U. D. Jentschura and V. G. Serbo, Generation of High-Energy Photons with Large Orbital Angular Momentum by Compton Backscattering, Phys. Rev. Lett. **106**, 013001 (2011).
[88] U. D. Jentschura and V. G. Serbo, Compton upconversion of twisted photons: Backscattering of particles with non-planar wave functions, Eur. Phys. J. C **71**, 1571 (2011).
[89] S. V. Abdrashitov, O. V. Bogdanov, P. O. Kazinski, and T. A. Tukhfatullin, Orbital angular momentum of channeling radiation from relativistic electrons in thin Si crystal, Phys. Lett. A **382**, 3141 (2018).
[90] V. Epp, J. Janz, and M. Zotova, Angular momentum of radiation at axial channeling, Nucl. Instrum. Methods Phys. Res. Sect. B **436**, 78 (2018).
[91] A. B. Migdal, O. A. Markin, and I. I. Mishustin, The pion spectrum in nuclear matter and pion condensation, Zh. Eksp. Teor. Fiz. **66**, 443 (1974) [Sov. Phys. JETP **39**, 212 (1974)].
[92] D. Fursaev and D. Vassilevich, *Operators, Geometry and Quanta: Methods of Spectral Geometry in Quantum Field Theory* (Springer, Heidelberg, 2011).
[93] I. S. Kalinichenko and P. O. Kazinski, High-temperature expansion of the one-loop effective action induced by scalar and Dirac particles, Eur. Phys. J. C **77**, 880 (2017).
[94] S. Weinberg, *The Quantum Theory of Fields, Vol. 1: Foundations* (Cambridge University Press, Cambridge, 1996).
[95] S. Weinberg, Infrared photons and gravitons, Phys. Rev. **140**, B516 (1965).
[96] K. Gottfried and T.-M. Yan, *Quantum Mechanics: Fundamentals* (Springer, New York, 2003).
[97] R. Jáuregui and S. Hacyan, Quantum-mechanical properties of Bessel beams, Phys. Rev. A **71**, 033411 (2005).
[98] I. Bialynicki-Birula and Z. Bialynicka-Birula, Beams of electromagnetic radiation carrying angular momentum: The Riemann-Silberstein vector and the classical-quantum correspondence, Opt. Commun. **264**, 342 (2006).
[99] I. P. Ivanov, Colliding particles carrying nonzero orbital angular momentum, Phys. Rev. D **83**, 093001 (2011).
[100] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, Oxford, 1980), Pt. 2.
[101] V. M. Grichine and S. S. Sadilov, Radiation energy loss of an accelerated charge in an absorbing medium, Phys. Lett. B **559**, 26 (2003).
[102] R. J. Glauber, Coherent and incoherent states of the radiation field, Phys. Rev. **131**, 2766 (1963).
[103] V. N. Baier, V. M. Katkov, and V. S. Fadin, *Radiation from Relativistic Electrons* (Atomizdat, Moscow, 1973).
[104] I. P. Ivanov and D. V. Karlovets, Detecting Transition Radiation from a Magnetic Moment, Phys. Rev. Lett. **110**, 264801 (2013).
[105] I. P. Ivanov and D. V. Karlovets, Polarization radiation of vortex electrons with large orbital angular momentum, Phys. Rev. A **88**, 043840 (2013).
[106] M. N. Polyanskiy, http://refractiveindex.info.
[107] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon, Oxford, 1991).
[108] P. G. de Gennes and J. Prost, *The Physics of Liquid Crystals* (Clarendon, Oxford, 1993).
[109] E. Pisanty, L. Rego, J. San Román, A. Picón, K. M. Dorney, H. C. Kapteyn, M. M. Murnane, L. Plaja, M. Lewenstein, and C. Hernández-García, Conservation of Torus-Knot Angular Momentum in High-Order Harmonic Generation, Phys. Rev. Lett. **122**, 203201 (2019).